Obstacle Problems and Optimal Control

Exercise sheet 2

Dr. Amal Alphonse (amal.alphonse@wias-berlin.de)

1. Let $f \in L^2(\Omega)$. Consider the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ \nabla u \cdot \nu &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where ν is outward unit normal vector, and derive its weak form:

$$\text{find } u \in H^1(\Omega) : \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi \quad \forall \varphi \in H^1(\Omega).$$

Hint: recall Green's first identity from the last sheet.

2. In the last sheet, we found out that the space

$$X := \left\{ u \in H^1(\Omega) : \int_\Omega u = 0 \right\}$$

of H^1 functions with mean value zero possesses the Poincare inequality:

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad \forall u \in X.$$

(a) Show that the PDE (1) has a weak solution $u \in H^1(\Omega)$ if and only if $\int_\Omega f = 0$.

(b) Explain if the problem (1) is well posed (i.e., does there exist a unique solution).

3. Let $V := H^1_0(\Omega)$. Given $b \in L^\infty(\Omega)$, define $A\colon V \to V^*$ by

$$\langle Au, v \rangle = \int_\Omega b(x) \nabla u(x) \cdot \nabla v(x),$$

i.e., $Au = -\nabla \cdot (b\nabla u)$. Is A bounded and coercive? Explain your answer. If it's not bounded and/or coercive, what further assumptions can you add to make it so?

4. Assuming the result of the Stampacchia theorem, deduce Lax–Milgram.

5. For a given non-negative function $\psi \in L^2(\Omega)$, define

$$K := \{ v \in H^1_0(\Omega) : |\nabla v| \leq \psi \text{ a.e. in } \Omega \}.$$

Given a source term $f \in H^{-1}(\Omega)$ and the bilinear form

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v,$$

explain if the VI

$$u \in K : a(u, u - v) \leq \langle f, u - v \rangle \qquad \forall v \in K$$

is well posed via the Stampacchia theorem or not. If it is not, can you see a way to make it well posed by strengthening or adding an extra assumption?

6. The same set up and question as above, except now

$$K := \{ v \in H^1_0(\Omega) : \psi_1 \leq v \leq \psi_2 \text{ a.e. in } \Omega \}$$

where $\psi_1, \psi_2 \in C^0(\bar{\Omega})$.

7. Let $V := H_0^1(\Omega)$. We are given a bounded, linear and coercive operator $A\colon V \to V^*$, and suppose we have $f_n \to f$ in V^* and $\psi_n \to \psi$ in V.

For each n, define

$$K_n := \{v \in V : v \le \psi_n \text{ a.e. in } \Omega\}$$

and define u_n as the solution of the VI

$$u_n \in K_n : \langle Au_n - f_n, u_n - v\rangle \le 0 \qquad \forall v \in K_n.$$

(a) Prove that there exists some $u \in V$ such that

$$u_n \rightharpoonup u \text{ in } V$$

(at least for a subsequence).

(b) Prove that in fact u is the solution of the VI

$$u \in K : \langle Au - f, u - v\rangle \le 0 \qquad \forall v \in K$$

where

$$K := \{v \in V : v \le \psi \text{ a.e. in } \Omega\}.$$

(c) Can we say that the entire sequence $\{u_n\}$ converges to u (and not just that a subsequence converges)?

Hint: The fact that (weakly and strongly) convergent sequences are uniformly bounded and Minty's lemma might help. You may need to construct a clever test function to use in the VI for u_n when passing to the limit.
