Aging for 1D transient RWRE in the sub-ballistic regime

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The model

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  $P \equiv$ law of ω. $E \equiv$ expectation under P.

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$$\begin{aligned} P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) &= \omega_x, \\ P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right) &= 1-\omega_x. \end{aligned}$$

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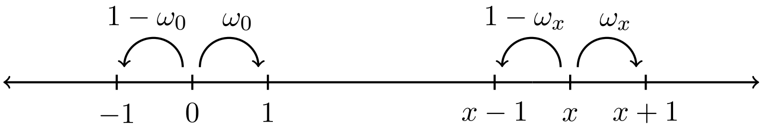
  $P_\omega \equiv$ law of X in the environment ω : **quenched law**.

- $\mathbb{P} \equiv$ joint law of $(\omega,\ (X_n))$: **annealed law**. $\mathbb{E} \equiv$ expectation under $\mathbb{P}$.

Transition probabilites



Transience-recurrence criterion

Notations :

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Theorem (Solomon, 1975)

If $E[\log \rho_0]$ is defined, $(X_n,\ n \geq 0)$ is recurrent iff $E[\log \rho_0] = 0$.

Law of large numbers

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There exists $v \in [-1, 1]$, *which depends only on the environment, such that,* $\mathbb{P}$*-a.s.,*

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where v *satisfies*

$$v := \begin{cases} \frac{1-E[\rho_0]}{1+E[\rho_0]} > 0 & \text{if } E[\rho_0] < 1, \\ 0 & \text{if } (E[\rho_0^{-1}])^{-1} \le 1 \le E[\rho_0], \\ \frac{E[\rho_0^{-1}]-1}{E[\rho_0^{-1}]+1} < 0 & \text{if } 1 < (E[\rho_0^{-1}])^{-1}. \end{cases}$$

The recurrent case : Sinai's walk

Theorem (Sinai, 1982)

If $E[\log \rho_0] = 0$ *(and technical conditions), then*

$$\frac{\sigma^2}{(\log n)^2} X_n \xrightarrow{law} b_\infty ,$$

where $\sigma^2 := \mathrm{Var}[\log \rho_0] > 0.$

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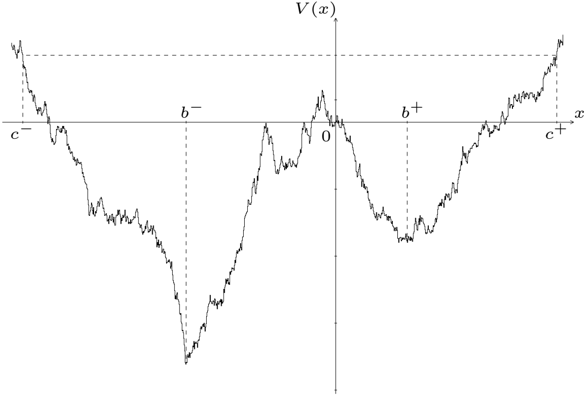
Example of potential



Valleys and localization

- Valleys : (a, b, c) such that $a < b < c$ and :

$$\begin{aligned} \min_{a \leq x \leq c} V(x) &= V(b), \\ \max_{a \leq x \leq b} V(x) &= V(a), \\ \max_{b \leq x \leq c} V(x) &= V(c). \end{aligned}$$

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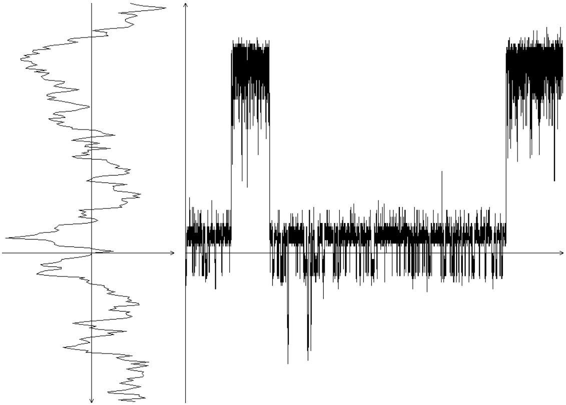
- **Golosov (1984)** : Exit time $\simeq \mathrm{e}^{H}$.

