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Energy estimates for electro-reaction-diffusion systems with partly fast kinetics

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Electro-reaction-diffusion system

notation:

X_i	species, $i = 1, \dots, m$	$\zeta_i = v_i + q_i v_0$	electrochem. pot.
v_i	chem. pot.	$u_i = \widehat{u}_i\, e^{v_i}$	part. densities
q_i	charge numbers	$\widehat{u}_i$	reference densities
v_0	electrostat. pot.	$u_0 = \sum_{i=1}^m q_i u_i$	charge density

mass fluxes:

$$j_i = -D_i\, u_i \nabla \zeta_i$$

reactions:

$$\alpha_1 X_1 + \cdots + \alpha_m X_m \rightleftharpoons \beta_1 X_1 + \cdots + \beta_m X_m, \quad (\alpha, \beta) \in \mathcal{R}$$

reaction rates:

$$R_i = \sum_{(\alpha,\beta)\in\mathcal{R}} k_{\alpha\beta} \Big(\prod_{k=1}^m e^{\alpha_k \zeta_k} - \prod_{k=1}^m e^{\beta_k \zeta_k} \Big) (\alpha_i - \beta_i)$$

differential equations:

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} + \nabla \cdot j_i + R_i &= 0 && \text{on } \mathbb{R}_+ \times \Omega \\
\nu \cdot j_i &= 0 && \text{on } \mathbb{R}_+ \times \Gamma \\
u_i(0) &= U_i && \text{on } \Omega, \qquad i = 1, \ldots, m, \\
-\nabla \cdot (\varepsilon \nabla v_0) &= f + u_0 && \text{on } \mathbb{R}_+ \times \Omega \\
\nu \cdot (\varepsilon \nabla v_0) + \tau v_0 &= 0 && \text{on } \mathbb{R}_+ \times \Gamma_N \\
v_0 &= 0 && \text{on } \mathbb{R}_+ \times \Gamma_D
\end{aligned}
$$

Weak formulation

variables:

$$v = (v_0, v_1, \dots, v_m) \in X = H_0^1(\Omega \cup \Gamma_N) \times H^1(\Omega, \mathbb{R}^m)$$

$$u = (u_0, u_1, \dots, u_m) \in X^*, \quad U = \Big(\sum_{i=1}^m q_i U_i, U_1, \dots, U_m\Big)$$

operators:

$A: \; X \cap L^\infty(\Omega, \mathbb{R}^{m+1}) \to X^*$, $E = (E_0, \dots, E_m): \; X \to X^*$

$$\langle A\, v, \bar{v}\rangle_X := \int_\Omega \Big\{ \sum_{i=1}^m D_i \widehat{u}_i \mathrm{e}^{v_i} \nabla \zeta_i \cdot \nabla \bar{\zeta}_i + \sum_{(\alpha,\,\beta)\in\mathcal{R}} k_{\alpha\beta} \Big(\prod_{k=1}^m \mathrm{e}^{\alpha_k \zeta_k} - \prod_{k=1}^m \mathrm{e}^{\beta_k \zeta_k} \Big) (\alpha - \beta) \cdot \bar{\zeta} \Big\} \,\mathrm{d}x$$

$$\zeta_i = v_i + q_i v_0$$

$$\langle E\, v, \bar{v}\rangle_X := \langle E_0 v_0, \bar{v}_0\rangle + \int_\Omega \sum_{i=1}^m \widehat{u}_i \, \mathrm{e}^{v_i} \bar{v}_i \,\mathrm{d}x, \quad \bar{v} \in X,$$

$$\langle E_0 v_0, \bar{v}_0\rangle_{H_0^1(\Omega\cup\Gamma_N)} := \int_\Omega (\varepsilon \nabla v_0 \cdot \nabla \bar{v}_0 - f\, v_0) \,\mathrm{d}x + \int_{\Gamma_N} \tau\, v_0 \bar{v}_0 \,\mathrm{d}\Gamma$$

