Weierstraß-Institut für Angewandte Analysis und Stochastik

A. Glitzky (joint work with J. A. Griepentrog)

Discrete Sobolev-Poincaré inequalities using the $W^{1,p}$-seminorm in the setting of Voronoi finite volume approximations

Outline of the talk

- ▷ Notation in finite volume methods
- ▷ Assumptions
- ▷ Potential theoretical lemmas
- ▷ Main result
- ▷ Ideas of the proof of the discrete Sobolev-Poincaré inequality
- ▷ Concluding remarks

Motivation

Sobolev imbedding result

$$\|u\|_{L^q} \leq c_{q,p}\|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega)$$

for $q \in [1, \infty)$ if $p = n$, for $q \in [1, \frac{pn}{n-p}]$ if $p < n$.

Discrete imbedding results in the context of finite volume schemes

	zero boundary values	general boundary values
	YES	NO
$p = 2$	[1], [2]	WIAS-Preprint 1429 (2009)
non Hilbertian case	[3], [4]	talk

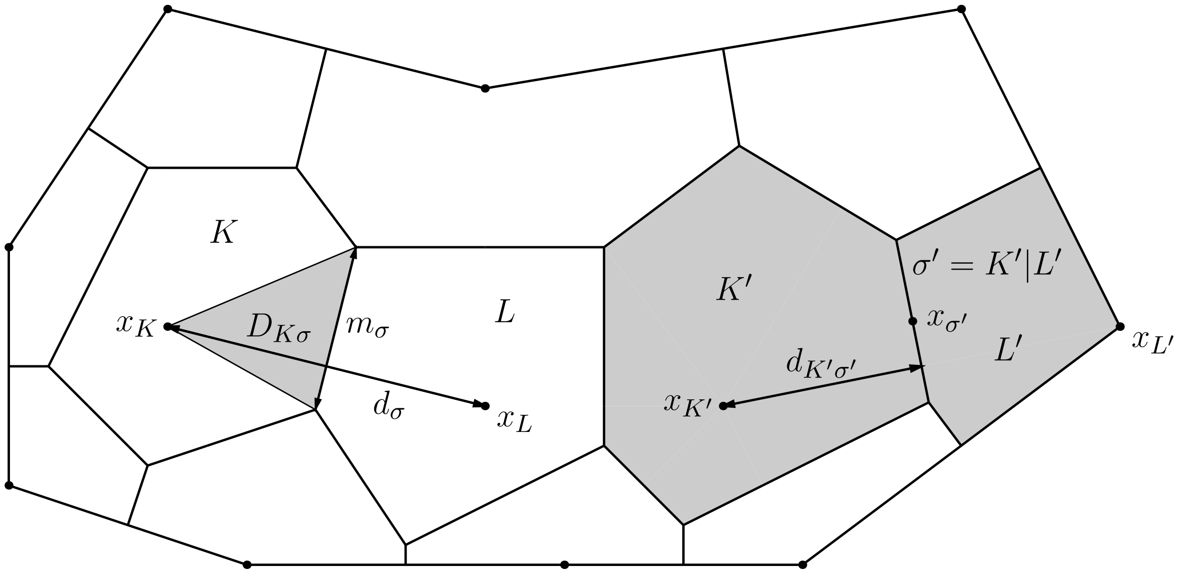
[1] Eymard, Gallouët, Herbin, in Handbook of Numerical Analysis VII 2000.
[2] Coudière, Gallouët, Herbin, M2AN **35** (2001).
[3] Droniou, Gallouët, Herbin, SINUM **41** (2003).
[4] Eymard, Gallouët, Herbin to appear in IMA JMA.

Notation

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded, polyhedral domain.

- A Voronoi mesh of Ω denoted by $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$ is formed by
  - a family $\mathcal{P}$ of grid points in $\bar{\Omega}$,
  - a family $\mathcal{T}$ of Voronoi control volumes,
  - a family $\mathcal{E}$ of parts of hyperplanes in $\mathbb{R}^n$ (faces of the Voronoi boxes).
- For $x_K \in \mathcal{P}$ the control volume K of the Voronoi mesh is defined by

