Freie Universität Berlin
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Master Thesis

The Leray-α Model of Turbulence

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Declaration: Herewith I declare that this thesis is the result of my independent work. All sources and auxiliary materials used by me in this thesis are cited completely.

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1 Introduction

Flows that are turbulent occur in many situations in nature. The simulation of turbulent flows is a very active area of research. Nevertheless, turbulence is not a mathematically defined term.

All incompressible, viscous flows are governed by the Navier-Stokes equations. The structure of the equations does not change when describing turbulent or non-turbulent flow. One way to distinguish these two types of flow was discovered by the physicist Osborne Reynolds who, after performing a series of experiments, extracted from the data of the flows he experimented on a dimensionless number, the Reynolds number, that for each type of flow experiment can be used to mark the onset of turbulence. This number can be interpreted as the ratio of inertial forces to viscous forces and the higher this number is, the more likely it is for the flow it describes to be turbulent. This interpretation of turbulence is compatible with the interpretation of a turbulent flow as being one where the nonlinearity in the Navier-Stokes equations is dominant.

It seems easier to describe non-turbulent flow. This is laminar flow, which can be described as flowing in layers that are not mixing. Generally, one pictures a turbulent flow as consisting of lots of vortices of different sizes, as for example in a tornado or in smoke rising from a burning cigarette. A large range for the size of the structures that make up the flow is characteristic for turbulent flows and herein lies the difficulty when representing them numerically. Generally, if a grid has enough grid points to accurately represent the smallest flow structures of turbulent flows, then this number of grid points will be too large for direct numerical simulation (DNS). Due to time and memory limitations, it is simply not feasible. From a very simplified numerical point of view, one could then interpret turbulence simply as the occurrence of structures in the flow that is to be simulated that are too small as to be represented on the given grid.

One way to deal with structures that are too small to be represented on a given grid is to introduce a turbulence model, or subgrid model. The goal is to model, based on physical considerations, the influence of the small scales on the large scales without actually representing the small scales. This method is called Large Eddy Simulation. One disadvantage of this method seems to be the fact that different subgrid models lead to widely different results, making it difficult to choose the right result.

This work concerns itself with the properties of the Leray-α model of turbulence. It was introduced in 2005 by Cheskidov et al. (2005) and it traces back to the method for proving the existence of weak solutions of the Navier-Stokes equations used by Leray (1934).

In the first section, the Navier-Stokes equations that model incompressible flows will be derived. Whether or not a flow is turbulent cannot be seen from the structure of the equations themselves. This information is encoded in the Reynolds number, which will also be introduced in the first section. In the second chapter the Leray-α model and its origins will be presented. A first reason why this model seems more suitable for turbulence modeling then previous models is discussed: an upper bound for the dimension of its attractor is much lower than would be expected for 3D models. A derivation of this upper bound following Cheskidov et al. (2005) is presented.

The third chapter consists of a numerical analysis of the Leray-α model discretized by the Crank-Nicolson scheme using Q_2/P_1^{disc} finite elements and following Layton et al. (2008).

The last chapter is concerned with actual turbulence modeling. In it, following Geurts and Holm (2003), the Leray-α model is recast as a Large Eddy Simulation, which actually implies a subgrid model. Finally, the results of numerical simulations of a turbulent channel flow at $Re_\tau = 180$ performed with the code MooNMD from John and Matthies (2004) are presented.

2 Introduction to the Navier-Stokes Equations

In this section we give a concise derivation of the Navier-Stokes equations for incompressible viscous fluids. It mostly follows Landau and Lifshitz's Fluid Mechanics course, see Landau and Lifshitz (1959). After that the Reynolds number will be introduced.

2.1 A Brief Derivation of the Navier-Stokes Equations

Let V_0 be a small control volume in the body under consideration Ω. Let $\rho = \rho(x, y, z, t)$ denote the density, $p = p(x, y, z, t)$ the pressure and $u = u(x, y, z, t)$ the velocity of the fluid at point $x = (x, y, z)$ in space and t in time. Let $n(x, y, z)$ be the outer unit normal to V_0. Then, the mass of fluid in our control volume at time $t = t_1$ is $\int_{V_0} \rho(x, t_1)\, dx$ and the mass of fluid flowing through the boundary at time $t = t_1$ is $\int_{\partial V_0} (\rho u \cdot n)(x, t_1)\, dx$. As the difference in mass between two distinct points in time t_1 and t_2 must equal the sum of flow over the boundary, we have, using the Gaussian formula on the boundary integral,

$$\int_{V_0} \rho(x, t_2) - \rho(x, t_1)\, dx = -\int_{t_1}^{t_2} \int_{V_0} \nabla \cdot (\rho u(x, t))\, dx dt.$$

Assuming $\rho \in C^1(\Omega \times [0, \infty))$, we can divide by $t_2 - t_1$ and, letting $t_2 \to t_1$ and using the fundamental theorem of calculus, we get

$$\int_{V_0} \partial_t \rho(x, t_1)\, dx = -\int_{V_0} \nabla \cdot (\rho u(x, t_1))\, dx.$$

As our control volume $V_0 \subset\subset \Omega$ was chosen arbitrarily and so were $t_1, t_2 \in [0, \infty)$, we get that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot u + u \cdot \nabla \rho = 0 \tag{1}$$

pointwise. This is called *equation of continuity* and it describes conservation of mass.

The pressure being denoted by p, an expression for the outside forces acting on our test volume V_0 is $-\int_{\partial V_0} pn\, dS$. Considering that p is a scalar valued function, we use the Gaussian formula on each component of the vector pn to get

$$-\int_{\partial V_0} pn\, dS = -\int_{V_0} \nabla p\, dV.$$

We can interpret this to mean that the fluid surrounding the control volume exerts on it a force $-\nabla p\, dV$ and considering V_0 to be a unit volume, this force is $-\nabla p$. Now, this total outside force is responsible for a change in movement (i.e. acceleration) inside the unit control volume, and considering ρ to be the mass per unit volume, we have

$$\rho \frac{du}{dt} = -\nabla p, \tag{2}$$

where $\frac{du}{dt}$ represents the acceleration of a fluid particle moving in the volume. What we need, however, is an expression of $\frac{du}{dt}$ in terms of fixed points in space. To this end, we consider u along the parametrized curve $c(t) = (x(t), y(t), z(t), t)$. Then

$$\begin{aligned}
\frac{du}{dt} &= u'(c(t)) \cdot c'(t) \\
&= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial t}\right)|_{c(t)} \cdot \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}, 1\right)|_t \\
&= (u \cdot \nabla u)|_{c(t)} + \frac{\partial u}{\partial t}|_{c(t)}.
\end{aligned}$$

Plugging this into our above equation (2), we get

$$(u \cdot \nabla u) + \frac{\partial u}{\partial t} = -\frac{1}{\rho}\nabla p. \tag{3}$$

This is *Euler´s equation* or *equation of motion.*
Using Euler´s equation (3), we can now get an expression for the change of momentum in our fluid. Momentum is defined as mass times velocity, so in our control unit volume it would be ρu. The rate of change of momentum in componentwise notation then is

$$\frac{\partial}{\partial t}(\rho u_i) = \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t}. \tag{4}$$

Using the same notation, the Euler's equation (3) and the continuity equation (1) become

$$\begin{aligned}
\frac{\partial u_i}{\partial t} &= -\sum_k u_k\frac{\partial u_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \qquad \text{Euler,} \\
\frac{\partial \rho}{\partial t} &= -\sum_k \frac{\partial\,(\rho u_k)}{\partial x_k} \qquad \text{Continuity.}
\end{aligned} \tag{5}$$

Plugging the equations (5) into the equation for the rate of change of momentum (4), we get

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial p}{\partial x_i} - \sum_k \frac{\partial(\rho u_i u_k)}{\partial x_k} = -\sum_k \frac{\partial}{\partial x_k}\underbrace{(\delta_{ik}p + \rho u_i u_k)}_{\Pi_{ik}} = -\sum_k \frac{\partial \Pi_{ik}}{\partial x_k}. \tag{6}$$

Π_{ik} is called the *momentum flux tensor*. It represents the transfer of momentum due to the mechanical transport of fluid particles and the pressure acting on the volume.
Up to this point, we have not concerned ourselves with any distinguishing properties our fluid might have. As the introduction says, it is supposed to be incompressible and viscous. Incompressibility means that there is no expansion or compression in the fluid, meaning that the density ρ is basically constant. This assumption changes nothing in Euler´s equation but it considerably simplifies the equation of continuity.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \qquad \text{then becomes} \qquad \nabla \cdot u = 0, \tag{7}$$

meaning the velocity in an incompressible fluid is divergence-free.
Viscosity is the term for internal friction. It depends on the size and shape of the particles

comprising the fluid and the attraction between them. Honey, for example, has a higher viscosity than water. This will obviously change the transfer of momentum in our fluid and therefore the momentum flux density tensor. Therefore, when including viscosity in our model, we will have to change the equation of motion for our model.
We start by adding a term $-\sigma_{ik}$ to the momentum flux density tensor. The tensor $-\sigma_{ik}$ is supposed to model internal friction. This occurs only when fluid particles move with different velocities. Therefore, $-\sigma_{ik}$ must depend on the spatial derivatives of u. There can be no term in $-\sigma_{ik}$ without these derivatives as $-\sigma_{ik}$ must vanish for constant u. So $-\sigma_{ik}$ will be a linear combination of these derivatives. It must also vanish in the case of uniform rotation. Let ω be the angular velocity and let r be the displacement vector. Then, $u = \omega \times r$. Consider as an example uniform rotation in the x-y-plane:

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \qquad r = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \quad \Rightarrow \quad u = \begin{pmatrix} -y\omega \\ x\omega \\ 0 \end{pmatrix}.$$

Then, $\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} = 0$. Seeing this, we make the ansatz

$$\sigma_{ik} = a \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) + b \sum_l \frac{\partial u_l}{\partial x_l} \delta_{ik},$$

where a and b are functions independent of u. It turns out to be convenient to write this in the form

$$\sigma_{ik} = \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \sum_l \frac{\partial u_l}{\partial x_l} \right) + \xi \delta_{ik} \sum_l \frac{\partial u_l}{\partial x_l},$$

where $\eta > 0$ and $\xi > 0$ are called *coefficients of viscosity.*
Now we plug the new Π_{ik} back into the expression for the rate of change of momentum (6):

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho u_i) &= -\sum_k \frac{\partial \Pi_{ik}}{\partial x_k} = -\sum_k \frac{\partial}{\partial x_k}(\delta_{ik} p + \rho u_i u_k - \sigma_{ik}) \\
&= -\frac{\partial p}{\partial x_i} - \rho \sum_k u_k \frac{\partial u_i}{\partial x_k} - u_i \sum_k \frac{\partial(\rho u_k)}{\partial x_k} + \sum_k \frac{\partial}{\partial x_k} \sigma_{ik} \\
&= \underbrace{\rho \left(-\sum_k u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \right)}_{\text{former Euler}} - u_i \underbrace{\nabla \cdot (\rho u)}_{= -\frac{\partial \rho}{\partial t}} + \sum_k \frac{\partial}{\partial x_k} \sigma_{ik} \\
&\overset{!}{=} \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t},
\end{aligned}$$

which means we have to have

$$\frac{\partial u_i}{\partial t} = -\sum_k u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \sum_k \frac{\partial}{\partial x_k} \sigma_{ik}$$

as our new equation of motion. If we now expand the derivative of σ, we get

$$\begin{aligned}\sum_k \frac{\partial}{\partial x_k}\sigma_{ik} &= \eta\sum_k \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \eta\frac{\partial}{\partial x_i}\sum_k \frac{\partial u_k}{\partial x_k} - \frac{2}{3}\eta\sum_k \frac{\partial u_k}{\partial x_k} + \xi\frac{\partial}{\partial x_i}\sum_k \frac{\partial u_k}{\partial x_k}\\ &= \eta\sum_k \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \left(\frac{1}{3}\eta + \xi\right)\frac{\partial}{\partial x_i}\sum_k \frac{\partial u_k}{\partial x_k}\\ &= \eta\Delta u_i + \left(\frac{1}{3}\eta + \xi\right)\frac{\partial}{\partial x_i}(\nabla\cdot u).\end{aligned}$$

Putting all of the above together our new Euler equation in vector form is

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = -\nabla p + \eta\Delta u + \left(\frac{1}{3}\eta + \xi\right)\nabla(\nabla\cdot u).$$

We have already seen in (7) that incompressible fluids have divergence-free velocities. Therefore the equations of motion and mass conservation for incompressible, viscous fluids are

$$\left.\begin{aligned}\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) - \frac{\eta}{\rho}\Delta u = -\frac{1}{\rho}\nabla p,\\ \nabla\cdot u = 0.\end{aligned}\right\} \tag{8}$$

The set of equations (8) are the *Navier-Stokes equations for incompressible, viscous flow*. We see that the new equation of motion differs from the old by the term $\frac{\eta}{\rho}\Delta v$, which models viscosity. The ratio $\nu := \frac{\eta}{\rho}$ is called *kinematic viscosity*, as opposed to η, which is called *dynamic viscosity*.

2.2 The Reynolds Number

Dimensional analysis is an important tool in model theory and experimental physics. The goal is to extract from the model a dimensionless number which captures the essence of the model in the sense that comparing these numbers allows us to evaluate the extent to which different processes governed by the same model are similar.
If as an example we wanted to study the flow past an object in a given space, the type of the flow will be determined by the shape of the object, the velocity of the flow and the kinematic viscosity $\nu = \frac{\eta}{\rho}$. If we take a sphere, its shape is then completely determined by one number, the radius, and the unit of this number is length (L). The units of u and ν are, respectively $\frac{L}{T}$ and $\frac{L^2}{T}$, where T represents the unit of time. Then, the only dimensionless number that can be formed from these quantities is

$$Re = \frac{uL}{\nu}. \tag{9}$$

This is the Reynolds number. The fraction Re can also be interpreted as a ratio of inertial to viscous forces:

$$Re = \frac{uL}{\nu} = \frac{\rho u L}{\eta} = \frac{\text{inertial forces}}{\text{viscous force}}.$$

This allows for the interpretation that the Reynolds number measures the turbulence in a flow. The higher the Reynolds number, the more turbulent the flow.
The two interesting cases are $Re \to 0$ and $Re \to \infty$. For $Re \to 0$, the non-linear term in the Navier-Stokes equations can be neglected and they reduce to the well-understood Stokes problem. It describes creeping flow. In the other case, the flow exhibits turbulent behavior and this is the form of behavior we are interested in.
Whether a flow is turbulent or not cannot be seen from the Navier-Stokes equations directly, which are the model for all kinds of flow. Turbulence is not a mathematically defined term. One could therefore think of turbulence as the opposite of non-turbulence. Non-turbulence would then be the behavior of a flow which is laminar, meaning its layers are basically not mixing. Each flow experiment will have a Reynolds number below which the flow is considered not turbulent and above which it is considered turbulent. Distinguishing flows in this way goes back to Osborne Reynolds, see Pope (2000). He performed an experiment of injecting dye into a flow through a pipe and then realized that the behavior of the flow could be described by one number: the ratio $\frac{uL}{\nu}$, which is the Reynolds number.

3 The Leray-α Model

Following Layton (2008), we first motivate the interest in a priory estimates of the L^2 and H^1 norms of our possible solution.
The total kinetic energy is defined by $\frac{1}{2}$mass$*$velocity^2. This means that the total kinetic energy of a fluid with constant density ρ and velocity v on a domain Ω is

$$\frac{1}{2}\rho \int_\Omega |u|^2 \, dx.$$

The space $L^2(\Omega)$ is defined as the set of all functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^2(\Omega)} = \int_\Omega |u|^2 \, dx < \infty,$$

where the norm is induced by the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u\overline{v} \, dx,$$

where in this instance $\overline{v}$ is the complex conjugate of v. Please note that in the following, $\overline{v}$ will have a different meaning. This can of course be extended to functions $u : \Omega \to \mathbb{R}^d$:

$$L^2(\Omega)^d = \left\{ u = (u_1, u_2, ..., u_d) : \Omega \to \mathbb{R}^d, \quad u_j \in L^2(\Omega) \text{ for all } j \right\}$$

$$\text{and} \qquad \|u\|_{L^2(\Omega)^d} = \left(\sum_{j=1}^d \|u_j\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

The space $L^2(\Omega)^d$ is therefore the set of all velocity fields with finite kinetic energy. From now on, for convenience we drop the L^2 subscript and denote by $(\cdot,\cdot)$ the L^2 scalar product and by $\|\cdot\|$ the L^2 norm. But it is not only the velocity itself that is important. As we have seen in Section 2.1, changes in velocity are what cause one layer of fluid to exert force on another layer of fluid. This force must be finite for the velocity to be physically relevant. The space that models this is the Hilbert space $H_0^1(\Omega)$. It is defined as the closure of

$$\left\{u : \Omega \to \mathbb{R}^d : u \in C^1(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega\right\} \quad \text{in} \quad \|.\|_{H_0^1(\Omega)}\,.$$

This norm is defined as follows:

$$\|u\|_{H_0^1(\Omega)} = \left(\|u\|^2 + \|\nabla u\|^2\right)^{\frac{1}{2}}.$$

Suppose we have already found (u, p) smooth, such that (u, p) solves the Navier-Stokes equations. We multiply the equation of motion by u itself and integrate over the domain Ω. We get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega (u\cdot\nabla)\,u\;u - \nu\int_\Omega \Delta u\;u + \int_\Omega \nabla p\;u = \int_\Omega f\;u.$$

After integrating by parts, the second term and the pressure term vanish because of $\nabla\cdot u = 0$. We are left with

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\,\|\nabla u\|^2 = (f, u).$$

After integrating this in time on $(0, t)$ and using the fundamental theorem of calculus, we get

$$\frac{1}{2}\,|u(t)|^2 + \nu\int_0^t \|\nabla u\|^2 = \frac{1}{2}\,\|u(0)\|^2 + \int_0^t (f, u).$$

With our interpretation of the norms as above, this states (see Layton (2008), Chapter 8):

$$\begin{aligned}\text{kinetic energy}(t) + {}& \text{ total energy dissipated over } [0,t] \\ &= \text{initial kinetic energy } + \text{ total power input.}\end{aligned}$$

This is the *energy equality* that Leray in Leray (1934) called *energy dissipation relation*. We can see from this relation what an appropriate space for our solution might look like. The terms on the left-hand side should be finite and so we define:

$$\begin{aligned} L^2(0,T;H_0^1(\Omega)) &:= \left\{u(\cdot,t) \in H_0^1(\Omega) \quad \forall t \in [0,T] : \quad \int_0^T \|u\|_{H_0^1}\,dt < \infty\right\}, \\ L^\infty(0,T;L^2(\Omega)) &:= \left\{u(\cdot,t) \in L^2(\Omega) \quad \forall t \in [0,T] : \quad \text{ess sup}_{0<t<T}\,\|u\| < \infty\right\}. \end{aligned}$$

More generally, for a Hilbert space V we can define

$$L^p(0,T;V(\Omega)) := \left\{ u(\cdot,t) \in V(\Omega) \quad \forall t \in [0,T] \quad \left(\int_0^T \|\nabla u\|_V^p \right)^{1/p} dt < \infty \right\}.$$

After deriving this integral form of our equations, we can already see that there is a way to interpret the equations that does not require the solution u to be in C^2 any more. In fact, after introducing the notion of weak derivative, we do not need our solution to be differentiable in the classical sense at all. Leray first introduced the notion of weak derivative in Leray (1934). The modern definition is derived by way of partial integration of smooth functions and the realization that the expression

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi \, dx,$$

where α is a multiindex, still makes sense for $u, D^\alpha u \in L^1_{loc}(\Omega)$ and so called test functions $\phi \in C_c^\infty(\Omega)$. Leray showed the existence and uniqueness of so-called *regular* solutions for the Navier-Stokes equations, but only for an interval of time $[0,T)$. He could not show that these solutions would not become irregular in finite time and he states in his introduction in Leray (1934): "In fact it is not paradoxical to suppose that the thing which regularizes the motion- dissipation of energy - does not suffice to keep the second derivatives of the velocity components bounded and continuous." However, when in the original Navier-Stokes equations he replaced $(u \cdot \nabla)u$ by $(\overline{u} \cdot \nabla)u$, where $\overline{u}$ is a mollified version of u, he could show that the resulting solution to this mollified system of equations would not become irregular and in fact exists for all times, see Leray (1934), §26. The limit of these solutions of mollified systems, which he called *turbulent solution*, see Leray (1934), §26, is what is known today as a *weak solution*. There is more than one notion of weak solution, though. One can construct weak solutions by way of the Galerkin method. As it is not known whether weak solutions are unique, it is not known whether a solution obtained by passing to the limit with Leray´s method is the same weak solution as is obtained by the Galerkin method.

Jean Leray obtained his results by solving a mollified system and then showing that the limit of the solutions to such mollified systems is in some sense a solution of the Navier-Stokes equations. The Leray-α model now attempts to do the same. Instead of a convolution, it uses an approximation of this convolution. The mollified system is

$$\left.\begin{aligned} \frac{\partial}{\partial t} u^\alpha - \nu \Delta u^\alpha + (\overline{u}^\alpha \cdot \nabla) u^\alpha + \nabla(p^\alpha) &= f, \\ \nabla \cdot u^\alpha &= 0, \\ \overline{u}^\alpha &= \Phi^\alpha * u^\alpha, \end{aligned}\right\}$$

where u^α is the solution for that specific α, $\overline{u}^\alpha$ is the mollified version of u^α and as $\overline{u}^\alpha \to u^\alpha$ for $\alpha \to 0$, this system converges to the Navier-Stokes equations.

We now fix $\alpha > 0$ and chose $\overline{u}^\alpha = (I - \alpha^2 \Delta)^{-1} u^\alpha$, the smoothing kernel associated with the Green function of the Helmholtz operator, as is done in Cheskidov et al. (2005), the

presentation which we are following in this chapter.
The Leray-α model for some $\alpha > 0$ fixed then is the following:

$$\left.\begin{aligned} \frac{\partial}{\partial t}u - \nu\Delta u + (\overline{u}\cdot\nabla)u + \nabla p = f,\\ \nabla\cdot u = 0,\\ \overline{u} = u - \alpha^2\Delta u,\\ \text{v periodic, with periodic box } \Omega = [0,2\pi L]^3,\\ + \text{ initial condition.} \end{aligned}\right\} \tag{10}$$

We are looking for a solution in the periodic box $\Omega = [0, 2\pi L]^3$ so as to not have to concern ourselves with boundary values. Looking for a periodic solution also allows us to use Fourier analysis in some of the proofs. We will instead require our solution to have the property $\int_\Omega v\,dx = 0$. This will take care of solutions that would only differ by an additive constant. We define

$$H = \left\{ v : v \in L^2(\Omega), \quad \nabla\cdot v = 0, \quad \text{v is periodic in } \Omega, \quad \int_\Omega v\,dx = 0 \right\}$$

and

$$V = \left\{ v : v \in H^1(\Omega), \quad \nabla\cdot v = 0, \quad \text{v is periodic in } \Omega, \quad \int_\Omega v\,dx = 0 \right\}.$$

For $v \in H$, one can show $\nabla\cdot v \in L^2(\Omega)$, see Temam (1984). This is true by assumption for $v \in V$. Further, one can define a scalar product on V by

$$((u, v)) = (\nabla u, \nabla v).$$

It induces a norm on V denoted by $\|u\|_V := ((u, u))^{1/2}$. Because the Poincaré inequality is valid on V we have

$$\|v\|^2 \leq \frac{1}{\lambda_1}\|v\|_V^2 \tag{11}$$

and the norm induced by the scalar product $((\cdot,\cdot))$ is equivalent to the H^1-norm on V. As V is a Hilbert space, because of the theorem of Riesz we can associate with $u \in V$ an operator $A : V \to V'$ such that

$$((u, v)) = \langle Au, v\rangle \quad \forall v \in V.$$

Since for $u \in V \cap H^2$ we have

$$\langle Au, v\rangle = ((u, v)) = (\nabla u, \nabla v) = -(\Delta u, v) = -\langle \Delta u, v\rangle,$$

we can interpret A as the Laplace operator $-\Delta$ in the distributional sense. In order for Au to actually be a function, we need $u \in V \cap H^2$. We define the domain of A in V as $D(A) := V \cap H^2$. We then have the relations

$$D(A) \subset V \subset H \subset V',$$

where the embeddings are continuous and the embedding $V \hookrightarrow H$ is compact because the embedding $H^1 \hookrightarrow L^2$ is compact by the Rellich Lemma. These relations are needed to fulfill the assumptions of the Aubin compactness theorem, which is needed in the proof for existence of a solution to the Navier-Stokes equations.
With the help of the norm in V, we can define dimensionally homogeneous norms in $H^1(\Omega)$ and $H^2(\Omega)$ as follows:

$$\begin{aligned}\|u\|_{H^1}^2 &:= \lambda_1 \left(\|u\|^2 + \alpha^2 \|u\|_V^2 \right), \\ \|u\|_{H^2}^2 &:= \lambda_1^2 \left(\|u\|^2 + 2\alpha^2 \|u\|_V^2 + \alpha^4 \|\Delta u\|^2 \right).\end{aligned}$$

Using this definition of the H^2-norm, $\|u\|^2 = \left\|\overline{u} - \alpha^2 \Delta \overline{u}\right\|^2$ and $((u,v)) = -(u, \Delta v)$ for $v \in H^2$, we immediately get for $\overline{u} \in H^2$

$$\|\overline{u}\|_{H^2}^2 = \lambda_1^2 \|u\|^2 ,$$

and therefore

$$\lambda_1 \|u\| \leq \|\overline{u}\|_{H^2} \leq 2\lambda_1 \|u\| , \tag{12}$$

meaning the norm of $\overline{u}$ in H^2 is equivalent to $\lambda_1 \|u\|$. We now make more precise the notion of weak solution.
If u is a smooth solution of (10), we take the L^2 scalar product in V with a function $v \in V$ to arrive at

$$\begin{aligned}&\left(\frac{\partial}{\partial t} u, v \right) + \nu((u,v)) + b(\overline{u}, u, v) = (f, v) \quad \forall v \in V, \\ \text{where} \qquad &b(u,v,w) = \sum_{i,j=1}^{n} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j,\end{aligned} \tag{13}$$

where herein, we just assume that the functions in this expression are smooth enough for b to make sense. By setting

$$(B(u,v), w) = b(u,v,w) \quad \forall u,v,w \in V,$$

we define a bilinear operator from $V \times V$ into V'. By partial integration, this operator has the properties

$$\begin{aligned}&\langle B(u,v), w \rangle_{V'} = - \langle B(u,w), v \rangle_{V'} \\ \Rightarrow &\langle B(u,v), v \rangle_{V'} = 0.\end{aligned} \tag{14}$$

Interpreting $\frac{du}{dt}$ as the distributional derivative, (13) is equivalent to the functional equality

$$\frac{d}{dt} u - \nu A v + B(\overline{u}, u) = f.$$

This allows us to interpret the equation as an equation in V'. We now only have to assume $f \in V'$ for this to make sense. The Leray-α-model (10) then becomes

$$\left.\begin{aligned} \frac{d}{dt}u + \nu A u + B(\overline{u}, u) &= f, \\ (I + \alpha^2 A)^{-1} u &= \overline{u}, \\ \nabla \cdot u &= 0, \\ u(0) &= u_0, \end{aligned}\right\} \tag{15}$$

where $u \in V$. Note that it is not necessarily clear what is meant by $u(0) = u_0$. One needs to clarify why a function from V can be evaluated in a point. This is usually clarified through an analysis of the problem, see Temam (1988), Chapter 2 for a discussion.
This is the weakest possible formulation of the Leray-α model. There are several different formulations for weak and strong solutions of the Navier-Stokes equations. They mostly differ in the function spaces that are used. Generally, a weak solution is a solution of a functional differential equation like (15), making the solution a function with values in a dual space. By strong solution, one then usually means a more regular solution with values in the function space obtained by stronger assumptions on the data of the equation. For an extended introduction to the modern theory of existence and uniqueness of solutions of the Navier-Stokes equations, see Temam (1984). For a shorter overview of the methods used in the proofs and the function spaces, see Temam (1988).
Finally, we present Leray´s results from 1934 as stated in Cheskidov et al. (2005):

Theorem 1. *Let $T > 0$, $\nu > 0$, $\alpha > 0$ be given.*
If $f \in V'$, $u_0 \in H$ then (15) has a unique weak solution on $[0, T]$. That is, there a function

$$u \in L^\infty((0,T); H) \cap L^2((0,T); V) \cap C([0,T]; H) \quad \textit{and} \quad \frac{d}{dt}u \in L^2((0,T); V')$$

such that

$$\left\langle \frac{d}{dt}u, \phi \right\rangle + \nu \left\langle Au, \phi \right\rangle + \left\langle B(\overline{u}, u), \phi \right\rangle = \left\langle f, \phi \right\rangle$$

for every $\phi \in V$, where $\overline{u} = (I + \alpha A)^{-1} u$ and $u_0 = u(0)$.
If $f \in H$, $u_0 \in V$ then this unique weak solution is a strong solution on $(0, T)$. That is

$$u \in L^2((0,T); D(A)) \cap C([0,T]; V) \quad \textit{and} \quad \frac{d}{dt}u \in L^2((0,T); H)$$

such that

$$\left(\frac{d}{dt}u, \phi \right) + \nu(\nabla u, \nabla \phi) + (B(\overline{u}, u), \phi) = \int_0^t (f, \phi)$$

for every $\phi \in V$ where $\overline{u} = (I + \alpha A)^{-1} u$ and $u_0 = u(0)$.

