Parametric and Stochastic Problems — an Overview of Computational Methods

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Overview

1. Motivation and goals
2. Stochastic model problem
3. Possible computational approaches
4. Parametric problems, linear maps, and approximations
5. Computational techniques
6. Outlook

Motivation

- Mathematical model of some physical process / system (often described by PDEs) may contain uncertain or random parameters (e.g. random coefficient fields)
- Solution of PDE (state of system) is also a function of parameters / a random field
- Of interest are functionals of the solution (Quantities of Interest / QoI)
- It is advantageous to apply theory and computational methodology which abstractly looks like deterministic method. This allows abstractly similar error estimation, etc.
- Observe: computational challenge is high dimensionality.

Why probabilistic or stochastic models?

Systems may contain uncertain elements, as some details are not precisely known.

- Incompletely known parameters, processes or fields.
- Heterogeneous, not completely known material.
- Small or unresolved scales, a kind of background noise.
- Systems with imprecisely known components.
- Action from the surrounding environment, noisy signals.
- Loading of the system, e.g. due to wind, waves, etc.

All these items introduce some uncertainty in the model.

Ontology and uncertainty modelling

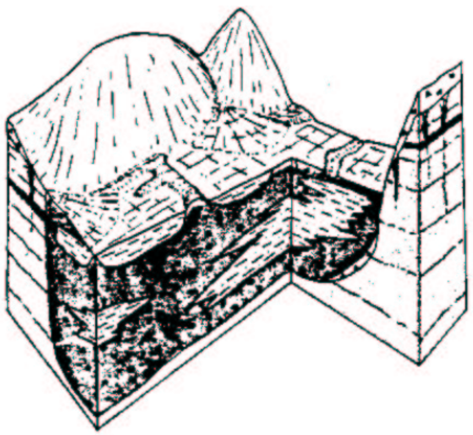
A bit of ontology: Uncertainty may be

- aleatoric, which means random and not reducible, or
- epistemic, which means due to incomplete knowledge.

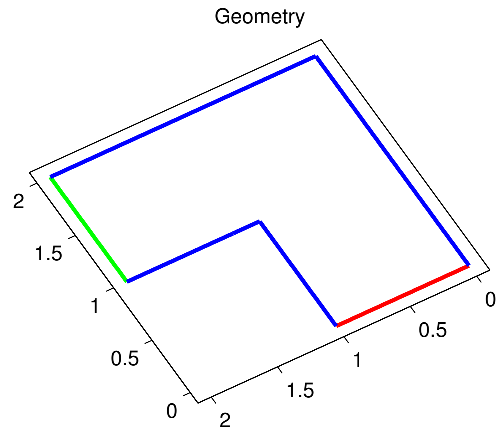
Stochastic models give quantitative information about uncertainty, they are used for both types of uncertainty.

Possible areas of use: Reliability, heterogeneous materials, upscaling, incomplete knowledge of details, uncertain [inter-]action with environment, random loading, etc.

Model problem



Aquifer



2D Model

simple stationary model of groundwater flow (Darcy)

$$-\nabla \cdot (\kappa(x) \cdot \nabla v(x)) = f(x), \qquad x \in \mathcal{G} \subset \mathbb{R}^d,$$

$$v(x) = 0 \text{ for } x \in \partial\mathcal{G}.$$

v hydraulic head, κ conductivity, f sinks and sources.

Theory reminder: variational form

Diffusion problem:

Solution $v \in \mathcal{W} = \mathring{H}(\mathcal{G})$ satisfies variational equation (weak form):

for all test functions $w \in \mathcal{W}$:

$$a(w, v) := \int_{\mathcal{G}} \nabla w(x) \cdot \kappa(x) \cdot \nabla v(x) \, \mathrm{d}x = \int_{\mathcal{G}} f(x) w(x) \, \mathrm{d}x =: \langle f, w \rangle.$$

Here equivalently: solution $v \in \mathcal{W}$ minimises Φ over $\mathcal{W}$, where

$$\Phi(v) = \frac{1}{2} \int_{\mathcal{G}} \nabla v(x) \cdot \kappa(x) \cdot \nabla v(x) \, \mathrm{d}x - \int_{\mathcal{G}} f(x) v(x) \, \mathrm{d}x.$$