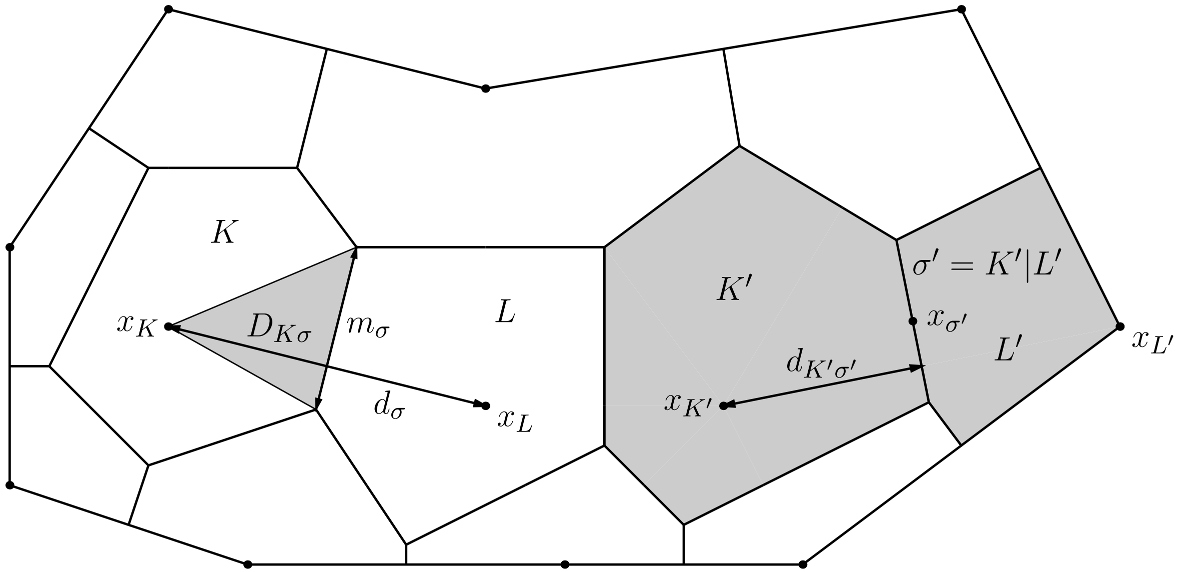
$$K = \{x \in \Omega : |x - x_K| < |x - x_L| \quad \forall x_L \in \mathcal{P},\ x_L \neq x_K\}, \quad K \in \mathcal{T}.$$



Notation

The set $\mathcal{E}$ and subsets

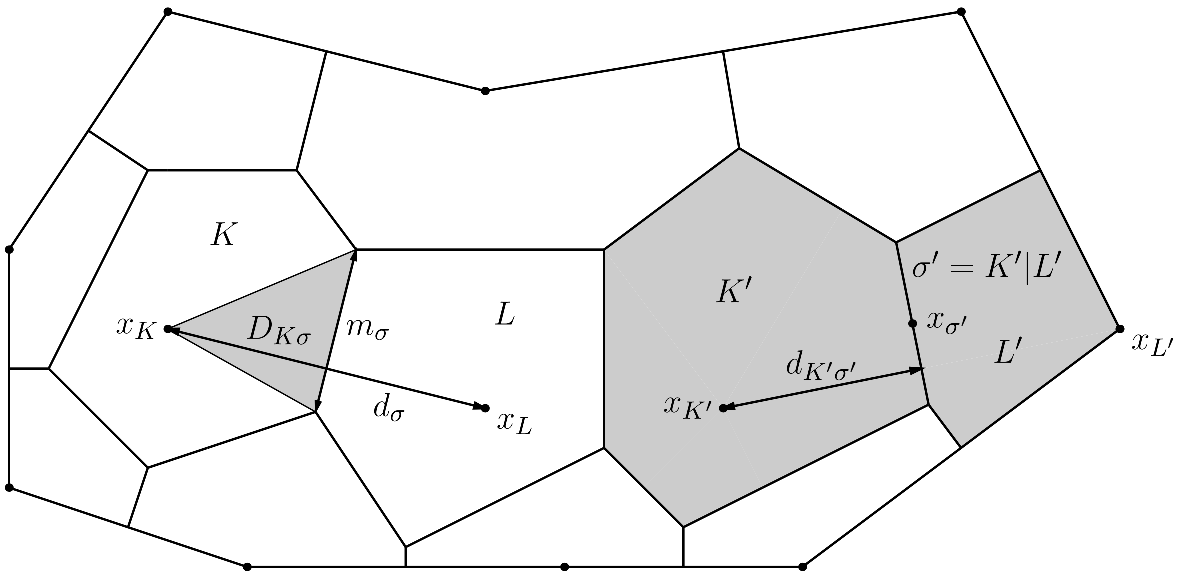
- For $K, L \in \mathcal{T}$ with $K \neq L$ either the $(n-1)$ dimensional Lebesgue measure of $\bar{K} \cap \bar{L}$ is zero or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}$.
- $\sigma = K|L$ denotes the Voronoi face between K and L.
- $\mathcal{E}_{int}$ denotes the set of interior Voronoi faces.
- $\mathcal{E}_{ext}$ denotes the set of external Voronoi faces.
- For $K \in \mathcal{T}$: $\mathcal{E}_K$ is the subset of $\mathcal{E}$ such that $\partial K = \bar{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$.