We are going to assume that f is independent of time, which makes (15) an autonomous dynamical system.

3.1 A Priori Estimates

As motivated by the introduction, next we are going to establish a priori L^2 and H^1 estimates for solutions of these equations. The presentation follows Cheskidov et al. (2005).

3.1.1 L^2-Estimate

Taking the inner product of (15) with u itself and using the property of the operator B (14) gives

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|u\|_V^2 = (f,u). \tag{16}$$

We continue using the Cauchy-Schwarz inequality and Young´s inequality with $p = q = 2$ and the estimate of the L^2-norm (11):

$$\begin{aligned}(f,v) &\le \|f\|\,\|u\| = \frac{1}{\sqrt{\nu\lambda_1}}\|f\|\sqrt{\nu\lambda_1}\,\|u\| \le \frac{1}{2\nu\lambda_1}\|f\|^2 + \frac{\nu\lambda_1}{2}\|u\|^2 \\ &\le \frac{1}{2\nu\lambda_1}\|f\|^2 + \frac{\nu}{2}\|u\|_V .\end{aligned}$$

Therefore, after using the estimate of the norm (11) in the last step once more, (16) reads

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|u\|_V^2 \le \frac{1}{2\nu\lambda_1}\|f\|^2 + \frac{\nu}{2}\|u\|_V \\ \Leftrightarrow\quad &\frac{d}{dt}\|u\|^2 + \nu\|u\|_V^2 \le \frac{1}{\nu\lambda_1}\|f\|^2 \\ \Rightarrow\quad &\frac{d}{dt}\|u\|^2 + \lambda_1\nu\|u\|^2 \le \frac{1}{\nu\lambda_1}\|f\|^2 . \end{aligned} \tag{17}$$

We would like to bring this into a form on which we can use the Grönwall inequality. Therefore, we define

$$\frac{d}{dt}\|u\|^2 \le \frac{\|f\|^2}{\nu\lambda_1} - \nu\lambda_1\|u\|^2 =: -\tilde{u}. \tag{18}$$

We now derive an inequality for $\tilde{u}$ by differentiating. Note that we assume f to be independent of t. Therefore, we get

$$\frac{d}{dt}\tilde{u} = \nu\lambda_1\frac{d}{dt}\|u\|^2 \le -\nu\lambda_1\tilde{u}.$$

Now setting $g = -\nu\lambda_1$, we can use the Grönwall inequality to obtain

$$\tilde{u}(t) \le \tilde{u}(0)\exp\left(\int_0^t -\nu\lambda_1\,ds\right) = \tilde{u}(0)\exp\left(-\nu\lambda_1 t\right). \tag{19}$$

Resubstituting the definition of $\tilde{u}$ (18) in (19) yields

$$\begin{aligned} &\frac{-\|f\|^2}{\nu\lambda_1} + \nu\lambda_1\|u(t)\|^2 \le \left(\frac{-\|f\|^2}{\nu\lambda_1} + \nu\lambda_1\|u(0)\|^2\right)\exp\left(-\nu\lambda_1 t\right). \\ \Rightarrow\quad &\|u(t)\|^2 \le \left(1-\exp(-\nu\lambda_1 t)\right)\frac{\|f\|^2}{(\nu\lambda_1)^2} + \|u(0)\|^2\exp(-\nu\lambda_1 t) =: R(t). \end{aligned} \tag{20}$$

From this, we see that

$$\limsup_{t\to\infty} \|u(t)\| \leq \frac{1}{\nu\lambda_1} \|f\| =: R.$$

From (12) it immediately follows that

$$\limsup_{t\to\infty} \left\|\overline{u(t)}\right\|_{H^2} \leq \lambda_1 R.$$

After integrating the middle equation in (17) on $(0,t)$ we can also verify

$$\|u(t)\|^2 + \nu \int_0^t \|u(s)\|_V^2 \, ds \leq \|u(0)\|^2 + t\frac{\|f\|^2}{\nu\lambda_1} \quad \forall t,$$

which means we have $u \in L^2((0,t),V)$ for all $t > 0$.

3.1.2 H^1-Estimate

To establish an a priori H^1-estimate, we take the inner product of (15) with Au:

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} \|u\|_V^2 + \nu \|Au\|^2 + (B(\overline{u},u),Au) = (f,Au) \\ \Leftrightarrow \quad & \frac{1}{2}\frac{d}{dt} \|u\|_V^2 + \nu \|Au\|^2 = (f,Au) - (B(\overline{u},u),Au). \end{aligned}$$

We proceed by estimating the right-hand side from above. We get

$$(f,Au) - (B(\overline{u},u),Au) \leq |(f,Au)| + |(B(\overline{u},u),Au)|,$$

and by using the Cauchy-Schwarz inequality and Young´s inequality with $p = q = 2$ we get

$$|(f,Au)| \leq \|f\| \, \|Au\| = \frac{\sqrt{2}}{\sqrt{\nu}} \|f\| \frac{\sqrt{\nu}}{\sqrt{2}} \|Au\| \leq \frac{\|f\|^2}{\nu} + \nu\frac{1}{4} \|Au\|^2 .$$

Using Cauchy-Schwarz, $\|(u\cdot\nabla)v\| \leq \|u\|_{L^\infty} \|\nabla v\|$ whenever $u \in L^\infty$, Young´s inequality with $p = q = 2$, and the Sobolev inequality in three dimensions, we also have

$$\begin{aligned} & |(B(\overline{u},u),Au)| \\ \leq & \|B(\overline{u},u)\| \, \|Au\| \leq \|\overline{u}\|_{L^\infty} \|\nabla u\| \, \|Au\| \\ \leq & \frac{\sqrt{2}}{\sqrt{\nu}} \|\overline{u}\|_{L^\infty} \|u\|_V \|Au\| \frac{\sqrt{\nu}}{\sqrt{2}} \leq \frac{1}{\nu} \|\overline{u}\|_{L^\infty}^2 \|u\|_V^2 + \frac{\nu}{4} \|Au\|^2 \\ \leq & \frac{c^2}{\nu\lambda_1^{\frac{1}{2}}} \|\overline{u}\|_{H^2}^2 \|u\|_V^2 + \frac{\nu}{4} \|Au\|^2 . \end{aligned}$$

So, for the right-hand side we have

$$|(f,Au) - (B(\overline{u},u),Au)| \leq |(f,Au)| + |(B(\overline{u},u),Au)| \leq \frac{\|f\|^2}{\nu} + \nu\frac{1}{2} \|Au\|^2 + \frac{c^2}{\nu\lambda_1^{\frac{1}{2}}} \|\overline{u}\|_{H^2}^2 \|u\|_V^2 .$$

In total, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_V^2 + \frac{1}{2}\nu\|Au\|^2 \leq \frac{\|f\|^2}{\nu} + \frac{c^2}{\nu\lambda_1^{\frac{1}{2}}}\|\overline{u}\|_{H^2}^2\|u\|_V^2 . \tag{21}$$

Using the estimate for the H^2-norm of $\overline{u}$ from equation (12) and the L^2-estimate already established for u in (20), we get

$$\|\overline{u}\|_{H^2}^2 \leq 4\lambda_1^2\|u\|^2 \leq 4\lambda_1^2(\|u(0)\|^2 e^{-\lambda_1\nu t} + \frac{\|f\|^2}{(\lambda_1\nu)^2}(1-e^{-\lambda_1\nu t})) =: 4\lambda_1^2 R(t)^2.$$

Using this in (21) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_V^2 + \frac{1}{2}\nu\|Au\|^2 \\
\leq&\frac{\|f\|^2}{\nu} + \frac{c^2}{\nu\lambda_1^{\frac{1}{2}}}\|u\|_V^2 4\lambda_1^2 R(t)^2 = \frac{\|f\|^2}{\nu} + \frac{4c^2\lambda_1^{\frac{3}{2}}}{\nu}\|u\|_V^2 R(t)^2 \\
\Rightarrow \quad &\frac{d}{dt}\|u\|_V^2 + \nu\|Au\|^2 \leq \frac{2\|f\|^2}{\nu} + \frac{8c^2\lambda_1^{\frac{3}{2}}}{\nu}\|u\|_V^2 R(t)^2.
\end{aligned}$$

Setting $K(t) = \max\left\{\frac{2\|f\|^2}{\nu}, \quad \frac{8c^2\lambda_1^{\frac{3}{2}}}{\nu}R(t)^2\right\}$ we get

$$\begin{aligned}
&\frac{d}{dt}\|u\|_V^2 + \underbrace{\nu\|Au\|^2}_{\geq 0} \leq K(t)(1+\|u\|_V^2). \\
\Rightarrow \quad &\frac{d}{dt}\|u\|_V^2 \leq K(t)(1+\|u\|_V^2) \\
\Rightarrow \quad &\frac{d}{dt}(1+\|u\|_V^2) \leq K(t)(1+\|u\|_V^2).
\end{aligned}$$

After setting $w(t) = (1+\|u\|^2)$, we again use the Grönwall Lemma. This then gives

$$\begin{aligned}
w(t) &\leq w(s)\exp\left(\int_s^t K(\tau)\,d\tau\right). \\
\Rightarrow \quad 1+\|u(t)\|_V^2 &\leq (1+\|u(s)\|_V^2)\exp\left(\int_s^t K(\tau)\,d\tau\right).
\end{aligned} \tag{22}$$

Since K is integrable on $(0,T)$ for all T by the definition of R, this means if we suppose $u(0) \in V$, we have $u \in L^\infty([0,T];V)$.

3.2 Estimate of the Global Attractor

We consider the dynamical system

$$\frac{d}{dt}u = F(u), \quad u_0 = u(0). \tag{23}$$

Note that with equations (15), we have a formulation of the Leray-α model like this with $F(u) = f - \nu Au + B(\overline{u}, u)$.
We say the state of the system is described by u which is an element of a metric space H, meaning $u : [0, T] \to H$, $u(t) \in H$ for all t for which u is defined.
If f does not depend on time, this is an autonomous dynamical system. In general, the operator F and with it the solutions of the system (23) depend on a parameter λ. In our case, this is the Reynolds number. Experiments (see: Taylor experiment) suggest that if λ is small in a certain sense (dependent on the geometry of Ω and the viscosity of the fluid), then for $t \to \infty$ the flow will converge to a unique stationary solution of the problem (23), meaning to the solution of the problem $F_\lambda = 0$. As λ gets larger and larger, there will be more than one stationary solution. Solutions of problem (23) will then converge to one of these solutions dependent on the initial values they belong to. For even larger λ the stationary solutions disappear and we have periodic solutions, then quasi-periodic solutions and finally fully turbulent solutions. This is usually taken to mean that the Fourier expansion of the solution u does not consist of discrete frequencies anymore, see Landau and Lifshitz (1959), §26/27. One can still examine the behavior of solutions $u(t)$ as $t \to \infty$ and if $u(t) \to X$ as $t \to \infty$ for some subspace $X \subset H$ which in a sense to be made precise later is invariant under the dynamics of the system (23), one can still extract information on the long time behavior from the structure of the set X. This set X is a so-called attractor. It describes the long-time behavior of a flow. A comprehensive text on attractors is Temam (1988) and in it Temam writes: “It is our understanding here that the number of degrees of freedom of a turbulent phenomenon is the dimension of the attractor which represents it”. In this sense, the dimension of the attractor of the Leray-α model is a measure of its complexity.
Let L denote the typical length scale of the flow under consideration and l_d the viscous dissipation length scale, which is the smallest length scale (dependent on ν) that one needs to resolve, see Chapter 3 for a more detailed explanation. Cheskidov et al. (2005) calculate that the dimension of the attractor of the Leray-α model is proportional to $(L/l_d)^{12/7}$, whereas the number of degrees of freedom for the 3D Navier-Stokes equations is assumed to be proportional to $(L/l_d)^3$. This suggests that the Leray-α model is less complex and therefore easier to simulate numerically.

3.2.1 Attractors

Before we can begin estimating the dimension of the attractor of the Leray-α model, we need to first establish its existence and this requires some definitions.
The evolution of the system (23) is described by the operators $S(t)$ which are defined by

$$u(t) = S(t)u(0), \qquad S(0) = Id,$$
$$S(t + s) = S(t)S(s),$$

and therefore form a semi-group. Writing $u(t) = S(t)u(0)$ of course only makes sense if there is a unique solution of the problem (23).
The *orbit* or *trajectory* of u_0 is $\cup_{t \geq 0} S(t)u_0$, and an *ω-limit set* for u_0 is the set $\omega(u_0) = \cap_{s \geq 0} \cup_{t \geq s} S(t)u_0$. We can interpret this to mean that the ω-limit set is the set where the

trajectory of $u(0)$ leads. An ω-limit set can of course be defined for whole sets of initial values, the name ω-limit *set* is a lot more fitting then.
We call $X \subset H$ an *invariant set* for the semi-group S if $X = S(t)X$ for all $t \geq 0$.

Definition 1. *Let $B \subset U$, U open in H, be a subset of the metric space H. B is absorbing in U if the orbit of any bounded set $B_0 \subset U$ enters B after a certain time, in other words:*

$$\forall B_0 \subset U, \quad B_0 \textit{ bounded } \quad \exists t_1(B_0) : S(t)B_0 \subset B \quad \forall t > t_0(B_0).$$

Finally, we can define what an attractor is:

Definition 2. *Let H be a metric space. An attractor is a set $X \subset H$ that has the following properties:*
(i) $\quad S(t)X = X, \quad \forall t \geq 0$
(ii) $\quad$ *X possesses an open neighborhood U such that for every u_0 in U*

$$dist(S(t)u_0, X) \to 0 \quad \textit{as } t \to \infty \quad \textit{where} \quad dist(x, X) = \inf_{y \in X} dist(x, y).$$

(iii) We say that $X \subset H$ is a global attractor for the semi-group $\{S(t)\}_{t \geq 0}$ if X is a compact attractor that attracts the bounded sets of H.

Next, we need a tool for proving the existence of a global attractor. Before we do that, we need one more definition:

Definition 3. *We call the operators $S(t)$ uniformly compact for large t, if for every bounded set B_0 there is a t_0 such that*

$$\bigcup_{t \geq t_0} S(t)B_0$$

is relatively compact.

Now we have the following theorem for the existence of an attractor:

Theorem 2. *Let H be a metric space. Assume the operators $S(t)$ are uniformly compact. Further assume there is an open set U and a bounded set $B \subset U$ such that B is absorbing in U. Then the ω-limit set of B,*

$$X = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B},$$

is a compact attractor that attracts the bounded sets in U. It is also the maximal bounded attractor in U.

One now generally first shows existence and uniqueness of solutions for the problem (23). Then, one finds an absorbing set and shows that $S(t)$ is a compact operator. Using Theorem 2, one can then establish the existence of an attractor for the system.

For the Leray-α model (15) we already have existence and uniqueness of solutions from Theorem 1. Therefore, the solution operators $S(t)$ exist. From the L^2-estimate (20) we already deduced

$$\limsup_{t\to\infty} \|u(t)\| \leq \frac{1}{\nu\lambda_1} \|f\| =: R.$$

Therefore,

$$B_1(H) := \{w \in H : \|w\| \leq R\}$$

is an absorbing ball for the solution $u(t)$: the trajectory of any $u_0 \in B_0$ where B_0 is a bounded set in H will eventually after a time $t_0 = t_0(B_0)$ end up in $B_1(H)$.
From (12) we immediately get that

$$B_2(H) := \{w \in H : \|w\|_{H^2} \leq 2\lambda_1 R\}$$

is an absorbing ball for $\overline{u(t)}$. We now use the H^1 estimate to prove compactness of the operator $S(t)$. First, integrating the middle equation in (17) on (t, r) gives us the estimate

$$\begin{aligned}\int_t^{t+r} \|u(\tau)\|_V^2 \, d\tau &\leq \|u(t)\|^2 + \frac{\|f\|^2}{\nu\lambda_1} r \\ &\leq 2rR_0 \qquad \text{for } t \geq t_0(B_0) \text{ and } u_0 \in B_0.\end{aligned} \tag{24}$$

Then, we use this estimate and the uniform Grönwall Lemma on the inequality for the H^1-norm

$$1 + \|u(t)\|_V^2 \leq (1 + \|u(s)\|_V^2) \exp\left(\int_s^t K(\tau)\, d\tau\right)$$

derived in (22). Using notation from the uniform Grönwall Lemma, we then have $g = K(t)$, $y = (1 + \|u\|_V)^2$, $h = 0$ and because $K(t)$ is integrable and because of (24) we get the estimates

$$\begin{aligned}&\int_t^{t+r} y(s)\, ds = r + \int_t^{t+r} \|u(s)\|_V^2 \, ds \leq r(1 + 2R_0) =: a_3 \quad \text{for } t > t_0 B_0, \\ &\int_t^{t+r} g(s)\, ds = \int_t^{t+r} K(s)\, ds =: a_1.\end{aligned}$$

Now, the uniform Grönwall Lemma gives us the estimate

$$\|u(t + r)\|_V \leq \frac{a_3}{r} \exp\,(a_1) = (1 + 2R_0) \exp\,(a_1) =: R_2,$$

and we see that the ball $B_2(V)$, by which we will denote the ball with radius R_2 in V, is an absorbing set. But we have also bounded u in the V-norm depending on u_0 being bounded in the H-norm. This means if u_0 is bounded in H, after a certain time u will be bounded in V. In other words, for any bounded set B_0 in H there will be a time t_0 such that $S(t)B_0 \subset B_2(V)$. But $B_2(V)$ is bounded in H_0^1 and as we know from the Rellich Lemma, the embedding $H^1 \hookrightarrow L^2$ is compact. Therefore, the embedding $V \hookrightarrow H$ is also compact and $B_2(V)$ is relatively compact in H. Then, $S(t)$ is a uniformly compact operator. Now Theorem 2 guarantees the existence of a global attractor:

Theorem 3. *Let $u_0 \in H$ and also $f \in H$. Then, the Leray-α model possesses the unique global attractor X in H and*

$$X = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t) B_1}.$$

Note that in a similar way an absorbing set and attractor in V can be derived from the a priori H^1-estimates.

3.2.2 The Estimate

Following Cheskidov et al. (2005) we will now estimate the Hausdorff dimension of the global attractor. This is done by tracking the evolution of a small volume element in the attractor.
In any Hilbert space H, because it is endowed with a scalar product, one can for each element φ of this Hilbert space define a linear operator

$$L_\varphi : H \to \mathbb{R}, \quad L_\varphi(v) := (\varphi, v)_H.$$

Analogously, if H is an (at least) m-dimensional Hilbert space, one can, using m elements $\varphi_1, \ldots, \varphi_m$ of this Hilbert space, define an m-linear operator by

$$\begin{aligned} &\varphi_1 \otimes \ldots \otimes \varphi_m : H^m \to \mathbb{R}, \\ &\varphi_1 \otimes \ldots \otimes \varphi_m := \prod_{i=1}^{m} (\varphi_i, v_i)_H \quad \forall v_1, \ldots, v_m \in H. \end{aligned}$$

Now one can define the so-called *m-exterior product of H*, usually denoted by $\bigwedge^m H$, which is the space spanned by all the sums

$$\varphi_1 \wedge \ldots \wedge \varphi_m := \sum_\sigma (-1)^\sigma \varphi_{\sigma(1)} \otimes \ldots \otimes \varphi_{\sigma(m)},$$

where σ is a permutation and $\varphi_1 \wedge \ldots \wedge \varphi_m$ is called *wedge product.* On this space, one can define a scalar product as follows:

$$(\varphi_1 \wedge \ldots \wedge \varphi_m, \psi_1 \wedge \ldots \wedge \psi_m)_{\bigwedge^m H} := \det \left\{ (\varphi_i, \psi_i)_H \right\}_{1 \leq i,j \leq m},$$

where of course $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_m \in H$. This scalar product in the usual way induces a norm. As the norm of the determinant of a matrix equals the volume of the parallelepiped spanned by the columns of the matrix, $|\varphi_1 \wedge \ldots \wedge \varphi_m|_{\bigwedge^m H}$ can be interpreted as the volume of the volume element spanned by $\varphi_1, \ldots, \varphi_m$. Therefore, if we want to get an idea of how a small volume element develops, we need an estimate for the norm of the wedge product of the functions spanning it.
Setting $F = I + \alpha^2 A$, we first suppose that $u(t) \subset X$ for some $t > t_0$ and that it is a solution of

$$\left.\begin{aligned} \frac{d}{dt} u + \nu A u + B(\overline{u}, u) = f, \\ \overline{u} = F^{-1} u. \end{aligned}\right\}$$

Let $\xi \in V$ denote a small disturbance that we add to $u \in V$. Then we get

$$\begin{aligned} &\frac{d}{dt}(u+\xi) + \nu A(u+\xi) + B(F^{-1}(u+\xi), u+\xi) = f + \mathcal{O}(\xi^2) \\ \Leftrightarrow \quad &\frac{d}{dt}\xi + \nu A\xi + B(F^{-1}u, \xi) + B(F^{-1}\xi, u) = 0. \end{aligned}$$

This means the disturbance evolves according to

$$\left.\begin{aligned} \frac{d}{dt}\xi + \Lambda\xi = 0, \\ F^{-1}\xi = \eta, \\ \xi(0) = \xi_0, \end{aligned}\right\}$$

where $\Lambda(t)\xi = \nu A\xi + B(\overline{u}, \xi) + B(\eta, u)$.
Let $\xi_j(t) = (\xi_{j1}(t), \xi_{j2}(t), \xi_{j3}(t))$ denote solutions of this system corresponding to initial values $\xi_j(0) = \xi_j^0$ and let $Q_N(t)$ be the L^2-orthogonal projection from L^2 to the N-dimensional subspace spanned by $\xi_j(t)_{1\leq j\leq N}$. Then, assuming $\Lambda(t)$ Fréchet-differentiable and denoting by Tr the trace of an operator, analogously to Temam (1988) Section V.2.3, we can derive an equation governing the evolution of the N-dimensional volume element:

$$\begin{aligned} &|(\xi_1 \wedge \ldots \wedge \xi_N)(t)|_{\wedge^N L^2} \\ = &|(\xi_1 \wedge \ldots \wedge \xi_N)(0)|_{\wedge^N L^2} \exp\left(-\int_0^t \mathrm{Tr}(Q_N(\tau)\circ\Lambda(\tau)\circ Q_N(\tau))\right) d\tau. \end{aligned}$$

If we can now find a constant $K = K(N)$ dependent on the dimension N such that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{Tr}(Q_N(\tau)\circ\Lambda(\tau)\circ Q_N(\tau))\, d\tau \geq K(N) > 0,$$

then we have shown that

$$|(\xi_1 \wedge \ldots \wedge \xi_N)(t)|_{\wedge^N L^2} \leq |(\xi_1 \wedge \ldots \wedge \xi_N)(0)|_{\wedge^N L^2} \exp\left(-K(N)t\right),$$

for some $t > t_0$, meaning we have exponential decay of the volume element inside the attractor. This already suggests that the attractor must be finite-dimensional. A theorem from Temam (1988), Chapter V.3.3, Theorem 3.3 then gives the result that the Hausdorff dimension of the attractor will be less or equal to N if also N>1. Therefore, this estimate is what we are trying to establish.
We begin by selecting $\psi_1(t), \ldots, \psi_N(t)$, an L^2-orthonormal basis of $\mathrm{span}\{\xi_1(t), \ldots, \xi_N(t)\}$ and we let $\zeta_j = F^{-1}(\psi_j)$. Using the definition of trace and $(B(u,v),v) = 0$ for $u, v \in V$

we then estimate for a fixed time t:

$$\begin{aligned}
\mathrm{Tr}(Q_N(t)\circ\Lambda(t)\circ Q_N(t)) &= \sum_{j=1}^{\infty}((Q_N(t)\circ\Lambda(t)\circ Q_N)\psi_j,\psi_j) = \sum_{j=1}^{N}(\Lambda(t)\psi_j,\psi_j)\\
&= \sum_{j=1}^{N}\nu\left\|\psi_j\right\|_V^2 + (B(\overline{u},\psi_j),\psi_j) + (B(\zeta_j,u),\psi_j)\\
&= \sum_{j=1}^{N}\nu\left\|\psi_j\right\|_V^2 + (B(\zeta_j,u),\psi_j)\\
&\geq \sum_{j=1}^{N}\nu\left\|\psi_j\right\|_V^2 - \left|\sum_{j=1}^{N}(B(\zeta_j,u),\psi_j)\right|.
\end{aligned} \tag{25}$$

Further, using the definition of B, we have

$$\begin{aligned}
\left|\sum_{j=1}^{N}(B(\zeta_j,u),\psi_j)\right| = \left|\sum_{j=1}^{N}((\zeta_j\cdot\nabla)u,\psi_j)\right| &= \left|\int_\Omega\sum_{j=1}^{N}\sum_{i,k=1}^{3}\zeta_{j_i}D_iu_k\psi_{j_k}\right|\\
&\leq \int_\Omega|\nabla u|\sum_{j=1}^{N}\sum_{i,k=1}^{3}|\zeta_{j_i}\psi_{j_k}|\\
&= \int_\Omega|\nabla u|\sum_{j,l=1}^{N}\sum_{i,k=1}^{3}|c_l\psi_{l_i}\psi_{j_k}| \quad \text{using } \zeta_{j_i}=\sum_{l=1}^{N}c_l\psi_{l_i}\\
&= \int_\Omega|\nabla u|\sum_{j=1}^{N}\sum_{i,k=1}^{3}|c_j\psi_{j_i}\psi_{j_k}| \quad \text{using orthonormality}\\
&\leq \int_\Omega|\nabla u|\underbrace{\left(\sum_{j=1}^{N}|\zeta_j|^2\right)^{1/2}}_{=:(\rho_N(x))^{1/2}}\underbrace{\sum_{j=1}^{N}\sum_{i=1}^{N}|\psi_{j_i}|^2}_{=\sum_{j=1}^{N}|\psi_j|^2}\\
&\leq \left\|\rho_N\right\|_{L^\infty}\int_\Omega|\nabla u|\sum_{j=1}^{N}|\psi_j|^2\\
&\leq \left\|\rho_N\right\|_{L^\infty}\left(\int_\Omega|\nabla u|\right)^{1/2}\left(\sum_{j=1}^{N}\underbrace{\int_\Omega|\psi_j|^2}_{=1}\right)^{1/2}\\
&= \left\|\rho_N\right\|_{L^\infty}\left\|u\right\|_V N^{1/2}.
\end{aligned}$$

We will need two more Propositions before we can proceed with estimating the trace.