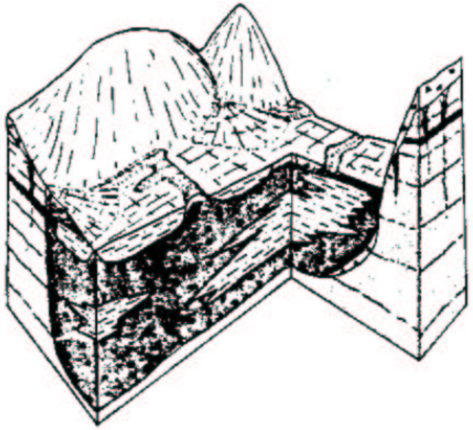
PDE in weak form is stationarity condition (Euler-Lagrange eq.) for Φ:

$$\forall w \in \mathcal{W} : \quad \langle \delta\Phi(v), w \rangle = 0,$$

with Gâteaux derivative denoted by $\delta\Phi(v)$.

Lax-Milgram lemma shows well-posedness.

Model stochastic problem



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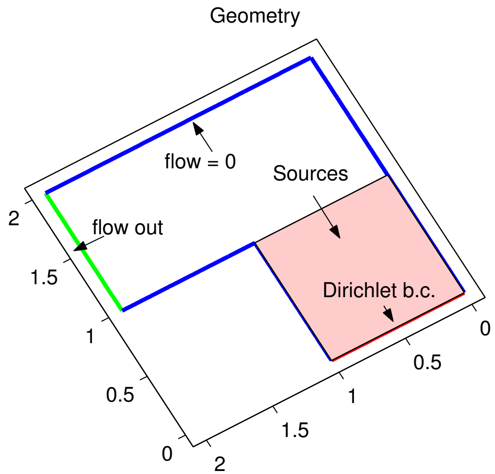


2D Model

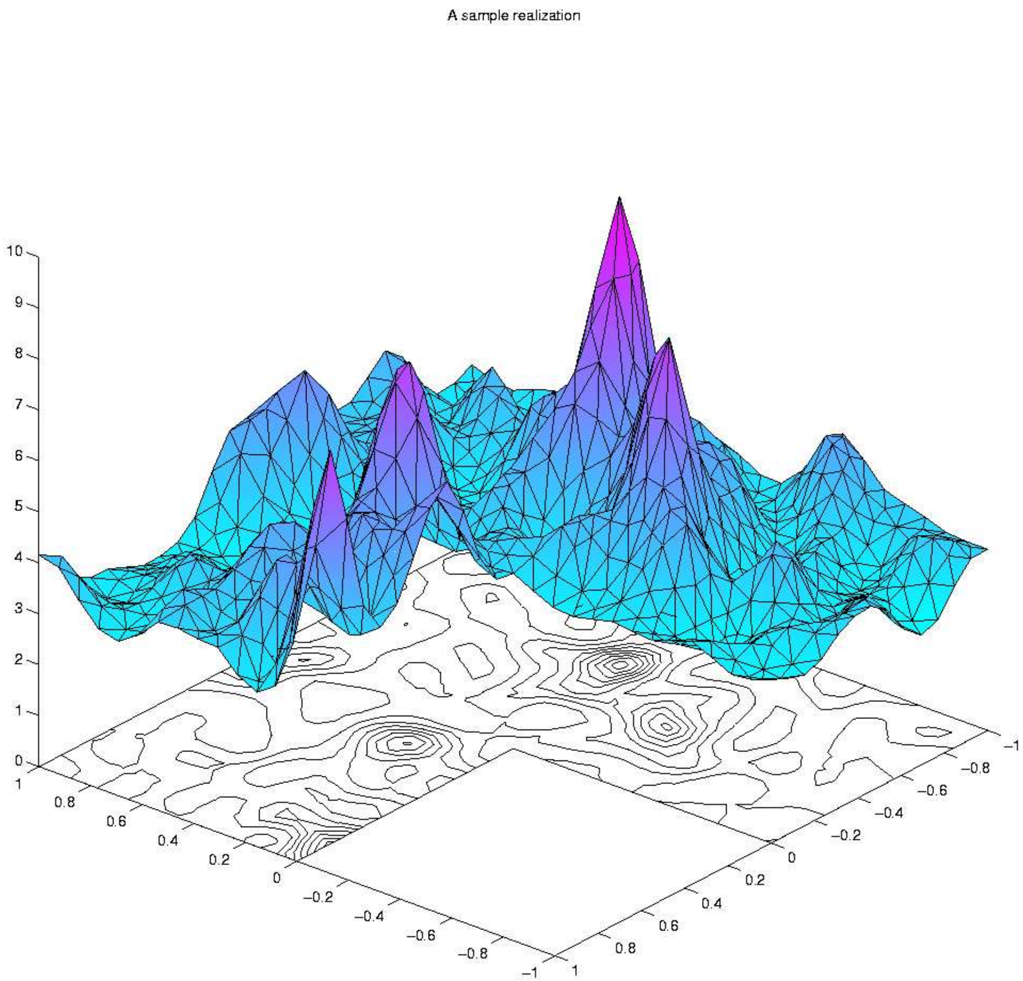
same model with stochastic data, $\mathbb{P}$-a.s. in $\omega \in \Omega$

