Weierstraß-Institut für Angewandte Analysis und Stochastik

A. Glitzky (joint work with J. A. Griepentrog)

Discrete Sobolev-Poincaré inequalities for Voronoi finite volume approximations

Outline of the talk

- Notation in finite volume methods
- Assumptions
- Potential theoretical lemmas
- Main result
- Ideas of the proof of the discrete Sobolev-Poincaré inequality
- Concluding remarks

Sobolev imbedding result

$$\|u\|_{L^q} \leq c_q \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)$$

for $q \in [1,\infty)$ if $n = 2$, for $q \in [1, \frac{2n}{n-2}]$ if $n \geq 3$.

Discrete imbedding results in the context of finite volume schemes

	zero boundary values	general boundary values
	YES [1], [2]	NO present talk
add. assumpt.	$d_{K,\sigma} > \theta d_\sigma, d_{K,\sigma} > \theta \mathrm{diam}(K)$	Voronoi finite volume

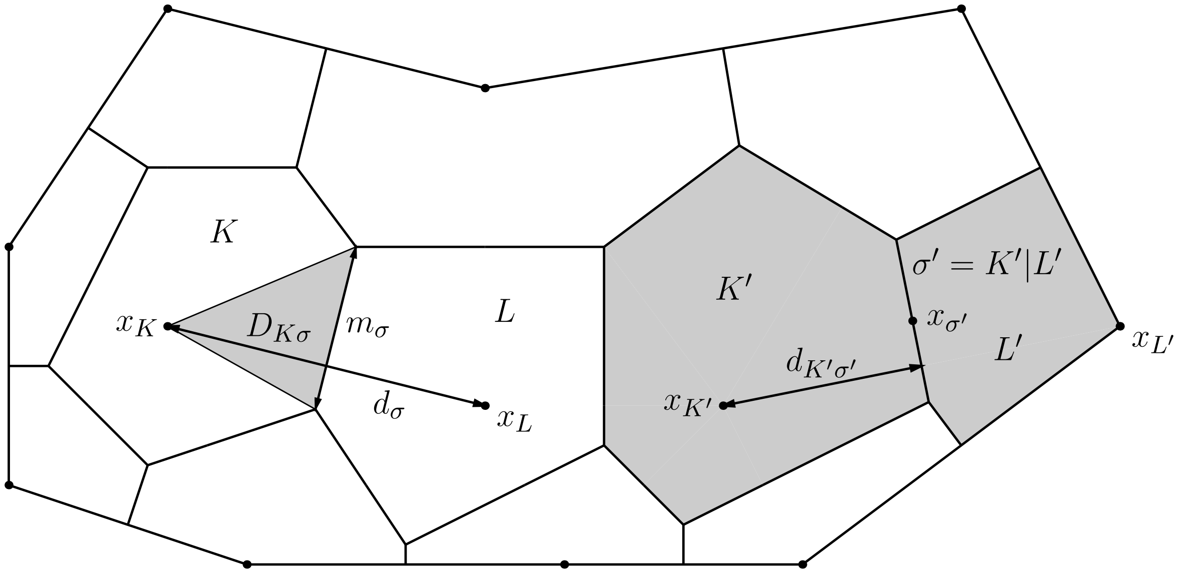
[1] Eymard, Gallouët, Herbin, in Handbook of Numerical Analysis VII 2000.
[2] Coudière, Gallouët, Herbin, M2AN **35**.

Notation

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded, polyhedral domain.

- A Voronoi mesh of Ω denoted by $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$ is formed by
  - a family $\mathcal{P}$ of grid points in $\bar{\Omega}$,
  - a family $\mathcal{T}$ of Voronoi control volumes,
  - a family $\mathcal{E}$ of parts of hyperplanes in $\mathbb{R}^n$ (surfaces of the V. boxes).
- For $x_K \in \mathcal{P}$ the control volume K of the Voronoi mesh is defined by

