Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Asymptotics beats Monte Carlo: The case of correlated local vol baskets

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Outline

$$u(t, S_t) = e^{-r(T-t)} E\left[f(S_T) \mid S_t \right]$$

Example (Example treated in this work)

- $f(\mathbf{S}) = \left(\sum_{i=1}^{n} w_i S_i - K \right)^+$, at least one weight positive
- n large (e.g., $n = 500$ for SPX)

- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas

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- PDE methods

  Pros: fast, general

  Cons: curse of dimensionality, path-dependence may or may not be easy to include
- (Quasi) Monte Carlo method
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- n large (e.g., $n = 500$ for SPX)

- PDE methods
- (Quasi) Monte Carlo method

  Pros: very general, easy to adapt, no curse of dimensionality

  Cons: slow, quasi MC may be difficult in high dimensions
- Fourier transform based methods
- Approximation formulas

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  Pros: very fast to evaluate ("explicit formula")

  Cons: only available for affine models, difficult to generalize, curse of dimensionality
- Approximation formulas

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  Pros: very fast evaluation

  Cons: derived on case by case basis, therefore very restrictive

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- (Quasi) Monte Carlo method
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- Approximation formulas
- Work horse methods: PDE methods and (in particular) (Q)MC
- Particular models allowing approximation formulas (e.g., *SABR formula*) or FFT (Heston model) very popular

- Expansions in large/small strike or large/small maturity

Example (Large strike expansion, Lee formula)

- $\partial_K \text{Call}(S_0, T, K) = -P(S_T \geq K)$
- For $K \gg 1$, this is a rare event (*large deviation*)
- Lee formula: $m := \log(S_0/K)$, β related to moment explosion

$$\lim_{m \to \pm\infty} \frac{T}{m} \sigma_I^2(S_0, T, K) = \beta_\pm$$

- Extensions e.g., by P. Friz et al.

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- Extensions e.g., by P. Friz et al.

- Consider the (one-dimensional) model

$$dS_t = \sigma(t, S_t)dW_t, \quad S_0 \in \mathbb{R}$$

- Expansion: $S_t^\epsilon = S_0 + \epsilon S_{1,t} + \frac{1}{2}\epsilon^2 S_{2,t} + o(\epsilon^2)$, with

$$S_{1,t} = \int_0^t \sigma(s, S_0)dW_s,$$
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- Wiener chaos decomposition
- $E[f(S_T)] \approx f(S_0) + \epsilon f'(S_0)E[S_{1,T}] + + \frac{1}{2}\epsilon^2\left(f'(S_0)E[S_{2,T}] + f''(S_0)E\left[S_{1,T}^2\right]\right) + \cdots$
- For non-smooth payoffs, extensions possible by Malliavin weights.

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- Ansatz $u^\epsilon = u_0 + \epsilon u_1 + \frac{1}{2}\epsilon^2 u_2 + \cdots$ gives (regular perturbation)

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- Formally, we get

$$u_0(T, S_0) = f(S_0), \quad L_0 u_0 = 0, \quad L_0 u_1 = 0, \quad \frac{1}{2}L_0 u_2 + L_2 u_0 = 0, \ldots$$

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Small noise expansion (PDE approach)

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Setting

- Local volatility model for forward prices

$$dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n,$$
$$\left\langle dW_i(t)\,,\, dW_j(t)\right\rangle = \rho_{ij}dt$$

- Generalized spread option with payoff $\left(\sum_{i=1}^{n} w_i F_i - K\right)^+$, at least one w_i positive
- Goal: fast and accurate approximation formulas, even for high n
- $n = 100$ or $n = 500$ not uncommon (index options)

Example

- Black-Scholes model: $\sigma_i(F_i) = \sigma_i F_i$
- CEV model: $\sigma_i(F_i) = \sigma_i F_i^{\beta_i}$

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$$d\sum_{i=1}^n w_i F_i(t) = \sum_{i=1}^n w_i \sigma_i(F_i(t)) dW_i(t)$$