$$-\nabla \cdot (\kappa(x,\omega) \cdot \nabla u(x,\omega)) = f(x,\omega) \quad x \in \mathcal{G} \subset \mathbb{R}^d$$

$$u(x,\omega) = 0 \text{ for } x \in \partial\mathcal{G}, \quad \omega \in \Omega$$

κ stochastic conductivity, f stochastic sinks and sources.

Realisation of $\kappa(x, \omega)$



Stochastic model

- Uncertainty of system parameters—e.g. $\kappa(x,\omega) = \bar{\kappa}(x) + \tilde{\kappa}(x,\omega)$, $f(x,\omega)$ are stochastic fields, $(\Omega, \mathfrak{A}, \mathbb{P})$ probability space of all realisations, with probability measure $\mathbb{P}$, and expectation functional

$$\bar{\phi} := \langle \phi \rangle := \mathbb{E}(\phi) := \int_{\Omega} \phi(\omega)\, \mathbb{P}(\mathrm{d}\omega)$$

- Input quantities (e.g. fields κ) are functions of
  - Space: $\kappa(\cdot,\omega) \in \mathcal{X}_x$ as a function of x,
  - Sample: $\kappa(x,\cdot) \in \mathcal{S}_\omega$ as a function of ω,
  - Together $\kappa \in \mathscr{K} := \mathcal{X}_x \otimes \mathcal{S}_\omega$ in a tensor product space $:= \{\varkappa \mid \varkappa(x,\omega) = \sum_\ell \varphi_\ell(x)\xi^{(\ell)}(\omega),\ \varphi_\ell \in \mathcal{X}_x, \xi^{(\ell)} \in \mathcal{S}_\omega\}$
- Example: approximate $\varphi_\ell(x)$ by FEM, and $\xi^{(\ell)}(\omega)$ by Wiener's polynomial chaos expansion (PCE).

Theory: Stochastic PDE and variational form

Stochastic diffusion problem:

Stochastic solution $u(x,\omega)$ is a stochastic field—in tensor product space

$$\mathscr{W} := \mathcal{W} \otimes \mathcal{S} \ni u(x,\omega) = \sum_m v_m(x)\eta^{(m)}(\omega); \quad \text{e.g. } \mathcal{S} = L_2(\Omega).$$

Variational formulation: $u \in \mathscr{W} = \mathcal{W} \otimes \mathcal{S}$ satisfies $\forall w \in \mathscr{W}$

$$\boldsymbol{a}(w,u) := \int_\Omega \int_\mathcal{G} \nabla w(x,\omega) \cdot (\kappa(x,\omega) \cdot \nabla u(x,\omega)) \, \mathrm{d}x \, \mathbb{P}(\mathrm{d}\omega) = \mathbb{E}\left(a(u,w)\right)$$

$$= \mathbb{E}\left(\langle f, w \rangle\right) = \int_\Omega \left[\int_\mathcal{G} f(x,\omega) w(x,\omega) \, \mathrm{d}x \right] \mathbb{P}(\mathrm{d}\omega) =: \langle\!\langle f, w \rangle\!\rangle.$$

Here equivalently u minimises $\boldsymbol{\Phi}$ over $\mathscr{W}$:

$$\boldsymbol{\Phi}(u) = \mathbb{E}\left(\Phi(u)\right) = \int_\Omega \Phi(u(\cdot,\omega)) \, \mathbb{P}(\mathrm{d}\omega).$$

Weak form of SPDE is stationarity condition for $\boldsymbol{\Phi}$.