Notation

For $\sigma \in \mathcal{E}$:

m_σ	-	$(n-1)$-dimensional measure of the Voronoi face σ.
x_σ	-	center of gravity of σ.
$d_{K,\sigma}$	-	Euclidean distance between x_K and σ, if $\sigma \in \mathcal{E}_K$.
d_σ	=	$\lvert x_K - x_L \rvert$ if $\sigma = K\lvert L \in \mathcal{E}_{int}$.



half-diamonds

$$D_{K\sigma} = \{tx_K + (1-t)y : t \in (0,1),\ y \in \sigma\}, \quad \mathrm{mes}(D_{K\sigma}) = \frac{1}{n} m_\sigma d_{K,\sigma}$$

Definition.

Let $\mathcal{M}$ be a Voronoi finite volume mesh of Ω.

1. $X(\mathcal{M})$ = set of functions from Ω to $\mathbb{R}$ which are constant on each $K \in \mathcal{T}$.

   u_K = value of $u \in X(\mathcal{M})$ on K.

2. Discrete $W^{1,p}$-seminorm of $u \in X(\mathcal{M})$, $p \in [1, \infty)$

$$|u|_{1,p,\mathcal{M}} = \Big(\sum_{\sigma \in \mathcal{E}_{int}} \Big(\frac{D_\sigma u}{d_\sigma}\Big)^p m_\sigma d_\sigma \Big)^{1/p},$$

where $D_\sigma u = |u_K - u_L|$ for $\sigma = K|L$.

Aim of the talk:

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_{q,p} |u|_{1,p,\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \quad m_\Omega(u) = \frac{1}{\mathrm{mes}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x.$$

Assumptions on the geometry and the mesh

(A1) $\Omega \subset B(0, \widetilde{R}) \subset \mathbb{R}^n$ open, polyhedral, star shaped w.r.t. some ball $B(0, R)$.

Let $\varrho : \mathbb{R}^n \to [0, \infty)$, $\quad \varrho(y) = \begin{cases} \exp\left\{ -\frac{R^2}{R^2 - |y|^2} \right\} & \text{if } |y| < R \\ 0 & \text{if } |y| \geq R \end{cases}.$

define $\varrho^{\mathcal{M}} \in X(\mathcal{M})$ as $\quad \varrho_K^{\mathcal{M}}(x) = \min_{y \in \overline{K}} \varrho(y) \quad$ for $x \in K$.

(A2) Let $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$ be a Voronoi finite volume mesh with $\int_\Omega \varrho^{\mathcal{M}}(x)\,\mathrm{d}x \geq \rho_0$ ($\rho_0 > 0$) and with the property that $\mathcal{E}_K \cap \mathcal{E}_{ext} \neq \emptyset \implies x_K \in \partial\Omega$.

(A3) The geometric weights fulfill $0 < \frac{\mathrm{diam}(\sigma)}{d_\sigma} \leq \kappa_1$ for all $\sigma \in \mathcal{E}_{int}$.

(A4) There exists a constant $\kappa_2 \geq 1$ such that

$$\max_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \max_{x \in \overline{\sigma}} |x_K - x| \leq \kappa_2 \min_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} d_{K,\sigma} \text{ for all } x_K \in \mathcal{P}.$$

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$\max_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \max_{x \in \overline{\sigma}} |x_K - x| \leq \kappa_2 \min_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} d_{K,\sigma}$ for all $x_K \in \mathcal{P}$.