- Ito's formula formally implies that
- Let $p(\mathbf{F}_0, \mathbf{F}, t) := P(\mathbf{F}(t) \in d\mathbf{F} \mid \mathbf{F}(0) = \mathbf{F}_0)$ and H_{n-1} be the Hausdorff measure on $\mathcal{E}(K)$, then we have the *Carr-Jarrow formula*

$$C(\mathbf{F}_0, K, T) = \left(\sum_{i=1}^n w_i F_i(0) - K\right)^+ +$$

$$+\frac{1}{2}\int_0^T \frac{1}{|w|}\int_{\mathcal{E}(K)} \sum_{i,j=1}^n w_i w_j \sigma_i(F_i)\sigma_j(F_j)\rho_{ij} p(\mathbf{F}_0, \mathbf{F}, u) H_{n-1}(d\mathbf{F}) du.$$

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$$+ \sum_{i=1}^{n} w_i \int_0^T \mathbf{1}_{\sum w_i Fi(u) > K} dF_i(u) + \frac{1}{2}\int_0^T \delta_{\sum w_i F_i(u) = K} \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}(u)) du$$

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- Heat kernel expansion (to be discussed in detail later):

$$\sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F})p(\mathbf{F}_0,\mathbf{F},t) \approx \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2t} - C(\mathbf{F}_0,\mathbf{F})\right)$$

- By change of variables $F_n = \frac{1}{w_n}\left(K - \sum_{i=1}^{n-1} w_i F_i\right)$ on $\mathcal{E}_K$:

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$$= t^{\frac{n-1}{2}} e^{-\frac{d(\mathbf{F}_0,\mathbf{F}^*)^2}{2t} - C(\mathbf{F}_0,\mathbf{F}^*)} \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{\det Q}}$$

We rely on the *principle of not feeling the boundary*.

Theorem

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+ +$$
$$+\frac{1}{2\sqrt{2\pi}\,|w_n|\,d(\mathbf{F}_0,\mathbf{F}^*)^2\sqrt{\det Q}}e^{-C(\mathbf{F}_0,\mathbf{F}^*)-\frac{d(\mathbf{F}_0,\mathbf{F}^*)}{2T}}T^{3/2}+o(T^{3/2}),\ \text{as } T \to 0.$$

- Bachelier implied vol (with $\overline{F}_0 = \sum_{i=1}^n w_i F_{0,i}$):

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- Goal: sensitivity w. r. t. model parameter κ of the option price

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) \approx C_{BS}(\overline{F}_0, K, \sigma_{BS}, T)$$

- Sensitivity: $\underbrace{\partial_\kappa C_{BS}}_{\text{BS greek}}(\overline{F}_0, K, \sigma_{BS}, T) + \underbrace{\nu_{BS}}_{\text{BS vega}}(\overline{F}_0, K, \sigma_{BS}, T)\partial_\kappa \sigma_{BS}$
- Recall that $\sigma_{BS,0}, \sigma_{BS,1}$ explicit up to $\mathbf{F}^*$
- By the minimizing property: $\partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\big|_{\mathbf{G}=\mathbf{G}^*} = 0$
- Differentiating with respect to κ gives

$$\partial_\kappa \partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\big|_{\mathbf{G}^*} + \sum_{l=1}^{n-1} \partial_{F_l}\partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\big|_{\mathbf{G}^*} \partial_\kappa F_l^* = 0$$

Up to the above system of linear equations for $\partial_\kappa \mathbf{F}^*$, there are explicit expression for the sensitivities of the approximate option prices.

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Outline

Heat kernels and geometry

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

$$L = \frac{1}{2}a^{i,j}\frac{\partial^2}{\partial x^i \partial x^j} + b^i \frac{\partial}{\partial x^i}, \quad a = \sigma^T \sigma$$

- Heat kernel: fundamental solution $p(\mathbf{x}, \mathbf{y}, t)$ of $\frac{\partial}{\partial t}u = Lu$
- Transition density of $\mathbf{X}_t$

"Can you hear the shape of the drum?"(Kac '66)

Take $L = \Delta$ on a domain D and relate:

