Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

A regularity structure for rough volatility

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Joint work with: P. Friz, P. Gassiat, J. Martin, B. Stemper

Jim Gatheral's 60'th birthday conference

Some years ago, Jim Gatheral (et al.) kicked off exciting new development in rough volatility. Let $K(s,t) := \sqrt{2H}\,|t-s|^{H-1/2}\,\mathbf{1}_{t>s}$.

Example (Rough Bergomi model)

$$dS_t = \sqrt{v_t} S_t \left(\rho dW_t + \overline{\rho} dW_t^{\perp}\right), \quad v_t = \xi(t)\mathcal{E}\left(\eta \widehat{W}_t\right), \quad \widehat{W}_t := \int_0^t K(s,t) dW_s$$

$$\rightsquigarrow \int_0^t f(s, \widehat{W}_s) dW_s$$

Example (Rough Heston model)

$$dS_t = \ldots, \quad v_t = v_0 + \int_0^t (a - b v_s) K(s,t) ds + \int_0^t c\sqrt{v_s} K(s,t) dW_s$$

$$\rightsquigarrow Z_t = z + \int_0^t K(s,t) v(Z_s) ds + \int_0^t K(s,t) u(Z_s) dW_s$$

Objectives

Provide unified analytic (i.e., pathwise) framework for rough volatility models as above.

- Stratonovich version of rough volatility models
- Existence and uniqueness and stability of solutions
- Numerical approximation based on approximation of the driving Brownian motion W
- Large deviation principle for analyzing behaviour of implied volatility

Requirements

- Smoothness of coefficient functions
- Structure adapted to Hurst index H – more detailed structure needed for $H \ll \frac{1}{2}$

Theorem

Let $dS_t = f(\widehat{W}_t)\,(\rho dW_t + \overline{\rho} dW_t^{\perp})$, W^{ε} *approximation (at scale* ε*) of* W.

1. *There is* $\mathscr{C}^{\varepsilon} = \mathscr{C}^{\varepsilon}(t)$ *s.t.* $\widetilde{S}^{\varepsilon} \to S$ *(in probability, on* $[0,T]$*) with*

$$\frac{d}{dt}\widetilde{S}^{\varepsilon} = f\left(\widehat{W}^{\varepsilon}\right)\left(\rho \dot{W}^{\varepsilon} + \overline{\rho}\dot{W}^{\perp,\varepsilon}\right) - \rho \mathscr{C}^{\varepsilon} f'\left(\widehat{W}^{\varepsilon}\right) - \frac{1}{2} f^2\left(\widehat{W}^{\varepsilon}\right).$$

   For $H < \frac{1}{2}$, $\int_0^T \mathscr{C}^{\varepsilon}(t)dt \xrightarrow{\varepsilon\to 0} \infty$.

2. *Let* $\Psi(I,V) := C_{BS}\left(S_0 \exp\left(\rho I - \frac{\rho^2}{2}V\right), K, \overline{\rho}^2 V\right)$ *and*

$$\widetilde{\mathscr{I}}^{\varepsilon} := \int_0^T f\left(\widehat{W}_t^{\varepsilon}\right) dW_t^{\varepsilon} - \int_0^T \mathscr{C}^{\varepsilon}(t) f'\left(\widehat{W}_t^{\varepsilon}\right) dt, \quad \mathscr{V}^{\varepsilon} := \int_0^T f^2\left(\widehat{W}_t^{\varepsilon}\right) dt.$$

   Then $E\left[(S_T - K)^+\right] = \lim_{\varepsilon\to 0} E\left[\Psi(\widetilde{\mathscr{I}}^{\varepsilon}, \mathscr{V}^{\varepsilon})\right]$.

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   Then $E\left[(S_T - K)^+\right] = \lim_{\varepsilon\to 0} E\left[\Psi(\widetilde{\mathscr{I}}^\varepsilon, \mathscr{V}^\varepsilon\right]$.