Mathematical results

If κ and κ^{-1} are in $L_\infty(\mathcal{G} \times \Omega)$, finding a solution $u \in \mathscr{W} = \mathcal{W} \otimes \mathcal{S}$

- is guaranteed by Lax-Milgram lemma, problem is well-posed in the sense of Hadamard (existence, uniqueness, continuous dependence on data f in L_2- and on κ in L_∞-norm).
- Numerical solution may be achieved by Galerkin methods, convergence established with Céa's lemma
- Galerkin methods are stable, if no variational crimes are committed

Good approximating subspaces of $\mathscr{W} = \mathcal{W} \otimes \mathcal{S}$ have to be found, as well as efficient numerical procedures worked out.

Note that as $\mathcal{W} \otimes \mathcal{S} \cong L_2(\Omega; \mathcal{W})$, solutions are automatically measurable w.r.t. ω.

Possible difficulties

The condition that κ and κ^{-1} are in $L_\infty(\mathcal{G} \times \Omega)$ may sometimes be too strong,
e.g. a lognormal field $\kappa = \exp(g)$ — with g a Gaussian field — does not satisfy it.

Such (and other cases) can be covered by using other spaces $\mathcal{W}$ and $\mathcal{S}$ in the tensor product $\mathscr{W} = \mathcal{W} \otimes \mathcal{S}$.

Especially $\mathcal{S} = L_2(\Omega)$ with norm $\|\cdot\|_2$ has to be replaced by completions w.r.t. norm $\|u\|_{2,s} := \|A^s u\|_2$,
where A is a suitable s.p.d. operator, related to covariance opertor.

Similar to usual Sobolev spaces, where norm on $H^s(\Omega)$ comes from above construction with $A = (I - \Delta)^{1/2}$.

Quantities of interest

Desirable: Uncertainty quantification (UQ) or
probabilistic information on solution / state $u \in \mathscr{W}$:

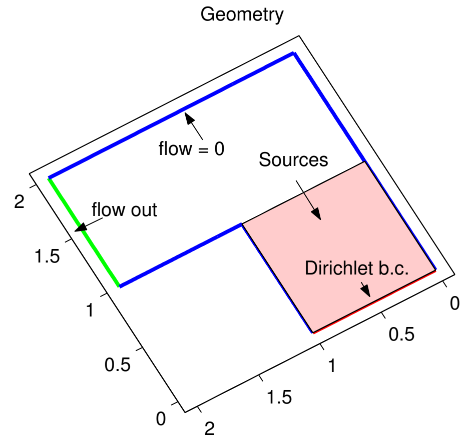
The goal is to compute functionals of the solution:
quantities of interest (QoI)

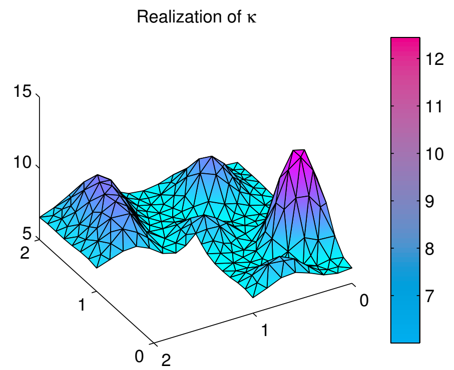
$$\boldsymbol{\Psi}_u = \langle \boldsymbol{\Psi}(u) \rangle := \mathbb{E}\left(\boldsymbol{\Psi}(u)\right) := \int_\Omega \int_\mathcal{G} \Psi(u(x,\omega), x, \omega)\, \mathrm{d}x\, \mathbb{P}(\mathrm{d}\omega)$$

