Research Statement of Oleg Butkovsky

My research interests lie in the areas of probability, stochastic analysis, and partial differential equations (PDEs). Broadly speaking, I study how random noise changes the behavior of a complex deterministic system. More precisely, I investigate *regularization–by-noise* and *stabilization–by-noise* for ordinary and partial differential equations, *ergodicity* and *stability* for finite and infinite–dimensional Markov processes (stochastic PDEs, stochastic delay equations) as well as the *long-term behavior* of non-Markov processes (stochastic differential equations driven by fractional Brownian motion, McKean-Vlasov processes). I am also interested in construction of theoretically justified numerical algorithms related to the processes mentioned above.

Let me describe my accomplishments in each of these research areas as well as explain my research plans.

1. Regularization by noise. It has been known since 1970s that the addition of a noise source into an ill-posed deterministic system could make it well-posed. This phenomenon is called "regularization by noise". To illustrate this effect consider an ordinary differential equation (ODE)

$$dX_t = b(X_t)dt, \quad t \geq 0, \tag{1}$$

where b is a bounded vector field. If b is not Lipschitz continuous, then ODE (1) might have multiple solutions and if b is not continuous, then (1) might have no solutions. Nevertheless, Zvonkin [44] and Veretennikov [42] showed that the the corresponding *stochastic* differential equation (SDE)

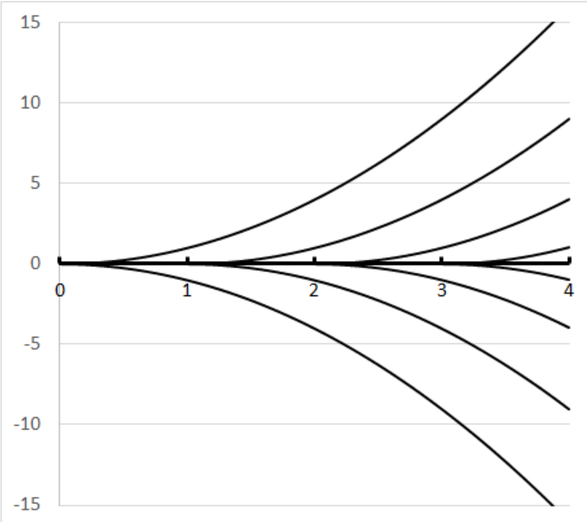
$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \tag{2}$$

driven by a Brownian motion B has a unique strong solution when b is merely bounded measurable and σ is uniformly elliptic. Thus, the presence of noise not only produces the solution in the situations where there was none, but also selects a unique physical solution in the situations where there were multiple, see Figure 1.

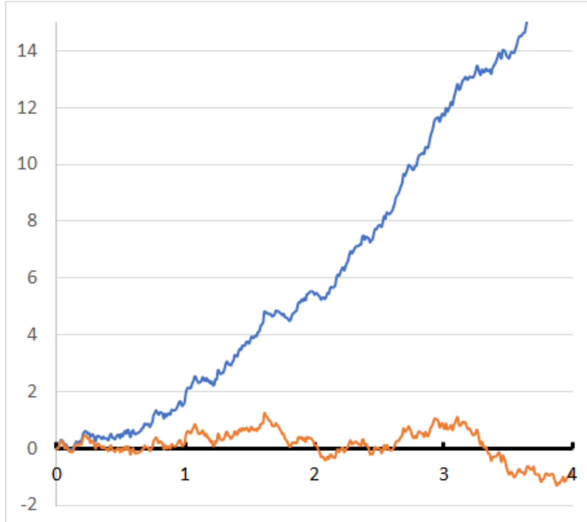
The pioneering results of Zvonkin and Veretennikov were later extended in the works of Krylov and Röckner [26], Figalli [11], Röckner and Zhao [35] and others, where well-posedness of (2) for possibly unbounded drifts satisfying only a certain integrability condition was established. Furthermore, it turned out that equation (2) also makes sense even if b is not a function but a Schwarz distribution (such equations arise in various models of mathematical physics). In this case, the term $b(X_t)$ is a priori not defined and one has to properly define the notion of a solution to this equation. This was done by Bass and Chen [1] who has shown that in the one-dimensional case (2) has a unique strong solution if $b \in \mathcal{C}^\gamma$, $\gamma > -1/2$. Note that in this case the solution X is not a semimartingale but a Dirichlet process. The Bass–Chen result was further extended in our paper [ABM20], where we obtained well-posedness of equation