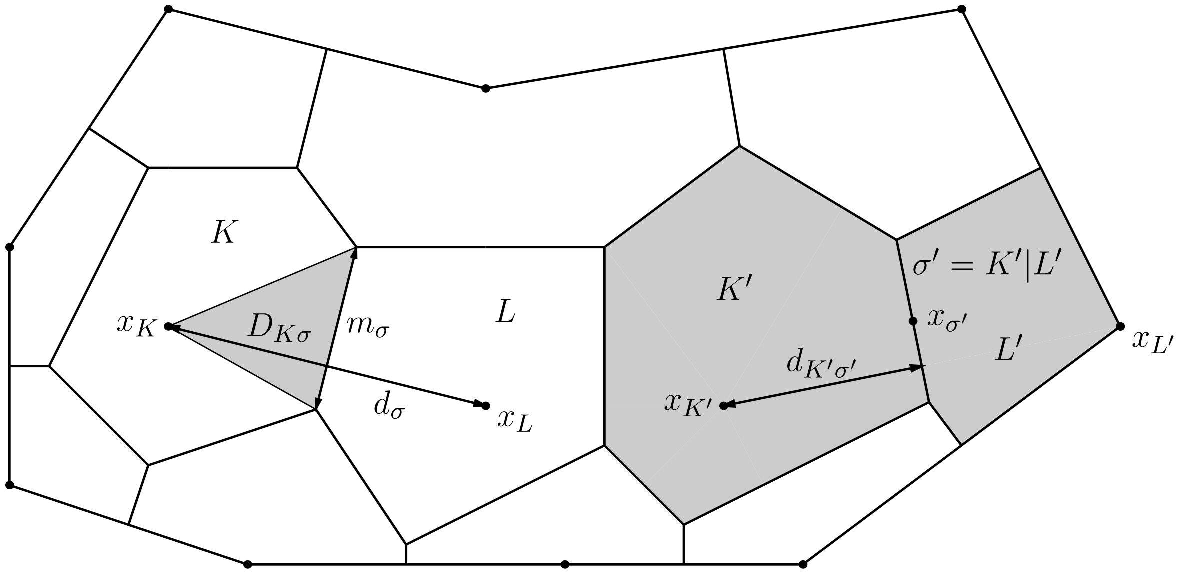
$$K = \{x \in \Omega : |x - x_K| < |x - x_L| \quad \forall x_L \in \mathcal{P},\ x_L \neq x_K\}, \quad K \in \mathcal{T}.$$