- Model space $\mathcal{T} := \left\langle \left\{ \mathbf{1}, X, X^2, \ldots, X^M \right\} \right\rangle$ with degrees $\left| X^k \right| := k$
- Describes jet of local expansions at any point
- Model (Π, Γ). $\Pi_x : \mathcal{T} \to \mathcal{S}'(\mathbb{R})$ local expansion around $x \in \mathbb{R}$
- $(\Pi_x X^k)(z) := (z - x)^k$, $z \in \mathbb{R}$
- $\Gamma_{xy} : \mathcal{T} \to \mathcal{T}$ translates a "local expansion" around y to one around x, i.e., $\Pi_y = \Pi_x \Gamma_{xy}$
- Canonical choice: $\Gamma_{xy} X^k := (X + (y - x)\mathbf{1})^k$
- Modelled distribution $F : \mathbb{R} \to \mathcal{T}$ is in $\mathcal{D}^\gamma$ if it is "regular" in the sense that $F(x) - \Gamma_{xy} F(y)$ "small" at each level
- "Jets" of local expansions in terms of defining symbols
- Reconstruction operator $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{S}'(\mathbb{R})$ such that $\mathcal{R}F - \Pi_x F(x)$ is "small" when tested against test functions centered in $x \in \mathbb{R}$

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- For a degree β and $\tau \in \mathcal{T}$, let $|\tau|_k$ be the modulus of the coefficient X^β
- Modelled distributions: $F \in \mathcal{D}_K^\gamma$ for $K > 0$ iff

$$||F||_{\mathcal{D}_K^\gamma} := \sup_{\beta<\gamma,\, |x|\le K} |F(x)|_\beta + \sup_{\beta<\gamma,\, |x|,|y|\le K,\, x\ne y} \frac{\left|F(x) - \Gamma_{xy} F(y)\right|_\beta}{|x-y|^{\gamma-\beta}}$$

- Example: $f \in C^\alpha(\mathbb{R})$ (in the Lipschitz sense), then

$$F : x \mapsto \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{1}{k!} f^{(k)}(x) X^k \in \mathcal{D}_K^\alpha.$$

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For $\varphi \in C_c^M$ (compactly supported in a fixed set), let

$$\varphi_x^\lambda(z) := \frac{1}{\lambda}\varphi\left(\frac{z-x}{\lambda}\right), \quad \lambda > 0,\ x \in \mathbb{R}$$

Theorem and definition

Reconstruction operator $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{S}'(\mathbb{R})$ defined by the property that

$$\forall x : \ \left|\mathcal{R}F(\varphi_x^\lambda) - (\Pi_x F(x))(\varphi_x^\lambda)\right| \lesssim \lambda^\gamma$$

In the polynomial regularity structure, with $F \in \mathcal{D}^\gamma$ constructed from $f \in C^\gamma$, we get $\mathcal{R}F = f$.

Goal: pathwise definition of $\int_0^t f(W_s)dW_s$, $t \in [0, T]$, $W \in \mathbb{R}$ Bm

- Symbol Ξ representing $\dot{W}$ (in distributional sense)
- Operator $\mathcal{I}$ representing antiderivative — kernel $K(s,t) = \mathbf{1}_{t>s}$
- No need to add polynomials as objects will not be smooth

- Fix $0 < \kappa$ small ("regularity" measured in $(1/2 - \kappa)$-Hölder space, *small*)
- $\mathcal{T} := \left\langle \{ \Xi, \Xi\mathcal{I}(\Xi), \ldots, \Xi\mathcal{I}(\Xi)^M, \mathbf{1}, \mathcal{I}(\Xi), \ldots, \mathcal{I}(\Xi)^M \} \right\rangle$
- $|\Xi| := -\frac{1}{2} - \kappa$, $|\mathcal{I}(\Xi)| := \frac{1}{2} - \kappa$, $|\tau \cdot \tau'| := |\tau| + |\tau'|$, M s. t. $\left|\Xi\mathcal{I}(\Xi)^M\right| > 0$
- Models will contain true distributions, modelled distributions are local expansions around special distributions
- Will define many models, as models will depend on ω
- Different natural model classes: Itô, Stratonovich, ...