Proposition 1. *Let $\gamma = \alpha/L$. Then for every function $\zeta \in H^2(\Omega)$*

$$\|\zeta\|_{L^\infty} \leq C(\gamma)(2\pi L)^{-\frac{3}{2}} \left\|(\zeta + \alpha^2 A\zeta)\right\|.$$

Proof. We will use the Fourier transform of a function in this proof. Note that there are several definitions of the Fourier transform. Here, it will be defined by

$$\hat{\zeta}_k = \left(\frac{1}{2\pi L}\right)^3 \int_\Omega \zeta(x) \exp\left(-ik\frac{x}{L}\right)\, dx.$$

Therefore, we can represent ζ as

$$\zeta(x) = \sum_{k\in\mathbb{Z}^3} \hat{\zeta}_k \exp\left(ik\frac{x}{L}\right). \tag{26}$$

We can calculate the L^2-norm of ζ in terms of its Fourier coefficients $\hat{\zeta}$, using on each component of the exponential function the fact that for $n \in \mathbb{N}$ we have

$$\int_0^{2\pi L} \exp(int)\, dt = \begin{cases} 2\pi L & \text{for } n = 0 \\ \frac{1}{in}\left(\exp\left(in2\pi L - 1\right)\right) = 0 & \text{for } nL \in \mathbb{N}, \quad n \neq 0. \end{cases}$$

We then immediately get

$$\begin{aligned} \|\zeta\|^2 &= \left\|\sum_{k\in\mathbb{Z}^3} \hat{\zeta}_k \exp\left(ik\frac{x}{L}\right)\right\|^2 = \sum_{k,l\in\mathbb{Z}^3} \hat{\zeta}_k \hat{\zeta}_l \int_\Omega \exp\left(ik\frac{x}{L}\right) \overline{\exp\left(il\frac{x}{L}\right)}\, dx \\ &= \sum_{k,l\in\mathbb{Z}^3} \hat{\zeta}_k \hat{\zeta}_l \int_\Omega \exp\left(i(k-l)\frac{x}{L}\right)\, dx = \sum_{k\in\mathbb{Z}^3} \left|\hat{\zeta}_k\right|^2 \underbrace{|\Omega|}_{(2\pi L)^3}. \end{aligned} \tag{27}$$

Using the definition of A, we calculate

$$\begin{aligned} A\zeta = -\Delta\zeta &= -\sum_{k\in\mathbb{Z}^3} \hat{\zeta}_k \sum_{j=1}^3 \partial_j \partial_j \Big(\exp\left(ik\frac{x}{L}\right)\Big) = -\sum_{k\in\mathbb{Z}^3} \hat{\zeta}_k \sum_{j=1}^3 \left(\frac{i}{L} k_j\right)^2 \exp\left(ik\frac{x}{L}\right) \\ &= \sum_{k\in\mathbb{Z}^3} \frac{|k|^2}{L^2} \hat{\zeta}_k \exp\left(ik\frac{x}{L}\right). \end{aligned} \tag{28}$$

Now using (28), (26), $\gamma = \frac{\alpha}{L}$ and (27), we can also calculate

$$\begin{aligned} \left\|\zeta + \alpha^2 A\zeta\right\|^2 &= \left\|\sum_{k\in\mathbb{Z}^3} \left(1 + \frac{\alpha^2}{L^2}|k|^2\right) \hat{\zeta}_k \exp\left(ik\frac{x}{L}\right)\right\|^2 \\ &= \left|\sum_{k\in\mathbb{Z}^3} \left(1 + \frac{\alpha^2}{L^2}|k|^2\right)\right|^2 \|\zeta\|^2 = \left|\sum_{k\in\mathbb{Z}^3} \left(1 + \frac{\alpha^2}{L^2}|k|^2\right)\right|^2 \left(\sum_{k\in\mathbb{Z}^3} \left|\hat{\zeta}_k\right|^2\right) (2\pi L)^3 \\ &\geq \sum_{k\in\mathbb{Z}^3} \left|\hat{\zeta}_k\right|^2 \left(1 + \gamma |k|^2\right)^2 (2\pi L)^3. \end{aligned} \tag{29}$$

We now use all of this to derive the bound we actually want. By using the Fourier expansion of ζ, the fact that $|\exp(i\varphi)| = 1$ for all $\varphi \in \mathbb{R}$, multiplying by 1 and using the inequality $\sum_k (a_k b_k)^{1/2} \le (\sum_k a_k)^{1/2} (\sum_k b_k)^{1/2}$, we get

$$\begin{aligned}
|\zeta(x)| &= \left| \sum_{k\in\mathbb{Z}^3} \hat{\zeta}_k \exp\left(ik\frac{x}{L}\right) \right| \le \sum_{k\in\mathbb{Z}^3} \left|\hat{\zeta}_k\right| \\
&= \sum_{k\in\mathbb{Z}^3} \left(\left|\hat{\zeta}_k\right|^2 \left(1+\gamma^2 |k|^2\right)^2 \left(1+\gamma^2 |k|^2\right)^{-2} \right)^{\frac{1}{2}} \\
&\le \left(\sum_{k\in\mathbb{Z}^3} \left|\hat{\zeta}_k\right|^2 \left(1+\gamma^2 |k|^2\right)^2 \right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}^3} \left(1+\gamma^2 |k|^2\right)^{-2} \right)^{\frac{1}{2}}.
\end{aligned} \tag{30}$$

Now we prove that there is $C = C(\gamma)$ such that

$$\sum_{k\in\mathbb{Z}^3} \left(1+\gamma^2 |k|^2\right)^{-2} \le C^2(\gamma). \tag{31}$$

We start by choosing $c_1^2 > 0$ such that $c_1^2 < \frac{|k|}{p^{4/3}}$ and therefore

$$\sum_{k\in\mathbb{Z}^3} \left(1+\gamma^2 |k|^2\right)^{-2} \le \sum_{p=0}^{\infty} \left(1 + c_1^2\gamma^2 p^{4/3}\right)^{-2}.$$

For a number $a \in \mathbb{R}$ we denote by $[a]$ the smallest number $n \in \mathbb{N}$ such that $n \ge a$. We then split the sum as

$$\sum_{p=0}^{\infty} \left(1 + c_1^2\gamma^2 p^{4/3}\right)^{-2} = \sum_{p=0}^{\left[(c_1\gamma)^{-3/2}\right]} \left(1 + c_1^2\gamma^2 p^{4/3}\right)^{-2} + \sum_{p=\left[(c_1\gamma)^{-3/2}\right]}^{\infty} \left(1 + c_1^2\gamma^2 p^{4/3}\right)^{-2}. \tag{32}$$

Writing $P := \left(1 + c_1^2\gamma^2 p^{4/3}\right)^{-2}$ and expanding P, we see that we can estimate

$$P \le 1 \quad \text{and} \quad P \le \frac{1}{c_1^4\gamma^4 p^{8/3}}.$$

Using the first estimate for the first sum and the second estimate for the second sum in (32), we obtain

$$\begin{aligned}
&\sum_{p=0}^{\left[(c_1\gamma)^{-3/2}\right]} 1 + \sum_{p=\left[(c_1\gamma)^{-3/2}\right]}^{\infty} \frac{1}{c_1^4\gamma^4 p^{8/3}} \\
\le& 1 + (c_1\gamma)^{-(3/2)} + \int_{(c_1\gamma)^{-3/2}}^{\infty} \frac{1}{c_1^4\gamma^4 p^{8/3}}\, dp = 1 + \left(\frac{1}{c_1^2\gamma^2}\right)^{3/4} + \frac{8}{5}\left(\frac{1}{c_1^2\gamma^2}\right)^{3/4} = 1 + \frac{13}{5}\left(\frac{1}{c_1^2\gamma^2}\right)^{3/4} \\
=:& C^2(\gamma),
\end{aligned}$$

and have therefore achieved our goal. Immediately using this estimate (31) and (29) in (30), we get

$$\|\zeta\|_{L^\infty} \leq C(\gamma) \sum_{k\in\mathbb{Z}^3} \left(\left|\hat{\zeta}_k\right|^2 \left(1+\gamma^2 |k|^2\right)^2 \right)^{\frac{1}{2}} = C(\gamma)(2\pi L)^{-3/2} \left\|\zeta + \alpha^2 A\zeta\right\|,$$

which is the desired estimate. □

Proposition 2. *Let $\{\psi_1 \dots \psi_N\}$ be orthonormal in the L^2 inner product. Let $\zeta_j \in H^2(\Omega)$ such that $\zeta_j = (I+\alpha^2 A)^{-1}(\psi_j), \quad j = 1, 2, \dots, N$. Let $|\rho_N(x)|^2 = \sum_{j=1}^N |\zeta_j(x)|^2$. Then there exists a constant $C_F(\gamma)$ independent of N such that*

$$\|\rho_N\|_{L^\infty} \leq C_F(\gamma)(2\pi L)^{-\frac{3}{2}}.$$

Proof. Let $\Theta_1, \dots, \Theta_N \in \mathbb{R}$ be such that $\sum_{j=1}^N \Theta_j^2 = 1$. Then, using the result of Proposition 1 on each ζ_j individually, using the definition of ψ_j, the assumption that $\sum_{j=1}^N \Theta_j^2 = 1$ and finally the orthogonality of $(\psi)_{j=1}^N$ we get

$$\begin{aligned}
\left|\sum_{j=1}^N \Theta_j \zeta_j(x)\right| &\leq C(\gamma)(2\pi L)^{-3/2} \left\|\sum_{j=1}^N \Theta_j \left(\zeta_j + \alpha^2 A\zeta_j\right)\right\| \\
&= C(\gamma)(2\pi L)^{-3/2} \left\|\sum_{j=1}^N \Theta_j \psi_j\right\| = C(\gamma)(2\pi L)^{-3/2} \left(\sum_{j=1}^N \Theta_j^2\right)^{1/2} \\
&= C(\gamma)(2\pi L)^{-3/2}
\end{aligned}$$

for all $x \in \Omega$. Squaring the inequality and denoting by ζ_j^k the k-th component of the vector ζ_j, we get

$$\left(\sum_{j=1}^N \Theta_j \zeta_j^1(x)\right)^2 + \left(\sum_{j=1}^N \Theta_j \zeta_j^2(x)\right)^2 + \left(\sum_{j=1}^N \Theta_j \zeta_j^3(x)\right)^2 \leq C^2(\gamma)(2\pi L)^{-3}, \quad x \in \Omega.$$

Choosing first $\Theta_j = (\sum_{j=1}^N \zeta_j^1(x)^2)^{-1/2} \zeta_j^1(x)$, we obtain

$$\sum_{j=1}^N (\zeta_j^1(x))^2 \leq C(\gamma)^2 (2\pi L)^{-3}, \quad x \in \Omega.$$

Next, choosing $\Theta_j = (\sum_{j=1}^N \zeta_j^2(x)^2)^{-1/2} \zeta_j^1(x)$ and then $\Theta_j = (\sum_{j=1}^N \zeta_j^1(x)^3)^{-1/2} \zeta_j^1(x)$, we obtain analogous estimates for $\sum_{j=1}^N (\zeta_j^2(x))^2$ and $\sum_{j=1}^N (\zeta_j^3(x))^2$. Finally, we have

$$|\rho_N(x)|^2 = \sum_{j=1}^N |\zeta_j(x)|^2 = \sum_{j=1}^N \zeta_j^1(x)^2 + \sum_{j=1}^N \zeta_j^2(x)^2 \sum_{j=1}^N \zeta_j^3(x)^2 \leq 3C(\gamma)^2 (2\pi L)^{-3}, \quad x \in \Omega.$$

□

Plugging the results of Proposition 1 and Proposition 2 into (25) and then using the Sobolev-Lieb-Thirring Inequality in three dimensions ($n = 3$, $m = 1$), we obtain

$$\begin{aligned}
\mathrm{Tr}(Q_N(t)\Lambda(t)\circ Q_N(t)) &\geq \nu \sum_{j=1}^{N} \|\psi_j\|_V^2 - \|u(t)\|_V \, \|\rho_N\|_{L^\infty} \, N^{1/2} \\
&\geq \nu \sum_{j=1}^{N} \|\psi_j\|_V^2 - \|u(t)\|_V \, C_F(\gamma)(2\pi L)^{-3/2} N^{1/2} \\
&\geq \nu c_1 \left((2\pi L)^3\right)^{-2/3} N^{5/3} - \|u(t)\|_V \, C_F(\gamma)(2\pi L)^{-3/2} N^{1/2} \\
&= \nu c_2 L^{-2} N^{5/3} - \|u(t)\|_V \, C_F(\gamma)(2\pi L)^{-3/2} N^{1/2}.
\end{aligned}$$

We now have

$$\mathrm{Tr}(Q_N(t)\circ\Lambda(t)\circ Q_N(t)) \geq \nu c_2 L^{-2} N^{5/3} - \|u(t)\|_V \, C_F(\gamma)(2\pi L)^{-3/2} N^{1/2},$$

which means

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{Tr}(Q_N(t)\circ\Lambda(t)\circ Q_N(t))\, dt \\
&\qquad \geq \nu c_2 L^{-2} N^{5/3} - C_F(\gamma)(2\pi L)^{-3/2} N^{1/2} \limsup_{T\to\infty} \left(\frac{1}{T}\int_0^T \|u(t)\|_V^2 \, dt\right)^{1/2}.
\end{aligned}$$

Defining $\epsilon_{\mathrm{Leray}}$, the mean rate of dissipation of energy, and l_d, the viscous dissipation length scale, by

$$\epsilon_{\mathrm{Leray}} := \frac{\nu}{(2\pi L)^3} \sup_{u(0)\in X} \limsup_{t\to\infty} \left(\frac{1}{T}\int_0^T \|u(t)\|_V^2 \, dt\right) \quad \text{and} \quad l_d := \left(\frac{\nu^3}{\epsilon_{\mathrm{Leray}}}\right)^{1/4}$$

we then have

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{Tr}(Q_N(t)\circ\Lambda(t)\circ Q_N(t))\, dt \\
&\qquad \geq \nu c_2 L^{-2} N^{5/3} - C_F(\gamma)(2\pi L)^{-3/2} N^{1/2} \left(\frac{(2\pi L)^3}{\nu}\epsilon_{\mathrm{Leray}}\right)^{1/2} \\
&\qquad > 0 \\
&\Leftrightarrow \quad N > \left(\frac{L}{l_d}\right)^{12/7} \left(\frac{C_F(\gamma)}{c_2}\right)^{6/7}.
\end{aligned}$$

Therefore, according to the so called trace formula explained in the beginning of this chapter, we have

$$\text{Hausdorff dimension of X} =: d_H(X) \leq \left(\frac{L}{l_d}\right)^{12/7} \left(\frac{C_F(\gamma)}{c_2}\right)^{6/7}.$$

4 Numerical Analysis

4.1 Finite Element Method

At its heart, the finite element method relies on the weak formulation of a problem. Transforming the problem into a problem formulated in the right function spaces, in this case a Hilbert space, one can use theorems from functional analysis such as Riesz, Lax-Milgram or Babuska's extension of the Lax-Milgram theorem, see for example Layton (2008), Chapter 2 Theorem 9, to prove existence and uniqueness of a solution for the weak formulation of a problem. In classical PDE theory, one would then further use regularity theorems to obtain conditions under which such a weak solution is actually regular. The aim of the finite element method, however, is to approximate a solution given by the above theorems. To this end, one has to assume that the Hilbert space H in which the solution lives has a countable orthonormal basis. Then, one can select finite-dimensional subspaces $H_1, H_2, ... \subset H$ that approximate H from within. This means that for each $v \in H$, there will be a sequence $(v_k)_{k\in\mathbb{N}}$ with $v_k \in H_k$ for each k that converges to v. In the case that we can formulate our problem so that it only involves the scalar product of a certain Hilbert space, we can use the theorem of Riesz to get existence and uniqueness of a solution. In the case that the bilinear form involved is not the scalar product but is still continuous and *coercive*, meaning there exists a constant $C > 0$ independent of v such that $C\left\|v\right\|_H^2 \leq a(v,v)$, we can use the Lax-Milgram Theorem.

In the case of the Navier-Stokes equations, however, there is one more difficulty. The equations contain two unknowns, the velocity u and the pressure p, that also live in two different function spaces. By choosing a divergence-free subspace from which the approximation functions come, one can eliminate the pressure from the weak formulation. In practice, the construction of a divergence-free subspace is not that easy (see for example the Argyris finite element). One then has to use what is called a mixed method. Instead of eliminating the pressure from the equation, one of the bilinear forms involved will take variables from two different spaces, namely the pressure space Q and the velocity space X. But this means that the coercivity condition, that guaranteed existence of a solution, needs to be modified.

If a is coercive on H, it follows that

$$C\left\|v\right\|_H \leq \sup_{w\in H} \frac{a(w,v)}{\left\|w\right\|_H} \quad \forall v \in H.$$

This weaker condition now makes sense for a bilinear form $b : H \times \Pi \to \mathbb{R}$, where Π is another function space. The condition

$$C\left\|p\right\|_\Pi \leq \sup_{v\in H} \frac{b(v,p)}{\left\|v\right\|_H} \quad \forall p \in \Pi$$

is called *inf-sup condition*. The discrete version of this, meaning the inequality that has to hold for the approximation functions from the finite element spaces, is the *discrete inf-sup condition* or $LBB^h - condition$, named after its inventors:

$$0 < C \leq \inf_{q\in\Pi^h} \sup_{v\in H^h} \frac{|b(v,p)|}{\left\|v\right\|_H \left\|q\right\|_\Pi}.$$

The LBB^h-condition can not only guarantee the existence and uniqueness of a pressure p^h for the velocity v^h, but also ensures that the choices for the finite element pressure space and the finite element velocity space fit, so to speak. An example for what can go wrong if LBB^h is not satisfied is given in Layton (2008), Section 4.3.
Another important question to consider when choosing a finite element space is whether it is *conforming*, meaning it is actually contained in the function space it is supposed to approximate. In the following analysis, we will always suppose the finite element space to be conforming and to satisfy the LBB^h condition. An extensive treatment of finite element spaces and their properties can be found in Brenner and Scott (1994).
A common choice of finite element spaces for fluid dynamics are the so-called Q_2/P_1^{disc} *elements*. They are used in our numerical experiments and they are defined as follows: Let T be a triangulation of Ω. Then, the velocity-space X^h is defined by

$$X^h = Q_2 := \left\{ v^h \in H_0^1(\Omega) : v^h\big|_\tau \text{ is triquadratic} \quad \forall \tau \in T \right\}.$$

The pressure space is defined by

$$Q^h = P_1^{disc} := \left\{ q^h \in L_0^2(\Omega) : q^h\big|_\tau \text{ is linear} \quad \forall \tau \in T \right\}.$$

Note that the pressure space is not presumed to be continuous across cells.
If we choose $X = H_0^1$ and $Q = L_0^2$ we obviously have $X^h \subset X$ and $Q^h \subset Q$. Therefore, Q_2/P_1^{disc} is conforming. It also satisfies the LBB^h condition, see Matthies and Tobiska (2002).

4.1.1 The Stokes Problem as an Example

The Stokes equations are given by

$$-\Delta u + \nabla p = f, \quad \nabla \cdot u = 0,$$

where as usual u is the velocity of the fluid and p is the pressure and we suppose u vanishes on $\partial\Omega$. The variational formulation then is

$$\int_\Omega \nabla u : \nabla v - \nabla \cdot vp = \int_\Omega f \cdot v \quad \forall v \in X, \qquad \int_\Omega \nabla \cdot uq = 0 \quad \forall q \in Q,$$

where X and Q are some appropriate function spaces. Setting

$$a(u,v) := \int_\Omega \nabla u : \nabla v \quad \text{and} \quad b(u,q) := \int_\Omega \nabla \cdot uq,$$

this becomes

$$a(u,v) - b(v,p) = F(v) \quad \forall v \in X, \qquad b(u,q) = 0 \quad \forall q \in Q. \tag{33}$$

These expressions now make sense for $u \in H_0^1(\Omega) = X$ and $p \in Q$ where

$$Q = \left\{ q \in L^2 : \int_\Omega q = 0 \right\}.$$

The bilinear form a is coercive on $H_0^1(\Omega)$, as can be seen by using the Poincaré-Friedrichs inequality:

$$\begin{aligned} 2a(v,v) = 2\int_\Omega |\nabla v|^2 &= \|\nabla v\|^2 + \|\nabla v\|^2 \\ &\geq \|\nabla v\|^2 + C\|v\|^2 \geq \max\{1, C\} \|\nabla v\|_{H_0^1}^2 . \end{aligned}$$

As a consequence of the continuity of b, see Layton (2008), Chapter 4, the space

$$V = \left\{ v \in H_0^1(\Omega) : \quad b(v,q) = 0 \quad \forall q \in Q \right\}$$

is a closed subspace of $H_0^1(\Omega)$ and therefore itself a Hilbert space. Since $b(u,q) = 0$ in (33), we are looking for a solution $u \in V$. The problem of finding u then becomes the problem

$$\text{Find } u \in V, \text{ such that } a(u,v) = F(v). \tag{34}$$

As a is coercive, Lax-Milgram guarantees the existence of a unique solution to this part of the problem. Having determined $u \in V$, the equation

$$b(v,p) = a(u,v) - F(v) =: \tilde{F}(v) \quad \forall v \in H_0^1(\Omega) \tag{35}$$

now must hold for the pressure p. If we assume the inf-sup-condition to hold for b, we can then use Babuska's extension of the Lax-Milgram Theorem, see Layton (2008), or prove existence directly as is done in Brenner and Scott (1994). This also means that finding $(u,p) \in (H_0^1, Q)$ such that (33) holds is equivalent to finding a solution $u \in V$ such that (34) holds and then finding $p \in Q$, such that (35) holds.

4.2 Numerical Analysis of the Leray-α Model

This analysis is based entirely on Layton et al. (2008). In their paper higher order deconvolution models are studied of which the Leray-α model is a special case. The computations herein are their calculations restricted to that special case. Herein, we will restrict ourselves to the non periodic case, as the analysis does not differ from the periodic case except for notation.

4.2.1 Preliminaries

In the non-periodic case the spaces used are

$$Q = L_0^2, \quad X = H_0^1 = \left\{ v \in H^1(\Omega), \quad v = 0 \text{ on } \partial\Omega \right\}.$$

The space of weakly divergence-free functions is

$$V := \{v \in X, \quad (\nabla \cdot v, q) = 0 \quad \forall q \in Q\}.$$

The finite element spaces X^h and Q^h are assumed to be conforming, i.e., $X^h \subset X$ and $Q^h \subset Q$ and to satisfy the LBB^h condition. We can for example think of Q_2/P_1^{disc} elements. The discretely divergence-free subset V^h of X^h is

$$V^h = \left\{ v^h \in X^h, (\nabla \cdot v^h, q^h) = 0 \quad \forall q^h \in Q^h \right\}.$$

As usual, the L^2-norm will simply be denoted by $\|\cdot\|$ and the L^2 scalar product by $(\cdot,\cdot)$. In $H^k = W^{k,2}$ the norm is denoted by $\|\cdot\|_k$, the semi-norm by $|\cdot|_k$. The spaces $L^p((0,T);H^k)$ are defined by having finite norms which are defined by

$$\|v\|_{p,k} := \left(\int_0^T \|v(t,\cdot)\|_k^p\right)^{\frac{1}{p}} \quad \text{for } 1 \le p < \infty, \tag{36}$$

and

$$\|v\|_{\infty,k} := \operatorname{ess}\sup_{0<t<T} \|v(t,\cdot)\|_k \quad \text{for } p = \infty. \tag{37}$$

The analogues for the discrete spaces are

$$|||v|||_{p,k} := \left(\sum_{n=0}^{M} \Delta t \, \|v_n\|_k^p\right)^{\frac{1}{p}}, \quad \left|\left|\left|v_{\frac{1}{2}}\right|\right|\right|_{p,k} := \left(\sum_{n=1}^{M} \Delta t \left\|v_{n-\frac{1}{2}}\right\|_k^p\right)^{\frac{1}{p}}, \tag{38}$$

and

$$|||v|||_{\infty,k} := \sup_{0\le n\le M} \|v_n\|_k \,, \quad \left|\left|\left|v_{\frac{1}{2}}\right|\right|\right|_{\infty,k} := \sup_{1\le n\le M} \left\|v_{n-\frac{1}{2}}\right\|_k. \tag{39}$$

For an operator $f: V_1 \to V_2$ the operator norm is defined as follows:

$$\|f\|_* := \sup_{v\in V_1} \frac{\|f(v)\|_{V_2}}{\|v\|_{V_1}}.$$

The following approximation properties of finite element spaces such as the ones above are used repeatedly:

Lemma 1.

$$\inf_{v\in X^h} \|u - v\| \le C h^{k+1} |u|_{k+1}\,, \quad u \in H^{k+1}(\Omega), \tag{40}$$

$$\inf_{v\in X^h} \|u - v\|_1 \le C h^{k} |u|_{k+1}\,, \quad u \in H^{k+1}(\Omega), \tag{41}$$

$$\inf_{r\in Q^h} \|p - r\| \le C h^{s+1} |p|_{s+1}\,, \quad p \in H^{s+1}(\Omega). \tag{42}$$

This is a corollary of the Bramble-Hilbert Lemma, see for example Brenner and Scott (1994), Chapter 4.
Next, as the Leray-α model involves a filtering operation, we provide some definitions and results for it.