Notation

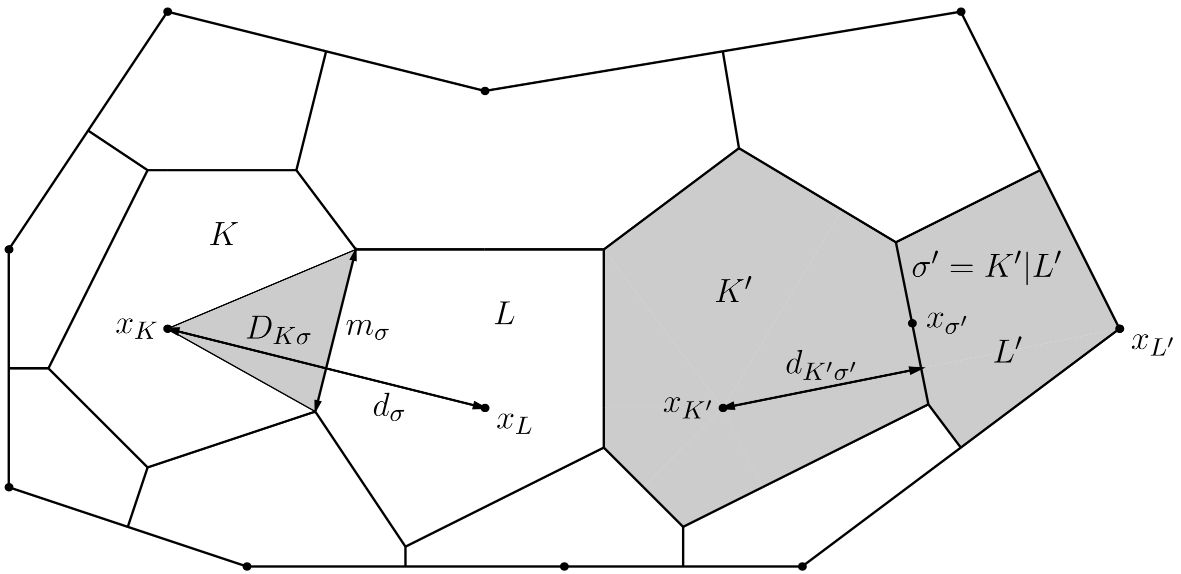
The set $\mathcal{E}$ and subsets

- For $K, L \in \mathcal{T}$ with $K \neq L$ either the $(n-1)$ dimensional Lebesgue measure of $\bar{K} \cap \bar{L}$ is zero or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}$.
- $\sigma = K|L$ denotes the Voronoi surface between K and L.
- $\mathcal{E}_{int}$ denotes the set of interior Voronoi surfaces
- $\mathcal{E}_{ext}$ denotes the set of external Voronoi surfaces
- For $K \in \mathcal{T}$: $\mathcal{E}_K$ is the subset of $\mathcal{E}$ such that $\partial K = \bar{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$.



Notation

For $\sigma \in \mathcal{E}$: m_σ - $(n-1)$-dimensional measure of the Voronoi surface σ.
x_σ - center of gravity of σ.
$d_{K,\sigma}$ - Euclidean distance between x_K and σ, if $\sigma \in \mathcal{E}_K$,
d_σ = $|x_K - x_L|$ if $\sigma = K|L \in \mathcal{E}_{int}$.



half-diamonds

$$D_{K\sigma} = \{tx_K + (1-t)y : t \in (0,1),\ y \in \sigma\}, \quad \mathrm{mes}(D_{K\sigma}) = \frac{1}{n} m_\sigma d_{K,\sigma}$$

Notation

Definition.

Let $\mathcal{M}$ be a Voronoi finite volume mesh of Ω.

1. $X(\mathcal{M})$ = set of functions from Ω to $\mathbb{R}$ which are constant on each $K \in \mathcal{T}$.

   u_K = value of $u \in X(\mathcal{M})$ on K.

2. Discrete H^1-seminorm of $u \in X(\mathcal{M})$

$$|u|_{1,\mathcal{M}}^2 = \sum_{\sigma \in \mathcal{E}_{int}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma},$$

where $D_\sigma u = |u_K - u_L|$ for $\sigma = K|L$.

Aim of the talk:

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_q |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \quad m_\Omega(u) = \frac{1}{\mathrm{mes}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x.$$

Assumptions on the geometry and the mesh

(A1) $\Omega \subset B(0,\widetilde{R}) \subset \mathbb{R}^n$ open, polyhedral, star shaped w.r.t. some ball $B(0,R)$.

Let $\varrho : \mathbb{R}^n \to [0,\infty)$, $\quad \varrho(y) = \begin{cases} \exp\left\{-\frac{R^2}{R^2-|y|^2}\right\} & \text{if } |y| < R \\ 0 & \text{if } |y| \geq R \end{cases}.$

define $\varrho^{\mathcal{M}} \in X(\mathcal{M})$ as $\quad \varrho_K^{\mathcal{M}}(x) = \min_{y\in\overline{K}} \varrho(y) \quad$ for $x \in K$.

(A2) Let $\mathcal{M} = (\mathcal{P},\mathcal{T},\mathcal{E})$ be a Voronoi finite volume mesh with $\int_\Omega \varrho^{\mathcal{M}}(x)\,\mathrm{d}x \geq \rho_0$ $(\rho_0 > 0)$ and with the property that $\mathcal{E}_K \cap \mathcal{E}_{ext} \neq \emptyset \implies x_K \in \partial\Omega$.

(A3) The geometric weights fulfill $0 < \frac{\mathrm{diam}(\sigma)}{d_\sigma} \leq \kappa_1$ for all $\sigma \in \mathcal{E}_{int}$.

(A4) There exists a constant $\kappa_2 \geq 1$ such that
$\max_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} \max_{x\in\overline{\sigma}} |x_K - x| \leq \kappa_2 \min_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} d_{K,\sigma}$ for all $x_K \in \mathcal{P}$.

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Lemma 1.
Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open, bounded, polyhydral and connected. Then there exists a $C_0 > 0$ such that for all Voronoi finite volume meshes $\mathcal{M}$

$$\|u - m_\Omega(u)\|_{L^2(\Omega)} \leq C_0 |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \quad m_\Omega(u) = \frac{1}{\mathrm{mes}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x.$$

(Eymard, Gallouët, Herbin, in Handbook of Numerical Analysis VII 2000, Gallouët, Herbin, Vignal, SIAM J. Numer. Anal. **37**.)