Weak formulation

Problem (P): $u'(t) + A\,v(t) = 0,\; u(t) = E\,v(t)$ f.a.a. $t \in \mathbb{R}_+$; $\quad u(0) = U,$

$u \in H^1_{\mathrm{loc}}(\mathbb{R}_+, X^*),\; v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\Omega, \mathbb{R}^{m+1}))$

stochiometric subspace: $\mathcal{S} := \mathrm{span}\Big\{\alpha - \beta : (\alpha, \beta) \in \mathcal{R}\Big\}$

$$\mathcal{U} := \Big\{u \in X^* : u_0 = \sum_{i=1}^{m} q_i u_i,\; (\langle u_1, 1\rangle_{H^1}, \ldots, \langle u_m, 1\rangle_{H^1}) \in \mathcal{S}\Big\}$$

$$\mathcal{U}^\perp := \Big\{v \in X : \langle u, v\rangle_X = 0 \quad \forall u \in \mathcal{U}\Big\}$$

Invariants:

(u, v) solution to (P) $\Longrightarrow \quad u(t) - U \in \mathcal{U} \quad \forall t \in \mathbb{R}_+$

Stationary problem (S): $(u^*, v^*) :\; Av^* = 0,\; u^* = Ev^*,$

$u^* - U \in \mathcal{U},\; v^* \in X \cap L^\infty(\Omega, \mathbb{R}^{m+1})$

Energy functionals

E strictly monotone potential operator

$$E(v) = \partial\Phi(v), \quad \Phi : X \to \mathbb{R},$$

$$\Phi(v) = \int_\Omega \Big\{ \frac{\varepsilon}{2} |\nabla v_0|^2 - f v_0 + \sum_{i=1}^m \widehat{u}_i (\mathrm{e}^{v_i} - 1) \Big\} \, \mathrm{d}x + \int_{\Gamma_N} \frac{\tau}{2} v_0^2 \, \mathrm{d}\Gamma$$

free energy $F : X^* \to \bar{\mathbb{R}}$, $F(u) := \Phi^*(u) = \sup_{v \in X} \Big\{ \langle u, v \rangle - \Phi(v) \Big\}$

Φ, F proper, convex, lower semi-continuous, $u \in H_0^1(\Omega \cup \Gamma_N)^* \times L_+^2(\Omega, \mathbb{R}^m) \Longrightarrow$

$$F(u) = \int_\Omega \Big\{ \frac{\varepsilon}{2} |\nabla v_0|^2 + \sum_{i=1}^m \big\{ u_i \ln \frac{u_i}{\widehat{u}_i} - u_i + \widehat{u}_i \big\} \Big\} \, \mathrm{d}x + \int_{\Gamma_N} \frac{\tau}{2} v_0^2 \, \mathrm{d}\Gamma$$

where $u_0 = E_0 v_0$

Theorem 1 (GH'97) (Steady states)

There are unique minimizers

$$u^* \in X^* \text{ of } F \text{ on } \mathcal{U} + U,$$

$$v^* \in X \text{ of } \Phi + \langle U, \cdot \rangle \text{ on } \mathcal{U}^\perp.$$

(u^*, v^*) is the unique solution to (S).

dissipation rate

$$D(v) = \langle Av, v \rangle_X \geq 0 \quad \forall v \in X \cap L^\infty(\Omega, \mathbb{R}^{m+1})$$

Brézis formula

(u, v) solution to (P) $\Longrightarrow$

$$v(t) \in \partial F(u(t)), \quad \frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = \langle u'(t), v(t) \rangle_X \text{ f.a.a. } t \in \mathbb{R}_+$$

$$\begin{aligned} e^{\lambda t_2}(F(u(t_2)) - F(u^*)) - e^{\lambda t_1}(F(u(t_1)) - F(u^*)) \\ = \int_{t_1}^{t_2} e^{\lambda s} \Big\{ \lambda (F(u(s)) - F(u^*)) - \langle Av(s), v(s) \rangle_X \Big\} \, \mathrm{d}s \end{aligned}$$

$$0 \le t_1 \le t_2, \ \lambda \in \mathbb{R}_+$$

Theorem 2 (GGH'96) (Boundedness of the free energy)