$$dX_t = b(X_t)dt + \sigma dL_t, \tag{3}$$

where L is an α-stable processes, $\alpha \in (1, 2)$, $b \in \mathcal{C}^\gamma$, $\gamma > 1/2 - \alpha/2$; this matches the result of Bass and Chen for $\alpha = 2$. The main challenge there was that the Bass–Chen strategy was inapplicable and an alternative strategy based on a new version of Zvonkin transformation method was developed.



(a)



(b)

Figure 1: Regularization by noise. (a): Multiple solutions of ODE $dX_t = \text{sign}(X_t)\sqrt{|X_t|}dt$. (b): The unique solution of SDE $dX_t = \text{sign}(X_t)\sqrt{|X_t|}dt + dB_t$ (in blue) and white noise B (in brown).

It follows from general considerations that the fact that the noise process is Markovian should not be too crucial for regularization-by-noise phenomenon. Yet, very few examples of regularization by non-Markov noise has been known until very recently. This is due to the fact that in the Markovian setting one can use an approach based on Itô calculus for diffusion processes. It allows to pass from the analysis of the original SDE with irregular drift, to the analysis of a new SDE whose drift and diffusion are easier to handle. Then well-posedness for the original SDE can be derived from the well-posedness of the new SDE. However, to implement this strategy, it is absolutely essential that the stochastic system has a good Itô formula, which is typically not the case for the non-Markov noises.

Therefore we are developing ([BDG20, ABLM20] and work in progress) a new general strategy to tackle regularization by both Markov and non-Markov noises. Our approach utilizes and extends a new tool in stochastic analysis (stochastic sewing lemma of Lê [28]), and is quite robust for the specific choice of random noise.

In [BDG20] we suggested a new method of analyzing the strong convergence of the Euler-Maruyama approximations

$$dX_t^n = b(X_{\kappa_n(t)}^n)dt + \sigma(X_{\kappa_n(t)}^n)dB_t, \tag{4}$$

with $\kappa_n(t) := \lfloor nt \rfloor / n$ to the solution of (2). We proved that the rate of convergence is actually 1/2-better than what it was thought to be (for Hölder continuous b). Furthermore, we also showed for the first time the convergence (and the rate of convergence) of the Euler-Maruyama approximation in the case of equations driven by fractional Brownian noise

$$dX_t = b(X_t)dt + \sigma dW_t^H, \tag{5}$$

where W^H is a fractional Brownian motion, $H \in (0,1)$. Our conditions on the regularity of the drift are optimal in the sense that they coincide with the conditions needed for the strong uniqueness of solutions of (5) from Catellier and Gubinelli [5].

In [ABLM20] we studied regularization by noise for parabolic PDEs. More precisely, we analyzed the stochastic reaction diffusion equation

$$\partial_t u_t(x) = \partial^2_{xx} u_t(x) + b(u_t(x)) + \dot{W}, \quad x \in D \subset \mathbb{R} \tag{6}$$

for the case where b is a generalized function and $\dot{W}$ is a space-time white noise on $\mathbb{R}_+ \times D$. Here again a good Itô formula is not available and we had to develop an alternative approach. We were able to show that (6) has a unique strong solution if b belongs to a certain class of Schwarz distributions which includes measures. In particular, we showed that (6) is well posed for the case when b is the Dirac delta function, which corresponds to the skewed stochastic heat equation. The latter equation is important for the stochastic interface models and appeared in Bounebache and Zambotti [3], where its well-posedness was left open. Thus, our results extend the work of Bass and Chen [1] mentioned above to the framework of stochastic partial differential equations.