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Itô model

- $\Pi_s \mathbf{1} := 1$, $\Pi_s \Xi := \dot{W}$, $\Gamma_{ts} \mathbf{1} := \mathbf{1}$, $\Gamma_{ts} \Xi := \Xi$ for $t, s \in [0, T]$
- $\Pi_s \mathcal{I}(\Xi)^m := (W_{\cdot} - W_s)^m$, $\Gamma_{ts} \mathcal{I}(\Xi)^m := (\mathcal{I}(\Xi) + (W_t - W_s)\mathbf{1})^m$
- $\Pi_s \Xi \mathcal{I}(\Xi)^m := t \mapsto \frac{\partial}{\partial t} \mathbb{W}^m_{s,t}$ with $\mathbb{W}^m_{s,t} := \int_s^t (W_r - W_s)^m dW_r$

Mollified model

Fix $\varepsilon > 0$, $\dot{W}^\varepsilon := \delta^\varepsilon * \dot{W}$ – or wavelet expansion at scale $\varepsilon = 2^{-N}$.
Hence, $\dot{W}^\varepsilon$ is a regular function.

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Abstract metric $|\!|\!|\cdot\,;\cdot|\!|\!|_{[0,T]}$ on models (Π, Γ): but $(\Pi^\varepsilon, \Gamma^\varepsilon) \xrightarrow{\varepsilon \to 0} (\Pi, \Gamma)$?

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- Evaluation against φ means $\int_0^\infty \cdots \varphi(s)ds \Rightarrow$ anticipative (Skorokhod) integrals!
- By classical results from Mallivin calculus:

$$\Pi_s \Xi \mathcal{I}(\Xi)^m(\varphi) = \int_0^\infty \varphi(t)\,(W_t - W_s)^m\,\delta W_t - m\int_0^\infty \varphi(t)K(s,t)(W_t - W_s)^{m-1}dt$$

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- Note: $D_t W_s = 0$ for $t > s$, but $D_t W_s^\varepsilon \neq 0$

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$\left(\widehat{\Pi}^{\varepsilon}, \widehat{\Gamma}^{\varepsilon}\right)$ *is a valid model and satisfies for any* $0 < \delta < 1$

$$E\left[\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert (\widehat{\Pi}^{\varepsilon}, \widehat{\Gamma}^{\varepsilon}) \,;\, (\Pi, \Gamma) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{[0,T]} \right]^{1/p} \lesssim \varepsilon^{\delta\kappa}.$$

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Reconstruction

- Let (for any model (Π, Γ)) $\mathscr{K}\Xi := t \mapsto \mathcal{I}(\Xi) + (K * \Pi_t \Xi)(t)\mathbf{1} \in \mathcal{D}_T^\infty$ satisfying $\mathcal{R}(\mathscr{K}F) = K * \mathcal{R}F$ for $F \in \mathcal{D}^\gamma$
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$$F^\Pi(t) := \sum_{m=0}^{M} \frac{1}{m!} f^{(m)}\left(\mathcal{R}^\Pi \mathscr{K}\Xi(s)\right) \mathcal{I}(\Xi)^m$$

Theorem

$$\mathscr{I}_f(t) := \mathcal{R}^\Pi \Xi F^\Pi (\mathbf{1}_{[0,t]}) = \int_0^t f(W_r) dW_r,$$

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For f smooth, we have for any $0 < \delta < 1$

$$E\left[\sup_{t\in[0,T]} \left|\widetilde{\mathscr{I}}_f^\varepsilon(t) - \mathscr{I}_f(t)\right|^p\right]^{1/p} \lesssim \varepsilon^{\delta/2}.$$

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For f smooth, we have for any $0 < \delta < 1$

$$E\left[\sup_{t\in[0,T]} \left|\widetilde{\mathscr{I}}_f^\varepsilon(t) - \mathscr{I}_f(t)\right|^p\right]^{1/p} \lesssim \varepsilon^{\delta/2}.$$

$$\int_0^t f\left(\widehat{W}_s\right) dW_s, \quad \widehat{W}_s = \int_0^s K(s,t) dW_t, \quad K(s,t) = \sqrt{2H}\,|t-s|^{H-\frac{1}{2}}\,\mathbf{1}_{t>s}$$