If (u, v) is a solution to (P) then

$$F(u(t)) \text{ decays monotonously,}$$

$$F(u(t)) \le F(U) \quad \forall t \ge 0.$$

Theorem 3 (GGH'96) (Nonlinear Poincaré like inequality)

For every $R > 0$ there is a $c_R > 0$ such that

$$F(Ev) - F(u^*) \le c_R \langle Av, v \rangle_X \quad \forall v \in M_R$$

where

$$M_R = \Big\{ v : \; F(Ev) - F(u^*) \le R, \; Ev - U \in \mathcal{U} \Big\}.$$

Theorem 4 (GGH'96) (Exponential decay of the free energy)

There exist $c, \lambda > 0$ such that for any solution (u, v) to (P)

$$F(u(t)) - F(u^*) \le e^{-\lambda t}(F(U) - F(u^*))$$

$$\|v_0(t) - v_0^*\|_{H^1} \le ce^{-\lambda t/2}$$

$$\|u_i(t) - u_i^*\|_{L^1} \le ce^{-\lambda t/2} \quad \forall t \ge 0.$$

Problems with some fast reactions

$$k_{\alpha\beta}(e^{\alpha\cdot\zeta} - e^{\beta\cdot\zeta})$$

$$v_{ch} = (v_1, \ldots, v_m)$$

$\mathcal{R}_0 \subset \mathcal{R}$, let for $(\alpha, \beta) \in \mathcal{R}_0$

$$k_{\alpha\beta} \to \infty \quad \Longrightarrow \quad 0 = e^{\alpha\cdot\zeta} - e^{\beta\cdot\zeta} = e^{\alpha\cdot v_{ch}} - e^{\beta\cdot v_{ch}}$$

occur only states (u, v) with

$$(\alpha - \beta)\cdot v_{ch} = 0 \quad \forall(\alpha, \beta) \in \mathcal{R}_0$$

$$v_{ch} \in \mathcal{S}_0^{\perp}, \quad \mathcal{S}_0 := \operatorname{span}\big\{\alpha - \beta : (\alpha, \beta) \in \mathcal{R}_0\big\}, \quad l := \dim \mathcal{S}_0^{\perp}$$

exist linear, injective mappings $\tilde{L} : \mathbb{R}^l \to \mathbb{R}^m$, $L : \mathbb{R}^{l+1} \to \mathbb{R}^{m+1}$

$$v_{ch} = \tilde{L}(\tilde{v}_1, \ldots, \tilde{v}_l) \quad \text{with} \quad \operatorname{Im} \tilde{L} = \mathcal{S}_0^{\perp}$$

$$v = L\tilde{v} = \big(v_0, \tilde{L}(\tilde{v}_1, \ldots, \tilde{v}_l)\big), \quad \tilde{v} = (v_0, \tilde{v}_1, \ldots, \tilde{v}_l)$$

Problems with some fast subprocesses

$L\,:\,\tilde{X} := H_0^1(\Omega \cup \Gamma_N) \times (H^1)^l \to H_0^1(\Omega \cup \Gamma_N) \times (H^1)^m$