Definition 4. *For $v \in L^2(\Omega)$ and $\alpha > 0$ the filtering operation on v is $\overline{v}$, where $\overline{v}$ is the unique solution of*

$$-\alpha^2 \Delta \overline{v} + \overline{v} = v. \tag{43}$$

$\overline{v}$ is called continuous differential filter. We denote by $F := (-\alpha^2\Delta + I)$ and write $F^{-1}v = \overline{v}$.

In order to define a discrete filter, we will need the two following definitions

Definition 5. *We define the L^2-projection $\Pi^h : L^2 \to X^h$ by*

$$(\Pi^h v - v, \xi) = 0 \quad \forall \xi \in X^h,$$

and the discrete Laplacian $\Delta_h : X \to X^h$ by

$$(\Delta_h v, \xi) = -(\nabla v, \nabla \xi) \quad \forall \xi \in X^h. \tag{44}$$

We set $F_h = -\alpha^2 \Delta_h + \Pi^h$.

Then, the discrete version of this filter can be defined:

Definition 6. *For $v \in L^2(\Omega)$ and $\alpha > 0$ the discrete filtering operation on v is $\overline{v}^h$, where $\overline{v}^h$ is the unique solution in X^h of*

$$\alpha^2 (\nabla \overline{v}^h, \nabla \xi) + (\overline{v}^h, \xi) = (v, \xi) \quad \forall \xi \in X^h. \tag{45}$$

We write $\overline{v}^h = F_h^{-1} v$.

Note that we only assume $v \in L^2$, which means an expression like $\nabla \overline{v^h}^h$, which will appear in the error analysis, makes sense. The next two lemmata provide us with some estimates on the discrete filter.

Lemma 2. *For $v \in X$, it holds*

$$\left\| \overline{v}^h \right\| \leq \|v\|, \quad \left\| \nabla \overline{v}^h \right\| \leq \|\nabla v\|. \tag{46}$$

Proof. In the definition of the discrete filter (45), set $\xi = \overline{v}^h$. We then have, using Cauchy-Schwarz and Young's inequality with $p = q = \frac{1}{2}$

$$\begin{aligned}
&\alpha^2 \left(\nabla \overline{v}^h, \nabla \overline{v}^h \right) + \left(\overline{v}^h, \overline{v}^h \right) = \left(v, \overline{v}^h \right) \\
\Rightarrow\ &\alpha^2 \left\| \nabla \overline{v}^h \right\|^2 + \left\| \overline{v}^h \right\|^2 \leq \left| \left(v, \overline{v}^h \right) \right| \leq \|v\| \left\| \overline{v}^h \right\| \leq \frac{1}{2} \|v\|^2 + \frac{1}{2} \left\| \overline{v}^h \right\|^2 \\
\Rightarrow\ &2 \left\| \overline{v}^h \right\|^2 \leq \|v\|^2 + \left\| \overline{v}^h \right\|^2 \\
\Leftrightarrow\ &\left\| \overline{v}^h \right\|^2 \leq \|v\|^2 .
\end{aligned}$$

For the second inequality,we use the definition of the discrete Laplacian (44) on the first term in the definition of the discrete filter (45), then set $\xi = \Delta_h \overline{v}^h$ and then use the definition of the discrete Laplacian again to obtain, in the same way as in the previous inequality,

$$\begin{aligned}
&-\alpha^2 \left\| \Delta_h \overline{v}^h \right\|^2 + \left(\overline{v}^h, \Delta_h \overline{v}^h \right) = \left(v, \Delta_h \overline{v}^h \right) \\
\Rightarrow\ &-\alpha^2 \left\| \Delta_h \overline{v}^h \right\|^2 - \left\| \nabla \overline{v}^h \right\|^2 = -\left(\nabla v, \nabla \overline{v}^h \right) \\
\Rightarrow\ &\alpha^2 \left\| \Delta_h \overline{v}^h \right\|^2 + \left\| \nabla \overline{v}^h \right\|^2 \leq \left| \left(\nabla v, \nabla \overline{v}^h \right) \right| \leq \frac{1}{2} \|\nabla v\|^2 + \frac{1}{2} \left\| \nabla \overline{v}^h \right\|^2 \\
\Rightarrow\ &\left\| \nabla \overline{v}^h \right\| \leq \|\nabla v\| .
\end{aligned}$$

$\square$

Lemma 3. *For $v \in X$, $\Delta v \in L^2(\Omega)$ it holds*

$$\alpha^2 \left\|\nabla(v - \overline{v}^h)\right\|^2 + \left\|v - \overline{v}^h\right\|^2 \leq C \sup_{\xi \in X^h} \left\{\alpha^2 \left\|\nabla(v-\xi)\right\|^2 + \|v-\xi\|^2\right\} + C\alpha^4 \|\Delta v\|^2 .$$

Proof. By definition of the discrete filter $\overline{v}^h$ satisfies

$$\alpha^2 \left(\nabla \overline{v}^h, \nabla \xi\right) + (\overline{v}^h, \xi) = (v, \xi) \quad \forall \xi \in X^h, \tag{47}$$

and by assumption v satisfies

$$\alpha^2 (\nabla v, \nabla \xi) + (v, \xi) = -\alpha^2 (\Delta v, \xi) + (v, \xi) \quad \forall \xi \in X^h. \tag{48}$$

Setting $e = v - \overline{v}^h$ and subtracting the first equation (47) from the second equation (48), we obtain

$$\alpha^2 (\nabla e, \nabla \xi) + (e, \xi) = -\alpha^2 (\Delta v, \xi). \tag{49}$$

Let $\tilde{v} \in X^h$ be arbitrary. Then, $e = (v - \tilde{v}) - (\overline{v}^h - \tilde{v}) =: \eta - \varphi$, where $\varphi \in X^h$ and we have split the error into a part that lies in X^h and into one that doesn't. The right-hand side of the inequality that we want to prove suggests that the strategy should be to keep the terms involving η on the right-hand side and to estimate from above terms involving φ. We then have

$$\begin{aligned} &\alpha^2(\nabla\eta, \nabla\xi) - \alpha^2(\nabla\varphi, \nabla\xi) + (\eta, \xi) - (\varphi, \xi) = -\alpha^2(\Delta v, \xi) \\ \Rightarrow\quad &\alpha^2(\nabla\varphi, \nabla\xi) + (\varphi, \xi) = \alpha^2(\Delta v, \xi) + \alpha^2(\nabla\eta, \nabla\xi) + (\eta, \xi). \end{aligned}$$

Setting $\xi = \varphi$ and using Cauchy-Schwarz, Young's inequality with $p = q = 2$ on the term in the middle and Young's inequality with $\epsilon = \frac{1}{4}$ on the first and the third term, we get

$$\begin{aligned} \alpha^2 \|\nabla\varphi\|^2 + \|\varphi\|^2 &\leq \alpha^2 \left|(\Delta v, \varphi)\right| + \alpha^2 \left|(\nabla\eta, \nabla\varphi)\right| + \left|(\eta, \varphi)\right| \\ &\leq \alpha^2 \|\Delta v\| \, \|\varphi\| + \alpha^2 \|\nabla\eta\| \, \|\nabla\varphi\| + \|\eta\| \, \|\varphi\| \\ &\leq \frac{1}{2} \|\varphi\|^2 + \alpha^4 \|\Delta v\|^2 + \alpha^2 \frac{1}{2} \|\nabla\eta\|^2 + \alpha^2 \frac{1}{2} \|\nabla\varphi\|^2 + \|\eta\|^2 . \end{aligned}$$

Multiplying by 2 and bringing every term involving φ to the left-hand side, we get

$$\alpha^2 \|\nabla\varphi\|^2 + \|\varphi\|^2 \leq 2\alpha^4 \|\Delta v\|^2 + \alpha^2 \|\nabla\eta\|^2 + 2\|\eta\|^2 .$$

We now use $v - \overline{v}^h = e = \eta - \varphi$ and the above calculation to obtain

$$\begin{aligned} \alpha^2 \left\|\nabla(v - \overline{v}^h)\right\|^2 + \left\|v - \overline{v}^h\right\|^2 &= \alpha^2 \|\nabla e\|^2 + \|e\|^2 \\ &\leq 2\left(\alpha^2 \|\nabla\eta\|^2 + \alpha^2 \|\nabla\varphi\|^2 + \|\eta\|^2 + \|\varphi\|^2\right) \\ &\leq C\left(\alpha^2 \|\nabla\eta\|^2 + \|\eta\|^2 + \alpha^4 \|\Delta v\|^2\right) \\ &\leq C\left(\alpha^2 \|\nabla(v - \tilde{v})\|^2 + \|v - \tilde{v}\|^2 + \alpha^4 \|\Delta v\|^2\right). \end{aligned}$$

As $\tilde{v}$ was arbitrary, we get

$$\alpha^2 \left\|\nabla(v - \overline{v}^h)\right\|^2 + \left\|v - \overline{v}^h\right\|^2 \leq C \inf_{\xi \in X^h} \left\{\alpha^2 \left\|\nabla(v - \xi)\right\|^2 + \left\|v - \xi\right\|^2\right\} + C\alpha^4 \left\|\Delta v\right\|^2 .$$

□

Lemma 4. *For $v \in H^{k-1}$ it holds*

$$\left\|v - \overline{v}^h\right\| \leq C(\alpha h^k + h^{k+1}) \left|\overline{v}\right|_{k+1} + C\alpha^2 \left\|\Delta F^{-1} v\right\| .$$

Proof. We begin by splitting the difference:

$$\left\|v - \overline{v}^h\right\| \leq \left\|v - \overline{v}\right\| + \left\|\overline{v} - \overline{v}^h\right\| .$$

From the definition of the filtering operation we immediately get

$$\left\|v - \overline{v}\right\| = \alpha^2 \left\|\Delta F^{-1} v\right\| .$$

For the second term, we use the definitions of the differential filter (43) and the discrete filter (45):

$$\begin{aligned} \alpha^2(\nabla \overline{v}, \nabla \xi) + (\overline{v}, \xi) &= (v, \xi) \quad \forall \xi \in X^h, \\ \alpha^2(\nabla \overline{v}^h, \nabla \xi) + (\overline{v}^h, \xi) &= (v, \xi) \quad \forall \xi \in X^h. \end{aligned}$$

Setting $e = \overline{v} - \overline{v}^h$ and subtracting the above equations, we get

$$\alpha^2(\nabla e, \nabla \xi) + (e, \xi) = 0 \quad \forall \xi \in X^h.$$

This is exactly the same situation as in (49) in the previous lemma, except we do not have a right-hand side. This means that from here, we can proceed in the same way as in the previous lemma to obtain an estimate, but we don´t have the contributions from the right-hand side. Also using inequalities (40) and (41) in the last step, we then arrive at

$$\begin{aligned} \alpha^2 \left\|\nabla(\overline{v} - \overline{v}^h)\right\|^2 + \left\|\overline{v} - \overline{v}^h\right\|^2 &\leq C \inf_{\xi \in X^h} \left\{\alpha^2 \left\|\nabla(\overline{v} - \xi)\right\|^2 + \left\|\overline{v} - \xi\right\|^2\right\} \\ \Rightarrow \left\|\overline{v} - \overline{v}^h\right\| &\leq C(\alpha \inf_{\xi \in X^h} \left\|\nabla(\overline{v} - \xi)\right\| + \inf_{\xi \in X^h} \left\|\overline{v} - \xi\right\|) \\ &\leq C(\alpha h^k \left|\overline{v}\right|_{k+1} + h^{k+1} \left|\overline{v}\right|_{k+1}) = C(\alpha h^k + h^{k+1}) \left|\overline{v}\right|_{k+1} . \end{aligned}$$

Putting everything together, we get

$$\begin{aligned} \left\|v - \overline{v}^h\right\| &\leq \left\|v - \overline{v}\right\| + \left\|\overline{v} - \overline{v}^h\right\| \\ &\leq \alpha^2 \left\|\Delta F^{-1} v\right\| + C(\alpha h^k + h^{k+1}) \left|\overline{v}\right|_{k+1} . \end{aligned}$$

□

Note that the estimate provided by Lemma 4 is only useful if $\overline{v}$ can actually be bounded independent of α. We just note that under the assumption that $\partial\Omega \in C^{k+3}$ and $v \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$, one can show $\overline{v} \in H_0^1(\Omega) \cap H^{k+3}(\Omega)$ and $\|\overline{v}\|_j \leq C \|v\|_j$ for $j = 0, 1, 2$, where C does not depend on α. This means an assumption of $v \in H_0^1$ would be enough for this estimate to make sense. Estimates for higher k are possible under additional assumptions, see Layton et al. (2008), Remark 2.15 and references therein.
Also note that in the definition of the discrete filter (45) the solution is sought in X^h and not in V^h, which means that the discrete divergence of the discrete filtering operation $\overline{v}^h$ is not necessarily zero and therefore incompressibility, which was modeled by the velocity being divergence-free, is not preserved. This can be rectified by modifying the model. In the Navier-Stokes equations the nonlinear term can be expressed as an instance of the trilinear form $b(u, v, w) = ((u \cdot \nabla)v, w)$ which, under the assumption that u is divergence-free, is skew-symmetric. If we now want to replace u by $\overline{u}$ as is done in the Leray-α model, and $\overline{u}$ is not divergence-free, we have to modify the equation to preserve skew-symmetry and with it incompressibility. One easily calculates

$$\begin{aligned} \frac{1}{2}b(\overline{u}, v, w) &= -\frac{1}{2}b(\overline{u}, w, v) - \frac{1}{2}((\nabla \cdot \overline{u})v, w), \\ \Rightarrow b(\overline{u}, v, w) + \frac{1}{2}((\nabla \cdot \overline{u})v, w) &= \underbrace{-\frac{1}{2}b(\overline{u}, w, v) + \frac{1}{2}b(\overline{u}, v, w).}_{=:b^*(u,v,w)} \end{aligned}$$

Now $b^*(u, v, w)$ is clearly skew-symmetric and we see that we have to modify our model by adding $\frac{1}{2}((\nabla \cdot \overline{u})v, w)$.

Definition 7. *The skew-symmetric trilinear form $b^* : X \times X \times X \to \mathbb{R}$ is defined by*

$$b^*(u, v, w) := \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v).$$

We note that for $v, w \in X$ and $u \in V$

$$\begin{aligned} b^*(u, v, w) &= ((u \cdot \nabla)v, w) \\ \text{and} \quad b^*(u, v, v) &= 0 \quad \forall u, v \in X. \end{aligned}$$

The Leray-α model modified to preserve incompressibility then is

$$\left.\begin{aligned} u_t + (\overline{u} \cdot \nabla)u + \frac{1}{2}(\nabla \cdot \overline{u}) - \nu\Delta u + \nabla p = f, \\ \nabla \cdot u = 0 \qquad \text{in } \Omega \times (0, T), \\ + \text{ boundary conditions and initial condition,} \end{aligned}\right\}$$

or in its weak formulation

$$\left.\begin{aligned} (u_t, v) + b^*(\overline{u}, u, v) + \nu(\nabla u, \nabla u) + (p, \nabla \cdot v) = (f, v) \quad \forall v \in X, \\ (\nabla \cdot u, q) = 0 \quad \forall q \in Q, \\ + \text{ initial condition.} \end{aligned}\right\}$$

We now list some properties of b^* to be used later.

Lemma 5. *For $u, v, w \in X$ and $v, \nabla v \in L^\infty(\Omega)$ the trilinear term b^* can be bounded:*

$$|b^*(u,v,w)| \leq \frac{1}{2}\left(\|u\| \, \|\nabla v\|_\infty \, \|w\| + \|u\| \, \|\nabla w\| \, \|v\|_\infty\right), \tag{50}$$

$$|b^*(u,v,w)| \leq C_0(\Omega) \, \|\nabla u\| \, \|\nabla v\| \, \|\nabla w\| , \tag{51}$$

$$|b^*(u,v,w)| \leq C_0(\Omega) \, \|u\|^{\frac{1}{2}} \, \|\nabla u\|^{\frac{1}{2}} \, \|\nabla v\| \, \|\nabla w\| . \tag{52}$$

Proof. We make use of the generalized Hölder's inequality for three functions:

$$\int_\Omega |u| \, |v| \, |w| \leq \|u\|_{L^q} \, \|v\|_{L^p} \, \|w\|_{L^r} \qquad \text{for } \frac{1}{q} + \frac{1}{p} + \frac{1}{r} = 1, \quad 1 \leq p, q, r < \infty.$$

For the first inequality (50) we get, using the triangle inequality and the assumptions v, $\nabla v \in L^\infty$,

$$\begin{aligned}
|b^*(u,v,w)| &\leq \frac{1}{2}\left(\int_\Omega |u| \, |\nabla v| \, |w| + \int_\Omega |u| \, |\nabla w| \, |v|\right) \\
&\leq \frac{1}{2}\left(\|\nabla v\|_\infty \int_\Omega |u| \, |w| + \|v\|_\infty \int_\Omega |u| \, |\nabla w|\right) \\
&\leq \frac{1}{2}\left(\|\nabla v\|_\infty \, \|\nabla u\| \, \|w\| + \|v\|_\infty \, \|\nabla u\| \, \|\nabla w\|\right).
\end{aligned}$$

For the second inequality (51) we use the generalized Hölder's inequality with $q = 2$ and $p = r = 4$. Using the Sobolev embedding theorem with $q = 4$, $p = \frac{1}{2}$ and therefore $m = \frac{3}{4}$ and then the interpolation inequality with $s_1 = 0$, $s_2 = 1$ and $\Theta = \frac{1}{4}$, we have the estimate for the L^4-norms

$$\|u\|_{L^4} \leq C \, \|u\|_{H^{\frac{3}{4}}} \leq C \, \|u\|_{L^2}^{\frac{1}{4}} \, \|u\|_{H^1}^{\frac{3}{4}} .$$

Due to the Poincaré inequality, the H^1 norm and $\|\nabla v\|_{L^2}$ are equivalent on H_0^1. We then get

$$\begin{aligned}
|b^*(u,v,w)| &\leq \|u\|_{L^4} \, \|\nabla v\|_{L^2} \, \|w\|_{L^4} + \|u\|_{L^4} \, \|\nabla w\|_{L^2} \, \|v\|_{L^4} \\
&\leq C \, \|u\|^{\frac{1}{4}} \, \|\nabla u\|^{\frac{3}{4}} \, \|\nabla v\| \, \|w\|^{\frac{1}{4}} \, \|\nabla w\|^{\frac{3}{4}} \\
&\quad + \|u\|^{\frac{1}{4}} \, \|\nabla u\|^{\frac{3}{4}} \, \|\nabla w\| \, \|v\|^{\frac{1}{4}} \, \|\nabla v\|^{\frac{3}{4}} .
\end{aligned}$$

One more application of the Poincaré inequality on the terms not containing gradients gets us the desired result. For the last inequality (52) we use the generalized Hölder´s inequality with $p = 3$, $q = 2$ and $r = 6$. Using the Sobolev embedding theorem with $q = 3$, $p = 2$ and therefore $m = \frac{1}{2}$ and then the interpolation inequality with $s_1 = 0$, $s_2 = 1$ and $\Theta = \frac{1}{2}$, we have the estimate for the L^3-norm

$$\|u\|_{L^3} \leq C \, \|u\|_{H^{\frac{1}{2}}} \leq C \, \|u\|_{L^2}^{\frac{1}{2}} \, \|u\|_{H^1}^{\frac{1}{2}} ,$$

and using the Sobolev embedding theorem with $q = 6$, $p = 2$, and therefore $m = 1$ we get the estimate for the L^6-norm

$$\|u\|_{L^6} \leq C\, \|u\|_{H^1}\,.$$

Again using the equivalence of norms mentioned above, we get

$$\begin{aligned}|b^*(u,v,w)| &\leq \|u\|_{L^3}\, \|\nabla v\|_{L^2}\, \|w\|_{L^6} + \|u\|_{L^3}\, \|\nabla w\|_{L^2}\, \|v\|_{L^6} \\ &\leq C\, \|u\|^{\frac{1}{2}}\, \|\nabla u\|^{\frac{1}{2}}\, \|\nabla v\|\, \|\nabla w\|\,.\end{aligned}$$

□

For the estimates in the next section, we will need the *discrete Grönwall inequality.*

Lemma 6. *Let $\Delta t, H, a_n, b_n, c_n, d_n > 0$ such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + H,$$

and suppose $\Delta t d_n < 1\ \forall n$. Then,

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq exp\left(\Delta t \sum_{n=0}^{l} \frac{d_n}{1-\Delta t d_n}\right)\left(\Delta t \sum_{n=0}^{l} c_n + H\right).$$

4.2.2 Analysis of the Crank-Nicolson Scheme

Throughout this text, the notation $v_{n+\frac{1}{2}} := \frac{(v_n+v_{n+1})}{2}$ is used.
The Crank-Nicholson method is a time-stepping method that can be derived by taking the average of a forward Euler at time $t = t_n$ and of a backward Euler in $t = t_{n+1}$. The discretization in space is done by finite element methods. However, there are two different notions of Crank-Nicolson method in the literature. To illustrate, let

$$\frac{\partial u}{\partial t} = F(u)$$

be some type of differential equation, where F is some kind of (differential) operator. Then, using forward Euler at time t_n and backward Euler at time t_{n+1} yields

$$\text{F.E.: } \frac{u_{n+1}-u_n}{\Delta t} = F(u_n), \quad \text{B.E.:} \frac{u_{n+1}-u_n}{\Delta t} = F(u_{n+1}),$$

and taking the average of these yields

$$\frac{u_{n+1}-u_n}{\Delta t} = \frac{1}{2}\left(F(u_{n+1}) + F(u_n)\right). \tag{53}$$

If F is linear, this is equivalent to

$$\frac{u_{n+1}-u_n}{\Delta t} = F(u_{n+\frac{1}{2}}). \tag{54}$$

One can now define the Crank-Nicolson method by the averaging which results in (53) or simply by defining it as (54). Herein, the numerical analysis will be performed for the version given by (54). The results are valid for the version (53) also, which is lacking some terms compared to (54). The version (53) will because of its simpler form be used in the computations.
As it will play a key role in the numerical analysis, the first lemma provides some general estimates regarding the quantity $v_{n+\frac{1}{2}}$.

Lemma 7. *Assume $u \in C^0(t_n, t_{n+1}; L^2(\Omega))$.*
If u is twice continuously differentiable in time and $u_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$, then

$$\left\| u_{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}}) \right\|^2 \leq C(\Delta t)^3 \int_{t^n}^{t^{n+1}} \|u_{tt}\|^2 \, dt. \tag{55}$$

Assume $u_t \in C^0(t_n, t_{n+1}; L^2(\Omega))$.
If u is three times continuously differentiable in time and $u_{ttt} \in L^2((t_n, t_{n+1}) \times \Omega)$, then

$$\left\| \frac{u_{n+1} - u_n}{\Delta t} - u_t(t_{n+\frac{1}{2}}) \right\|^2 \leq C(\Delta t)^3 \int_{t^n}^{t^{n+1}} \|u_{ttt}\|^2 \, dt. \tag{56}$$

Assume $\nabla u \in C^0(t_n, t_{n+1}; L^2(\Omega))$.
If the function ∇u_{tt} is continuous in time and $\nabla u_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$, then

$$\left\| \nabla(u_{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}})) \right\|^2 \leq C(\Delta t)^2 \int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^2 \, dt. \tag{57}$$

Proof. This proof is based on the Taylor expansion with remainder in mean value form. For $u \in C^2(t^n, t^{n+1})$ we can write

$$\begin{aligned} u(t_{n+1}) &= u(t_{n+\frac{1}{2}}) + u_t(t_{n+\frac{1}{2}}) \left(\frac{\Delta t}{2}\right) + \frac{u_{tt}(c_1)}{2} \left(\frac{\Delta t}{2}\right)^2 \quad \text{for } c_1 \in (t_{n+\frac{1}{2}}, t_{n+1}), \\ u(t_n) &= u(t_{n+\frac{1}{2}}) + u_t(t_{n+\frac{1}{2}}) \left(\frac{-\Delta t}{2}\right) + \frac{u_{tt}(c_2)}{2} \left(\frac{\Delta t}{2}\right)^2 \quad \text{for } c_2 \in (t_n, t_{n+\frac{1}{2}}). \end{aligned}$$

This gives us

$$\begin{aligned} \left\| u_{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}}) \right\|^2 = \left\| \frac{1}{4} \frac{(\Delta t)^2}{4} (u_{tt}(c_1) + u_{tt}(c_2)) \right\|^2 &= C(\Delta t)^4 \|u_{tt}(c_1) + u_{tt}(c_2)\|^2 \\ &\leq C(\Delta t)^3 \left(\frac{\Delta t}{2}\right) \left(\|u_{tt}(c_1)\|^2 + \|u_{tt}(c_2)\|^2\right), \end{aligned}$$

using $\frac{1}{2} \|u_{tt}(c_1) + u_{tt}(c_2)\|^2 \leq \|u_{tt}(c_1)\|^2 + \|u_{tt}(c_2)\|^2$. Because of the triangle inequality, the norm of a continuous function is a continuous function. As we assumed u to be twice continuously differentiable in time, this means the function $t \mapsto \|u_{tt}(t, \cdot)\|_{L^2}$ is continuous. Therefore, we can apply the midpoint rule. The midpoint rule for integrals states that there exists $\xi \in \left(t_n, t_{n+\frac{1}{2}}\right)$ such that $\left(\frac{\Delta t}{2}\right) \|u_{tt}(\xi)\|^2 = \int_{t_n}^{t_{n+\frac{1}{2}}} \|u_{tt}\|^2 \, dt$. We then get

$$\begin{aligned}
C(\Delta t)^3 \left(\frac{\Delta t}{2}\right) & (\|u_{tt}(c_1)\|^2 + \|u_{tt}(c_2)\|^2) \\
&= C(\Delta t)^3 \left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \|u_{tt}\|^2 \, dt + \int_{t_n}^{t_{n+\frac{1}{2}}} \|u_{tt}\|^2 \, dt \right) \\
&= C(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|u_{tt}\|^2 \, dt,
\end{aligned}$$

where the constant provided in the midpoint rule (here: ξ) and the ones we get from the Taylor expansion (here: c_1, c_2) are the same, because the constant in the remainder of the Taylor expansions comes precisely from the midpoint rule.
The next two proofs follow the exact same pattern. In order to prove (56), we expand u as

$$u(t_{n+1}) = u(t_{n+\frac{1}{2}}) + u_t(t_{n+\frac{1}{2}}) \left(\frac{\Delta t}{2}\right) + \frac{u_{tt}(t_{n+\frac{1}{2}})}{2} \left(\frac{\Delta t}{2}\right)^2 + \frac{u_{ttt}(c_1)}{6} \left(\frac{\Delta t}{2}\right)^3$$

for $c_1 \in (t_{n+\frac{1}{2}}, t_{n+1})$ and

$$u(t_n) = u(t_{n+\frac{1}{2}}) + u_t(t_{n+\frac{1}{2}}) \left(\frac{-\Delta t}{2}\right) + \frac{u_{tt}(t_{n+\frac{1}{2}})}{2} \left(\frac{\Delta t}{2}\right)^2 + \frac{u_{ttt}(c_1)}{6} \left(\frac{-\Delta t}{2}\right)^3$$

for $c_2 \in (t_n, t_{n+\frac{1}{2}})$ and proceed as above. The proof for (57) is exactly the same. $\square$

Because it is utilized so often in the error estimates, we present a quick estimate on the norm of $v_{n+\frac{1}{2}}$, although it is easy to derive. One uses the definition of a norm induced by a scalar product, the triangle inequality and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\left\|v_{n+\frac{1}{2}}\right\|^2 &= \frac{1}{4} \|v_{n+1} + v_n\|^2 \le \frac{1}{4} \left(\|v_{n+1}\|^2 + 2 \left|(v_{n+1}, v_n)\right| + \|v_n\|^2 \right) \\
&\le \frac{1}{4} \left(\|v_{n+1}\|^2 + 2 \|v_{n+1}\| \|v_n\| + \|v_n\|^2 \right) \\
&\le \frac{1}{4} \left(\|v_{n+1}\|^2 + \|v_{n+1}\|^2 + \|v_n\|^2 + \|v_n\|^2 \right) \\
&= \frac{1}{2} \left(\|v_{n+1}\|^2 + \|v_n\|^2 \right).
\end{aligned} \tag{58}$$

Therefore we have

$$\sum_{n=0}^{M-1} \left\|v_{n+\frac{1}{2}}\right\|^2 \le \sum_{n=0}^{M-1} \frac{1}{2} \left(\|v_{n+1}\|^2 + \|v_n\|^2 \right) \le \sum_{n=0}^{M} \|v_n\|^2 . \tag{59}$$

Finally we present the scheme that will be analyzed.