To obtain the results mentioned above we exploit the regularization effect of noise through a new strategy, which based on extensions of the stochastic sewing lemma [28] (which in turn was inspired by the deterministic sewing lemma of Gubinelli [14]). I am planning to further develop the stochastic sewing ideas and to advance this approach into novel directions related to regularization by noise for other type of systems.

In particular, I am planning to work on skew fractional Brownian motion, that is the solution to

$$dX_t = \kappa \delta_0(X_t) dt + dW_t^H, \tag{7}$$

where $\kappa \in \mathbb{R}$ and δ_0 is the Dirac delta function. It is known that (7) has a unique strong solution if $H < 1/4$ or if $H = 1/2$ and $|\kappa| \leq 1$. Such an apparent gap seems very strange: it is natural to conjecture that this equation has a unique strong solution for all $H < 1/2$. I am planning to work on this conjecture by refining stochastic sewing with random controls, a new tool developed in [ABLM20].

Another very interesting problem is obtaining the analogue of Bass and Chen result for the case of the fractional Brownian noise. Indeed, if $H = 1/2$, then in the one-dimensional setting equation (5) has a unique strong solution for $b \in \mathcal{C}^\gamma$, $\gamma > -1/2$, see [1]. However, for $H \neq 1/2$ strong existence and uniqueness is known only for $\gamma > 1 - 1/(2H)$, see [5], even in the 1-dimensional case, which is worse than the condition of Bass and Chen. A related question is *weak* existence and uniqueness of (5). This problem is very challenging and here again almost nothing is known for $H \neq 1/2$.

I am planning to further develop the error analysis of discrete approximations of SDEs with irregular drift and to extend the strategy of [BDG20] to new areas. I will study rate of convergence of strong approximations of SDEs driven by Lévy noise. I believe that this new approach would allow to improve the state of the art in this area [31]. Another important research area is weak approximations of SDEs. Surprisingly, the optimal weak rate of convergence is still unknown even for standard Brownian-driven SDEs with irregular coefficients and our results on strong convergence in [BDG20] already yields improvements compared to the previous state of the art, see [25]. The theory of weak approximations of fractional SDEs is even less developed, and I intend to explore this direction via the stochastic sewing approach as well.

With respect to regularization by noise for SPDEs, I am planning to complete the program initiated in [3] and show that SPDE (6) with Dirichlet boundary conditions for the case $b = \delta_0$ has a unique invariant measure and its transitional probabilities converge to this measure. This would require extending the ideas of [ABLM20] and establishing new stochastic sewing lemmas.

I also intend to study (6) with $b = \kappa\delta_0$ with $\kappa \to \infty$. We conjecture that it would converges to the solution of the Nualart-Pardoux equation (reflected stochastic heat equation). However, verifying this conjecture would require further development of the stochastic sewing techniques.

Finally, it is also important to analyze various properties of solutions to (6) (e.g., existence of local times) and relations between different notions of solutions.

So far we have discussed only strong and weak notions of uniqueness. Let me now mention a stronger notion of uniqueness of SDEs/SPDEs introduced by Krylov in 1980s, namely path–by–path uniqueness. The uniqueness is understood in the following sense: whether it is true that for almost all trajectories of Brownian motion the corresponding deterministic ODE

$$X_t = x_0 + \int_0^t b(X_s)ds + B_t, \quad t \geq 0$$

has a unique solution. The papers of Davie [9], Shaposhnikov [37], Catellier and Gubinelli [5], Priola [32] establihed path–by–path uniqueness for SDEs driven by Brownian motion, fractional Brownian motion, Levy processes. Moreover, it was shown very recently by Shaposhnikov and Wresch [38] that path-by-path uniqueness differs from the usual strong uniqueness however the relationship between these two notions is not yet fully understood.

In our article [BM19] we studied path–by–path uniqueness for SPDEs. Using a combination of deterministic and stochastic sewing arguments, we proved that this type of uniqueness holds for equation (6) if the drift b is bounded. This implies, in particular, that in this case SPDE (6) has a stochastic flow of solutions. While it is immediate that path-by-path uniqueness for an SDE/SPDE implies existence of a flow of solutions, less is known about the opposite statement (whether strong uniqueness and existence of a flow implies path-by-path uniqueness), note though recent work of Priola [33]. I hope to obtain this result using deterministic sewing with controls.