Lemma 1.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, polyhydral and connected. And let $n \geq 2$, $p \in (1, \infty)$. Then there exists a $C_{1,p} > 0$ such that for all Voronoi finite volume meshes $\mathcal{M}$

$$\|u - m_\Omega(u)\|_{L^1(\Omega)} \leq C_{1,p}|u|_{1,p\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \quad m_\Omega(u) = \frac{1}{\mathrm{mes}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x.$$

Idea:

Discrete Poincaré inequality + Hölder's inequality

$$\|u - m_\Omega(u)\|_{L^p(\Omega)} \leq C_{p,p}|u|_{1,p\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \quad p \in (1, 2].$$

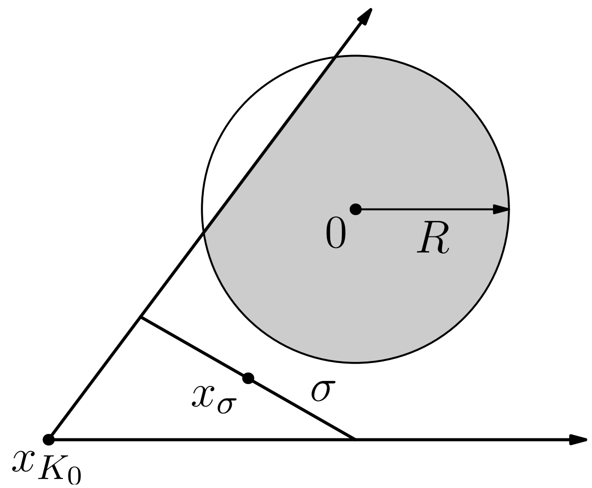
Prove for convex subdomains Ω_i that $\|u - m_\omega(u)\|_{L^p(\Omega_i)} \leq C_i|u|_{1,p\mathcal{M}}$ where $\omega \subset \Omega_i$, $mes(\omega) > 0$,
write $\Omega = \cup_{i=1}^r \Omega_i$, think of $\omega = \Omega_i$, $\omega = \Omega_i \cap \Omega_j$, compose the estimates.

Lemma 2.

Let $\mathcal{M}$ be a Voronoi finite volume mesh of Ω such that (A1) – (A3) are fulfilled. Let x_{K_0} be a fixed grid point and $\sigma \in \mathcal{E}_{int}$ an internal Voronoi face with gravitational center x_σ. Then

$$\begin{aligned}&\operatorname{mes}(\{x \in B(0,R) : [x_{K_0}, x] \cap \sigma \neq \emptyset\})\\&\quad \leq \frac{1}{n}\operatorname{diam}(\Omega)^n \max\{2,\, 4\,\kappa_1\}^{n-1} \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}} =: A_n \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}}\,.\end{aligned}$$

Idea:
Estimation of the solid angle, estimate $\operatorname{mes}(...)$ by the measure of the corresponding segment of the ball with radius $\operatorname{diam}(\Omega)$.



Lemma 3.

We assume (A1) – (A3). Let $p \in (1, n]$,

$$q \in \begin{cases} (p, \infty) & \text{if } p = n \\ (p, \frac{pn}{n-p}) & \text{if } p < n \end{cases}, \qquad 2\beta = \frac{n}{q} + \frac{n}{p'} - n + 1, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Let $x_{K_0} \in \mathcal{P}$ be a fixed grid point. Then

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{mes}(D_{K\sigma})}{|x_{K_0} - x_\sigma|^{n-p'\beta}} \le \max\{1+2\kappa_1, 2\}^{n-p'\beta} \frac{m_{n-1}}{p'\beta} (2\widetilde{R})^{p'\beta} =: \frac{B_n}{n},$$

where m_{n-1} denotes the measure of the $(n-1)$ dimensional unit sphere in $\mathbb{R}^n$.

Idea: Show $$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{mes}(D_{K\sigma})}{|x_{K_0} - x_\sigma|^{n-p'\beta}} \le c \int_\Omega \frac{\mathrm{d}x}{|x_{K_0} - x|^{n-p'\beta}} (< \infty).$$