Algorithm 1. *(Crank-Nicolson Finite Element Scheme for the Leray-α Model).* *Let $\Delta t > 0$, $(w_0, q_0) \in (X^h, Q^h)$, $f \in X^*$ and $T = M\Delta t$, where M is an integer. For $n = 0, \dots, M-1$, find (w^h_{n+1}, q^h_{n+1}) such that*

$$\frac{1}{\Delta t}\left(w^h_{n+1} - w^h_n, v^h\right) + b^*\left(\overline{w^h_{n+\frac{1}{2}}}^h, w^h_{n+\frac{1}{2}}, v^h\right) - (q^h_{n+\frac{1}{2}}, \nabla \cdot v^h) + \nu\left(\nabla w^h_{n+\frac{1}{2}}, \nabla v^h\right)$$
$$= \left(f_{n+\frac{1}{2}}, v^h\right) \quad \forall v^h \in X^h,$$
$$\left(\nabla \cdot w^h_{n+1}, \chi^h\right) = 0 \quad \forall \chi^h \in Q^h.$$

Since we assume the LBB^h condition to be satisfied, as explained in the previous section, this is equivalent to the problem to find a velocity in V^h:

$$\frac{1}{\Delta t}\left(w^h_{n+1} - w^h_n, v^h\right) + b^*\left(\overline{w^h_{n+\frac{1}{2}}}^h, w^h_{n+\frac{1}{2}}, v^h\right) + \nu\left(\nabla w^h_{n+\frac{1}{2}}, \nabla v^h\right) = \left(f_{n+\frac{1}{2}}, v^h\right) \quad \forall v^h \in V^h. \tag{60}$$

Now we do the analysis. First we note

Lemma 8. *The scheme* (60) *has a solution w^h_l for each time step $l = 1, \dots, M$. The scheme is also unconditionally stable and allows the a priori bound*

$$\left\|w^h_M\right\|^2 + \nu\Delta t\sum_{n=0}^{M-1}\left\|\nabla w^h_{n+\frac{1}{2}}\right\|^2 \le \left\|w^h_0\right\|^2 + \frac{C^2\Delta t}{\nu}\sum_{n=0}^{M-1}\left\|f_{n+\frac{1}{2}}\right\|_*^2, \tag{61}$$

where C is the constant from the Poincaré inequality.

Proof. The proof for the existence of a solution is an application of the Leray-Schauder principle, see for example Layton (2008). By replacing every w_{n+1} in the $w_{n+\frac{1}{2}}$-terms by a variable $z \in V^h$, we define a family of operators $B_n : V^h \to V^h$, setting $y = B_n(z)$, by

$$\begin{aligned}(y, v) = &-\Delta t b^*\left(\overline{\left(\frac{z + w^h_n}{2}\right)}^h, \frac{z + w^h_n}{2}, v\right) - \Delta t\nu\left(\nabla\left(\frac{z + w^h_n}{2}\right), \nabla v\right)\\ &+ (w^h_n, v) + \Delta t(f_{n+\frac{1}{2}}, v)\\ :=\ &(B_n(z), v).\end{aligned}$$

If the operators B_n are compact and if any solution $z_\lambda = \lambda B_n(z_\lambda)$ for $0 \le \lambda \le 1$, should it exist, allows a bound $\|z_\lambda\|_V \le r$ independent of λ, the Leray-Schauder fixed point theorem guarantees the existence of a solution to the fixed point problem $z = B_n(z)$. That means for an initial value w^h_n the existence of a bounded solution w^h_{n+1} is guaranteed. Therefore it is sufficient to prove the compactness of the operator B_0. For convenience of notation, we define $a(z) = \frac{z + w^h_0}{2}$.
In order to prove compactness, we define the operators

$$T : X^* \to V^h, \quad T(g) = z \quad \text{s.t.} \quad (z, v) + \Delta t\nu(\nabla a(z), \nabla v) = g(v) \quad \forall v \in V, \tag{62}$$

the solution operator of the Helmholtz equation, and

$$N : V^h \to X^*, \quad \text{s.t.} \quad (N(z), v) = (\Delta t f_{\frac{1}{2}} + w_0^h, v) - \Delta t b^*(\overline{a(z)}^h, a(z), v) \quad \forall v \in V. \tag{63}$$

We then have $B_0 = T \circ N$. T is obviously linear. It is also bounded as can be seen by setting $v = z$ in the Helmholtz equation (62). One then uses the definition of the operator norm on g, the Cauchy-Schwarz inequality and the fact that $\|v\|_X = \|v\|_{H_0^1} \leq (C+1)\|\nabla v\|$, which follows from the Poincaré inequality, to obtain

$$\|z\|^2 + \Delta t \nu \left(\nabla \left(\frac{z + w_0^h}{2} \right), \nabla z \right) = \|z\|^2 + \Delta t \nu \frac{1}{2} \|\nabla z\|^2 + \frac{1}{2} \Delta t \nu \left(\nabla w_0^h, \nabla z \right) = (g, z)$$

$$\begin{aligned}
\Rightarrow \quad \|\nabla z\|^2 &\leq \frac{2}{\Delta t \nu} |(g, z)| + \left| \left(\nabla w_0^h, \nabla z \right) \right| \leq \frac{2}{\Delta t \nu} \|g\|_* \|z\|_X + \left\| \nabla w_0^h \right\| \|\nabla z\| \\
&\leq \frac{2C}{\Delta t \nu} \|g\|_* \|\nabla z\| + \left\| \nabla w_0^h \right\| \|\nabla z\| \\
\Rightarrow \quad \|\nabla z\| &\leq \frac{2C}{\Delta t \nu} \|g\|_* + \left\| \nabla w_0^h \right\|.
\end{aligned}$$

Once more, we use that the H_0^1-norm is equivalent to $\|\nabla v\|$ and we see that T is a bounded operator.

As bounded linear operators are continuous, T is continuous. Because compositions of a continuous and a compact operator are compact, we now only have to prove that N is compact, as $B_0 = T \circ N$. But we have $V \subset H^1$ and by the Rellich Lemma, the embedding $H^1 \hookrightarrow H^{\frac{3}{4}}$ is compact. If we prove that N is continuous from $H^{\frac{3}{4}}$ to V^*, N is itself the composition of a compact and a continuous operator and therefore compact. The difficulty lies in bounding the nonlinearity b^*. Assuming $a(z) \in L^\infty \cap L^4$ and $\nabla a(z) \in L^\infty$, we can use the inequality (50), the Poincaré inequality, the property (46), the nestedness of L^p spaces and the Sobolev embedding theorem with $q = 4$, $p = 2$ and $m = \frac{3}{4}$ to obtain

$$\begin{aligned}
\left\| b^*(\overline{a(z)}^h, a(z), \cdot) \right\|_* &= \sup_{v \in V} \frac{\left| b^*(\overline{a(z)}^h, a(z), v) \right|}{\|v\|_V} \\
&\leq C \sup_{v \in V} \frac{\left\| \overline{a(z)}^h \right\| (\|\nabla a(z)\|_\infty \|v\| + \|\nabla v\| \|a(z)\|_\infty)}{\|\nabla v\|} \\
&\leq C \|a(z)\| \leq C \|a(z)\|_{L^4} \leq C \|a(z)\|_{H^{\frac{3}{4}}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|N\|_{V^*} &= \sup_{v \in V} \frac{|(N(z), v)|}{\|v\|_V} \\
&\leq \sup_{v \in V} \frac{\left| (\Delta t f_{\frac{1}{2}} + w_0^h, v) \right| + \Delta t \left| b^*(\overline{a(z)}^h, a(z), v) \right|}{\|\nabla v\|} \\
&\leq \Delta t \left\| f_{\frac{1}{2}} \right\|_* + \left\| w_0^h \right\|_* + \Delta t \|a(z)\|_{H^{\frac{3}{4}}} \leq C \|a(z)\|_{H^{\frac{3}{4}}},
\end{aligned}$$

where $C = C(\|f\|_*, \left\|w_0^h\right\|_*, \Delta t)$. Therefore, N is continuous and B_0 is compact. Finally, we check whether a z_λ satisfying $(z_\lambda, v) = \lambda(B_0(z_\lambda), v)$ for some $\lambda \in [0,1]$ is bounded independent of λ. We use $B_0 = T \circ N$, set $v = a(z)$ in the definition of T (62), use the property $b^*(u,v,v) = 0$ and the assumption $\lambda \leq 1$ to obtain

$$\begin{aligned}
&(z_\lambda, v) = \lambda(B(z_\lambda), v) \quad \forall v \in V \\
\Leftrightarrow\ &(z_\lambda, v) + \Delta t\nu(\nabla a(z_\lambda), \nabla v) = \lambda(N(z_\lambda), v) \quad \forall v \in V \\
\Rightarrow\ &(z_\lambda, a(z_\lambda)) + \Delta t\nu(\nabla a(z_\lambda), \nabla a(z_\lambda)) \leq \Delta t(f_{\frac{1}{2}} + w_0^h, a(z_\lambda)).
\end{aligned}$$

Taking the w_0^h-term to the left-hand side, using the definition of the operator norm on the f- term, the Poincaré inequality on $\|a(z)\|$ and the Peter-Paul inequality with $\epsilon = \nu$ we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|(z_\lambda)\|^2 - \left\|w_0^h\right\|^2\right) + \Delta t\nu \|\nabla a(z_\lambda)\|^2 \leq \Delta t C^2 \frac{\left\|f_{\frac{1}{2}}\right\|_*^2}{2\nu} + \Delta t\nu \frac{\|\nabla a(z_\lambda)\|^2}{2} \\
\Rightarrow\quad &\|\nabla a(z_\lambda)\|^2 \leq \frac{C^2}{\nu^2}\left\|f_{\frac{1}{2}}\right\|_*^2 + \frac{1}{\Delta t\nu}\left\|w_0^h\right\|^2 =: M^2 \\
\Rightarrow\quad &\|\nabla a(z_\lambda)\| \leq M.
\end{aligned} \tag{64}$$

Further, we have

$$\|\nabla z_\lambda\| = \left\|\nabla z_\lambda + \nabla w_0^h - \nabla w_0^h\right\| \leq \left\|\nabla z_\lambda + \nabla w_0^h\right\| + \left\|\nabla w_0^h\right\| = 2\|\nabla a(z_\lambda)\| \leq 2M + \left\|\nabla w_0^h\right\|.$$

Once more we use the Poincaré inequality to obtain

$$\|z_\lambda\|_X \leq (C+1)\|\nabla z\| \leq (C+1)\left(2M + \left\|w_0^h\right\|\right).$$

Therefore, z_λ is bounded independent from λ. This means that the operator B_0 possesses a fixed point which is the solution to our scheme. As mentioned above, this same proof works for B_n, whenever $\left\|f_{n+1/2}\right\|_* \leq C$ and $\left\|w_n^h\right\|_* \leq C$ for some C. For the a priori estimate (61), which is basically the same calculation as (64), one sets $v^h = w^h_{n+\frac{1}{2}}$ in the scheme (60). Using the definition of the operator norm, the Poincaré inequality and in the last step the Peter-Paul inequality with $\epsilon = \nu$ one obtains

$$\begin{aligned}
\frac{1}{2\Delta t}\left(\left\|w_{n+1}^h\right\|^2 - \left\|w_n^h\right\|^2\right) + \nu\left\|\nabla w^h_{n+\frac{1}{2}}\right\|^2 &\leq \left|\left(f_{n+\frac{1}{2}}, w_{n+\frac{1}{2}}\right)\right| \leq C\left\|f_{n+\frac{1}{2}}\right\|_*\left\|\nabla w^h_{n+\frac{1}{2}}\right\| \\
&\leq \frac{C^2}{2\nu}\left\|f_{n+\frac{1}{2}}\right\|_*^2 + \frac{\nu}{2}\left\|\nabla w^h_{n+\frac{1}{2}}\right\|^2 \quad \forall n.
\end{aligned}$$

Multiplying by two and bringing all terms involving $\nabla w^h_{n+\frac{1}{2}}$ to the left-hand side and finally summing from $n = 0$ to $n = M-1$ and multiplying by Δt completes the calculation. $\square$

Next, we present the main convergence result:

Theorem 4. *Let (u(t),p(t)) be a strong solution of the NSE with either no-slip boundary conditions or being periodic with zero mean such that for $k \in \mathbb{N}$ the bounding norms are finite. Suppose (w_0^h, q_0^h) are approximations of $(u(0), p(0))$ such that estimates (40) - (42) in Lemma 1 hold and $u \in L^\infty\left(H^{k+1}\right)$, $p \in L^\infty\left(H^{s+1}\right)$ for some $k, s \in \mathbb{N}$. Then for small enough Δt there is a constant $C = C(u,p)$ such that*

$$\left|\left|\left|u - w^h\right|\right|\right|_{\infty,0} \leq H(\Delta t, h, \alpha) + Ch^{k+1} \left|\left|\left|u\right|\right|\right|_{\infty,k+1},$$

$$\left(\nu \Delta t \sum_{n=0}^{M-1} \left\| \nabla\left(u_{n+\frac{1}{2}} - w^h_{n+\frac{1}{2}}\right)\right\|^2\right)^{\frac{1}{2}} \leq H(\Delta t, h, \alpha) + C\nu^{\frac{1}{2}}(\Delta t)^2 \left\|\nabla u_{tt}\right\|_{2,0} + C\nu^{\frac{1}{2}} h^k \left|\left|\left|u\right|\right|\right|_{2,k+1},$$

where

$$\begin{aligned} H(\Delta t, h, \alpha) := C^* \Bigg[& \nu^{-\frac{1}{2}} h^{k+\frac{1}{2}} \left(|||u|||^2_{4,k+1} + |||\nabla u|||^2_{4,0}\right) + \nu^{\frac{1}{2}} h^k |||u|||_{2,k+1} \\ & + \nu^{-\frac{1}{2}} h^k \left(|||u|||^2_{4,k+1} + \nu^{-\frac{1}{2}}\left(\left\|w_0^h\right\| + \nu^{-\frac{1}{2}} |||f|||_{2,*}\right)\right) \\ & + +\nu^{-\frac{1}{2}} h^{s+1} \left|\left|\left|p_{\frac{1}{2}}\right|\right|\right|_{2,s+1} + \nu^{-\frac{1}{2}} \alpha^2 \left\|\Delta F^{-1} u\right\|_{2,0} \\ & + \nu^{-\frac{1}{2}} \left(\alpha h^k + h^{k+1}\right) \|\overline{u}\|_{2,k+1} \\ & + (\Delta t)^2 \Bigg(\|u_{ttt}\|_{2,0} + \nu^{-\frac{1}{2}} \|p_{tt}\|_{2,0} + \|f_{tt}\|_{2,0} + \nu^{\frac{1}{2}} \|\nabla u_{tt}\|_{2,0} \\ & + \nu^{-\frac{1}{2}} \|\nabla u_{tt}\|^2_{4,0} + \nu^{-\frac{1}{2}} |||\nabla u|||^2_{4,0} + \nu^{-\frac{1}{2}} \left|\left|\left|\nabla u_{\frac{1}{2}}\right|\right|\right|^2_{4,0} \Bigg) \Bigg]. \end{aligned}$$

The constant C^* is dependent on ν: $C(\nu) = \exp(\nu^{-3} T)$. There is a smallness assumption on the time step needed in the Grönwall Lemma: $\Delta t < C\left(\nu^{-3} |||\nabla u|||^4_{\infty,0} + 1\right)^{-1}$.

Proof. At time $t_{n+\frac{1}{2}}$ a strong solution u of the Navier-Stokes equations satisfies

$$\left(\left(u_t, v^h\right) + \left((u \cdot \nabla)u, v^h\right) - \left(p, \nabla \cdot v^h\right) + \nu\left(\nabla u, \nabla v^h\right)\right)\Big|_{t=t_{n+\frac{1}{2}}} = (f, v^h)\Big|_{t=t_{n+\frac{1}{2}}}. \tag{65}$$

In order to approximate our scheme from Algorithm 1, we first write (65) as

$$\begin{aligned} &\left(\frac{u_{n+1} - u_n}{\Delta t}, v^h\right) + b^*\left(\overline{u_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, v^h\right) + \nu\left(\nabla u_{n+\frac{1}{2}}, \nabla v^h\right) - \left(p_{n+\frac{1}{2}}, \nabla \cdot v^h\right) \\ &= \left(f_{n+\frac{1}{2}}, v^h\right) + \mathrm{IntErr}\left(u_n, p_n; v^h\right) \quad \forall v^h \in V^h, \end{aligned} \tag{66}$$

where the usual interpolation error here takes the form

$$\begin{aligned}\operatorname{IntErr}\left(u_n, p_n; v^h\right) = &\left(\frac{u_{n+1}-u_n}{\Delta t} - u_t\big|_{t=t_{n+\frac{1}{2}}}, v^h\right) + \nu\left(\nabla u_{n+\frac{1}{2}} - \nabla u\big|_{t=t_{n+\frac{1}{2}}}, \nabla v^h\right)\\ &+ b^*\left(u_{n+\frac{1}{2}}, u_{n+\frac{1}{2}}, v^h\right) - b^*\left(u\big|_{t=t_{n+\frac{1}{2}}}, u\big|_{t=t_{n+\frac{1}{2}}}, v^h\right)\\ &- b^*\left(u_{n+\frac{1}{2}} - \overline{u_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, v^h\right)\\ &- \left(p_{n+\frac{1}{2}} - p\big|_{t=t_{n+\frac{1}{2}}}, \nabla\cdot v^h\right) + \left(f\big|_{t=t_{n+\frac{1}{2}}} - f_{n+\frac{1}{2}}, v^h\right).\end{aligned}$$

This is now in a form that can be compared to the Crank-Nicolson Algorithm (60). Note that

$$\begin{aligned}b^*\left(u_{n+\frac{1}{2}}, u_{n+\frac{1}{2}}, v^h\right) - b^*\left(u\big|_{t=t_{n+\frac{1}{2}}}, u\big|_{t=t_{n+\frac{1}{2}}}, v^h\right) - b^*\left(u_{n+\frac{1}{2}} - \overline{u_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, v^h\right)\\ = b^*\left(\overline{u_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, v^h\right) - b^*(u\big|_{t=t_{n+\frac{1}{2}}}, u\big|_{t=t_{n+\frac{1}{2}}}, v^h)\end{aligned}$$

and this contains terms of the form $b^*(\overline{u_{n+1}}^h, u_n, v^h) + b^*(\overline{u_n}^h, u_{n+1}, v^h)$ that would not be present in the simpler Crank-Nicolson scheme, see the beginning of this chapter. Setting $e_n = u_n - w_n^h$, we proceed by subtracting the Crank-Nicolson scheme (60) from (66) and obtain an equation for the error in each step:

$$\begin{aligned}\frac{1}{\Delta t}\left(e_{n+1} - e_n, v^h\right) + b^*\left(\overline{u_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, v^h\right) - b^*\left(\overline{w^h_{n+\frac{1}{2}}}^h, w^h_{n+\frac{1}{2}}, v^h\right) + \nu\left(\nabla e_{n+\frac{1}{2}}, \nabla v^h\right)\\ = \left(p_{n+\frac{1}{2}}, \nabla\cdot v^h\right) + \operatorname{IntErr}\left(u_n, p_n; v^h\right) \quad \forall v^h \in V^h.\end{aligned} \tag{67}$$

Denoting by U_n the L^2-projection of $u\big|_{t=t_n}$ in V^h, we can decompose the error e_n into the part that lies in V^h and the part that is orthogonal to V^h by writing $e_n = (u_n - U_n) - (w_n^h - U_n) := \eta_n - \varphi_n^h$. Here, $\varphi_n^h \in V^h$ and the strategy is again to keep the terms involving η and to estimate from above the terms involving φ. We now choose $v^h = \varphi^h_{n+\frac{1}{2}}$ in (67) and separating the parts of the error that lie in V^h and those that do not, we get for all $q^h \in Q^h$

$$\begin{aligned}&\left(\varphi^h_{n+1} - \varphi^h_n, \varphi^h_{n+\frac{1}{2}}\right) + \nu\Delta t\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 + \Delta t b^*\left(\overline{w^h_{n+\frac{1}{2}}}^h, e_{n+\frac{1}{2}}, \varphi^h_{n+\frac{1}{2}}\right)\\ &\quad + \Delta t b^*\left(\overline{e_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, \varphi^h_{n+\frac{1}{2}}\right)\\ &= \left(\eta_{n+1} - \eta_n, \varphi^h_{n+\frac{1}{2}}\right) + \Delta t\nu\left(\nabla\eta_{n+\frac{1}{2}}, \nabla\varphi^h_{n+\frac{1}{2}}\right) + \Delta t\left(p_{n+\frac{1}{2}} - q^h, \nabla\cdot\varphi^h_{n+\frac{1}{2}}\right)\\ &\quad + \operatorname{IntErr}(u_n, p_n; \varphi^h_{n+\frac{1}{2}}),\end{aligned}$$

and then, using $(\varphi^h_{n+1}-\varphi^h_n,\varphi^h_{n+\frac{1}{2}}) = (\varphi^h_{n+1}-\varphi^h_n,\frac{1}{2}(\varphi^h_{n+1}+\varphi^h_n)) = \frac{1}{2}\left(\left\|\varphi^h_{n+1}\right\|^2 - \left\|\varphi^h_n\right\|^2\right)$ and $\overline{e_{n+\frac{1}{2}}} = \overline{\eta_{n+\frac{1}{2}}} - \overline{\varphi^h_{n+\frac{1}{2}}}$ and $b^*(u,v,v)=0$ and $\left(\eta_{n+1}-\eta_n,\varphi^h_{n+\frac{1}{2}}\right)=0$ we get

$$
\begin{aligned}
&\frac{1}{2}\left(\left\|\varphi^h_{n+1}\right\|^2 - \left\|\varphi^h_n\right\|^2\right) + \nu\Delta t\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&= \nu\Delta t\left(\nabla\eta_{n+\frac{1}{2}},\nabla\varphi^h_{n+\frac{1}{2}}\right) - \Delta t b^*\left(\overline{\eta_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right) + \Delta t b^*\left(\overline{\varphi^h_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right) \\
&\quad - \Delta t b^*\left(\overline{w^h_{n+\frac{1}{2}}}^h, \eta_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right) + \Delta t\left(p_{n+\frac{1}{2}} - q^h, \nabla\cdot\varphi^h_{n+\frac{1}{2}}\right) \\
&\quad + \Delta t\ \mathrm{IntErr}\left(u_n,p_n;\varphi^h_{n+\frac{1}{2}}\right) \\
&\leq \left|\nu\Delta t\left(\nabla\eta_{n+\frac{1}{2}},\nabla\varphi^h_{n+\frac{1}{2}}\right)\right| + \left|\Delta t b^*\left(\overline{\eta_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right)\right| \\
&\quad + \left|\Delta t b^*\left(\overline{\varphi^h_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right)\right| + \left|\Delta t b^*\left(\overline{w^h_{n+\frac{1}{2}}}^h, \eta_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right)\right| \\
&\quad + \left|\Delta t\left(p_{n+\frac{1}{2}} - q^h, \nabla\cdot\varphi^h_{n+\frac{1}{2}}\right)\right| + \left|\Delta t\ \mathrm{IntErr}\left(u_n,p_n;\varphi^h_{n+\frac{1}{2}}\right)\right|.
\end{aligned}
\tag{68}
$$

We proceed by bounding the terms on the right-hand side. First, we use the Cauchy-Schwarz and the Peter-Paul inequality with $\epsilon = 10$ to obtain

$$
\begin{aligned}
\nu\Delta t\left(\nabla\eta_{n+\frac{1}{2}},\nabla\varphi^h_{n+\frac{1}{2}}\right) &\leq \nu\Delta t\left\|\nabla\eta_{n+\frac{1}{2}}\right\|\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\| \\
&\leq \frac{\nu\Delta t}{20}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 + C\nu\Delta t\left\|\nabla\eta_{n+\frac{1}{2}}\right\|^2,
\end{aligned}
$$

and then we again use Cauchy-Schwarz, the fact that $\|\nabla\cdot\varphi\| \leq \|\nabla\varphi\|$ and Peter-Paul with $\epsilon = \frac{10}{\nu}$ to obtain

$$
\begin{aligned}
\Delta t\left(p_{n+\frac{1}{2}} - q^h, \nabla\cdot\varphi^h_{n+\frac{1}{2}}\right) &\leq \Delta t\left\|p_{n+\frac{1}{2}} - q^h\right\|\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\| \\
&\leq \frac{\nu\Delta t}{20}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 + C\Delta t\nu^{-1}\left\|p_{n+\frac{1}{2}} - q^h\right\|^2.
\end{aligned}
$$