- Geometrical properties of the domain D
- Partition function $Z = \sum_{k \in \mathbb{N}} e^{\gamma_k t}$
- Heat kernel
- E.g. $-\gamma_k \sim C(n)(k/\operatorname{vol} D)^{2/n}$ (Weyl, '46)
- E.g. (for $n = 2$): $Z = \frac{\text{area}}{4\pi t} - \frac{\text{circ.}}{\sqrt{4\pi t}} + O(1)$ (McKean & Singer, '67)

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- On $\mathbb{R}^n$ (or a submanifold), introduce $g^{ij} := a^{ij}$, Riemannian metric tensor $(g_{ij}(\mathbf{x}))_{i,j=1}^n := \left((g^{ij}(\mathbf{x}))_{i,j=1}^n\right)^{-1}$
- Geodesic distance:

$$d(\mathbf{x}, \mathbf{y}) := \inf_{\mathbf{z}(0)=\mathbf{x}, \mathbf{z}(1)=\mathbf{y}} \int_0^1 \sqrt{\sum g_{ij}(\mathbf{z}(t))\dot{\mathbf{z}}^i(t)\dot{\mathbf{z}}^j(t)dt}$$

- inf attained by a smooth curve, the *geodesic*
- Laplace-Beltrami operator: $\Delta_g = \left(\det(g_{ij})\right)^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left(\det(g_{ij})\right)^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x^j}$

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The Riemannian metric associated to a diffusion

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Heat kernel expansion

$$p_N(\mathbf{x}_0, \mathbf{x}, T) = \sqrt{\det(g(\mathbf{x})_{ij})} U_N(\mathbf{x}_0, \mathbf{x}, T) \frac{e^{-\frac{d^2(\mathbf{x}_0, \mathbf{x})}{2T}}}{(2\pi T)^{\frac{n}{2}}}$$

- $U_N(\mathbf{x}_0, \mathbf{x}, T) = \sum_{k=0}^{N} u_k(\mathbf{x}_0, \mathbf{x}) T^k$, the heat kernel coefficients
- $u_0(\mathbf{x}_0, \mathbf{x}) = \sqrt{\Delta(\mathbf{x}_0, \mathbf{x})} e^{\int_z \langle h(z(t)), \dot{z}(t) \rangle_g dt}$
- Δ is the Van Vleck-DeWitt determinant: $\Delta(\mathbf{x}_0, \mathbf{x}) = \frac{1}{\sqrt{\det(g(\mathbf{x}_0)_{ij}) \det(g(\mathbf{x})_{ij})}} \det\left(-\frac{1}{2} \frac{\partial^2 d^2}{\partial \mathbf{x}_0 \partial \mathbf{x}}\right)$.
- $e^{\int_z \langle h(z(t)), \dot{z}(t) \rangle_g dt}$ is the exponential of the work done by the vector field h along the geodesic z joining $\mathbf{x}_0$ to $\mathbf{x}$ with $h^i = b^i - \frac{1}{2\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^j} \left[\sqrt{\det(g_{ij})} g^{ij} \right]$

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Assumption

The cut-locus of any point is empty, i.e., any two points are connected by a unique minimizing geodesic.

Theorem (Varadhan '67)

$b = 0$, σ uniformly Hölder continuous, system uniformly elliptic, then $\lim_{T\to 0} T \log p(\mathbf{x}, \mathbf{y}, T) = -\frac{1}{2} d(\mathbf{x}, \mathbf{y})^2$.

Theorem (Yosida '53)

On a compact Riemannian manifold, assume smooth vector fields and an ellipticity property. Then $p(\mathbf{x}, \mathbf{y}, T) - p_N(\mathbf{x}, \mathbf{y}, T) = O(T^N)$ *as* $T \to 0$.

Theorem (Azencott '84)

For a locally elliptic system in an open set $U \subset \mathbb{R}^n$, $\mathbf{x}, \mathbf{y} \in U$ *s. t.* $d(\mathbf{x}, \mathbf{y}) < d(\mathbf{x}, \partial U) + d(\mathbf{y}, \partial U)$, *we have* $p(\mathbf{x}, \mathbf{y}, T) - p_N(\mathbf{x}, \mathbf{y}, T) = O(T^N)$ *as* $T \to 0$.