Valley and localization in the recurrent case



The sub-ballistic regime

Assumptions

(a) There exists $0 < \kappa < 1$ such that $E\left[\rho_0^\kappa\right] = 1$ (and technical conditions).

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Theorem (Kesten-Kozlov-Spitzer, 1975)

Under (a), we have :

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} c_\kappa \mathcal{S}_\kappa^{ca}, \qquad n \to \infty,$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} c'_\kappa \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa, \qquad n \to \infty,$$

where $\mathcal{S}_\kappa^{ca}$ is a completely asymmetric stable law of index κ.

The sub-ballistic regime

Proof : Branching process in random environment with immigration.

No potential !

Main result : aging phenomenon

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Under assumption (a), we have, for all $h > 1$ and all $\eta > 0$,

$$\lim_{t\to\infty} \mathbb{P}(|X_{th} - X_t| \leq \eta \log t) = \frac{\sin(\kappa\pi)}{\pi} \int_0^{1/h} y^{\kappa-1}(1-y)^{-\kappa}\,\mathrm{d}y.$$

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Remark

Universality *of the Bouchaud's trap model.*

A renewal theorem of Dynkin

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- **Spent waiting time** and **residual waiting time** :

$$\begin{aligned} A_t &:= t - S_{N_t}, \qquad t \geq 0, \\ R_t &:= S_{N_t+1} - t, \qquad t \geq 0. \end{aligned}$$

A renewal theorem of Dynkin

Theorem (Dynkin)

For all $0 \leq x_1 < x_2 \leq 1$, *we have*

$$\lim_{t\to\infty} \mathbb{P}\left(x_1 \leq \frac{A_t}{t} \leq x_2\right) = \frac{\sin(\alpha\pi)}{\pi}\int_{x_1}^{x_2} \frac{x^{-\alpha}}{(1-x)^{\alpha-1}}\,\mathrm{d}x.$$

For all $0 \leq x_1 < x_2$, *we have*

$$\lim_{t\to\infty} \mathbb{P}\left(x_1 \leq \frac{R_t}{t} \leq x_2\right) = \frac{\sin(\alpha\pi)}{\pi}\int_{x_1}^{x_2} \frac{\mathrm{d}x}{x^{\alpha}(1+x)}.$$

The sub-ballistic regime : analysis of the potential

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(a) There exists $0 < \kappa < 1$ such that $E\left[\rho_0^\kappa\right] = 1$ (and technical conditions).

Potential

$$V(x) := \begin{cases} \displaystyle\sum_{i=1}^{x} \log \rho_i & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ \displaystyle -\sum_{i=x+1}^{0} \log \rho_i & \text{if } x \leq -1. \end{cases}$$

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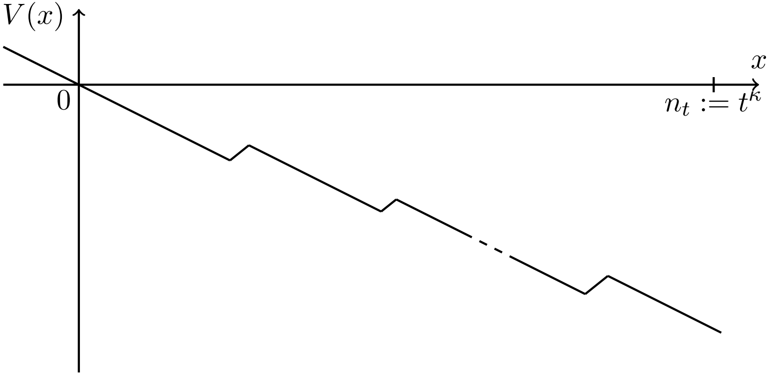
Remark : Assumption (a) implies $E[\log \rho_0] < 0$.