We proceed by treating the b^*-terms. Using the already established inequality (50) for b^* and Peter-Paul with $\epsilon = \frac{10}{\nu}$ and (46) we estimate:

$$
\begin{aligned}
\left|\Delta t b^*\left(\overline{\eta_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right)\right| &\leq C\Delta t\left\|\overline{\eta_{n+\frac{1}{2}}}^h\right\|^{\frac{1}{2}}\left\|\nabla\overline{\eta_{n+\frac{1}{2}}}^h\right\|^{\frac{1}{2}}\left\|\nabla u_{n+\frac{1}{2}}\right\|\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\| \\
&\leq \frac{\nu\Delta t}{20}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\quad + C\Delta t\nu^{-1}\left\|\nabla\eta_{n+\frac{1}{2}}\right\|\left\|\eta^h_{n+\frac{1}{2}}\right\|\left\|\nabla u_{n+\frac{1}{2}}\right\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\left|\Delta t b^*\left(\overline{w^h_{n+\frac{1}{2}}}^h, \eta_{n+\frac{1}{2}}, \varphi^h_{n+\frac{1}{2}}\right)\right| &\leq C\Delta t \left\|\overline{w^h_{n+\frac{1}{2}}}^h\right\|^{\frac{1}{2}} \left\|\nabla \overline{w^h_{n+\frac{1}{2}}}^h\right\|^{\frac{1}{2}} \left\|\nabla \eta_{n+\frac{1}{2}}\right\| \left\|\nabla \varphi^h_{n+\frac{1}{2}}\right\| \\
&\leq \frac{\nu \Delta t}{20} \left\|\nabla \varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\quad + C\Delta t \nu^{-1} \left\|\nabla w^h_{n+\frac{1}{2}}\right\| \left\|w^h_{n+\frac{1}{2}}\right\| \left\|\nabla \eta_{n+\frac{1}{2}}\right\|^2 .
\end{aligned}
$$

For the third term, we use the the same properties of b^*, the properties of the discrete filter from (46) and then Young's inequality with $\epsilon = \frac{\nu}{20}$ and $p = \frac{4}{3}$ and $q = 4$ to obtain

$$
\begin{aligned}
\Delta t b^*\left(\overline{\varphi^h_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, \varphi^h_{n+\frac{1}{2}}\right) &\leq C\Delta t \left\|\overline{\varphi^h_{n+\frac{1}{2}}}^h\right\|^{\frac{1}{2}} \left\|\overline{\nabla \varphi^h_{n+\frac{1}{2}}}^h\right\|^{\frac{1}{2}} \left\|\nabla u_{n+\frac{1}{2}}\right\| \left\|\nabla \varphi^h_{n+\frac{1}{2}}\right\| \\
&\leq C\Delta t \left\|\varphi^h_{n+\frac{1}{2}}\right\|^{\frac{1}{2}} \left\|\nabla \varphi^h_{n+\frac{1}{2}}\right\|^{\frac{3}{2}} \left\|\nabla u^h_{n+\frac{1}{2}}\right\| \\
&\leq \frac{\nu \Delta t}{20} \left\|\nabla \varphi^h_{n+\frac{1}{2}}\right\|^2 + C\Delta t \nu^{-3} \left\|\varphi^h_{n+\frac{1}{2}}\right\|^2 \left\|\nabla u_{n+\frac{1}{2}}\right\|^4 .
\end{aligned}
$$

Recall that we assume w^h_n to be the solution we computed using the scheme Algorithm 1 by starting with w^h_0 being the projection of u_0 into V^h. Then, by assumption $\left\|\varphi^h_0\right\| = 0$ and summing (68) from 0 to $M-1$ we obtain

$$
\begin{aligned}
&\frac{1}{2}\left\|\varphi^h_M\right\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \left\|\nabla \varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\leq C\nu^{-3}\Delta t \sum_{n=0}^{M-1} \left\|\varphi^h_{n+\frac{1}{2}}\right\|^2 \left\|\nabla u_{n+\frac{1}{2}}\right\|^4 + \frac{1}{4}\nu \Delta t \sum_{n=0}^{M-1} \left\|\nabla \varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\quad + C\nu \Delta t \sum_{n=0}^{M-1} \left\|\nabla \eta_{n+\frac{1}{2}}\right\|^2 + C\nu^{-1}\Delta t \sum_{n=0}^{M-1} \left\|\eta_{n+\frac{1}{2}}\right\| \left\|\nabla \eta_{n+\frac{1}{2}}\right\| \left\|\nabla u_{n+\frac{1}{2}}\right\|^2 \\
&\quad + C\nu^{-1}\Delta t \sum_{n=0}^{M-1} \left\|w^h_{n+\frac{1}{2}}\right\| \left\|\nabla w^h_{n+\frac{1}{2}}\right\| \left\|\nabla \eta_{n+\frac{1}{2}}\right\|^2 \\
&\quad + C\nu^{-1}\Delta t \sum_{n=0}^{M-1} \left\|p_{n+\frac{1}{2}} - q\right\|^2 + \Delta t \left|\sum_{n=0}^{M-1} \mathrm{IntErr}(u_n, p_n; \varphi^h_{n+\frac{1}{2}})\right| .
\end{aligned}
\tag{69}
$$

We will take care of the terms on the right-hand side involving φ later and proceed in further bounding the other terms on the right-hand side of (69). We use (59), the approximation property estimate (41) and the definition of the discrete norm (38):

$$
\begin{aligned}
C\nu \Delta t \sum_{n=0}^{M-1} \left\|\nabla \eta_{n+\frac{1}{2}}\right\|^2 &\leq C\nu \Delta t \sum_{n=0}^{M} \left\|\nabla \eta_n\right\|^2 \\
&\leq C\nu \Delta t \sum_{n=0}^{M} h^{2k} \left|u_n\right|^2_{k+1} \leq C\nu h^{2k} \left|\left|\left|u\right|\right|\right|^2_{2,k+1} .
\end{aligned}
\tag{70}
$$

For the next term, we use the approximation property estimates (40) and (41) and in the step after that we use Cauchy's inequality on each summand and then apply the same steps as in (58) just with the exponent 4. In the last step, one uses the definition of the discrete norm to obtain

$$
\begin{aligned}
&C\nu^{-1}\Delta t\sum_{n=0}^{M-1}\left\|\eta_{n+\frac{1}{2}}\right\|\left\|\nabla\eta_{n+\frac{1}{2}}\right\|\left\|\nabla u_{n+\frac{1}{2}}\right\|^2\\
&\quad\leq C\nu^{-1}\Delta t\sum_{n=0}^{M-1}\left(\|\eta_{n+1}\|+\|\eta_n\|\right)\left(\|\nabla\eta_{n+1}\|+\|\nabla\eta_n\|\right)\left\|\nabla u_{n+\frac{1}{2}}\right\|^2\\
&\quad\leq C\nu^{-1}\Delta t h^{2k+1}\sum_{n=0}^{M-1}\left(|u_{n+1}|_{k+1}^2+|u_{n+1}|_{k+1}|u_n|_{k+1}+|u_n|_{k+1}^2\right)\left\|\nabla u_{n+\frac{1}{2}}\right\|^2\\
&\quad\leq C\nu^{-1}h^{2k+1}\left(\Delta t\sum_{n=0}^{M}|u_n|_{k+1}^4+\Delta t\sum_{n=0}^{M}\|\nabla u_n\|^4\right)\\
&\quad= C\nu^{-1}h^{2k+1}\left(|||u_n|||_{4,k+1}^4+|||\nabla u_n|||_{4,0}^4\right).
\end{aligned}
\tag{71}
$$

For the next term, we start by using the already established a priori estimate (61) to absorb the $\left\|w^h_{n+\frac{1}{2}}\right\|$-terms into the constant. Note that we assume $\Delta t=\mathcal{O}(\nu)$. Otherwise, the constant we get from the a priori estimate is $C=C(\frac{1}{\nu})$. We then again use the approximation inequality (41), (58)-type calculations and in the last step again the a priori estimate, this time to estimate the gradient of $w^h_{n+\frac{1}{2}}$:

$$
\begin{aligned}
&C\nu^{-1}\Delta t\sum_{n=0}^{M-1}\left\|w^h_{n+\frac{1}{2}}\right\|\left\|\nabla w^h_{n+\frac{1}{2}}\right\|\left\|\nabla\eta_{n+\frac{1}{2}}\right\|^2\\
&\quad\leq C\nu^{-1}\Delta t\sum_{n=0}^{M-1}\left\|\nabla w^h_{n+\frac{1}{2}}\right\|\left\|\nabla\eta_{n+\frac{1}{2}}\right\|^2\\
&\quad\leq C\nu^{-1}\Delta t\sum_{n=0}^{M-1}\left(\|\nabla\eta_{n+1}\|^2+\|\nabla\eta_n\|^2\right)\left\|\nabla w^h_{n+\frac{1}{2}}\right\|\\
&\quad\leq C\nu^{-1}\Delta t h^{2k}\sum_{n=0}^{M-1}\left(|u_{n+1}|_{k+1}^2+|u_n|_{k+1}^2\right)\left\|\nabla w^h_{n+\frac{1}{2}}\right\|\\
&\quad\leq C\nu^{-1}h^{2k}\left(\Delta t\sum_{n=0}^{M}|u_n|_{k+1}^4+\Delta t\sum_{n=0}^{M-1}\left\|\nabla w^h_{n+\frac{1}{2}}\right\|^2\right)\\
&\quad\leq Ch^{2k}\nu^{-1}\left(|||u|||_{4,k+1}^4+\frac{1}{\nu}\left\|w^h_0\right\|^2+\frac{1}{\nu^2}\Delta t\sum_{n=0}^{M-1}\left\|f_{n+\frac{1}{2}}\right\|_*^2\right)\\
&\quad= Ch^{2k}\nu^{-1}\left(|||u|||_{4,k+1}^4+\frac{1}{\nu}\left\|w^h_0\right\|^2+\frac{1}{\nu^2}|||f|||_{2,*}^2\right).
\end{aligned}
\tag{72}
$$

For the pressure term we add a zero, use the approximation inequality (42) and the integral inequality (55) and the definition of the norms to get

$$
\begin{aligned}
C\nu^{-1}\Delta t\sum_{n=0}^{M-1}\left\|p_{n+\frac{1}{2}}-q^h\right\|^2 &\le C\nu^{-1}\Delta t\sum_{n=0}^{M-1}\left\|p(t_{n+\frac{1}{2}})-q^h\right\|^2+\left\|p_{n+\frac{1}{2}}-p(t_{n+\frac{1}{2}})\right\|^2 \\
&\le C\nu^{-1}\left(h^{2s+2}\Delta t\sum_{n=0}^{M-1}\left\|p(t_{n+\frac{1}{2}})\right\|_{s+1}^2+\Delta t\sum_{n=0}^{M-1}C(\Delta t)^3\int_{t_n}^{t_{n+1}}\|p_{tt}\|^2\,dt\right) \\
&\le C\nu^{-1}\left(h^{2s+2}\left|\left|\left|p_{\frac{1}{2}}\right|\right|\right|_{2,s+1}^2+(\Delta t)^4\,\|p_{tt}\|_{2,0}^2\right).
\end{aligned}
\tag{73}
$$

We proceed with the interpolation error. Using in the first step the Cauchy-Schwarz and Cauchy's inequalities and in the second step the integral inequalities (56) and (55) from Lemma 7 respectively, we get

$$
\begin{aligned}
\left|\left(\frac{u_{n+1}-u_n}{\Delta t}-u_t(t_{n+\frac{1}{2}}),\varphi^h_{n+\frac{1}{2}}\right)\right| &\le \frac{1}{2}\left\|\varphi^h_{n+\frac{1}{2}}\right\|^2+\frac{1}{2}\left\|\frac{u_{n+1}-u_n}{\Delta t}-u_t(t_{n+\frac{1}{2}})\right\|^2 \\
&\le \frac{1}{2}\left\|\varphi^h_{n+\frac{1}{2}}\right\|^2+C(\Delta t)^3\int_{t^n}^{t^{n+1}}\|u_{ttt}\|^2\,dt,
\end{aligned}
\tag{74}
$$

and

$$
\begin{aligned}
\left|\left(f(t_{n+\frac{1}{2}})-f_{n+\frac{1}{2}},\varphi^h_{n+\frac{1}{2}}\right)\right| &\le \frac{1}{2}\left\|\varphi^h_{n+\frac{1}{2}}\right\|^2+\frac{1}{2}\left\|f\left(t_{n+\frac{1}{2}}\right)-f_{n+\frac{1}{2}}\right\|^2 \\
&\le \frac{1}{2}\left\|\varphi^h_{n+\frac{1}{2}}\right\|^2+C(\Delta t)^3\int_{t^n}^{t^{n+1}}\|f_{tt}\|^2\,dt.
\end{aligned}
\tag{75}
$$

Using Cauchy-Schwarz and Peter-Paul with $\epsilon=\frac{8}{\nu}$ and in the second step Lemma 7 we obtain

$$
\begin{aligned}
\left|\left(p_{n+\frac{1}{2}}-p(t_{n+\frac{1}{2}}),\nabla\cdot\varphi^h_{n+\frac{1}{2}}\right)\right| &\le \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2+C\nu^{-1}\left\|p_{n+\frac{1}{2}}-p(t_{n+\frac{1}{2}})\right\|^2 \\
&\le \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2+C\nu^{-1}(\Delta t)^3\int_{t^n}^{t^{n+1}}\|p_{tt}\|^2\,dt,
\end{aligned}
\tag{76}
$$

and (with $\epsilon=8$, because this term already has a ν)

$$
\begin{aligned}
\nu\left|\left(\nabla u_{n+\frac{1}{2}}-\nabla u(t_{n+\frac{1}{2}}),\nabla\varphi^h_{n+\frac{1}{2}}\right)\right| &\le \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2+C\nu\left\|\nabla u_{n+\frac{1}{2}}-\nabla u(t_{n+\frac{1}{2}})\right\|^2 \\
&\le \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2+C\nu(\Delta t)^3\int_{t^n}^{t^{n+1}}\|u_{tt}\|^2\,dt.
\end{aligned}
\tag{77}
$$

For the b^* terms we get, using (51), Peter-Paul with $\epsilon=\frac{8}{\nu}$, the fact that $(|a|+|b|)^2\le 2\left(|a|^2+|b|^2\right)$ and Lemma 7 and then Cauchy´s inequality under the integral

$$
\begin{aligned}
& b^*\left(u_{n+\frac{1}{2}}, u_{n+\frac{1}{2}}, \varphi^h_{n+\frac{1}{2}}\right) - b^*\left(u(t_{n+\frac{1}{2}}), u(t_{n+\frac{1}{2}}), \varphi^h_{n+\frac{1}{2}}\right) \\
&\quad \leq \left| b^*\left(u_{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}}), u_{n+\frac{1}{2}}, \varphi^h_{n+\frac{1}{2}}\right) + b^*\left(u(t_{n+\frac{1}{2}}), u_{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}}), \varphi^h_{n+\frac{1}{2}}\right)\right| \\
&\quad \leq C\left\|\nabla\left(u_{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}})\right)\right\| \left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\| \left(\left\|\nabla u_{n+\frac{1}{2}}\right\| + \left\|u(t_{n+\frac{1}{2}})\right\|\right) \\
&\quad \leq \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 + C\nu^{-1}\left(\left\|\nabla u_{n+\frac{1}{2}}\right\| + \left\|u(t_{n+\frac{1}{2}})\right\|\right)^2 \left\|\nabla\left(u_{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}})\right)\right\|^2 \\
&\quad \leq C\nu^{-1}\left(\left\|\nabla u_{n+\frac{1}{2}}\right\|^2 + \left\|u(t_{n+\frac{1}{2}})\right\|^2\right)(\Delta t)^3 \int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^2\, dt + \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\quad \leq C\nu^{-1}(\Delta t)^3 \int_{t^n}^{t^{n+1}} \left\|\nabla u_{n+\frac{1}{2}}\right\|^2 \|\nabla u_{tt}\|^2 + \left\|\nabla u(t_{n+\frac{1}{2}})\right\|^2 \|\nabla u_{tt}\|^2\, dt + \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\quad \leq C\nu^{-1}(\Delta t)^3 \int_{t^n}^{t^{n+1}} \frac{1}{2}\left\|\nabla u_{n+\frac{1}{2}}\right\|^4 + \frac{1}{2}\|\nabla u_{tt}\|^4 + \frac{1}{2}\left\|\nabla u(t_{n+\frac{1}{2}})\right\|^4 + \frac{1}{2}\|\nabla u_{tt}\|^4\, dt \\
&\qquad + \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\quad = C\nu^{-1}(\Delta t)^4\left(\left\|\nabla u_{n+\frac{1}{2}}\right\|^4 + \left\|\nabla u(t_{n+\frac{1}{2}})\right\|^4\right) + C\nu^{-1}(\Delta t)^3 \int_{t^n}^{t^{n+1}} \|\nabla u_{tt}\|^4\, dt \\
&\qquad + \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2.
\end{aligned} \tag{78}
$$

For the last term, we start by using (50), then using the assumption that a strong solution of the Navier-Stokes equations is for each t_n a function in L^∞ and so is ∇u. Then, we use the Poincaré inequality on φ and then again Peter-Paul with $\epsilon = \frac{8}{\nu}$ and finally Lemma 4 to get

$$
\begin{aligned}
& b^*\left(u_{n+\frac{1}{2}} - \overline{u_{n+\frac{1}{2}}}^h, u_{n+\frac{1}{2}}, \varphi^h_{n+\frac{1}{2}}\right) \\
&\quad \leq \frac{1}{2}\left\|u_{n+\frac{1}{2}} - \overline{u_{n+\frac{1}{2}}}^h\right\| \left\|\nabla u_{n+\frac{1}{2}}\right\|_\infty \left\|\varphi^h_{n+\frac{1}{2}}\right\| + \frac{1}{2}\left\|u_{n+\frac{1}{2}} - \overline{u_{n+\frac{1}{2}}}^h\right\| \left\|u_{n+\frac{1}{2}}\right\|_\infty \left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\| \\
&\quad \leq C\left\|u_{n+\frac{1}{2}} - \overline{u_{n+\frac{1}{2}}}^h\right\| \left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\| \\
&\quad \leq \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 + C\nu^{-1}\left\|u_{n+\frac{1}{2}} - \overline{u_{n+\frac{1}{2}}}^h\right\|^2 \\
&\quad \leq \frac{\nu}{16}\left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 + C\nu^{-1}\alpha^4\left\|\Delta F^{-1}u\right\|^2 + C\nu^{-1}\left(\alpha^2 h^{2k} + h^{2k+2}\right)|\overline{u}|^2_{k+1}.
\end{aligned} \tag{79}
$$

Finally, we combine the estimates (74) - (79) to establish an estimate for the interpolation error:

$$\begin{aligned}
&\Delta t \sum_{n=0}^{M-1} \operatorname{IntErr}(u_n, p_n; \varphi^h_{n+\frac{1}{2}}) \\
&\leq \Delta t C \sum_{n=0}^{M-1} \left(\left\| \varphi^h_{n+\frac{1}{2}} \right\|^2 \right) + \frac{1}{4} \Delta t \nu \sum_{n=0}^{M-1} \left(\left\| \nabla \varphi^h_{n+\frac{1}{2}} \right\|^2 \right) \\
&+ C\nu^{-1}\alpha^4 \left\| \Delta F^{-1} u \right\|^2_{2,0} + C\nu^{-1} \left(\alpha^2 h^{2k} + h^{2k+2} \right) |\overline{u}|^2_{2,k+1} \\
&+ C(\Delta t)^4 \Big(\left\| u_{ttt} \right\|^2_{2,0} + \nu^{-1} \left\| p_{tt} \right\|^2_{2,0} + \left\| f_{tt} \right\|^2_{2,0} + \nu \left\| \nabla u_{tt} \right\|^2_{2,0} \\
&+ \nu^{-1} \left\| \nabla u_{tt} \right\|^4_{4,0} + \nu^{-1} \left\| | \nabla u | \right\|^4_{4,0} + \nu^{-1} \left\| \left| \nabla u_{\frac{1}{2}} \right| \right\|^4_{4,0} \Big).
\end{aligned} \tag{80}$$

Now finally, we can use the estimates (70) - (73) in (69) and also plug the estimate for the interpolation error (80) into it. Multiplying by two and taking the terms involving $\nabla \varphi$ from the right-hand side to the left-hand side, we get the estimate

$$\begin{aligned}
&\left\| \varphi^h_M \right\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \left\| \nabla \varphi^h_{n+\frac{1}{2}} \right\|^2 \\
&\leq \Delta t \sum_{n=0}^{M-1} C \left(\nu^{-3} \left\| \nabla u_{n+\frac{1}{2}} \right\|^4 + 1 \right) \left\| \varphi^h_{n+\frac{1}{2}} \right\|^2 \\
&+ C\nu^{-1} h^{2k+1} \left(\left\| | u | \right\|^4_{4,k+1} + \left\| | \nabla u | \right\|^4_{4,0} \right) + C\nu h^{2k} \left\| | u | \right\|^2_{2,k+1} \\
&+ C\nu^{-1} h^{2k} \left(\left\| | u | \right\|^4_{4,k+1} + \nu^{-1} \left(\left\| w^h_0 \right\|^2 + \nu^{-1} \left\| | f | \right\|^2_{2,*} \right) \right) \\
&+ C\nu^{-1} h^{2s+2} \left\| \left| p_{\frac{1}{2}} \right| \right\|^2_{2,s+1} + C\nu^{-1}\alpha^4 \left\| \Delta F^{-1} u \right\|^2_{2,0} \\
&+ C\nu^{-1} \left(\alpha^2 h^{2k} + h^{2k+2} \right) |\overline{u}|^2_{2,k+1} \\
&+ C(\Delta t)^4 \Big(\left\| u_{ttt} \right\|^2_{2,0} + \nu^{-1} \left\| p_{tt} \right\|^2_{2,0} + \left\| f_{tt} \right\|^2_{2,0} + \nu \left\| \nabla u_{tt} \right\|^2_{2,0} + \nu^{-1} \left\| \nabla u_{tt} \right\|^4_{4,0} \\
&+ \nu^{-1} \left\| | \nabla u | \right\|^4_{4,0} + \nu^{-1} \left\| \left| \nabla u_{\frac{1}{2}} \right| \right\|^4_{4,0} \Big).
\end{aligned}$$

Using (59) on the term involving φ on the right-hand side, we finally get

$$\begin{aligned}
&\left\| \varphi^h_M \right\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \left\| \nabla \varphi^h_{n+\frac{1}{2}} \right\|^2 \\
&\leq \Delta t \sum_{n=0}^{M-1} C \left(\nu^{-3} \left\| \nabla u_n \right\|^4 + 1 \right) \left\| \varphi^h_n \right\|^2 + \text{ terms involving various norms.}
\end{aligned}$$

In this form, we can now apply the discrete Grönwall Lemma. Using notation from the Lemma itself, we see $d_n = C \left(\nu^{-3} \left\| \nabla u_n \right\|^4 + 1 \right)$ and $c_n = 0$. If we suppose

$|||\nabla u|||^4_{\infty,0} < \infty$ and $\Delta t < C\left(\nu^{-3}|||\nabla u|||^4_{\infty,0}+1\right)^{-1}$, then $\Delta t d_n < 1$ for all times n. Further, we have

$$\Delta t \sum_{n=0}^{M-1} \frac{d_n}{1-\Delta t d_n} \le \tilde{C}\nu^{-3} \sum_{n=0}^{M-1} \Delta t = \tilde{C}\nu^{-3}T,$$

and the application of the Grönwall Lemma yields

$$\begin{aligned}
&\left\|\varphi_M^h\right\|^2 + \nu\Delta t \sum_{n=0}^{M-1} \left\|\nabla\varphi^h_{n+\frac{1}{2}}\right\|^2 \\
&\quad \le C\exp\left(\tilde{C}\nu^{-3}T\right)\Big[\nu^{-1}h^{2k+1}\left(|||u|||^4_{4,k+1} + |||\nabla u|||^4_{4,0}\right) + \nu h^{2k}\,|||u|||^2_{2,k+1} \\
&\qquad + \nu^{-1}h^{2k}\left(|||u|||^4_{4,k+1} + \nu^{-1}\left(\left\|w_0^h\right\|^2 + \nu^{-1}\,|||f|||^2_{2,*}\right)\right) + \nu^{-1}h^{2s+2}\left|\left|\left|p_{\frac{1}{2}}\right|\right|\right|^2_{2,s+1} \\
&\qquad + \nu^{-1}\alpha^4\left\|\Delta F^{-1}u\right\|^2_{2,0} + \nu^{-1}\left(\alpha^2h^{2k}+h^{2k+2}\right)\|\overline{u}\|^2_{2,k+1} \\
&\qquad + C(\Delta t)^4\Big(\|u_{ttt}\|^2_{2,0} + \nu^{-1}\|p_{tt}\|^2_{2,0} + \|f_{tt}\|^2_{2,0} + \nu\|\nabla u_{tt}\|^2_{2,0} + \nu^{-1}\|\nabla u_{tt}\|^4_{4,0} \\
&\qquad + \nu^{-1}|||\nabla u|||^4_{4,0} + \nu^{-1}\left|\left|\left|\nabla u_{\frac{1}{2}}\right|\right|\right|^4_{4,0}\Big)\Big] \\
&\quad =: H^2.
\end{aligned} \tag{81}$$

From this, we immediately get $\left\|\varphi_M^h\right\| \le H$. As M was not specified, this estimate holds for all times n. After one application of the triangle inequality and a last application of the approximation property (40) the proof is almost done:

$$\begin{aligned}
\|e_n\| = \left\|u_n - w_n^h\right\| &= \left\|(u_n - U_n) - \varphi_n^h\right\| \\
&\le \|u_n - U_n\| + \left\|\varphi_n^h\right\| \\
&\le Ch^{k+1}\,|u|_{k+1} + H,
\end{aligned}$$

As this holds for all $n = 0, \dots M$, it will hold for the maximum over all n and so we arrive at the desired estimate:

$$\left|\left|\left|u - w^h\right|\right|\right|_{\infty,0} \le H(\Delta t, h, \alpha) + Ch^{k+1}\,|||u|||_{\infty,k+1}\,.$$

For the estimate of the gradient we use the assumption that u is a strong solution. Therefore, we can estimate $\nabla u\big|_{t=t_{n+\frac{1}{2}}} \le C\nabla u_{n+\frac{1}{2}}$. Then, after adding some zeros and using the approximation inequality (41) and the integral inequality (57), we get

$$\begin{aligned}
\left\|\nabla\left(u_{n+\frac{1}{2}}-w^h_{n+\frac{1}{2}}\right)\right\|^2 &= \left\|\nabla\left(u_{n+\frac{1}{2}}-u(t_{n+\frac{1}{2}})+u(t_{n+\frac{1}{2}})-U_{n+\frac{1}{2}}-w^h_{n+\frac{1}{2}}+U_{n+\frac{1}{2}}\right)\right\|^2 \\
&\leq \left\|\nabla\left(u_{n+\frac{1}{2}}-u(t_{n+\frac{1}{2}})\right)\right\|^2 + C\left\|\nabla\eta_{n+\frac{1}{2}}\right\|^2 + \left\|\nabla\varphi_{n+\frac{1}{2}}\right\|^2 \\
&\leq C(\Delta t)^3\int_{t_n}^{t_{n+1}} |u_{tt}\|^2\, dt + Ch^k\, |u_n|^2_{k+1} + Ch^k\, |u_{n+1}|^2_{k+1} \\
&\quad + C\left|\nabla\varphi_{n+\frac{1}{2}}\right\|^2 .
\end{aligned}$$

Summing from $n=0$ to $n=M-1$, multiplying by $\nu\Delta t$ and using (81) on the gradient of φ, we get

$$\nu\Delta t\sum_{n=0}^{M-1}\left\|\nabla\left(u_{n+\frac{1}{2}}-w^h_{n+\frac{1}{2}}\right)\right\|^2 \leq C\nu\left(\Delta t^4\,\|u_{tt}\|^2_{2,0}+\Delta t H^2 + h^{2k}\,|||u|||^2_{2,k+1}\right).$$

Taking the square root, we arrive at the desired estimate. □

Under the assumptions of this last theorem and the assumption that the finite element space used is the Q_2/P_1^{disc} elements, it immediately follows

Theorem 5. *Suppose k=2 and s=1 and suppose $u \in H^3 \cap H^1_0$. Then, the error is of order*

$$\left|\left|\left|u-w^h\right|\right|\right|_{\infty,0}+\left(\nu\Delta t\sum_{n=0}^{M-1}\left\|\nabla\left(u_{n+\frac{1}{2}}-w^h_{n+\frac{1}{2}}\right)\right\|^2\right)^{\frac{1}{2}} = O(h^2+(\Delta t)^2+\alpha^2).$$

5 Turbulence Modeling

In this last section, we investigate the turbulence modeling qualities of the Leray-α model. In the same way as the Navier-Stokes equations model all kinds of flows, laminar or turbulent, in the Leray-α model itself there is no mention of turbulence. Following Geurts and Holm (2003), we will recast the Leray-α model as a typical Large Eddy Simulation, seeing that it automatically implies a subgrid model. Last, we show some simulations of a turbulent channel flow at $Re_\tau = 180$, following John and Roland (2007), and using the code MooNMD, see John and Matthies (2004).