Formally, nothing changes except that K is different – and (inside the integrand) $W \rightsquigarrow \widehat{W}$, $W^\varepsilon \rightsquigarrow \widehat{W}^\varepsilon$

- $\mathcal{I}(\Xi)$ represents $\widehat{W}_t$ and $|\mathcal{I}(\Xi)| = H - \kappa$
- $\mathcal{K}^\varepsilon$ is mollified version of K and explodes like $\mathcal{K}^\varepsilon(s,t) \lesssim \varepsilon^{H-\frac{1}{2}}$ as $\varepsilon \to 0$ — corresponding to exploding quadratic variation

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- Let (for any model (Π, Γ)) $\mathscr{K}\Xi := t \mapsto \mathcal{I}(\Xi) + (K * \Pi_t \Xi)(t)\mathbf{1} \in \mathcal{D}_T^\infty$.
- $f(\widehat{W}_t)$ encoded by $F^\Pi \in \mathcal{D}_T^\gamma$, $\frac{1}{2} + \kappa < \gamma < 1$ with

$$F^\Pi(t) := \sum_{m=0}^{M} \frac{1}{m!} f^{(m)}\left(\mathcal{R}^\Pi \mathscr{K}\Xi(s)\right) \mathcal{I}(\Xi)^m$$

Theorem

$$\widetilde{\mathscr{I}}_f^\varepsilon(t) := \mathcal{R}^{\widehat{\Pi}^\varepsilon} \Xi F^{\widehat{\Pi}^\varepsilon}\left(\mathbf{1}_{[0,t]}\right) = \int_0^t f\left(\widehat{W}_r^\varepsilon\right) dW_r^\varepsilon - \int_0^t \mathcal{K}^\varepsilon(r,r) f'\left(\widehat{W}_r^\varepsilon\right) dr,$$

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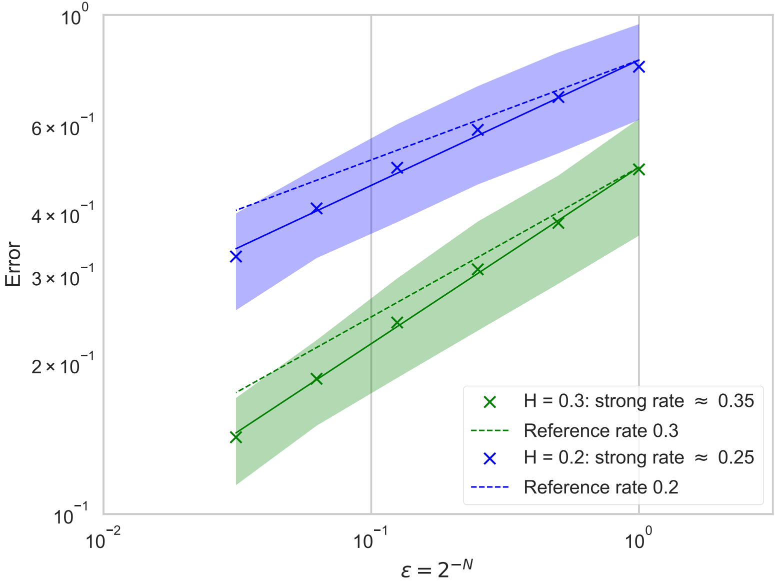
Numerical example: $f(x) = \exp(x)$, strong error



$$dS_t = S_t f(Z_t)\left(\rho dW_t + \overline{\rho} dW^{\perp}\right),$$
$$Z_t = z + \int_0^t K(s,t)v(Z_s)ds + \int_0^t K(s,t)u(Z_s)dW_s$$

- Special case: $u(z) = \sqrt{z}$ (rough Heston)
- Require f, v, u smooth
- For $H > \frac{1}{4}$ (for simplicity), then only need ($M = 1$):

$$\widehat{\mathcal{T}} := \langle \{ \Xi, \Xi\mathcal{I}(\Xi), \mathbf{1}, \mathcal{I}(\Xi), \overline{\Xi}, \overline{\Xi}\mathcal{I}(\Xi) \} \rangle$$