Potential and valleys



Potential and valleys

- Excursions of the potential above its past minimum

$$\begin{aligned} e_0 &:= 0, \\ e_i &:= \inf\{n > e_{i-1} : V(n) \leq V(e_{i-1})\}, \qquad i \geq 1. \end{aligned}$$

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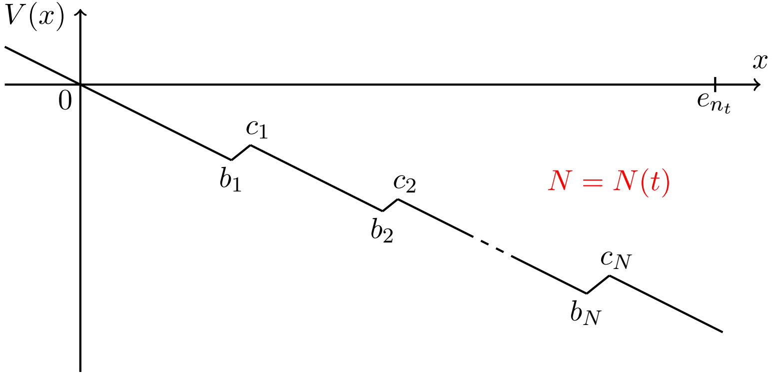
- Under (a), we have $E[e_1] < \infty$.
- Iglehart's result : $P\{H > h\} \sim C_I \mathrm{e}^{-\kappa h}$, $h \to \infty$.
- Deep valleys : boxes constructed around excursions higher than $h_t := \log t - \log\log t$.

Potential and valleys



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- The time spent between deep valleys is **negligible** :

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- The valleys are well separated : **"i.i.d." property**.

Occupation time

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- Exact computation : $\forall \lambda > 0$,

$$E_\omega\left[\mathrm{e}^{-\lambda \tau(b_k, d_k)}\right] \approx \frac{1}{1 + \lambda \mathrm{e}^{H_k} \underline{M}_k \overline{M}_k},$$

where

$$\underline{M}_k := \sum_{i=a_k}^{c_k} \mathrm{e}^{-(V(i) - V(b_k))},$$

$$\overline{M}_k := \sum_{i=b_k}^{d_k} \mathrm{e}^{V(i) - V(c_k)}.$$

Occupation time

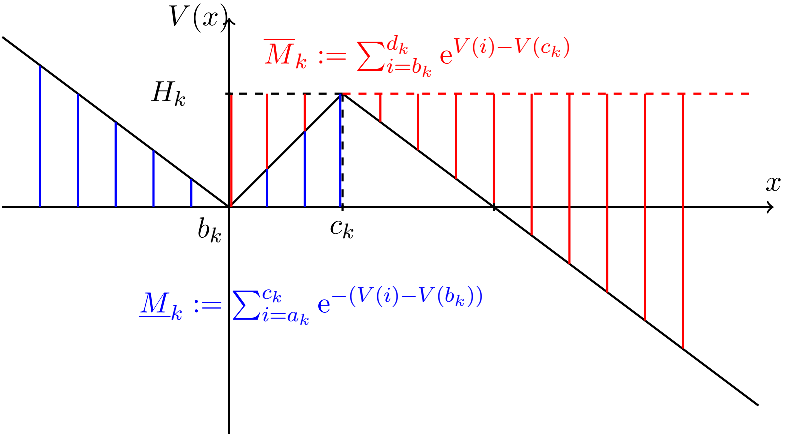

FIG.: $\underline{M}_k$ et $\overline{M}_k$.

Properties

- Occupation time : asymptotically (quenched result)

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- Asymptotic independence between e^{H_k}, $\underline{M}_k$ and $\overline{M}_k$: coupling arguments.
- Iglehart's result + $\underline{M}_k$ and $\overline{M}_k$ "nice" r.v. $\Rightarrow \tau(b_k, d_k)$ is **heavy tailed** under the annealed law.

Proof

- $\tau(b_1, d_1) + \tau(b_2, d_2) + \cdots + \tau(b_N, d_N)$ sum of "i.i.d." heavy-tailed random variables.

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- Occupation time : $T_i := \tau(b_i, d_i)$.
- Time between deep valleys negligible + "directed" property :

$$\{a_j \leq X_t \leq d_j\} = \left\{ \sum_{i=1}^{j-1} T_i \leq t < \sum_{i=1}^{j} T_i \right\}$$

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- **New** version of Dynkin's theorem !

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- Residual waiting time :

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- Then, we have, when $t \to \infty$,

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- Control around the bottom of the last visited deep valley : arguments of **invariant measure** for a Markov chain on a finite state space + geometrical properties of the valleys.