- L linear, injective, continuous
- Im L is closed in X
- $\mathcal{U}^\perp \subset$ Im L
- $L^*\,:\,H_0^1(\Omega \cup \Gamma_N)^* \times (H^1)^{*m} \to H_0^1(\Omega \cup \Gamma_N)^* \times (H^1)^{*l}$ surjective

similar for fast diffusion of some species

$$D_i \to \infty \quad \Longrightarrow \quad \zeta_i = \text{const}, \;\; i \in I_0$$

or some fast reactions and some fast diffusions we find spaces $\tilde{X}$, operators L fulfilling

$$L\,:\,\tilde{X} \to X \text{ linear, continuous, injective, } \text{Im } L \text{ closed in } X,\; \mathcal{U}^\perp \subset \text{ Im } L$$

$$v = L\tilde{v}, \quad \tilde{v} \in \tilde{X}$$

Projection scheme

Basic Problem (P):

$$u'(t) + Av(t) = 0, \quad u(t) = Ev(t), \quad u(0) = U$$

set $v = L\tilde{v}$ and apply L^* to all equations

define $\tilde{u} = L^*u$, $\tilde{U} = L^*U$ mass lumping

define $\tilde{A} = L^*AL$, $\tilde{E} = L^*EL$

Reduced Problem ($\tilde{\text{P}}$):

$$\tilde{u}'(t) + \tilde{A}\,\tilde{v}(t) = 0,\ \tilde{u}(t) = \tilde{E}\,\tilde{v}(t) \quad \text{f.a.a. } t \in \mathbb{R}_+, \quad \tilde{u}(0) = \tilde{U},$$

$$\tilde{u} \in H^1_{\text{loc}}(\mathbb{R}_+, \tilde{X}^*),\ \tilde{v} \in L^2_{\text{loc}}(\mathbb{R}_+, \tilde{X}), L\tilde{v} \in L^2_{\text{loc}}(\mathbb{R}_+, X) \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^\infty(\Omega, \mathbb{R}^{m+1}))$$

Transfer of the convex structure of (P) to ($\tilde{\mathrm{P}}$)

$\tilde{E} = \partial\tilde{\Phi}$, $\quad \tilde{\Phi}(\tilde{v}) = \Phi(L\tilde{v})$

free energy $\quad \tilde{F}(\tilde{u}) = \tilde{\Phi}^*(\tilde{u}) = \sup_{\tilde{v}\in\tilde{X}} \left\{\langle \tilde{u}, \tilde{v}\rangle_{\tilde{X}} - \Phi(L\tilde{v})\right\} = \inf\left\{F(u) : L^*u = \tilde{u}\right\}$

$$\text{if } \tilde{F} \text{ is subdiff. in } \tilde{u} = \tilde{E}\tilde{v} \implies \tilde{F}(\tilde{u}) = F(EL\tilde{v})$$

Stationary reduced problem ($\tilde{\mathrm{S}}$):

$$(\tilde{u}^*, \tilde{v}^*) : \ \tilde{A}\tilde{v}^* = 0,\ \tilde{u}^* = \tilde{E}\tilde{v}^*,$$
$$\tilde{u}^* - \tilde{U} \in L^*[\mathcal{U}],\ \tilde{v}^* \in \tilde{X},\ L\tilde{v}^* \in L^\infty(\Omega)^{m+1}$$

Theorem 5 (Steady states)

- (u^*, v^*) solution to (S) $\implies (L^*u^*, (L|_{(\tilde{X}\to \mathrm{Im}\, L)})^{-1}v^*)$ solution to ($\tilde{\mathrm{S}}$)
- $(\tilde{u}^*, \tilde{v}^*)$ solution to ($\tilde{\mathrm{S}}$) $\implies (EL\tilde{v}^*, L\tilde{v}^*)$ solution to (S)
- ($\tilde{\mathrm{S}}$) has a unique solution $(\tilde{u}^*, \tilde{v}^*)$.