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- Domain $\mathbb{R}^n_+$, $dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n$
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- Geodesics known in closed form
- CEV case: $\sigma_i(F_i) = \sigma_i F_i^{\beta_i}$, zeroth and first order heat kernel coefficients given explicitly

- Optimization problem for $\mathbf{F}^*$ is non-linear with a linear constraint
- With $q_i := \int_{F_{0,i}}^{F_i} \frac{du}{\sigma_i(u)}$, it is a quadratic optimization problem with non-linear constraint
- Fast convergence of Newton iteration
- Given $\mathbf{F}^*$, $C(\mathbf{F}_0, \mathbf{F}^*)$ is a line integral along the geodesic; this integral can be calculated in closed form in the CEV model.
- Formulas can be evaluated in less than 2 seconds for $n = 100$

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The initial guess in the Newton iteration

- Change of variable: $q_i = \frac{F_i^{1-\beta_i} - F_{0,i}^{1-\beta_i}}{1-\beta_i}$, $F_i = \left(F_{0,i}^{1-\beta_i} + (1-\beta_i) q_i\right)^{1/(1-\beta_i)}$
- $\Lambda^{-1} = (\sigma_i \sigma_j \rho_{ij})_{i,j=1}^n$
- Optimization problem: $\min \mathbf{q}^T \Lambda \mathbf{q} \ : \ \sum_{i=1}^n w_i F_i(q_i) = K$
- Linearized constraint: $\sum_{i=1}^{n} w_i \left(F_{0,i} + F_{0,i}^{\beta_i} q_i\right) = K$
- Minimizer $\mathbf{q}_0^* = \frac{K - \overline{F}_0}{\widetilde{\mathbf{F}}_0^T \Lambda^{-1} \widetilde{\mathbf{F}}_0} \Lambda^{-1} \widetilde{\mathbf{F}}_0$ with Lagrange multiplier $\lambda = 2 \frac{K - \overline{F}_0}{\widetilde{\mathbf{F}}_0^T \Lambda^{-1} \widetilde{\mathbf{F}}_0}$, where $\widetilde{F}_{0,i} = w_i F_{0,i}$
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- CEV model framework
- For CEV, the formulas are fully explicit apart from the minimizing configuration $\mathbf{F}^*$
- We observe very fast convergence of the iteration, but the initial guess is crucial.
- Reference values obtained using:
  - *Ninomiya Victoir* discretization
  - Quasi Monte Carlo based on Sobol numbers, Monte Carlo for very high dimensions ($n \approx 100$)
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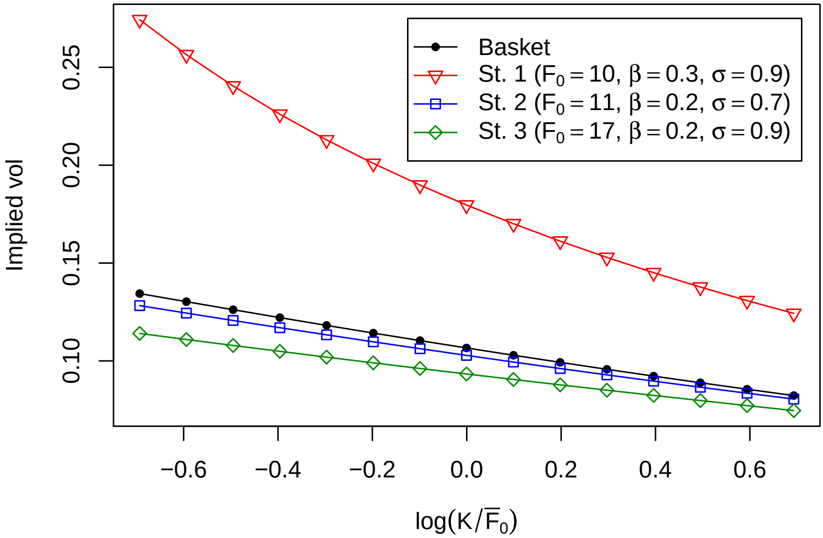
CEV index implied vol – three-dimensional visualization