2. Stabilization–by-noise and ergodicity. In the previous section we see that addition of noise can make an ill-posed deterministic system well-posed. Now we explore a complementary phenomenon and see how the addition of noise can make a well-posed but unstable deterministic system stable.

Consider the following toy yet very important example. Let $X = (X_t^{(x)})_{t\geq 0}$ be a solution to

$$dX_t = (X_t - X_t^3)dt, \quad X_0 = x.$$

It is clear that this equation has 3 stable points: if $x = 0$, then $X_t^{(0)} = 0$ for any $t \geq 0$; similarly $X^{(-1)} \equiv -1$ and $X^{(1)} \equiv 1$. Furthermore, if $x > 0$, then $X_t^{(x)} \to 1$ and if $x < 0$, then $X_t^{(x)} \to -1$ as $t \to \infty$. On the other hand, the corresponding *stochastic* system

$$dX_t = (X_t - X_t^3)dt + dB_t, \quad X_0 = x. \tag{8}$$

has a unique invariant measure; moreover $|X_t^{(x)} - X_t^{(y)}| \to 0$ in probability as $t \to \infty$ for any $x, y \in \mathbb{R}$, see Figure 2. One can say that in this case Markov process (8) is ergodic and *synchronization by noise* holds.

While ergodic properties of finite dimensional SDEs (and, generally speaking, finite dimensional Markov processes) are quite well understood by now, much less is known about infinite dimensional Markov processes. Indeed, the basic program for obtaining

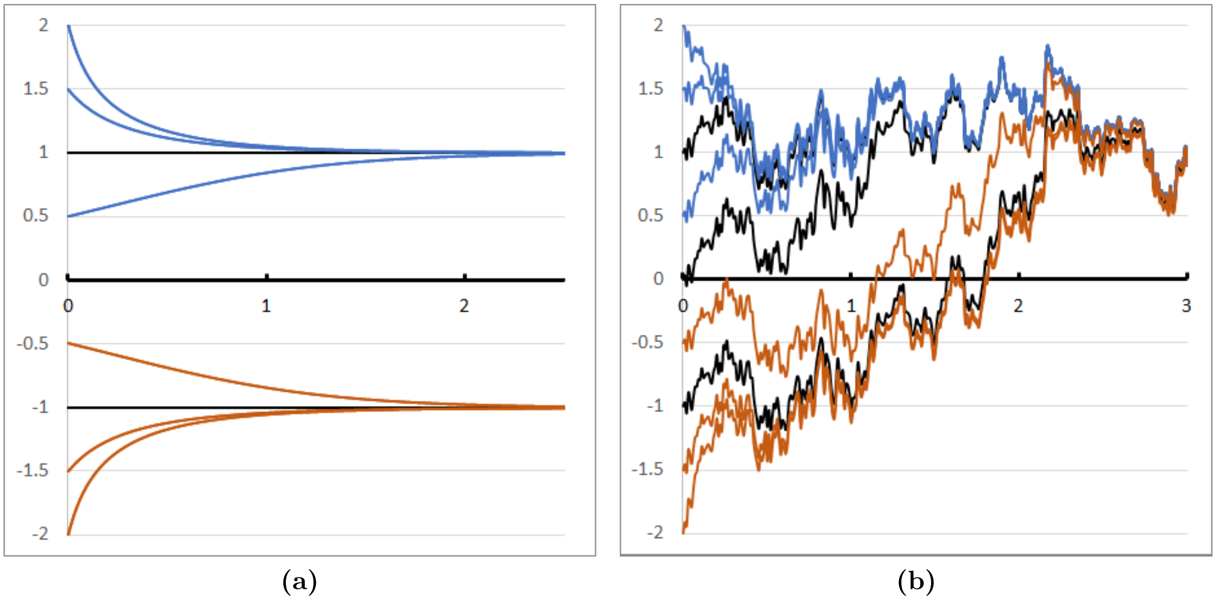


Figure 2: Stabilization by noise. (a): ODE $dX_t = (X_t - X_t^3)dt$ has multiple stable points. (b): SDE $dX_t = (X_t - X_t^3)dt + dW_t$ has a unique invariant measure.