Brézis formula

dissipation rate

$$\langle \tilde{A}\tilde{v}, \tilde{v}\rangle_{\tilde{X}} = \langle AL\tilde{v}, L\tilde{v}\rangle_X = D(L\tilde{v}) \geq 0 \quad \forall \tilde{v} \in \tilde{X}$$

$(\tilde{u}, \tilde{v})$ solution to $(\tilde{\mathrm{P}}) \Longrightarrow$

$$\tilde{v}(t) \in \partial\tilde{F}(\tilde{u}(t)), \quad \frac{\mathrm{d}}{\mathrm{d}t}\tilde{F}(\tilde{u}(t)) = \langle \tilde{u}'(t), \tilde{v}(t)\rangle_{\tilde{X}} \quad \text{f.a.a. } t \in \mathbb{R}_+$$

$$\mathrm{e}^{\lambda t_2}(\tilde{F}(\tilde{u}(t_2)) - \tilde{F}(\tilde{u}^*)) - \mathrm{e}^{\lambda t_1}(\tilde{F}(\tilde{u}(t_1)) - \tilde{F}(\tilde{u}^*))$$
$$= \int_{t_1}^{t_2} \mathrm{e}^{\lambda s}\Big\{\lambda(\tilde{F}(\tilde{u}(s)) - \tilde{F}(\tilde{u}^*)) - \langle \tilde{A}\tilde{v}(s), \tilde{v}(s)\rangle_{\tilde{X}}\Big\}\,\mathrm{d}s$$

$$0 \leq t_1 \leq t_2, \ \lambda \in \mathbb{R}_+$$

Energy estimates

$(\tilde{u}, \tilde{v})$ solution of $(\tilde{\mathrm{P}}) \Longrightarrow$

$$\tilde{F}(\tilde{u}) = F(EL\tilde{v}) \text{ a.e. on } \mathbb{R}_+, \quad \tilde{F}(\tilde{u}^*) = F(u^*)$$

$$\langle \tilde{A}\tilde{v}, \tilde{v}\rangle_{\tilde{X}} = \langle AL\tilde{v}, L\tilde{v}\rangle_X, \quad L\tilde{v} \in M_R,\ R = F(U) - F(u^*)$$

Theorem 6 (Energy estimates)

For any solution $(\tilde{u}, \tilde{v})$ to $(\tilde{\mathrm{P}})$

- $\tilde{F}(\tilde{u}(t))$ decays monotonously ,
- $\tilde{F}(\tilde{u}(t)) \le \tilde{F}(\tilde{U}) \le F(U) \quad \forall t \in \mathbb{R}_+,$
- $\tilde{F}(\tilde{u}(t)) - \tilde{F}(\tilde{u}^*) \le e^{-\lambda t}\big(\tilde{F}(\tilde{U}) - \tilde{F}(\tilde{u}^*)\big) \quad \forall t \in \mathbb{R}_+.$

Comparison of the models

$(\tilde{u}, \tilde{v})$ solution to $(\tilde{\mathrm{P}})$, (u, v) solution to (P)

prolongated quantities:

$$\breve{v} := L\tilde{v}, \quad \breve{u} := EL\tilde{v}$$

steady states:

$$\breve{v}^* := L\tilde{v}^* = v^*, \quad \breve{u}^* := EL\tilde{v}^* = u^*$$

on trajectories of $(\tilde{\mathrm{P}})$

$$\tilde{F}(\tilde{u}(t)) = F(\breve{u}(t)) \quad \text{f.a.a. } t \in \mathbb{R}_+$$

conclusions from energy estimates for (P) and $(\tilde{\mathrm{P}})$:

Theorem 7 (Asymptotics of prolongated quantities)

$$\begin{aligned}
&|F(\breve{u}(t)) - F(u(t))| \le ce^{-\lambda t}, \\
&\|\breve{u}_i(t) - u_i(t)\|_{L^1} \le ce^{-\lambda t/2}, \qquad i = 0, \ldots, m, \\
&\|\breve{v}_0(t) - v_0(t)\|_{H^1} \le ce^{-\lambda t/2} \qquad \text{f.a.a. } t \in \mathbb{R}_+.
\end{aligned}$$