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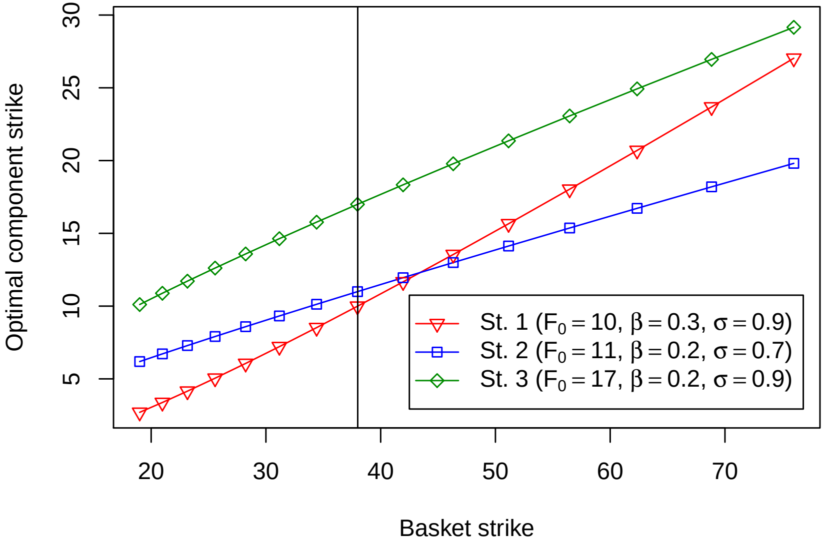

- Recall: $dF_i(t) = \sigma_i F_i(t)^{\beta_i} dW_i(t)$
- $\boldsymbol{\beta} = (0.7, 0.2, 0.8, 0.3, 0.5, 0.5, 0.6, 0.6, 0.3, 0.3)$
- $\boldsymbol{\sigma} = (0.8, 0.6, 0.9, 0.6, 0.8, 0.4, 0.9, 0.9, 0.3, 0.8)$
- $\mathbf{F}_0 = (10, 13, 11, 18, 9, 10, 17, 16, 13, 17)$
- $\mathbf{w} = (-1, -1, 1, 1, 1, -1, -1, 1, 1, 1)$

Spread option in dimension 10

T	$K = 32.9$	$K = 33.8$	$K = 34.1$	$K = 34.4$	$K = 35.3$
0.5	3.6352	3.1609	3.0123	2.8684	2.4649
1	4.8959	4.4332	4.2857	4.1416	3.7292
2	6.6912	6.2385	6.0924	5.9487	5.5322
5	10.2656	9.8261	9.6825	9.5408	9.1251
10	14.2385	13.8122	13.6726	13.5298	13.1204

Table : Quasi Monte Carlo prices.

T	$K = 32.9$	$K = 33.8$	$K = 34.1$	$K = 34.4$	$K = 35.3$
0.5	3.6306	3.1562	3.0076	2.8637	2.4601
1	4.8844	4.4214	4.2739	4.1297	3.7174
2	6.6640	6.2109	6.0648	5.9211	5.5046
5	10.2020	9.7617	9.6182	9.4763	9.0604
10	14.1930	13.7635	13.6229	13.4835	13.0728

Table : Zero order asymptotic prices.

Spread option in dimension 10

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1	4.8976	4.4348	4.2873	4.1431	3.7307
2	6.7015	6.2487	6.1027	5.9590	5.5423
5	10.3507	9.9112	9.7678	9.6260	9.2100
10	14.6137	14.1863	14.0461	13.9069	13.4960

Table : First order asymptotic prices.

