Obstacle Problems and Optimal Control

Exercise sheet 6

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1. Let U and H be Hilbert spaces. Suppose there is a bounded linear map $S\colon U \to H$ and let $y_d \in H$ be given. Fix $\nu \in \mathbb{R}_+$ and define the objective function

$$J(u) := \frac{1}{2}\left\|S(u) - y_d\right\|_H^2 + \frac{\nu}{2}\left\|u\right\|_U^2.$$

Prove that if $\nu > 0$ or S is injective, then J is strictly convex.

By Theorem 6.3 from the lectures, this gives uniqueness to the optimal control problem (6.1).

2. Let $J\colon X \to \mathbb{R}$ be Gateaux differentiable and convex, and let $K \subset X$ be a non-empty convex subset. If $x^* \in K$ satisfies

$$J'(x^*)(x^* - x) \leq 0 \qquad \text{for all } x \in K$$

prove that

$$J(x^*) \leq J(x) \text{ for all } x \in K.$$

3. Let $V \overset{d}{\hookrightarrow} H \overset{c}{\hookrightarrow} V^*$ be a Gelfand triple and define $S\colon V^* \to V$ as the VI solution mapping: $y = S(u)$ solves

$$y \in K : \langle Ay - u, y - v\rangle \leq 0 \qquad \forall v \in K$$

under all the usual assumptions guaranteeing well posedness. Suppose that

(a) $J\colon V \times H \to \mathbb{R}$ is bounded from below.

(b) If $y_n \to y$ in $V \times V$ and $u_n \rightharpoonup u$ in H, then

$$J(y, u) \leq \liminf_{n\to\infty} J(y_n, u_n).$$

(c) If $\{J(y_n, u_n)\}$ is bounded for a sequence $\{(y_n, u_n)\} \subset V \times U_{ad}$, then $\{u_n\}$ is bounded in H.

Consider the problem

$$\min_{u \in U_{ad}} J(y, u) \quad \text{where} \quad y = S(u) \tag{1}$$

where $U_{ad} \subset H$ is closed, convex and bounded.

Prove that there exists an optimal pair (y^*, u^*) of (1).

4. Take the setting of the previous question and let K be polyhedric.

We proved the following in the last lecture before time ran out: given a local minimiser (y^*, u^*) of (1), there exist multipliers $(p^*, \xi^*) \in V \times V^*$ satisfying

$$Ay^* - u^* + \xi^* = 0, \tag{2}$$

$$\xi^* \geq 0 \text{ in } V^*, \quad y^* \leq \psi, \quad \langle \xi^*, y^* - \psi\rangle = 0, \tag{3}$$

$$p^* = J_u(y^*, u^*). \tag{4}$$

Define

$$K_K := K_K(u^*, u^* - Ay^*) = T_K(u^*) \cap (u^* - Ay^*)^{\perp}.$$

The local minimiser u^* satisfies

$$J_y(y^*, u^*)S'(u^*)(h) + J_u(y^*, u^*)h \geq 0 \qquad \forall h \in H.$$

(a) Prove

$$\langle p^*, h \rangle_{V,V^*} \geq 0 \quad \forall h \in K_K^\circ. \tag{5}$$

Hint: (1) use the density of $H \subset V^*$, (2) recall the VI characterisation of S'.

(b) Define λ^* by

$$A^* p^* + \lambda^* = -J_y(y^* u^*). \tag{6}$$

Prove that

$$\langle \lambda^*, v \rangle \leq 0 \quad \forall v \in K_K. \tag{7}$$

(c) The system formed by (2), (3), (4), (5), (6), (7) is called a strong stationarity system. How does it compare to the weak C-stationarity system and the $\mathcal{E}$-almost C-stationarity system?