ergodicity of a Markov process involves identification of a certain subset of the state space (called coupling set), where the total variation distance between transition probabilities of the process has a non-trivial bound, and showing that the process will return to this coupling set often enough (usually this is done by means of Lyapunov functions), see the classical book of Meyn and Tweedie [30] or a nice brief summary in Hairer and Mattingly [21]. While in the finite dimensional case, Markov processes usually have a good coupling set, infinite dimensional Markov processes typically do not have any non-trivial coupling sets. Indeed, it is known that transition kernels of many infinite dimensional Markov processes are mutually singular when starting from different initial conditions: this is the case for some SPDEs [19, 18, BS20], stochastic delay equations [36, 34], piecewise deterministic Markov processes [7], linear autoregressive models [But14a] and so on. Moreover, these Markov processes do not converge to invariant measure in total variation (but converge to it weakly).

To overcome this problem and to study convergence of these "pathological" Markov processes Hairer, Mattingly and Scheutzow [16] suggested a different strategy. Instead of requiring two copies of a Markov process to couple at the coupling set (which might be impossible, if their probabilities are mutually singular), they allow them to be within $\varepsilon > 0$ distance of each other (the so-called "d-small set condition"). They show that combined with a certain global non-expansion condition as well as the recurrence condition, this implies exponential ergodicity of the Markov process in the Wasserstein metric. My work [But14a] extended these results and established subexponential bounds on convergence rate under weaker assumptions. Based on this article, as an application, we obtained in [BS17] general conditions for exponential and subexponential ergodicity of stochastic delay equations.

Note that verification of the "d-small set condition" mentioned above for a specific Markov process is usually highly non-trivial. To ease this process, in [BKS20] we developed a set of verifiable sufficient conditions for exponential or subexponential ergodicity in terms of generalized coupling conditions. Furthermore, we provided a special framework to ease the application of these conditions to stochastic partial differential equations and showed exponential ergodicity of various SPDEs in the effectively elliptic setting. This generalizes recent results of Constantin, Glatt-Holtz, Vicol [8] and Glatt-Holtz, Mattingly, Richards [13]. This technique was further developed in [BW19], where we established the connection between the asymptotic strong Feller property and the generalized coupling.

The results discussed above are quite general: they are applicable to a broad class of Markov processes. If a particular Markov process has further “nice” properties, then these general results can be refined. Thus, in [BS20] we studied ergodic properties of *order–preserving* Markov processes. We showed that under very mild additional assumptions these processes are exponentially ergodic. In particular, this implies exponential ergodicity of stochastic–reaction diffusion equation in the hypoelliptic setting, thus refining and complementing the corresponding results in Hairer, Mattingly [20]. Furthermore, we show that under the same conditions synchronization-by-noise takes place (recall Figure 2b). Recently, Gess and Tsatsoulis [12] used our framework to show that synchronization-by-noise holds for 2D and 3D stochastic quantization equation (the so-called Φ^4_d model), which is a singular SPDE defined using regularity structures of Hairer [15].

I am planning to use these techniques and develop new methods to study convergence of invariant measures corresponding to an approximated Markov kernel. This problem is very important for modern computational Bayesian statistics, see, e.g., [39]. The first step in this direction was done by Johndrow and Mattingly [23, 24], who showed that the distance between μ^ε (an invariant measure, corresponding to an approximated Markov kernel P^ε) and μ (an invariant measure, corresponding to the Markov kernel P under consideration) is Lipschitz at $\varepsilon = 0$ under certain assumptions; similar results were obtained by Cerrai and Glatt-Holtz [6] for some SPDE models. However, the problems of Bayesian statistics require more delicate analysis (in particular, finer results about smoothness of the distance between μ^ε and μ are needed). I hope that generalized couplings techniques developed in [BKS20] might be useful here.