Normalized errors

- Approximation error supposed to depend on "dimension-free" time to maturity $\sigma^2 T$
- Use $\overline{\sigma} := \sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0) / \left(\sum_{i=1}^{n} w_i F_{0,i} \right)$ as proxy in local vol framework
- Normalized error: $\frac{\text{Rel. error}}{\overline{\sigma}^2 T}$

T	Dim. 5	Dim. 10	Dim. 15	Dim. 100
0.5	0.1555	−0.0293	0.3085	−0.0143
1	0.1481	−0.0261	0.3162	−0.0105
2	0.1429	−0.0218	0.3222	−0.0075
5	0.1376	−0.0129	0.3252	
10	0.1328	−0.0035	0.3198	
$\overline{\sigma}$	0.1704	0.3187	0.1073	0.2964

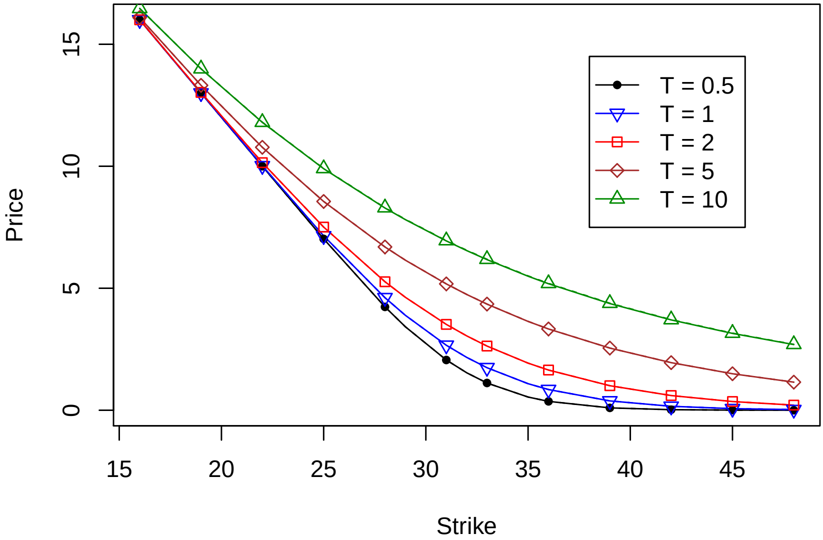
Table : Normalized relative error of the zero-order asymptotic prices.

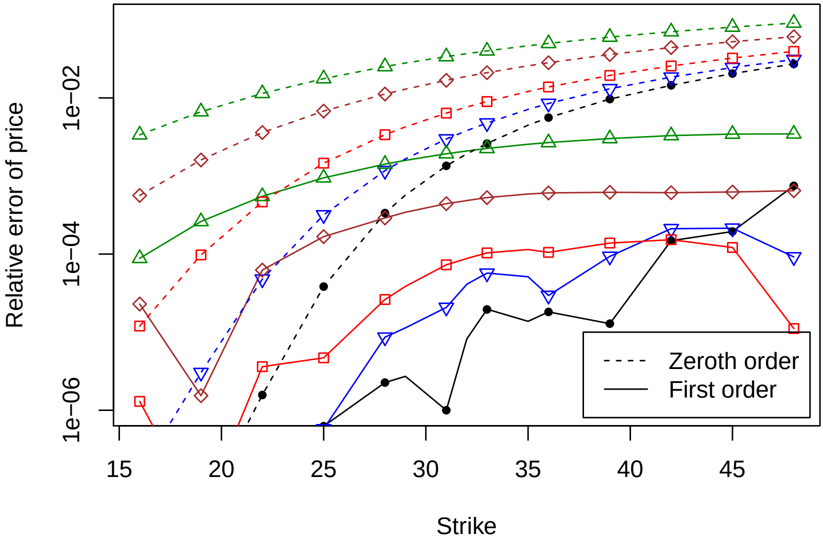
- Approximation error supposed to depend on "dimension-free" time to maturity $\sigma^2 T$
- Use $\overline{\sigma} := \sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0) / \left(\sum_{i=1}^{n} w_i F_{0,i} \right)$ as proxy in local vol framework
- Normalized error: $\frac{\text{Rel. error}}{\overline{\sigma}^2 T}$

T	Dim. 5	Dim. 10	Dim. 15	Dim. 100
0.5	-4.02×10^{-4}	1.76×10^{-4}	8.76×10^{-3}	5.06×10^{-5}
1	-9.47×10^{-4}	3.58×10^{-3}	1.53×10^{-3}	2.08×10^{-3}
2	-1.63×10^{-3}	8.09×10^{-3}	-3.92×10^{-3}	3.89×10^{-3}
5	-3.41×10^{-3}	1.71×10^{-2}	-1.33×10^{-2}	
10	-7.15×10^{-3}	2.67×10^{-2}	-2.82×10^{-2}	
$\overline{\sigma}$	0.1704	0.3187	0.1073	0.2964

Table : Normalized error of the first order asymptotic prices.