5.1 Large Eddy Simulation

Turbulent flows are characterized by having structures, called eddies, of various scales. Seeing as in reality, it will be far too expensive to compute (or resolve) all scales (large eddies and small eddies) of a turbulent flow, one needs to think about what scales are important to resolve.
Driven by viscosity, larger eddies break down into smaller eddies until the smallest structures are so small that they are ground down by dissipation. Their kinetic energy is finally dissipated by the viscosity of the fluid. This process is called the *energy cascade.*
The next question then is that of the smallest scales in a turbulent flow. Following Kolmogorov's hypothesis that at sufficiently high Reynolds numbers the small scale motions are *isotropic* i.e., invariant under shifts in space and time and invariant under rotations and reflections of the coordinate system, it is possible to estimate the smallest scales in a turbulent flow, as long as one knows two parameters of the flow: the rate of dissipation of turbulence energy, herein denoted by ϵ and the kinematic viscosity ν. Experiments show that ϵ is proportional to a ratio of velocity and length scales:

$$\epsilon \sim \frac{u^3}{L}. \tag{82}$$

One can then define the so-called Kolmogorov scales

$$\lambda = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}, \quad u_\lambda = (\epsilon\nu)^{\frac{1}{4}}, \quad t_\lambda = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}. \tag{83}$$

One can infer from this, as is done in John (2006), that these scales are actually dissipative scales, i.e., scales at which the kinetic energy of eddies possessing these scales is dissipated by viscosity. Plugging into the definition of smallest length scale λ from (83) the expression for the rate of dissipation (82), one easily calculates

$$\lambda \sim \mathrm{Re}^{\frac{3}{4}} L.$$

Using this similarity one can, depending on a given grid size, find a bound on the Reynolds number of a flow that can be accurately represented on this grid. An example is given in John (2006). This shows that Direct Numerical Simulation (DNS), which aims to represent all scales of a flow down to the smallest Kolmogorov scale, is an expensive endeavor. However, one can think of the large eddies as those eddies that contain enough energy to have a lot of influence on the overall development on the flow. Therefore it is mostly the large eddies that are important to resolve in a numerical simulation. Nevertheless, the smaller eddies do have influence on the larger eddies. So even if one does not fully resolve them, one needs to consider their influence on the larger eddies. In Large Eddy simulation (LES), the resolved large scales are defined by an average. Usually, one uses a convolution with an appropriate filter function g. The large scales $(\overline{u}, \overline{p})$ are therefore defined by

$$\overline{u}(y) = \frac{1}{\alpha(y)^d} \int_{\mathbb{R}^d} g\left(\frac{y-x}{\alpha(y)}\right) u(x)dx, \quad \text{and} \quad \overline{p}(y) = \frac{1}{\alpha(y)^d} \int_{\mathbb{R}^d} g\left(\frac{y-x}{\alpha(y)}\right) p(x)dx.$$

Seeing as the support of $\overline{u}$ lies in a ball of radius α if the support of the filter function lies in a ball of radius 1, it is the parameter α that defines the large scales. The small scales are then simply defined as the difference

$$u' = u - \overline{u} \quad \text{and} \quad p' = p - \overline{p}.$$

The general idea of a Large Eddy Simulation then goes as follows: we filter the whole set of equations, trying to get a new equation for only the large scales $\overline{u}$. If there are terms that still involve the small scales u' or equivalently the whole function u, we try to model that term using only terms involving $\overline{u}$. We will of course apply this line of thinking to the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f, \quad \nabla \cdot u,$$

here for simplicity in componentwise notation (where summation over repeated indicees is implied),

$$\partial_t u_i + u_j \partial_j u_i - \nu \partial_j \partial_j u_i + \partial_i p = f_i, \quad \partial_j u_j = 0,$$

which, using $\partial_j u_j = 0$, is equal to

$$\partial_t u_i + \partial_j (u_j u_i) - \nu \partial_j \partial_j u_i + \partial_i p = f_i, \quad \partial_j u_j = 0.$$

First, we filter the whole system of equations and make the assumption that the filtering operation and the differentiation operators commute to get

$$\left.\begin{aligned} \partial_t \overline{u_i} + \partial_j (\overline{u_j u_i}) - \nu \partial_j \partial_j \overline{u_i} + \partial_i \overline{p} = \overline{f_i}, \\ \partial_i \overline{u_i} = 0. \end{aligned}\right\} \tag{84}$$

Recognizing that in general $\overline{u_j u_i} \neq \overline{u_j}\ \overline{u_i}$, we introduce the so called *turbulent stress tensor*

$$\tau_{ij} = \overline{u_j u_i} - \overline{u_j}\ \overline{u_i}, \tag{85}$$

which allows us to reformulate problem (84) as

$$\left.\begin{aligned} \partial_t \overline{u_i} + \partial_j (\overline{u_j u_i}) - \nu \partial_j \partial_j \overline{u_i} + \partial_i \overline{p} = \overline{f_i} - \partial_j \tau_{ij}, \\ \partial_i \overline{u_i} = 0. \end{aligned}\right\} \tag{86}$$

Now, the left-hand side only contains the filtered (and therefore less expensive to compute) $\overline{u}$. The right-hand side still contains the whole function u, containing large and small scales. Through the appearance of the small scales on the right-hand side the effect of the small scales on the large scales is modeled. The function $\overline{u}$ is what we want to solve for. The right-hand side is the term that we need to model in order to get an equation for only the large scales. The choice of an expression only involving large scale terms for τ then is the choice of a *turbulence model*, sometimes called *subgrid model*. For an overview of models and the problems that come with them, see John (2006); for an extensive treatment of the topic, see Pope (2000) or Sagaut (2006).
There are of course problems associated with this kind of turbulence modeling. For starters, different turbulence models produce very different results, seemingly contradicting the assumption that the small scales are isotropic and therefore have an almost universal character, see Geurts and Holm (2006).
This is a problem that the Leray-α model solves. From the model itself, we can derive an LES-type model like the one in (86), where τ will be a slightly different expression. But by having a filter L and its inverse given explicitly, τ now automatically only contains $\overline{u}$-terms, so there is no need for additional subgrid modeling. The subgrid model is automatically implied by the Leray-α model itself!
To derive this, we proceed as is done in Geurts and Holm (2003) and first present the Leray-α model in componentwise notation. It then reads

$$\left.\begin{aligned} \partial_t u_i + \overline{u_j}\partial_j u_i - \nu\partial_j\partial_j u_i + \partial_i p = f_i, \\ \partial_j u_j = 0, \\ L\overline{u_i} = u_i, \end{aligned}\right\}$$

where for example $L = -\alpha^2\Delta + I$, as in the previous sections. We proceed by replacing every u by $L(\overline{u})$. Then, we use the assumption that $\partial_j\overline{u_j} = 0$ (alternatively, we impose zero boundary conditions) to get for the convective term $\overline{u_j}\partial_j(L\overline{u_i}) = \partial_j(\overline{u_j}L\overline{u_i})$. Subtracting this and adding $L\partial_j(\overline{u_j}\ \overline{u_i})$ on both sides, we get

$$L\partial_t\overline{u_i} + L\partial_j(\overline{u_j}\ \overline{u_i}) - \nu\partial_j\partial_j\overline{u_i} + \partial_i p = f_i - \partial_j(\overline{u_j}L\overline{u_i}) + L\partial_j(\overline{u_j}\ \overline{u_i}), \quad \partial_j u_j = 0.$$

We now assume the filtering operation L to be invertible and adopting the notation $L^{-1}p = \overline{p}$ and $L^{-1}f = \overline{f}$ and then applying the inverse operator L^{-1} to the equation, we get

$$\begin{aligned} \partial_t\overline{u_i} + \partial_j(\overline{u_j}\ \overline{u_i}) - \nu\partial_j\partial_j\overline{u_i} + \partial_i\overline{p} &= \overline{f_i} - L^{-1}\partial_j\left(\overline{u_j}L\overline{u_i}\right) + \partial_j(\overline{u_j}\ \overline{u_i}) \\ &= \overline{f_i} - \partial_j\left(\underbrace{L^{-1}(\overline{u_j}L\overline{u_i}) - \overline{u_j}\ \overline{u_i}}_{:=m_{ij}}\right). \end{aligned} \tag{87}$$

Comparing (87) to (86), it is immediately apparent that the Leray-α model directly implies a subgrid model, as the tensor m_{ij} only contains large scale terms and filtering operators that are explicitly given. So (87) is already the new equation for the large

scales $\overline{u}$ that can now be solved by discretizing with a method of choice.
Note that the assumptions used herein, namely the assumption that the operator L and the differential operators commute and the assumption that $\overline{u}$ is divergence-free, are not entirely unproblematic. In practice, commutation errors will occur and are non-negligible especially near the boundary, see John (2006). In general, $\overline{u}$ will not be divergence-free just because u is. This can certainly not be derived from the Leray-α model itself. However, this assumption does follow from imposing zero boundary values for u and $\overline{u}$ and applying the ∇ operator to the equation for $\overline{u}$.

5.2 Numerical Simulations

In our simulations, we consider the flow through a channel, using an anisotropic grid in wall normal direction. The simulations were performed on a coarse grid and a finer grid. Because the filter width parameter α should be connected to the width of the mesh h as $\alpha = Ch$, where C is the filter width constant, the focus of our numerical experiments laid on comparing different values of C on the same grid. Note that unlike in the numerical analysis, α is not constant on the grid, as the grid is anisotropic. As there are different ways of measuring the width of a mesh cell, we performed our experiments for three different measures, namely the geometric mean, the diameter of the cell and the shortest edge of the cell. The results were compared to results from Moser et al. (1999).

5.2.1 Channel Flow

We consider the flow through a rectangular channel Ω. The bottom wall is at $y = 0$, the top wall is at $y = 2H$ and the center line is at $y = H, z = 0$. The flow is predominantly in the x-direction, making it the stream wise direction. The velocity varies mainly in the y-direction, which is the cross stream direction. The z-direction, which is also called the span wise direction, is assumed to be large compared with the height of the channel. This allows us to assume that the flow is statistically independent of z, except of course at and near the walls. However, in our experiments homogeneous Dirichlet boundary conditions were only used in y-direction and periodic boundary conditions were used in the x- and z-directions, making the channel infinitely long and infinitely wide, so as to not have to deal with walls and the problems they pose. This will mean that statistics no longer vary with x, making the flow statistics only dependent on y, see also Pope (2000), Chapter 7, Wall flows for a similar set-up. This type of flow is, as one would expect, statistically symmetric around the mid-plane $y = H$.

5.2.2 Algorithm

The Leray-α model is first discretized in time by a simplified Crank-Nicolson method, see Chapter 4, section 4.2.2, (53) to yield

$$\left.\begin{aligned} u_{n+1} &+ \frac{1}{2}\Delta t\left(-\nu\Delta u_{n+1} + (\overline{u_{n+1}}\cdot\nabla)u_{n+1} + \nabla p_{n+1}\right)\\ &\qquad = u_n - \frac{1}{2}\Delta t\left(-\nu\Delta u_n + (\overline{u_n}\cdot\nabla)u_n + \nabla p_n\right) + \frac{1}{2}\Delta t(f_n + f_{n+1}),\\ \nabla\cdot u_{n+1} &= 0,\\ \left(\mathrm{Id} - \alpha^2\Delta\right)\overline{u_{n+1}} &= u_{n+1}. \end{aligned}\right\} \tag{88}$$

This is transformed into variational form and discretized by a finite element method using the Q_2/P_1^{disc} finite element. Finally, this is linearized to resemble an Oseen system, meaning in the nonlinearity one replaces $(\overline{u_{n+1}}\cdot\nabla)u_{n+1}$ by $(\overline{u_n}\cdot\nabla)u_{n+1}$. This is then solved by a fixed point iteration. Problem (88) then becomes:
Given $\overline{u_{n+1}}^0 = \overline{u_n}$, solve in each time step

$$\left.\begin{aligned} (u_{n+1}^{(k)}, v) &+ \frac{1}{2}\Delta t\left(\nu(\nabla u_{n+1}^{(k)}, \nabla v) + \left((\overline{u_{n+1}}^{(k-1)}\cdot\nabla)u_{n+1}^{(k)}, v\right) + (p_{n+1}^{(k)}, \nabla\cdot v)\right)\\ &\qquad = (u_n, v) - \frac{1}{2}\Delta t\Bigg(\nu(\nabla u_n, v) + ((\overline{u_n}\cdot\nabla)u_n, v)\\ &\qquad\qquad + (p_n, \nabla\cdot v)\Bigg) + \frac{1}{2}\Delta t(f_n + f_{n+1}, v),\\ (\nabla\cdot u_{n+1}^{(k)}, q) &= 0,\\ \left(\mathrm{Id} - \alpha^2\Delta\right)\overline{u_{n+1}}^{(k)} &= u_{n+1}^{(k)}. \end{aligned}\right\}$$

In the numerical experiments themselves, the deformation tensor formulation was used. Herein, the term $(\nabla u, \nabla v)$ is replaced by the term $2(D(u), D(v))$, where $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. This is solved using homogeneous Dirichlet boundary conditions. The Crank-Nicolson scheme was applied with an equidistant time step of $\Delta t = 0.002$. The grid that was used is uniform in the periodic directions x and z. It is anisotropic in the wall normal direction y, where the spacing of the nodes becomes finer and finer towards the wall. The nodes y_i are then given by

$$y_i = 1 - \cos\left(\frac{i\pi}{N_y}\right), \quad i = 0, \ldots, N_y,$$

where N_y is the number of cell layers in y-direction. Our coarse grid consists of 8×8 cells in the x- and z-directions and and 16 layers of these cells in y-direction. For the fine grid, we refined once more in the y-direction, resulting in 32 layers of cells.

5.2.3 Statistics of Interest

As we want to compare our results to a direct numerical simulation of the channel flow, there are some statistics that were computed.

Let $u^h(t,x,y,z) = (U(t,x,y,z), V(t,x,y,z), W(t,x,y,z))$ be the solution computed by using the method described above. Let N_a denote the number of grid points in the direction indicated by the subscript a. The first statistic of interest is the spatial average: as the grids used are uniform, the spatial mean velocity at time t_n in the plane y =const. is denoted by

$$\left\langle u^h(t_n,x,y,z)\right\rangle_s = \frac{1}{N_xN_z}\sum_{i=1}^{N_x}\sum_{j=1}^{N_z} u^h(t_n,x_i,y,z_j).$$

If we average these in time, again keeping in mind that the time steps are uniform too, we get the mean velocity profile depending on y:

$$u^h_{\text{mean}}(y) := \left\langle\left\langle u^h(t_n,x,y,z)\right\rangle_s\right\rangle_t = \frac{1}{N_t+1}\sum_{n=0}^{N_t}\left\langle u^h(t_n,x,y,z)\right\rangle_s. \tag{89}$$

The first secondary statistic used is the *Reynolds Stress Tensor*. It is derived from the Navier-Stokes equations in exactly the same way as the turbulent stress tensor τ, see (85) , except that we now interpret the bar over u as an averaging of our choice instead of as a given filter. If we denote this averaging of our choice by $\langle\cdot\rangle$, the equations (86) then read

$$\left.\begin{aligned}\partial_t\langle u_i\rangle + \partial_j(\langle u_j\rangle\langle u_i\rangle) - \nu\partial_j\partial_j\langle u_i\rangle + \partial_i\langle p\rangle = \langle f_i\rangle - \partial_j\mathbb{R}_{ij},\\ \partial_i\langle u_i\rangle = 0,\end{aligned}\right\}$$

where

$$\mathbb{R}_{ij} = \langle u_ju_i\rangle - \langle u_i\rangle\langle u_j\rangle$$

is the general form of the *Reynolds Stress Tensor*. From this equation, we can interpret the Reynolds Stress Tensor as the force that determines how the average $\langle u_i\rangle$ develops. As the averaging is one of our choice, there is more than one version of the Reynolds Stress Tensor used in the literature, depending on what kind of averaging was used. Here, we will present results for the component $\mathbb{R}_{uv} := \mathbb{R}_{u_1u_2}$ of the Reynolds Stress Tensor, which we define as

$$\mathbb{R}_{uv} = \langle\langle uv\rangle_s\rangle_t - \langle\langle u\rangle_s\rangle_t\langle\langle v\rangle_s\rangle_t. \tag{90}$$

The diagonal components of the stress tensor are called *normal stresses* and the off-diagonal components are the *shear stresses*. Naturally, what is shear stress and what is normal stress depends on the choice of coordinate system, but useful distinctions can be made between isotropic and anisotropic stresses. Seeing as the trace of a tensor is invariant under coordinate transformations, the term $\frac{1}{3}\sum_{i=1}^{3}\langle\mathbb{R}_{ii}\rangle$, which is the trace of the Reynolds Stress Tensor normalized by the dimension, is called *isotropic stress*. Therefore, the deviation from isotropy can be expressed by $\mathbb{R}_{ij} - \frac{1}{3}\sum_{i=1}^{3}\mathbb{R}_{ii}\delta_{ij}$.

In contrast to free shear flows, a channel flow is a wall flow, meaning that the viscosity of the fluid is an influential parameter at the walls of the channel. The total shear stress can be derived from the averaged momentum equation. This is solved for the pressure term. The other terms then model the stress and the term containing the viscosity ν

is called viscous contribution, see Pope (2000), Section 7.1.2 for details. If we assume zero boundary conditions, at the wall the so called *wall shear stress* is entirely due to the *viscous contribution*

$$\tau_w = \rho\nu \left(\frac{d \langle u \rangle}{dy} \right)_{y=0}.$$

This stress will influence the flow dependent on its distance from the wall. Therefore, the introduction of so called *viscous scales*, dependent on τ_w, is appropriate. These are

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} \quad \text{friction velocity}, \qquad \alpha_\nu = \nu \sqrt{\frac{\rho}{\tau_w}} \quad \text{viscous length scale},$$
$$Re_\tau = \frac{u_\tau \alpha}{\nu} \quad \text{friction Reynolds number}, \qquad y^+ = \frac{y}{\alpha_\nu} = \frac{u_\tau y}{\nu} \quad \text{wall units}.$$

The friction Reynolds number in our numerical experiments is $Re_\tau = 180$. Herein, the simulated friction velocity is approximated by a one-sided difference at each wall and then the average will be computed, giving

$$u_\tau^h = \frac{1}{2} \left(\frac{U_{\text{mean}}(y^+_{\text{min}})}{y^+_{\text{min}}} - \frac{U_{\text{mean}}(2 - y^+_{\text{min}})}{2 - y^+_{\text{min}}} \right),$$

where y^+_{min} is the minimum height of a cell. This will be used to normalize the secondary statistics.

Finally, we can give the definitions of the statistics used in the simulations. The first order statistic we calculate is the mean velocity. Results for this statistic will be shown for the first component $U_{\text{mean}}(y)$ of $u^h_{\text{mean}}(y)$ from equation (89). For clarity, we also show a plot of the difference of the mean to the reference mean. The Reynolds Stress Tensor component that is shown was calculated by

$$\mathbb{R}_{uv} := \mathbb{R}_{12} := \frac{\mathbb{R}^h_{12}}{(u^h_\tau)^2}.$$

The root mean square turbulence intensities were computed by

$$u_{rms} = \frac{\left| \mathbb{R}^h_{11} - \frac{1}{3} \sum_{j=1}^3 \mathbb{R}^h_{jj} \right|^{\frac{1}{2}}}{u^h_\tau}.$$

Finally, we note that when actually using the Large Eddy Formulation of the Leray-α model to compute a solution, there should be a slight modification when computing the statistics, as is pointed out by Winckelmans et al. (2002). As we want to compare our results to a Direct Numerical Simulation, we will have to compare the statistics of u^{DNS}, which is the result of the DNS calculation, to the statistics of $\overline{u}$, which is the result of Large Eddy Modeling. It is natural to assume that

$$\langle u \rangle \sim \langle \overline{u} \rangle,$$

as $\overline{u}$ is just a filtered version of u. Therefore, we can directly compare the first order statistics. For the second order statistics, we use the fact that the tensor m_{ij} from (87), after being filtered once more, can be written as

$$\overline{m_{ij}} = \overline{\overline{u_j}\ \overline{u_i}} - \overline{\overline{u_j u_i}}.$$

We also use the fact that averaging is homogeneous, meaning

$$\langle u \rangle \approx \langle \overline{u} \rangle \iff \langle u \rangle a \approx \langle \overline{u} \rangle a \iff \langle ua \rangle \approx \langle \overline{u}a \rangle .$$

The Reynolds Stress Tensors $\mathbb{R}_{ij}^{DNS}$ of u and $\mathbb{R}_{ij}^{LES}$ belonging to $\overline{u}$ are then related by

$$\begin{aligned}\mathbb{R}_{ij}^{DNS} &= \langle u_i \rangle \langle u_j \rangle - \langle u_i u_j \rangle \approx \langle \overline{u_i} \rangle \langle \overline{u_j} \rangle - \langle \overline{u_i u_j} \rangle \\ &\approx \langle \overline{u_i} \rangle \langle \overline{u_j} \rangle - \langle \overline{m_{ij}} \rangle - \left\langle \overline{\overline{u_j}\ \overline{u_i}} \right\rangle = \mathbb{R}_{ij}^{LES} - \langle \overline{m_{ij}} \rangle .\end{aligned}$$

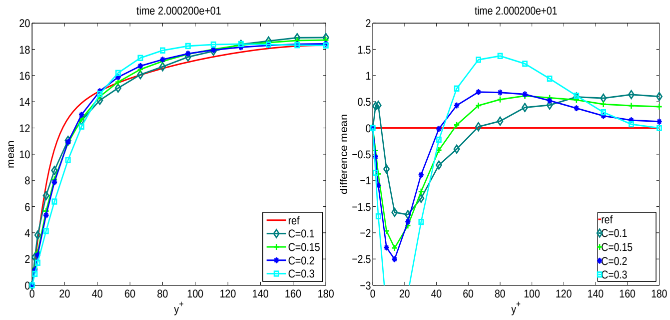
5.2.4 Results

According to the rule $\alpha = Ch_K$, where h_K is some measure of the cell K, the filter width α varies in wall normal direction.
For the cell measure h_K, three different options were tested. Denoting by $h_x(K)$, $h_y(K)$ and $h_z(K)$ the lengths of the edges of the cell K, we define

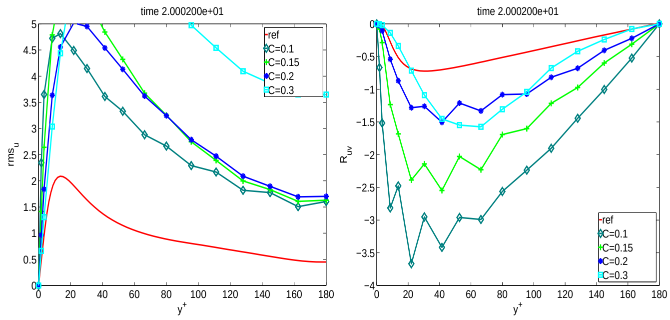
$$\mathrm{cubic}(K) := \sqrt[3]{h_x h_y h_z}, \quad \mathrm{diam}(K) := \sqrt{h_x^2 + h_y^2 + h_z^2}, \quad \mathrm{edge}(K) := \min\{h_x, h_y, h_z\} .$$

In the first stage of our experiment results for values of C in the range of $C = 0.2, 0.3, \ldots$ $0.9, 1.0, 1.2, 1.5, 2.0$ were computed for each of these cell measures on the coarse grid. Depending on the results, additional finer values were computed.
The results on the coarse grid are presented in Figures 1, 2 and 3. We classify the results by first comparing the mean profile and selecting acceptable values of C. Then from these values of C we choose the one that is nearest to the secondary statistics of he reference and declare it as the best value.
For the Cubic measure (Figure 1), values $C = 0.1$ and $C = 0.15$ were the closest to the mean velocity profile of the reference mean velocity profile from the DNS simulation. The computed root mean square intensities (rms_u) do have the correct form, but highly overpredict the values. The overprediction is smallest for $C = 0.1$. None of the results for the Reynolds Stress Tensor component (R_{uv}) are particularly good. The form of the curves is similar to the reference values for $C = 0.1$, $C = 0.15$ and $C = 0.2$, but the oscillations near the wall are quite pronounced. These are both effects that will be lessened considerably on the finer grid. All in all, for the cubic cell measure, the value $C = 0.1$ seems to be the best value.

For the diam cell measure, the values of the filter width constant for which the mean velocity profile was acceptable were somewhat lower than the values for the edge measure. As can be seen in Figure 2, only the value $C = 0.025$ produced reasonable results. Especially the root mean square intensities are far overpredicted for the other values.

