Obstacle Problems and Optimal Control

Exercise sheet 1

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1. On $\Omega = (1, \infty)$, consider the function

$$u(x) = \frac{1}{x}.$$

For which $p \geq 1$ does $u \in L^p(\Omega)$?

2. Let $x^* \in \Omega$ be given. If $k > 0$, prove that there is no $g \in L^2(\Omega)$ such that

$$\int_\Omega g\varphi = k\varphi(x^*) \qquad \forall \varphi \in C_c^\infty(\Omega).$$

3. Consider on the domain $\Omega = (0, 2)$ the two functions

$$u(x) = \begin{cases} x & : x \in (0,1) \\ 1 & : x \in [1,2) \end{cases}$$

and

$$v(x) = \begin{cases} x & : x \in (0,1) \\ 10 & : x \in [1,2). \end{cases}$$

(a) What is the best L^p space that u belongs to? That is, what is the largest p such that $u \in L^p(\Omega)$?

(b) Same question for v.

(c) Are u and v weakly differentiable? Prove your claims.

4. On $\Omega = (0, 1)$, consider the function $u(x) = \sqrt{x}$.

(a) Show that $u \in L^2(\Omega)$.

(b) In class, we defined the α^{th}-weak derivative of a function w in $L^2(\Omega)$ as the element $\partial^\alpha w \in L^2(\Omega)$ satisfying

$$\int_\Omega w(x)\partial^\alpha \varphi(x) = (-1)^{|\alpha|} \int_\Omega \partial^\alpha w(x)\varphi(x) \quad \forall \varphi \in C_c^\infty(\Omega).$$

In fact, we do not need to insist on $L^2(\Omega)$ for w and $\partial^\alpha w$; instead we can just replace all instances of $L^2(\Omega)$ with $L^1(\Omega)$ in the definition because the integrals still make sense. In this way, we can think about weak derivatives of $L^1(\Omega)$ functions.

With this in mind, what (if any) Sobolev space does u belong to?

5. For some given number c and a function $u \colon \mathbb{R} \to \mathbb{R}$ which is defined at the point c, define the Dirac delta functional

$$\delta_c(u) := u(c).$$

With $\Omega = (0, 1)$, prove that $\delta_c \in H^{-1}(\Omega)$.

Hint: in 1D, we have that $H^1(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is a continuous embedding.

6. Show that the norms given by the expressions

$$\|u\|_{H^1(\Omega)}^2 = \int_\Omega |u|^2 + |\nabla u|^2$$

and

$$\|u\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla u|^2$$

are equivalent on $H_0^1(\Omega)$.

Can you think of an example demonstrating why they cannot be equivalent on $H^1(\Omega)$?

7. Define $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$.

Explain if this is a bounded and/or coercive bilinear form and if so, derive the boundedness and/or coercivity constants.

8. Define the space

$$X := \left\{ u \in H^1(\Omega) : \int_\Omega u = 0 \right\}.$$

Prove that there exists a constant C such that

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in X.$$

Hence we have the Poincaré inequality for functions in X too.

Hint: argue by contradiction and use that

(a) $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ is a compact embedding

(b) if $\nabla u = 0$ a.e., u is constant.

9. Prove the following statement (as claimed in the lecture): let $a\colon H \times H \to \mathbb{R}$ be a bounded bilinear form, then there exists a unique bounded linear operator $A\colon H \to H$ such that

$$a(u,v) = (Au, v) \qquad \forall u, v \in H.$$

10. Let $f \in L^2(\Omega)$. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Using Green's first identity

$$\int_\Omega (\Delta \eta)\varphi = -\int_\Omega \nabla\eta \cdot \nabla\varphi + \int_{\partial\Omega} \varphi \nabla\eta \cdot \nu,$$

derive the weak form and argue well posedness by applying Lax–Milgram (state what the bilinear form and the linear functional are, etc.).