Potential theoretical lemmas

Lemma 4.
We assume (A1) – (A4). Let $p \in (1, n]$,

$$q \in \begin{cases} (p, \infty) & \text{if } p = n \\ (p, \frac{pn}{n-p}) & \text{if } p < n \end{cases}, \qquad 2\beta = \frac{n}{q} + \frac{n}{p'} - n + 1, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Let $\sigma \in \mathcal{E}_{int}$ be a fixed inner Voronoi face with gravitational center x_σ. Then

$$\sum_{K_0 \in \mathcal{T}} \sum_{\sigma_0 \in \mathcal{E}_{K_0}} \frac{\mathrm{mes}(D_{K_0\sigma_0})}{|x_{K_0} - x_\sigma|^{n-q\beta}} \leq \big(1 + \kappa_2(1 + 2\kappa_1)\big)^{n-q\beta} \frac{m_{n-1}}{q\beta} (2\widetilde{R})^{q\beta} =: D_n.$$

Idea: Show $$\sum_{K_0 \in \mathcal{T}} \sum_{\sigma_0 \in \mathcal{E}_{K_0}} \frac{\mathrm{mes}(D_{K_0\sigma_0})}{|x_{K_0} - x_\sigma|^{n-q\beta}} \leq c \int_\Omega \frac{\mathrm{d}x}{|x - x_\sigma|^{n-q\beta}} \; .$$

Main result: Discrete Sobolev-Poincaré inequality

Theorem 1.
Let Ω be an open bounded polyhedral subset of $\mathbb{R}^n$ and let $\mathcal{M}$ be a Voronoi finite volume mesh such that (A1) – (A4) are fulfilled. Let $p \in (1, n]$, and $q \in (p, \infty)$ for $p = n$ and $q \in (p, \frac{pn}{n-p})$ for $p < n$, respectively. Then there exists a constant $c_{q,p} > 0$ only depending on n, p, q, Ω and the constants in (A1) – (A4) such that

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_{q,p}|u|_{1,p,\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \quad m_\Omega(u) = \frac{1}{\mathrm{mes}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x.$$

Glitzky, Griepentrog, WIAS-Preprint 1429 (2009) for $p = 2$.

Proof of the discrete Sobolev-Poincaré inequality, 1

Let $\mathcal{T}_0 = \{K \in \mathcal{T} : \overline{K} \subset B(0,R)\}$.

$$I_1 := \int_\Omega (u(x) - m_\Omega(u))\varrho^{\mathcal{M}}(x)\,\mathrm{d}x = \sum_{K' \in \mathcal{T}_0} \int_{K'} (u(x) - m_\Omega(u))\varrho^{\mathcal{M}}_{K'}\,\mathrm{d}x.$$

Let $K_0 \in \mathcal{T}$ be arbitrarily fixed. For all $K' \in \mathcal{T}_0$, f.a.a. $x \in K'$ write

$$u(x) - m_\Omega(u) = u_{K_0} - m_\Omega(u) + \sum_{\sigma = K_i|K_j} (u_{K_i} - u_{K_j})\chi_\sigma(x_{K_0}, x)$$

use correct order!

where

$$\chi_\sigma(x,y) = \begin{cases} 1 & \text{if } x, y \in \bar{\Omega} \text{ and } [x,y] \cap \sigma \neq \emptyset, \\ 0 & \text{if } x \notin \bar{\Omega} \text{ or } y \notin \bar{\Omega} \text{ or } [x,y] \cap \sigma = \emptyset. \end{cases}$$

and $[x,y]$ denotes the line segment $\{sx + (1-s)y,\ s \in [0,1]\}$.

Proof of the discrete Sobolev-Poincaré inequality, 2

Discrete Sobolev's integral representation

$$I_1 = (u_{K_0} - m_\Omega(u)) \int_\Omega \varrho^{\mathcal{M}} \, \mathrm{d}x + \sum_{K' \in \mathcal{T}_0} \int_{K'} \sum_{\sigma = K_i | K_j} (u_{K_i} - u_{K_j}) \varrho^{\mathcal{M}}_{K'} \chi_\sigma(x_{K_0}, x) \, \mathrm{d}x.$$

By (A2) $\Longrightarrow$

$$|u_{K_0} - m_\Omega(u)| \le \frac{|I_1|}{\rho_0} + \frac{I_2(K_0)}{\rho_0},$$

$$I_2(K_0) := \sum_{K' \in \mathcal{T}_0} \int_{K'} \sum_{\sigma = K_i | K_j \in \mathcal{E}_{int}} D_\sigma u \, \varrho^{\mathcal{M}}_{K'} \chi_\sigma(x_{K_0}, x) \, \mathrm{d}x.$$

$$\begin{aligned} |I_1| &\le \left| \int_\Omega (u(x) - m_\Omega(u)) \varrho^{\mathcal{M}}(x) \, \mathrm{d}x \right| \\ &\le \|u - m_\Omega(u)\|_{L^1(\Omega)} \\ &\le C_{1,p} \, |u|_{1,p,\mathcal{M}} \end{aligned}$$