First order prices




Relative error of price
1e−02
1e−04
1e−06
15
20
25
30
35
40
45
Strike
Zeroth order
First order

Delta

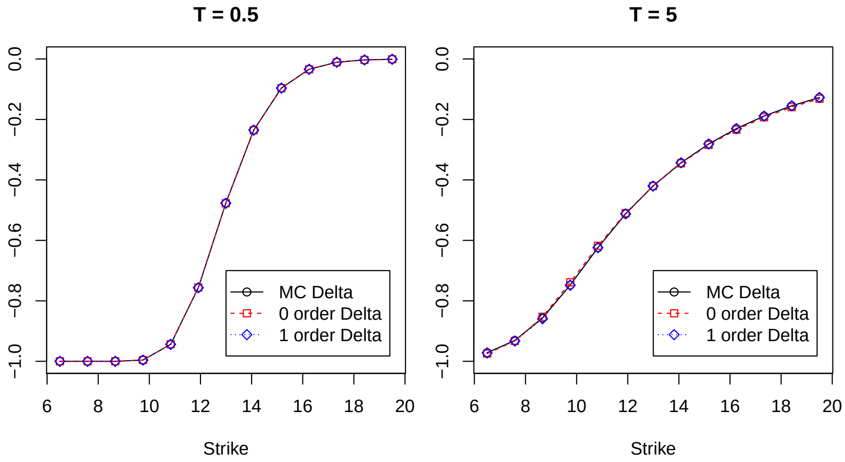
$$\mathbf{F}_0 = \begin{pmatrix} 13 \\ 9 \\ 9 \end{pmatrix}, \ \boldsymbol{\xi} = \begin{pmatrix} 0.1 \\ 0.7 \\ 0.6 \end{pmatrix}, \ \boldsymbol{\beta} = \begin{pmatrix} 0.3 \\ 0.7 \\ 0.5 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1.0000 & 0.9142 & 0.7706 \\ 0.9142 & 1.0000 & 0.8429 \\ 0.7706 & 0.8429 & 1.0000 \end{pmatrix}.$$

- Objective: Compute the sensitivity (delta) w.r.t.$F_{0,3}$.
- Note that the option payoff is

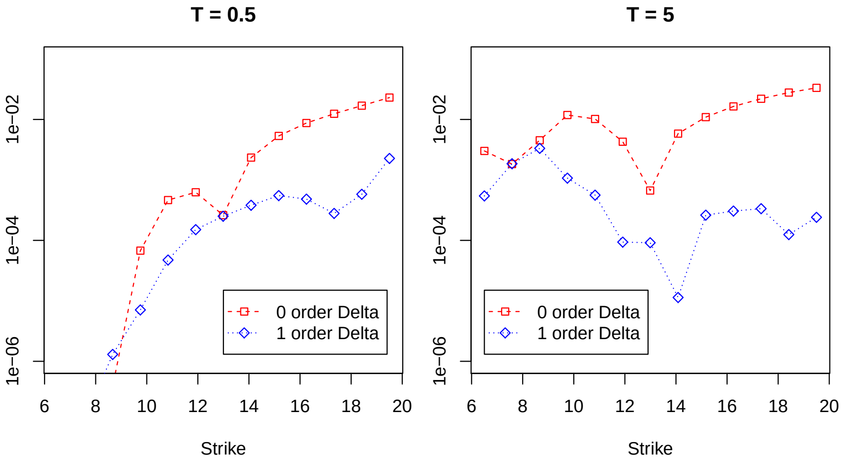
$$P(\mathbf{F}) = (F_1 + F_2 - F_3 - K)^+$$

Delta



T = 0.5
T = 5
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
6
8
10
12
14
16
18
20
Strike
MC Delta
0 order Delta
1 order Delta

Relative error of delta



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