Another very interesting problem is studying ergodicity for hidden Markov processes, see the pioneering work of Tong and van Handel [41]. In particular, let us consider again an SDE driven by fractional Brownian motion W^H, $H \in (0, 1)$

$$dX_t = b(X_t)dt + dW^H_t, \quad t \geq 0.$$

Recall that we have the following representation

$$W^H_t = \int_{-\infty}^t K(t, s)dB_s,$$

for a certain kernel K and a Brownian motion B. Let $B_{(-\infty,t]}$ denotes the whole trajectory of B up to time t. The process X is not Markovian; however it is possible to show that X is a hidden Markov process and the pair $(X_t, B_{(-\infty,t]})$ forms a Markov process. Unfortunately, investigating its ergodic properties turned out to be a very challenging problem. Hairer [17] established existence and uniqueness of invariant measure and rather slow (just polynomial) convergence of transition probabilities. I am planning to apply the recently developed by Eberle [10] technique of mirror coupling to establish exponential ergodicity of X. In the case $d = 1$, the order-preservation property might be very helpful in obtaining this result.

3. **McKean–Vlasov equations and financial applications**.

Finally, let us discuss McKean–Vlasov equations (MVE). Those are SDEs whose drift and diffusion coefficients depend not only on the current *state* of the process but also on the current *distribution* of the process. These processes were introduced by McKean [29] to model plasma dynamics; for later developments of the theory we refer to the book of Sznitman [40]. MVEs arise naturally as the limit of a large number of weakly interacting Markov processes.

It is known that MVEs have unusual ergodic properties. For instance, even under some very natural conditions, they might nonetheless have multiple invariant measures, see [22]. The discrete time analogue of MVEs was analyzed in [But12] where sufficient conditions optimal in a certain sense for existence and uniqueness of an invariant measure and uniform ergodicity were presented. The paper [But12] contains also further examples of peculiar ergodic properties of discrete-time MVEs. In particular, in contrast to the Markovian case, positivity of the elements of the one-step transition matrix does not imply even weak convergence to the invariant measure.

One of the first attempts to obtain general enough conditions for unique ergodicity of MVEs dates back to Veretennikov [43]. This work was extended in [But14a], where we derived sufficient conditions for exponential convergence of solutions of MVEs to stationarity in the total variation metric. More precisely, we show that if an SDE is exponentially ergodic, then its small perturbation which adds McKean–Vlasov component is also exponentially ergodic (the fact that a large perturbation might not be ergodic is known from [22]). To obtain this result we developed a new approach and transferred the ideas of Hairer and Mattingly [21] to the McKean–Vlasov setting.

Recently, I began working on well-posedness of the following McKean–Vlasov equation which arises from the models of financial mathematics:

$$\begin{aligned} dS_t &= b_1(S_t)dt + \sigma_1(S_t)\frac{g(S_t)}{\sqrt{\mathsf{E}[g(S_t)|X_t]}}dW_t^{(1)};\\ dX_t &= b_2(X_t)dt + \sigma_2(X_t)dW_t^{(2)}, \end{aligned} \tag{9}$$

where $W^{(1)}, W^{(2)}$ are independent Brownian motions and b_1, b_2, σ_1, σ_2, g are sufficiently nice functions. It is interesting to note that despite the widespread use of this equation in practice for calibration of the local volatility model, the theoretical results about this equation are very scarce. In particular, even existence or uniqueness of solutions to this equation is not known. The problems come from the fact that the diffusion coefficient of S depends on the joint law (S, X) in a very singular way; therefore standard techniques for establishing well–posedness of MVEs from, e.g., [4] are not applicable here. First results in this direction were obtained by Lacker, Shkolnikov and Zhang [27], where existence of a stationary solution to (9) under certain rather restrictive conditions on b_1, b_2 was obtained. We hope to utilize the strategy of [2] to show well–posedness of (9) in a more broad setting.

List of works of Oleg Butkovsky

A. Regularization by noise.

[BM19] O. Butkovsky, L. Mytnik (2019). Regularization by noise and flows of solutions for a stochastic heat equation. *Annals of Probability*, **47**(1), 165–212.

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C. McKean–Vlasov equations.

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