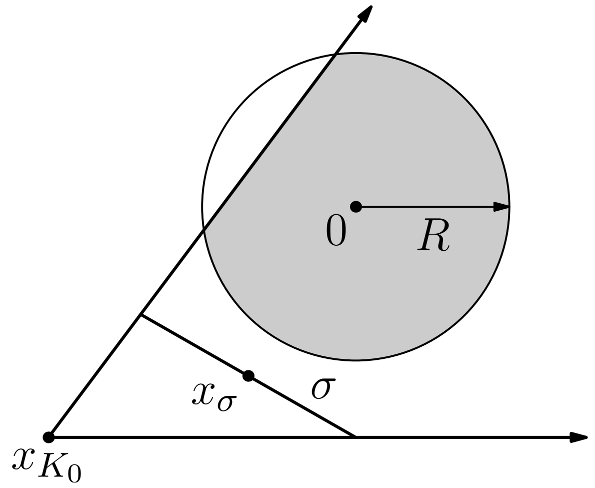
Lemma 2.

Let $\mathcal{M}$ be a Voronoi finite volume mesh of Ω such that (A1) – (A3) are fulfilled. Let x_{K_0} be a fixed grid point and $\sigma \in \mathcal{E}_{int}$ an internal Voronoi surface with gravitational center x_σ. Then

$$\mathrm{mes}(\{x \in B(0,R) : [x_{K_0}, x] \cap \sigma \neq \emptyset\})$$
$$\leq \frac{1}{n}\mathrm{diam}(\Omega)^n \max\{2,\, 4\,\kappa_1\}^{n-1} \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}} =: A_n \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}} \,.$$

Idea:

Estimation of the solid angle, estimate $\mathrm{mes}(...)$ by the measure of the corresponding segment of the ball with radius $\mathrm{diam}(\Omega)$.



Lemma 3.
We assume (A1) – (A3). Let

$$q \in \begin{cases} (2,\infty) & \text{if } n = 2 \\ (2, \frac{2n}{n-2}) & \text{if } n \geq 3 \end{cases}, \qquad 2\beta = \frac{n}{q} - \frac{n}{2} + 1.$$

Let $x_{K_0} \in \mathcal{P}$ be a fixed grid point. Then

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{mes}(D_{K\sigma})}{|x_{K_0} - x_\sigma|^{n-2\beta}} \leq \max\{1+2\kappa_1, 2\}^{n-2\beta} \frac{m_{n-1}}{2\beta} (2\widetilde{R})^{2\beta} =: \frac{B_n}{n},$$

where m_{n-1} denotes the measure of the $(n-1)$ dimensional unit sphere in $\mathbb{R}^n$.

Idea: Show
$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{mes}(D_{K\sigma})}{|x_{K_0} - x_\sigma|^{n-2\beta}} \leq c \int_\Omega \frac{\mathrm{d}x}{|x_{K_0} - x|^{n-2\beta}} (< \infty).$$

Lemma 4.

We assume (A1) – (A4). Let

$$q \in \begin{cases} (2,\infty) & \text{if } n = 2 \\ (2, \frac{2n}{n-2}) & \text{if } n \geq 3 \end{cases}, \quad 2\beta = \frac{n}{q} - \frac{n}{2} + 1.$$

Let $\sigma \in \mathcal{E}_{int}$ be a fixed inner Voronoi surface with gravitational center x_σ. Then

$$\sum_{K_0 \in \mathcal{T}} \sum_{\sigma_0 \in \mathcal{E}_{K_0}} \frac{\mathrm{mes}(D_{K_0\sigma_0})}{|x_{K_0} - x_\sigma|^{n-q\beta}} \leq \big(1 + \kappa_2(1 + 2\kappa_1)\big)^{n-q\beta} \frac{m_{n-1}}{q\beta} (2\widetilde{R})^{q\beta} =: D_n.$$