Lemma 1

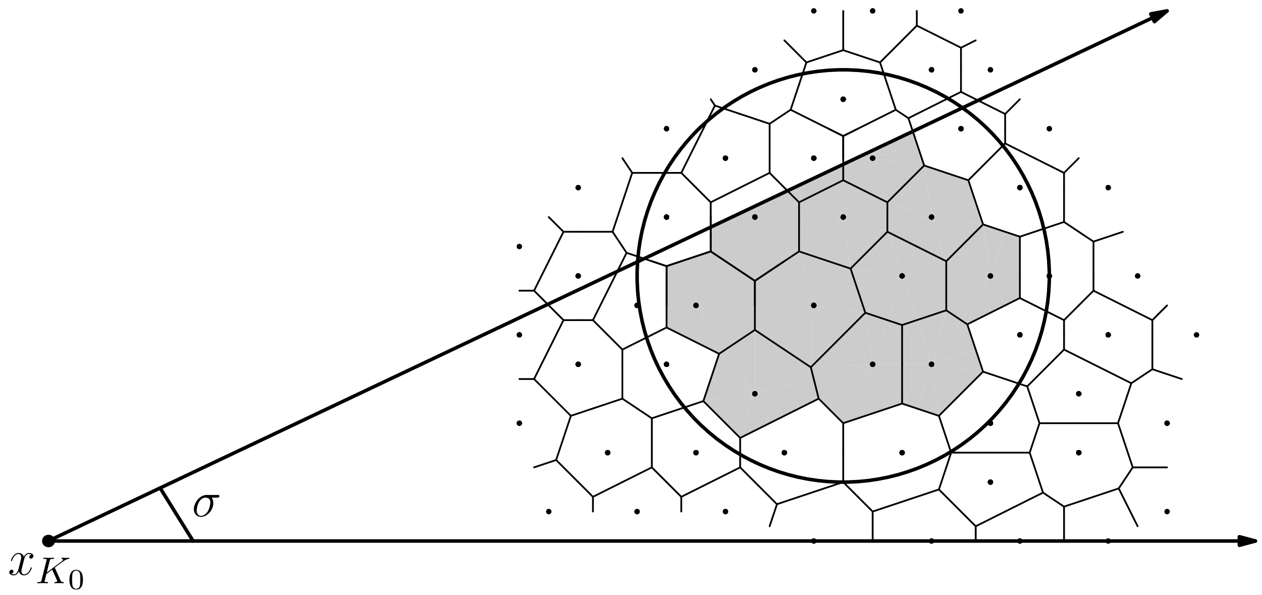
Proof of the discrete Sobolev-Poincaré inequality, 3

$$I_2(K_0) = \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u \sum_{K'\in\mathcal{T}_0} \int_{K'} \varrho_{K'}^{\mathcal{M}}\, \chi_\sigma(x_{K_0}, x)\,\mathrm{d}x$$

$$\leq \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u \ \mathrm{mes}(\{x \in B(0,R) : \sigma \cap [x_{K_0}, x] \neq \emptyset\})$$

$$\leq A_n \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}}$$

Lemma 2



Proof of the discrete Sobolev-Poincaré inequality, 4

Hölder's inequality for $\alpha_1 = q$, $\alpha_2 = pq/(q-p)$, $\alpha_3 = p'$, let $2\beta = \frac{n}{q} + \frac{n}{p'} - n + 1$

$$\frac{I_2(K_0)}{A_n} \leq \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u |x_{K_0} - x_\sigma|^{1-n} m_\sigma$$

$$\leq \Big(\sum_{\sigma\in\mathcal{E}_{int}} \Big(\frac{D_\sigma u}{d_\sigma}\Big)^p |x_{K_0} - x_\sigma|^{-n+q\beta} m_\sigma d_\sigma \Big)^{1/q} \Big(\sum_{\sigma\in\mathcal{E}_{int}} \Big(\frac{D_\sigma u}{d_\sigma}\Big)^p m_\sigma d_\sigma \Big)^{\frac{q-p}{pq}}$$

$$\times \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} |x_{K_0} - x_\sigma|^{-n+p'\beta} m_\sigma d_{K,\sigma} \Big)^{1/p'}$$

$$\leq B_n^{1/p'} |u|_{1,p,\mathcal{M}}^{1-p/q} \Big(\sum_{\sigma\in\mathcal{E}_{int}} \Big(\frac{D_\sigma u}{d_\sigma}\Big)^p |x_{K_0} - x_\sigma|^{-n+q\beta} m_\sigma d_\sigma \Big)^{1/q}$$