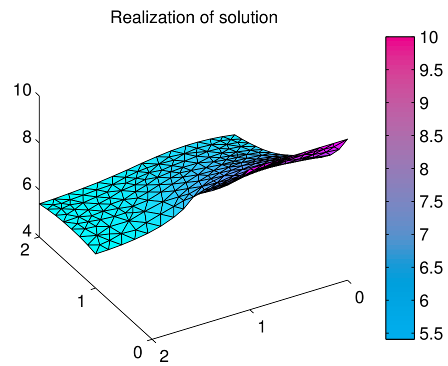
e.g.: $\bar{u} = \mathbb{E}\left(u\right)$, or $\mathrm{var}_u = \mathbb{E}\left((\tilde{u})^2\right)$, where $\tilde{u} = u - \bar{u}$,
or $\mathbb{P}\{u \le u_0\} = \mathbb{P}(\{\omega \in \Omega | \, u(\omega) \le u_0\}) = \mathbb{E}\left(\chi_{\{u \le u_0\}}\right)$

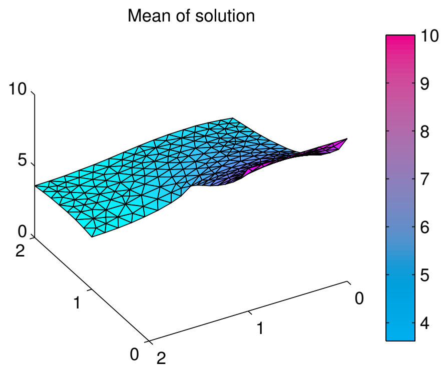
All desirables are usually expected values of some functional, to be computed via (high dimensional) integration over Ω.