- Generally, fixed point arguments require more operations, need to add symbols like

$$\Xi\mathcal{I}\left(\Xi(\mathcal{I}(\Xi))^m\right), \quad \mathcal{I}\left(\Xi\mathcal{I}\left(\Xi\mathcal{I}(\Xi)^k\right)^m\right), \ldots$$

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Let U and V denote the “lifts” of u and v to modelled distributions. Then

$$\mathcal{Z} = z\mathbf{1} + \mathcal{K}\left(U(\mathcal{Z}) \cdot \Xi + V(\mathcal{Z})\right).$$

Theorem

1. *For u, v smooth, there is a unique solution $\mathcal{Z} \in \mathcal{D}^\gamma(\mathcal{T})$, $\frac{1}{2} + \kappa < \gamma < 1$, $(u, v, \Pi) \mapsto \mathcal{Z}$ is (loc.) Lipschitz.*
2. *If (Π, Γ) is the Itô model, then $Z := \mathcal{R}\mathcal{Z}$ solves the Itô equation.*
3. *The extended renormalized model $(\widehat{\Pi}^\varepsilon, \widehat{\Gamma}^\varepsilon)$ converges to (Π, Γ), implying that $Z = \lim_{\varepsilon \to 0} Z^\varepsilon$, where*

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- $\overline{\mathcal{T}} := \mathcal{T} + \langle \{ \overline{\Xi}, \overline{\Xi}\mathcal{I}(\Xi), \ldots, \overline{\Xi}\mathcal{I}(\Xi)^M \} \rangle$
- Canonical model (Π, Γ) extended by
  $\Pi_s \overline{\Xi}\mathcal{I}(\Xi)^m := t \mapsto \frac{\partial}{\partial t} \int_s^t \left(\widehat{W}_u - \widehat{W}_s \right)^m d\overline{W}_u$
- Small noise model: for $\delta > 0$

$$\Pi_s^\delta \mathcal{I}(\Xi)^m := \delta^m \Pi_s \mathcal{I}(\Xi)^m, \; \Pi_s^\delta \Xi \mathcal{I}(\Xi)^m := \delta^{m+1} \Pi_s \Xi \mathcal{I}(\Xi)^m, \ldots$$

- Fix $h := (h^1, h^2) \in \mathcal{H}^2$ for $\mathcal{H} := L^2([0,T])$ and let

$$\Pi_s^h \Xi := h^1, \; \Pi_s^h \overline{\Xi} := h^2, \; \Pi_s^h \mathcal{I}(\Xi)(t) := \int_0^{t \vee s} (K(u,t) - K(u,s))\, h^1(u) du, \ldots$$

Theorem

The models $(\Pi^\delta, \Gamma^\delta)$ satisfy an LDP in the space of models with speed δ^2 and rate function

$$J(\Pi) := \begin{cases} \frac{1}{2} \|h\|_{\mathcal{H}}^2, & \Pi = \Pi^h, \\ +\infty, & \textit{else.} \end{cases}$$

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Let $X_t := \log S_t$,

$$z^h(t) = z + \int_0^t K(s,t)u\left(z^h(s)\right)h(s)ds.$$

Corollary

f smooth (without boundedness assumption). Then $t^{H-\frac{1}{2}}X_t$ satisfies an LDP with speed t^{2H} and rate function

$$I(x) := \inf_{h\in\mathcal{H}} \left\{ \frac{1}{2}\|h\|^2_{\mathcal{H}} + \frac{\left(x - I_1^z(h)\right)^2}{2I_2^z(h)} \right\},$$

$$I_1^z(h) := \rho \int_0^1 f(z^h(s))h(s)ds, \quad I_2^z(h) := \int_0^1 f(z^h(s))^2 ds.$$

- Choose $\delta \equiv t^H$ in the theorem.

A regularity structure for rough volatility models allows:

- Unified analysis (existence, uniqueness, stability) — *sketched, but not completely worked out in paper*
- Numerical approximation by wavelet approximation to underlying Brownian motion
- Large deviation principle

Example of a regularity structure:

- Simple one-dimensional regularity structure with genuine need for renormalization.

A regularity structure for rough volatility — preprint available soon!

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