Idea: Show
$$\sum_{K_0 \in \mathcal{T}} \sum_{\sigma_0 \in \mathcal{E}_{K_0}} \frac{\mathrm{mes}(D_{K_0\sigma_0})}{|x_{K_0} - x_\sigma|^{n-q\beta}} \leq c \int_\Omega \frac{\mathrm{d}x}{|x - x_\sigma|^{n-q\beta}} .$$

Main result: Discrete Sobolev-Poincaré inequality

Theorem 1.
Let Ω be an open bounded polyhedral subset of $\mathbb{R}^n$ and let $\mathcal{M}$ be a Voronoi finite volume mesh such that (A1) – (A4) are fulfilled. Let $q \in (2, \infty)$ for $n = 2$ and $q \in (2, \frac{2n}{n-2})$ for $n > 2$, respectively. Then there exists a constant $c_q > 0$ only depending on n, q, Ω and the constants in (A1) – (A4) such that

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \le c_q |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \quad m_\Omega(u) = \frac{1}{\mathrm{mes}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x.$$

Glitzky, Griepentrog, WIAS-Preprint 1429 (2009)

Proof of the discrete Sobolev-Poincaré inequality, 1

Let $\mathcal{T}_0 = \{K \in \mathcal{T} : \overline{K} \subset B(0,R)\}$.

$$I_1 := \int_\Omega (u(x) - m_\Omega(u))\varrho^{\mathcal{M}}(x)\,\mathrm{d}x = \sum_{K' \in \mathcal{T}_0} \int_{K'} (u(x) - m_\Omega(u))\varrho^{\mathcal{M}}_{K'}\,\mathrm{d}x.$$

Let $K_0 \in \mathcal{T}$ be arbitrarily fixed. For all $K' \in \mathcal{T}_0$, f.a.a. $x \in K'$ write

$$u(x) - m_\Omega(u) = u_{K_0} - m_\Omega(u) + \sum_{\sigma = K_i | K_j} (u_{K_i} - u_{K_j})\chi_\sigma(x_{K_0}, x)$$

use correct order!

where

$$\chi_\sigma(x,y) = \begin{cases} 1 & \text{if } x, y \in \bar{\Omega} \text{ and } [x,y] \cap \sigma \neq \emptyset, \\ 0 & \text{if } x \notin \bar{\Omega} \text{ or } y \notin \bar{\Omega} \text{ or } [x,y] \cap \sigma = \emptyset. \end{cases}$$

and $[x,y]$ denotes the line segment $\{sx + (1-s)y,\ s \in [0,1]\}$.

Proof of the discrete Sobolev-Poincaré inequality, 2

Discrete Sobolev's integral representation

$$I_1 = (u_{K_0} - m_\Omega(u)) \int_\Omega \varrho^{\mathcal{M}} \, \mathrm{d}x + \sum_{K' \in \mathcal{T}_0} \int_{K'} \sum_{\sigma = K_i | K_j} (u_{K_i} - u_{K_j}) \varrho^{\mathcal{M}}_{K'} \chi_\sigma(x_{K_0}, x) \, \mathrm{d}x.$$

By (A2) $\Longrightarrow$

$$|u_{K_0} - m_\Omega(u)| \le \frac{|I_1|}{\rho_0} + \frac{I_2(K_0)}{\rho_0},$$

$$I_2(K_0) := \sum_{K' \in \mathcal{T}_0} \int_{K'} \sum_{\sigma = K_i | K_j \in \mathcal{E}_{int}} D_\sigma u \, \varrho^{\mathcal{M}}_{K'} \chi_\sigma(x_{K_0}, x) \, \mathrm{d}x.$$