Lemma 3, discrete $W^{1,p}$-seminorm

Proof of the discrete Sobolev-Poincaré inequality, 5

$$
\begin{aligned}
\|I_2\|^q_{L^q(\Omega)} &= \sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} I_2(K_0)^q \mathrm{mes}(D_{K_0\sigma_0}) \\
&\le A_n^q B_n^{q/p'} |u|^{q-p}_{1,p,\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{int}} \left(\frac{D_\sigma u}{d_\sigma}\right)^p m_\sigma d_\sigma \sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} |x_{K_0}-x_\sigma|^{-n+q\beta} \mathrm{mes}(D_{K_0\sigma_0}) \\
&\le A_n^q B_n^{q/p'} D_n |u|^q_{1,p,\mathcal{M}} \qquad \text{Lemma 4, discrete } W^{1,p}\text{-seminorm}
\end{aligned}
$$

In summary, for $u \in X(\mathcal{M})$

$$
\begin{aligned}
\|u - m_\Omega(u)\|_{L^q(\Omega)} &\le \frac{1}{\rho_0}\left[\|I_1\|_{L^q(\Omega)} + \|I_2\|_{L^q(\Omega)}\right] \\
&\le \frac{1}{\rho_0} \mathrm{mes}(\Omega)^{1/q}\, C_{1,p}\, |u|_{1,p,\mathcal{M}} + \frac{A_n}{\rho_0} B_n^{1/p'} D_n^{1/q} |u|_{1,p,\mathcal{M}}
\end{aligned}
$$

Concluding remarks

- For $q \in [1,p]$ and $n \geq p$, the discrete Sobolev-Poincaré inequalities

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_{q,p}|u|_{1,p,\mathcal{M}} \quad \forall u \in X(\mathcal{M})$$

are a direct consequence of Theorem 1 and Hölder's inequality.

- Corollary. Assume (A1) – (A4). Let $q \in [1,\infty)$ for $n = p$ and $q \in [1, \frac{pn}{n-p})$ for $n > p$, respectively. Then there exists a constant $c_{q,p} > 0$ only depending on n, q, p, Ω and the constants in (A1) – (A4) such that

$$\|u\|_{L^q(\Omega)} \leq c_{q,p}\,|u|_{1,p,\mathcal{M}} + \operatorname{mes}(\Omega)^{\frac{1}{q}-1}\Big|\int_\Omega u\,\mathrm{d}x\Big| \quad \forall u \in X(\mathcal{M}).$$

- More general domains: Discrete Sobolev inequalities remain true if Ω is a finite union of δ-overlapping star shaped domains Ω_i, $i = 1, \ldots, N$.

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- More general domains: Discrete Sobolev inequalities remain true if Ω is a finite union of δ-overlapping star shaped domains Ω_i, $i = 1, \ldots, N$.

Concluding remarks

- Exponential estimate for $p = n$:
  Under the assumptions (A1) – (A4) there exist constants $\Sigma > 0$ and $\gamma > 0$ only depending on n, Ω and the constants in (A1) – (A4) such that

$$\int_\Omega \mathrm{e}^{r|u|}\,\mathrm{d}x \le \gamma \exp\left\{ r|m_\Omega(u)| + \frac{(r|u|_{1,n,\mathcal{M}})^n}{n(n'\Sigma)^{\frac{n}{n'}}} \right\} \quad \forall u \in X(\mathcal{M}),\ \forall r \in (0,\infty).$$

- The case $p > n$:
  - $\|u - m_\Omega(u)\|_{L^\infty(\Omega)} \le c_{\infty,p}|u|_{1,p,\mathcal{M}} \quad \forall u \in X(\mathcal{M}).$
  - Discrete analog to the imbedding of $W^{1,p}(\Omega)$ in $C^{0,\alpha}(\bar{\Omega})$ is under discussion.

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