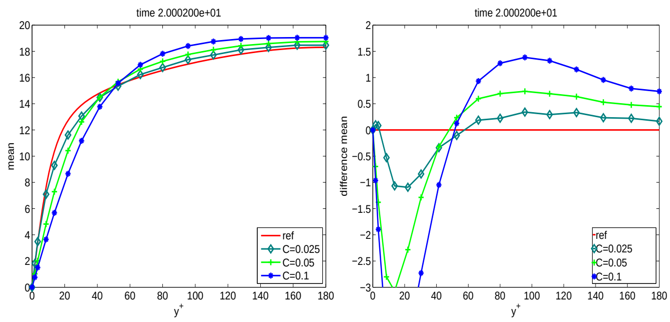


(a) Mean and Difference Mean

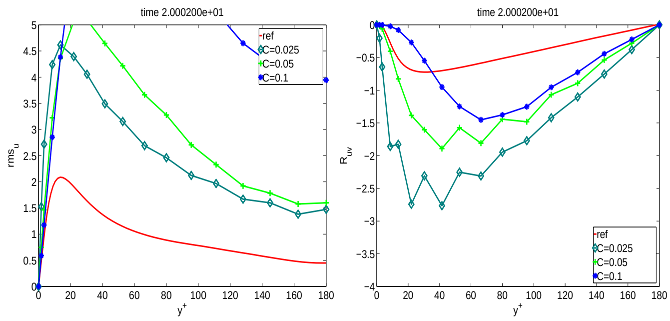


(b) rms and R_{uv}

Figure 1: Cubic on coarse grid



(a) Mean and Difference Mean



(b) rms and R_{uv}

Figure 2: Diam on coarse grid

For the cell measure edge, we show the full range of values for C, see Figure 3. In terms of the quality of their mean velocity profile, none of the results are particularly good, but the values $C = 0.3$ through $C = 0.8$ are comparable. As the root mean square intensity and the Reynolds Stress Tensor are best for $C = 0.8$, this is the value we select as best result.

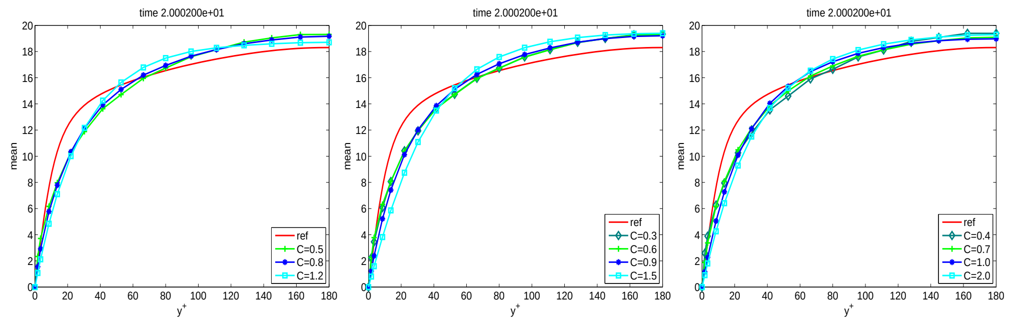
The results for the fine grid are presented in Figures 4, 5 and 6.
On the fine grid, the only two values of C that produce a reasonably close velocity profile for the measure Cubic are $C = 0.2$ and $C = 0.3$, see Figure 4. These two values are also the only two values that produce curves that have the correct shape for rms_u and the Reynolds Stress Tensor. Of these, $C = 0.3$ far overpredicts rms_u. We select $C = 0.2$ as the best value. We see that the overpredictions and oscillations are lessened.

For the measure Diam on the fine grid, similar to the situation on the coarse grid, only $C = 0.025$ produces acceptable results for all considered statistics, see Figure 5.

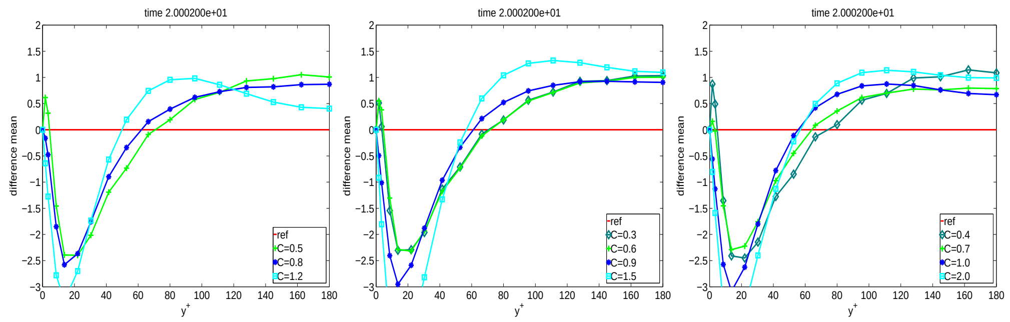
For the measure Edge, the differences between the results for different C were minimal, especially in the range $[0.2, 0.8]$. Of these, $C = 0.8$ yields the best results for the second order statistics, which is why we pick it as the best value.

A comparison shows that the best result on the coarse grid is produced for the measure Diam and the value $C = 0.025$, see Figure 7, but this is also the only value that Diam produces a reasonable result for. On the fine grid, Figure 8 shows that the best result is also achieved by selecting the diam measure.

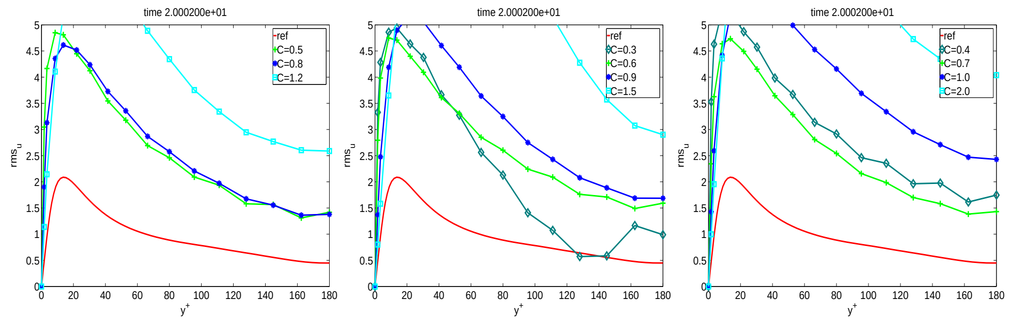
All in all, the dependence on α can clearly be seen. For the measures Cubic and Diam the range of C that produced acceptable results is relatively small. No choice of C was a perfect fit for the reference value profile.



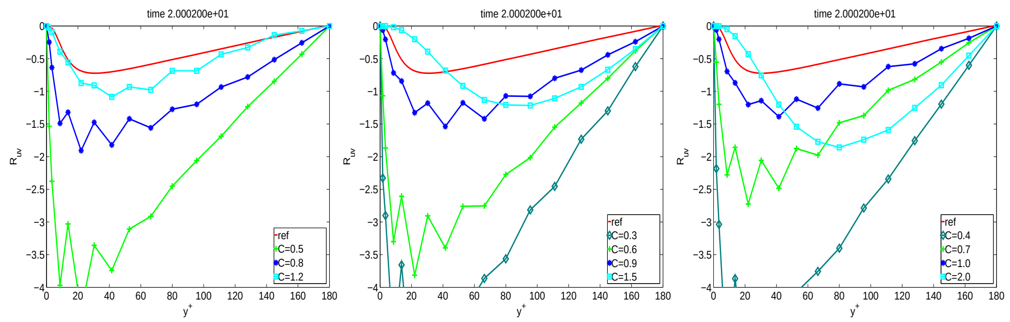
(a) Mean



(b) Difference to Mean

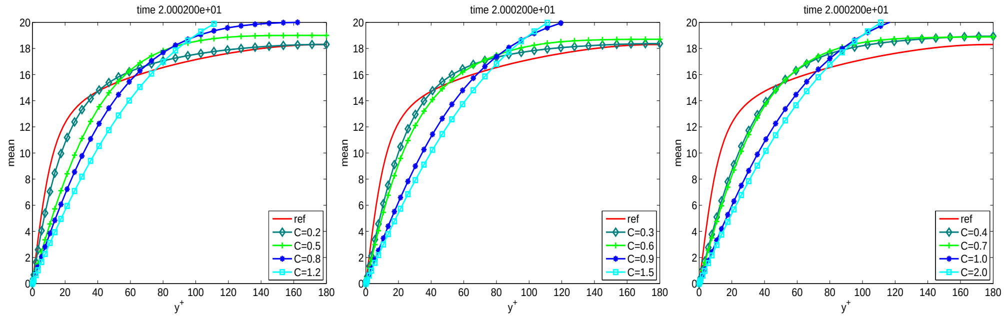


(c) Root Mean Square

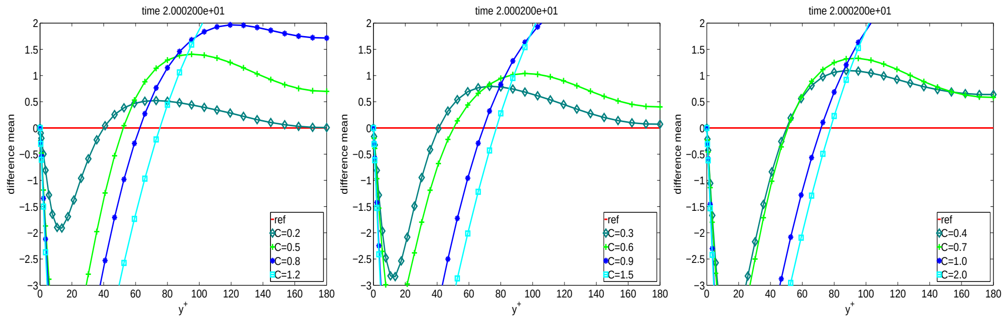


(d) R_{uv}

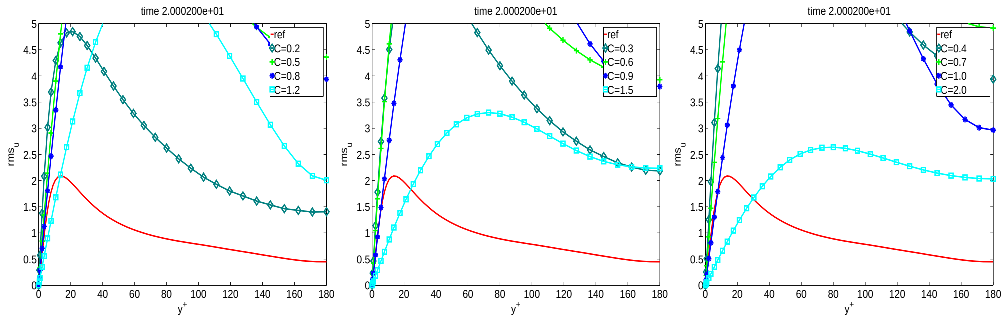
Figure 3: Edge on coarse grid



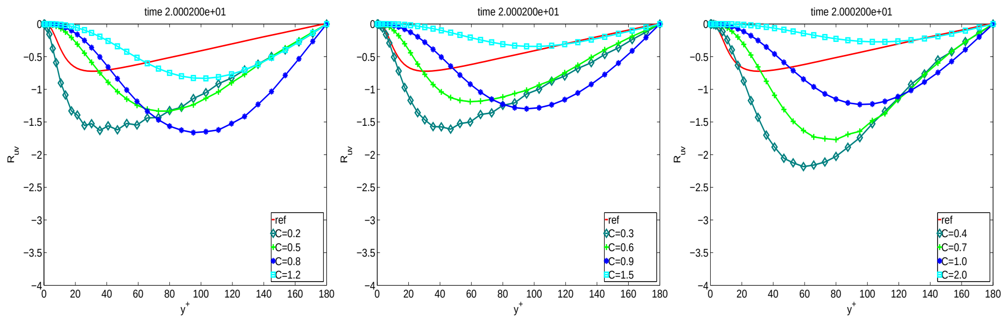
(a) Mean



(b) Difference to Mean

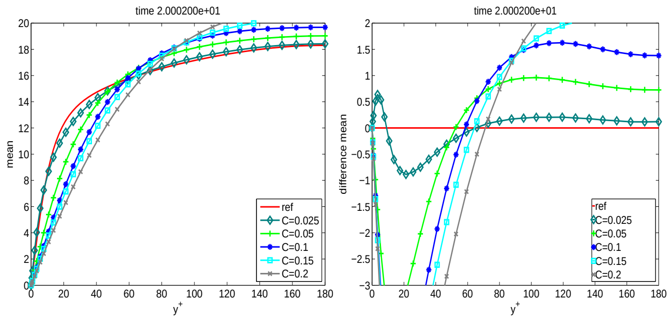


(c) Root Mean Square

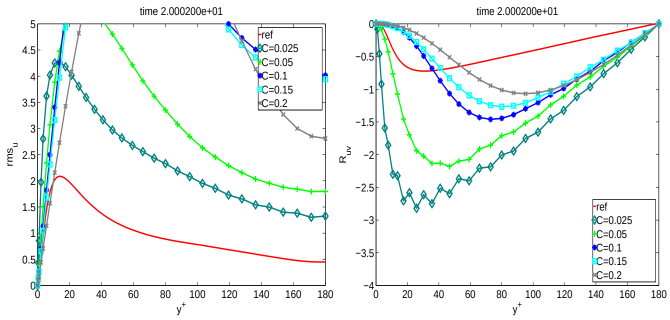


(d) R_{uv}

Figure 4: cubic on fine grid

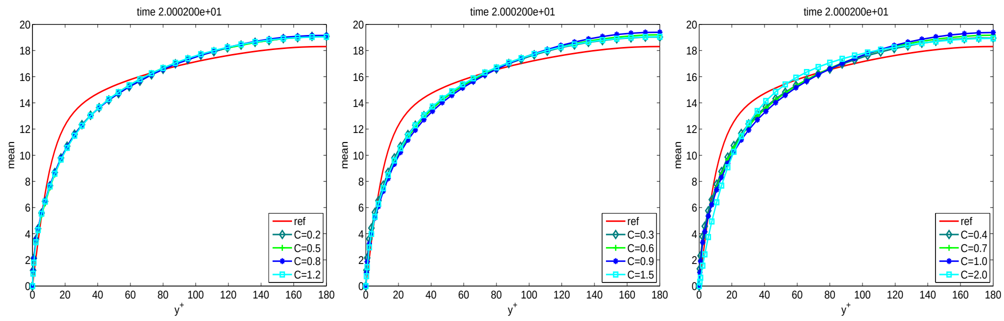


(a) Mean and Difference Mean

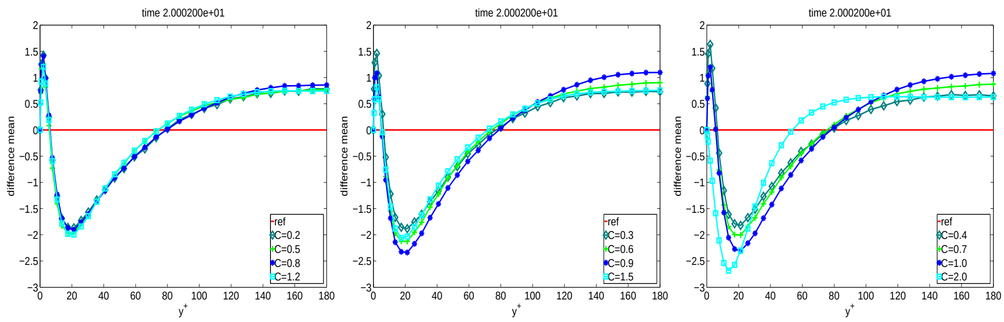


(b) rms and R_{uv}

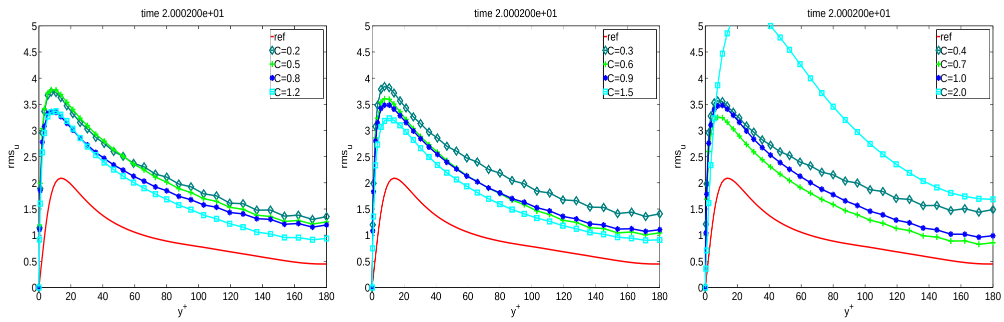
Figure 5: Diam on fine grid



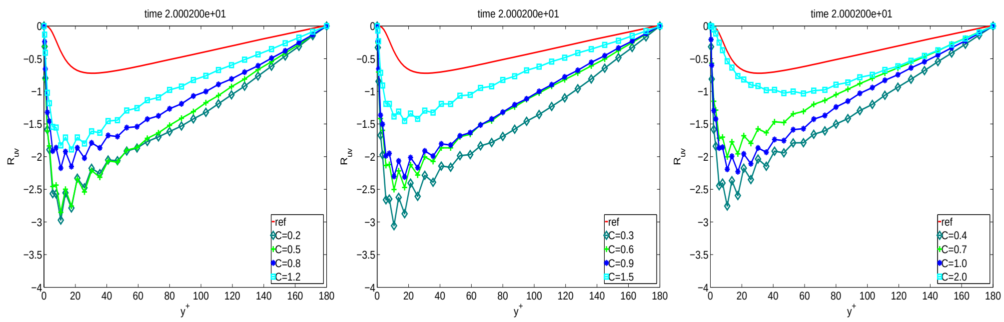
(a) Mean



(b) Difference to Mean

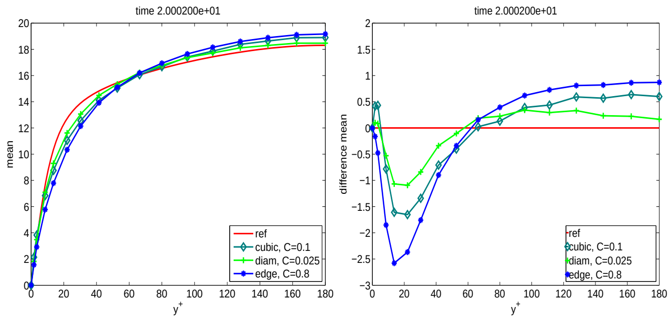


(c) Root Mean Square

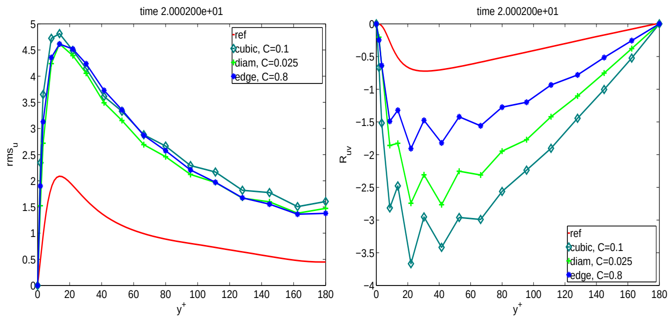


(d) R_{uv}

Figure 6: Edge on fine grid

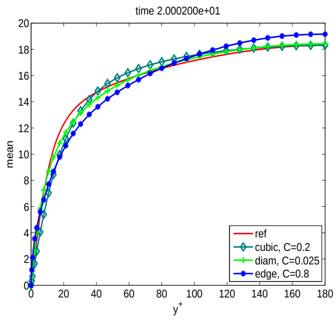


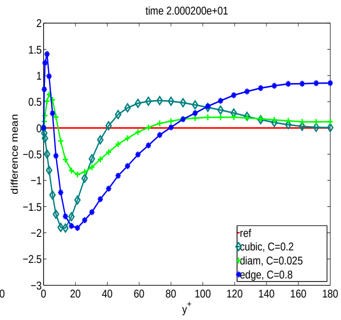
(a) Mean and Difference Mean



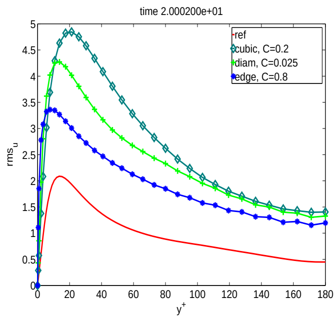
(b) rms and $\mathrm{R_{uv}}$

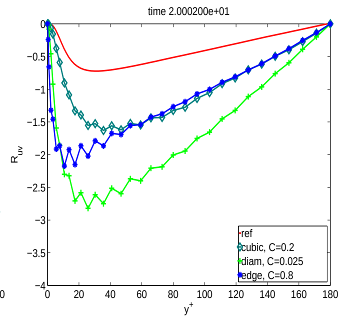
Figure 7: Comparison of best results on coarse grid





(a) Mean and Difference Mean





(b) rms and $\mathrm{R_{uv}}$

Figure 8: Comparison of best results on fine grid

6 Summary and Outlook

We have seen that due to the smaller dimension of its attractor, theory suggests that the Leray-α model of turbulence is easier to simulate numerically than other 3d models of turbulence. Our own numerical experiments did not produce a value for the filter width α that produces perfect results. It might be interesting to consider the question of what would be a perfect result. After all, even if it implies its own subgrid model, the Leray-α model is still just a model of turbulence. The question of how accurately it can depict turbulence and what accuracy means in this context might be interesting to consider. Still, some experiments do suggest that the Leray-α model might be better at depicting the turbulent character of a flow than the usual Large Eddy Simulations, see for example Figure 1 in Geurts and Holm (2003).

Nevertheless, more numerical experiments using the Leray-α model and studies that compare the results of experiments using the Leray-α model to the results obtained by other turbulence models are needed.

An interesting candidate for an even better turbulence model seems to be the so-called LANS-α Model, which is an extension of the Leray-α model. These two models are investigated in Geurts and Holm (2006), where the authors record the result that "the Leray model is more robust but also slightly less accurate than the LANS-α model. The LANS-α model retains more of the small-scale variability in the resolved solution". The authors do note that this increase in accuracy requires a corresponding increase in spatial resolution. This of course will result in an increase in memory consumption and computational time compared to the Leray-α model. Again, more experiments are needed to quantify whether this increase in computational cost is justified compared to the increase in accuracy.

7 Appendix

As notation and nomenclature are not consistent throughout the literature, this will be a list of standard inequalities that are used throughout the text.

Cauchy's inequality

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

Cauchy's inequality with ϵ

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

Cauchy-Schwarz Let $(\cdot, \cdot)$ denote a scalar product and let $\|\cdot\|$ denote the norm induced by this scalar product. Then

$$|(a,b)| \leq \|a\| \, \|b\| \, .$$

Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{q} + \frac{1}{p} = 1.$$

Young's inequality with ϵ

$$ab \leq \epsilon a^p + C(\epsilon) b^q, \quad C(\epsilon) = (\epsilon p)^{\frac{-q}{p}} q^{-1}.$$

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$$ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}.$$

Set $\epsilon = \frac{\tilde{\epsilon}}{2}$ in Cauchy with ϵ to obtain this.

Hölder

$$\|uv\|_{L^1} \leq \|u\|_{L^p} \|v\|_{L^q}, \quad 1 \leq p, q, \leq \infty \quad \frac{1}{q} + \frac{1}{p} = 1.$$

Grönwall´s inequality, differential form Let g, h be non-negative, summable functions on $[0,T]$. Let y be a non-negative, absolutely continuous function on $[0,T]$ that for a.e. $t \in [0,T]$ satisfies

$$\frac{dy}{dt} \leq gy + h.$$

Then, y also satisfies

$$y \leq \exp\left(\int_0^t g(s)\, ds\right)\left(y(0) + \int_0^t h(s)\, ds\right).$$

Uniform Grönwall Lemma, see Temam (1988). Let g, h, y be three positive locally integrable functions on (t_0, ∞) such that y' is locally integrable on (t_0, ∞) and which satisfy

$$\frac{dy}{dt} \leq gy + h \quad \text{for } t \geq t_0,$$

and

$$\int_1^{t+r} g(s)\,ds \le a_1, \quad \int_1^{t+r} h(s)\,ds \le a_2, \quad \int_1^{t+r} y(s)\,ds \le a_3, \quad \text{for } t \ge t_0,$$

where r, a_1, a_2, a_3 are positive real constants. Then,

$$y(t+r) \le \Big(\frac{a_3}{r} + a_2\Big)\exp(a_1), \quad \forall t \ge t_0.$$

Poincaré inequality Let $u \in W^{1,p}(\Omega)$, Ω bounded with Lipschitz boundary and $1 \le p \le \infty$. Then there is a constant $C = C(p, \Omega)$ such that

$$\|u - u_\Omega\|_{L^2} \le C \|\nabla u\|_{L^2},$$
$$\text{where} \quad u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(y)dy.$$

On bounded domains $\Omega \subset \mathbb{R}^n$ a **Poincaré-type** inequality of the form

$$\|v\|^2 \le \frac{1}{\lambda_1}\|\nabla v\|^2,$$

where λ_1 is the smallest eigenvalue of the Stokes operator $-\Delta$, holds for functions in H_0^1 and for functions in H^1 with periodic boundary conditions. This can be derived using the Fredholm alternative and the fact that the Rayleigh quotient of the negative Laplacian can be minimized by λ_1: $\min \frac{\|\nabla v\|^2}{\|v\|^2} = \lambda_1$.

Interpolation inequality for Sobolev spaces Let $s_1 < s_2$ and $s = \Theta s_1 + (1-\Theta)s_2$, where $0 < \Theta < 1$. Then

$$\|v\|_{H^s} \le \|v\|_{H^{s_1}}^{\Theta} \|v\|_{H^{s_2}}^{1-\Theta} \quad \forall v \in H^{s_2}.$$

Sobolev embedding theorem

$$\|u\|_{L^q(\mathbb{R}^n)} \le C \|u\|_{W^{m,p}(\mathbb{R}^n)} \quad \text{for } \frac{1}{q} = \frac{1}{p} - \frac{m}{N}, \quad p \ne \frac{N}{m},$$

N being the dimension.

Sobolev inequality in 3 dimensions

$$\|u\|_{L^\infty} \le \frac{c}{\lambda_1^{1/4}}\|u\|_{H^2},$$

where λ_1 is again the smallest eigenvalue of the Stokes operator.

Generalization of Sobolev-Lieb-Thirring inequality, see Temam (1988). Let Ω be a bounded set of $\mathbb{R}^n$ with regular boundary. Let φ_j, $1 \le j \le N$ be a family in $H^m(\Omega)^k$ which is finite and orthonormal in $L^2(\Omega)^k$. We then have

$$|\Omega|^{2m/n}\sum_{j=1}^{N}\int_\Omega |D^m\varphi_j|^2\,dx \ge c_1 N^{1+2m/n},$$

where c_1 depends on n, m and the shape of Ω.

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