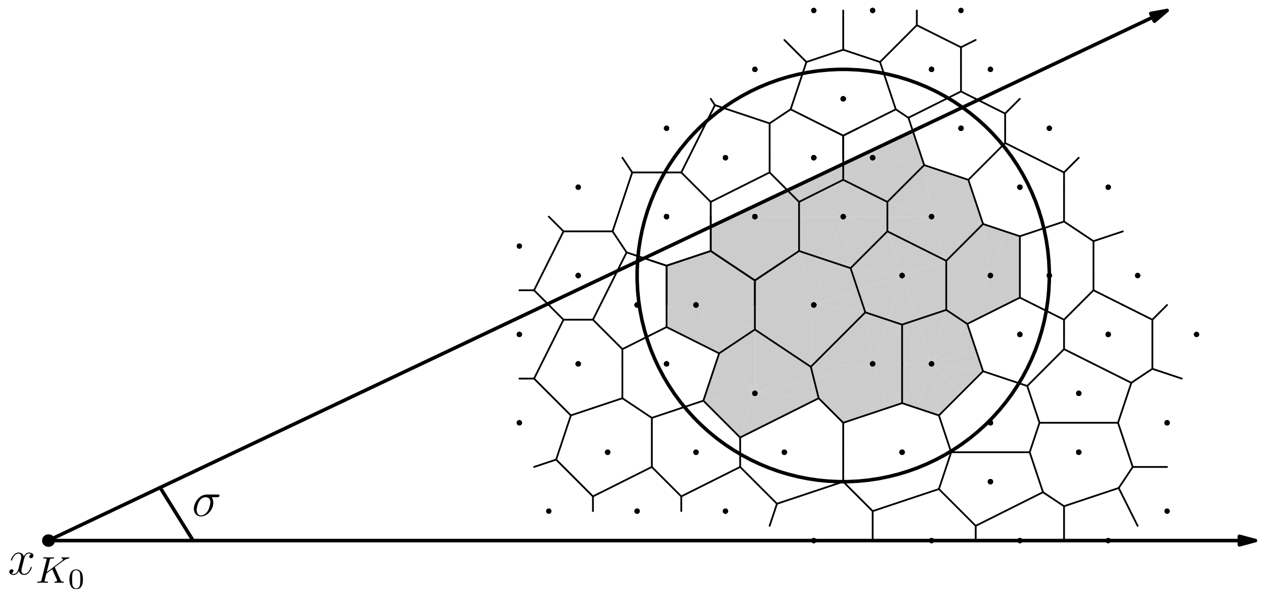
$$\begin{aligned} |I_1| &\le \left| \int_\Omega (u(x) - m_\Omega(u)) \varrho^{\mathcal{M}}(x) \, \mathrm{d}x \right| \\ &\le \mathrm{mes}(\Omega)^{1/2} \|u - m_\Omega(u)\|_{L^2(\Omega)} \\ &\le \mathrm{mes}(\Omega)^{1/2} \, C_0 \, |u|_{1,\mathcal{M}} \end{aligned}$$

(discrete Poincaré inequality)

Proof of the discrete Sobolev-Poincaré inequality, 3

$$
\begin{aligned}
I_2(K_0) &= \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u \sum_{K'\in\mathcal{T}_0} \int_{K'} \varrho_{K'}^{\mathcal{M}}\, \chi_\sigma(x_{K_0}, x)\,\mathrm{d}x \\
&\le \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u\ \mathrm{mes}(\{x\in B(0,R)\,:\,\sigma\cap[x_{K_0},x]\neq\emptyset\}) \\
&\le A_n \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u\, \frac{m_\sigma}{|x_{K_0}-x_\sigma|^{n-1}}
\end{aligned}
$$

Lemma 2



Proof of the discrete Sobolev-Poincaré inequality, 4

Hölder's inequality for $\alpha_1 = q$, $\alpha_2 = 2q/(q-2)$, $\alpha_3 = 2$, let $2\beta = \frac{n}{q} - \frac{n}{2} + 1$

$$\frac{I_2(K_0)}{A_n} \le \sum_{\sigma \in \mathcal{E}_{int}} |D_\sigma u| |x_{K_0} - x_\sigma|^{1-n} m_\sigma$$

$$\le \Big(\sum_{\sigma \in \mathcal{E}_{int}} |D_\sigma u|^2 |x_{K_0} - x_\sigma|^{-n+q\beta} \frac{m_\sigma}{d_\sigma} \Big)^{1/q} \Big(\sum_{\sigma \in \mathcal{E}_{int}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma} \Big)^{\frac{q-2}{2q}}$$

$$\times \Big(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} |x_{K_0} - x_\sigma|^{-n+2\beta} m_\sigma d_{K,\sigma} \Big)^{1/2}$$

$$\le B_n^{1/2} |u|_{1,\mathcal{M}}^{1-2/q} \Big(\sum_{\sigma \in \mathcal{E}_{int}} |D_\sigma u|^2 |x_{K_0} - x_\sigma|^{-n+q\beta} \frac{m_\sigma}{d_\sigma} \Big)^{1/q}$$