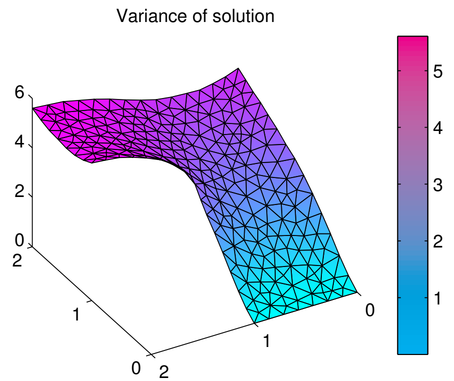
Example solution

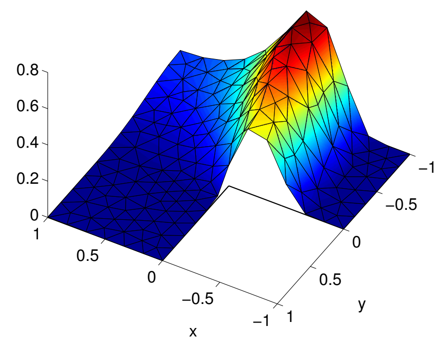








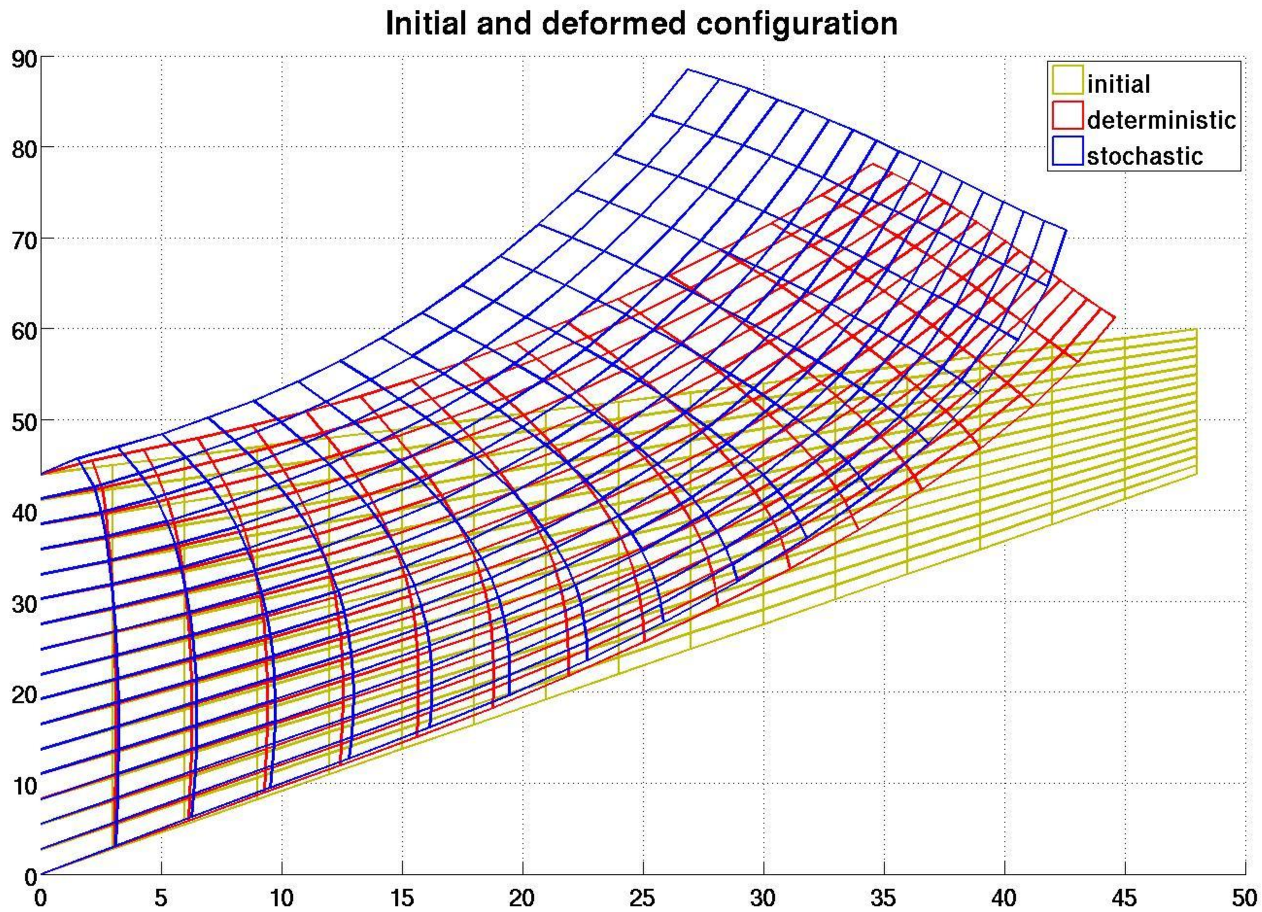




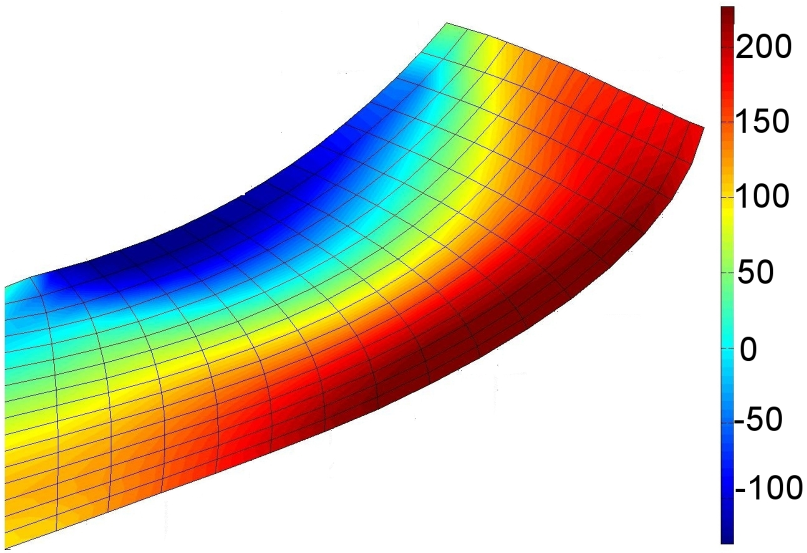
$\Pr\{u(x) > 8\}$

Example: Cook's membrane

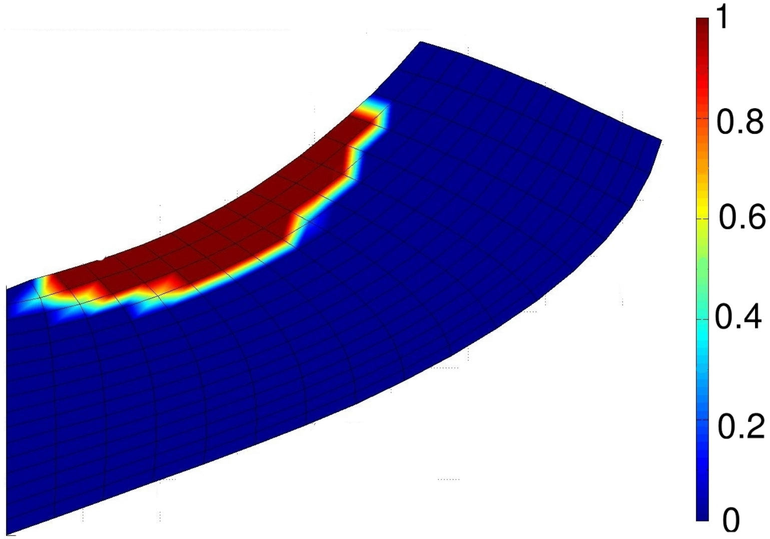
Large strain elasto-plasticity: uncertain shear modulus