Fully implicit discrete-time version of $(\tilde{\mathrm{P}})$

partition

$$Z_n = \left\{t_n^0, t_n^1, \ldots, t_n^k, \ldots\right\}, \; t_n^0 = 0, \; t_n^k \in \mathbb{R}_+, \; t_n^{k-1} < t_n^k, \; t_n^k \to +\infty \text{ as } k \to \infty$$

$$h_n^k := t_n^k - t_n^{k-1}, \; \bar{h}_n := \sup_{k \in \mathbb{N}} h_n^k$$

spaces of piecewise constant functions

$$C_n(\mathbb{R}_+, B) := \left\{\tilde{u} : \mathbb{R}_+ \longrightarrow B \; : \; \tilde{u}(t) = \tilde{u}^k \quad \forall t \in (t_n^{k-1}, t_n^k], \; \tilde{u}^k \in B, \; k \in \mathbb{N}\right\}$$

$$\Delta_n : C_n(\mathbb{R}_+, \tilde{X}^*) \longrightarrow C_n(\mathbb{R}_+, \tilde{X}^*), \quad (\Delta_n \tilde{u})^k := \frac{1}{h_n^k}(\tilde{u}^k - \tilde{u}^{k-1}), \quad \tilde{u}^0 := \tilde{U}$$

Problem $(\tilde{\mathrm{P}}_n)$:

$$\Delta_n \tilde{u}_n(t) + \tilde{A}\tilde{v}_n(t) = 0, \quad \tilde{u}_n(t) = \tilde{E}\,\tilde{v}_n(t) \;\; \forall t \in \mathbb{R}_+,$$

$$\tilde{v}_n \in C_n(\mathbb{R}_+, \tilde{X}), \quad L\,\tilde{v}_n \in C_n(\mathbb{R}_+, X) \cap C_n(\mathbb{R}_+, L^\infty(\Omega, \mathbb{R}^{m+1}))$$

Energy estimates for the fully implicit discrete-time problem $(\tilde{\mathrm{P}})$

invariants: (u_n, v_n) solution to $(\tilde{\mathrm{P}}_n) \Longrightarrow \quad \tilde{u}_n(t) - \tilde{U} \in L^*[\mathcal{U}] \quad \forall t \in \mathbb{R}_+$

steady states: $(\tilde{\mathrm{P}}_n)$ has the same steady state $(\tilde{u}^*, \tilde{v}^*)$ as $(\tilde{\mathrm{P}})$

Theorem 8 (Energy estimates for the fully implicit discrete-time version)
Let $h > 0$ be given and let Z_n be any partition of $\mathbb{R}_+$ with $\bar{h}_n \le h$. Then the free energy $\tilde{F}$ decreases monotonously and exponentially along any solution $(\tilde{u}_n, \tilde{v}_n)$ to $(\tilde{\mathrm{P}}_n)$, i.e.,

$$\tilde{F}(\tilde{u}_n(t_2)) \le \tilde{F}(\tilde{u}_n(t_1)) \le F(U) \quad \text{for} \quad t_2 \ge t_1 \ge 0,$$

$$\tilde{F}(\tilde{u}_n(t)) - F(u^*) \le \mathrm{e}^{-\lambda t}(F(U) - F(u^*)) \quad \forall t \ge 0.$$

Discrete Brézis formular

$$e^{\lambda t_n^k}\big(\tilde F(\tilde u_n^k) - F(u^*)\big) - e^{\lambda t_n^j}\big(\tilde F(\tilde u_n^j) - F(u^*)\big)$$

$$= \sum_{l=j+1}^k \Big\{\big(e^{\lambda t_n^l} - e^{\lambda t_n^{l-1}}\big)\big(\tilde F(\tilde u_n^l) - F(u^*)\big) + e^{\lambda t_n^{l-1}}\big(\tilde F(\tilde u_n^l) - \tilde F(\tilde u_n^{l-1})\big)\Big\}$$