Lemma 3, discrete H^1-seminorm

Proof of the discrete Sobolev-Poincaré inequality, 5

$$\|I_2\|^q_{L^q(\Omega)} = \sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} I_2(K_0)^q \mathrm{mes}(D_{K_0\sigma_0})$$

$$\leq A_n^q B_n^{q/2} |u|^{q-2}_{1,\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma} \sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} |x_{K_0} - x_\sigma|^{-n+q\beta} \mathrm{mes}(D_{K_0\sigma_0})$$

$$\leq A_n^q B_n^{q/2} D_n |u|^q_{1,\mathcal{M}}$$

Lemma 4, discrete H^1-seminorm

In summary, for $u \in X(\mathcal{M})$

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq \frac{1}{\rho_0}\left[\|I_1\|_{L^q(\Omega)} + \|I_2\|_{L^q(\Omega)}\right]$$

$$\leq \frac{1}{\rho_0} \mathrm{mes}(\Omega)^{1/q+1/2}\, C_0\, |u|_{1,\mathcal{M}} + \frac{A_n}{\rho_0} B_n^{1/2} D_n^{1/q} |u|_{1,\mathcal{M}}$$

Concluding remarks

- For $q \in [1,2]$ and $n \geq 2$, the discrete Sobolev-Poincaré inequalities

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_q |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M})$$

are a direct consequence of Theorem 1 and Hölder's inequality.

- Corollary. Assume (A1) – (A4). Let $q \in [1,\infty)$ for $n = 2$ and $q \in [1, \frac{2n}{n-2})$ for $n \geq 3$, respectively. Then there exists a constant $c_q > 0$ only depending on n, q, Ω and the constants in (A1) – (A4) such that

$$\|u\|_{L^q(\Omega)} \leq c_q\, |u|_{1,\mathcal{M}} + \mathrm{mes}(\Omega)^{\frac{1}{q}-1} \Big| \int_\Omega u \,\mathrm{d}x \Big| \quad \forall u \in X(\mathcal{M}).$$

- More general domains: Discrete Sobolev inequalities remain true if Ω is a finite union of δ-overlapping star shaped domains Ω_i, $i = 1, \ldots, r$.

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- More general domains: Discrete Sobolev inequalities remain true if Ω is a finite union of δ-overlapping star shaped domains Ω_i, $i = 1, \ldots, r$.

Concluding remarks

- Critical exponent in higher space dimensions:
  For $n \geq 3$, the discrete version of the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ (the critical Sobolev exponent in n dimensions) can not be obtained by the presented technique using the Sobolev integral representation. This is exactly the same situation as for the continuous case.

- Discrete Sobolev-Poincaré inequalities for $p < n$, $q \in [1, \frac{np}{n-p})$:

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_q |u|_{1,p,\mathcal{M}} \quad \forall u \in X(\mathcal{M})$$

- requires a Poincaré like inequality using the discrete $W^{1,p}$-seminorm

$$|u|_{1,p,\mathcal{M}} = \Big(\sum_{\sigma \in \mathcal{E}_{int}} \Big(\frac{D_\sigma u}{d_\sigma} \Big)^p m_\sigma d_\sigma \Big)^{1/p}.$$

- adapt technique of the proof of Theorem 1 for $p \neq 2$.

Concluding remarks

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  For $n \geq 3$, the discrete version of the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ (the critical Sobolev exponent in n dimensions) can not be obtained by the presented technique using the Sobolev integral representation. This is exactly the same situation as for the continuous case.

- Discrete Sobolev-Poincaré inequalities for $p < n$, $q \in [1, \frac{np}{n-p})$:

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_q |u|_{1,p,\mathcal{M}} \quad \forall u \in X(\mathcal{M})$$

  - requires a Poincaré like inequality using the discrete $W^{1,p}$-seminorm

$$|u|_{1,p,\mathcal{M}} = \Big(\sum_{\sigma \in \mathcal{E}_{int}} \Big(\frac{D_\sigma u}{d_\sigma} \Big)^p m_\sigma d_\sigma \Big)^{1/p}.$$

  - adapt technique of the proof of Theorem 1 for $p \neq 2$.