$$\leq \sum_{l=j+1}^k \Big\{e^{\lambda t_n^{l-1}}\big(e^{\lambda h_n^l} - 1\big)\big(\tilde F(\tilde u_n^l) - F(u^*)\big) + e^{\lambda t_n^{l-1}}\langle \tilde u_n^l - \tilde u_n^{l-1}, \tilde v_n^l\rangle_{\tilde X}\Big\}$$

$$\leq \sum_{l=j+1}^k \Big\{e^{\lambda t_n^{l-1}}\, e^{\lambda h}\, \lambda\, h_n^l\big(\tilde F(\tilde u_n^l) - F(u^*)\big) - e^{\lambda t_n^{l-1}}\, h_n^l\langle \tilde A\tilde v_n^l, \tilde v_n^l\rangle_{\tilde X}\Big\}$$

$$\leq \sum_{l=j+1}^k h_n^l\, e^{\lambda t_n^{l-1}}\Big\{e^{\lambda h}\, \lambda\, \big(F(EL\tilde v_n^l) - F(u^*)\big) - D(L\tilde v_n^l)\Big\}$$

1. $\lambda = 0 \Longrightarrow \quad \tilde F(\tilde u_n^k) \leq \tilde F(\tilde u_n^j) \leq \tilde F(L^*U) \leq F(U) \quad \forall k \geq j \geq 0$

2. fix $R > F(U) - F(u^*)$, for $\tilde u_n = \tilde E\tilde v_n$ we have $L\tilde v_n^l \in M_R$, $l \in \mathbb{N}$,
choose $\lambda > 0$ such that $\lambda\, e^{\lambda h} c_R \leq 1$, set $j = 0$, Poincaré like inequality $\Longrightarrow$

$$\tilde F(\tilde u_n^k) - F(u^*) \leq e^{-\lambda t_n^k}\big(F(U) - F(u^*)\big) \quad \forall k \in \mathbb{N}$$

Assumptions

$\Omega \subset \mathbb{R}^2$ bounded Lipschitzian domain, $\Gamma := \partial\Omega,\ \Gamma_D,\ \Gamma_N$ disjoint open subsets of Γ, $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ consists of finitely many points;

$q_i \in \mathbb{Z},\ \widehat{u}_i,\ U_i,\ D_i \in L^\infty(\Omega),\ \widehat{u}_i,\ U_i,\ D_i \ge c > 0,\ U_0 := \sum_{i=1}^m q_i U_i$,
$f \in L^2(\Omega),\ \varepsilon \in L^\infty(\Omega),\ \varepsilon \ge c > 0,\ \tau \in L_+^\infty(\Gamma_N),\ \mathrm{mes}\,\Gamma_D + \|\tau\|_{L^1(\Gamma_N)} > 0$;

$\mathcal{R}$ finite subset of $\mathbb{Z}_+^m \times \mathbb{Z}_+^m,\ (\alpha - \beta)\cdot(q_1, \dots, q_m) = 0\ \ \forall(\alpha, \beta) \in \mathcal{R}$;
for $(\alpha, \beta) \in \mathcal{R}$ we define $R_{\alpha\beta} := k_{\alpha\beta}(x, y)\,(\mathrm{e}^{\alpha\cdot\zeta} - \mathrm{e}^{\beta\cdot\zeta})$,
$x \in \Omega,\ y = (y_0, y_1, \cdots, y_m) \in \mathbb{R}^{m+1},\ \zeta_i := y_i + q_i y_0,\ i = 1, \dots, m,$ where
$k_{\alpha\beta} \colon \Sigma \times \mathbb{R}^{m+1} \to \mathbb{R}_+$ Carathéodory functions,
$k_{\alpha\beta}(x, y) \ge c_R > 0$ f.a.a. $x \in \Omega,\ \forall y \in [-R, R] \times \mathbb{R}^m$;

there are no "false" equilibria in the sense of Prigogine & Defay’54

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