Results Cook's membrane



a) $\mathbb{E}\left(\sigma_{\mathsf{VM}}\right)$

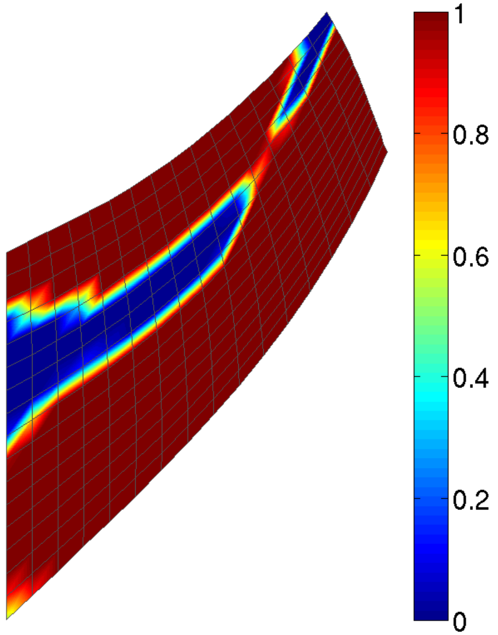


b) $\mathbb{P}(\{\sigma_{\mathsf{VM}} \leq 50\})$

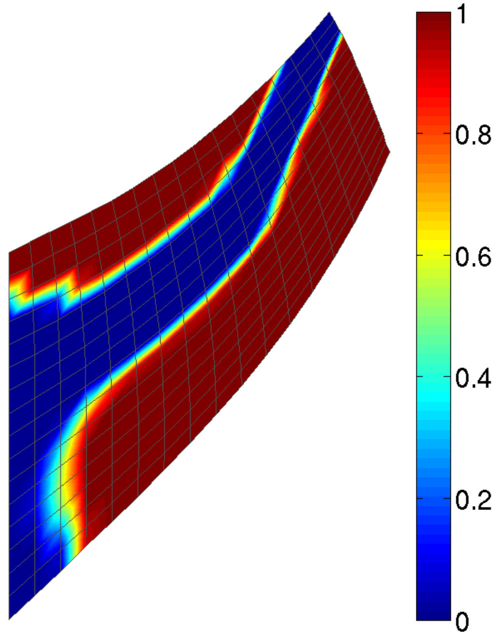
Shear modulus is uncertain (coefficient of variation 10%).
Material is a Saint Venant-Kirchhoff model.

Results Cook's membrane II

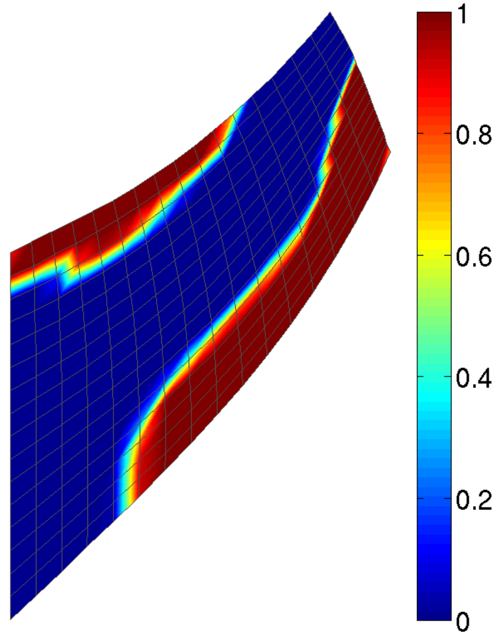
$\mathbb{P}(\{\sigma_{\mathsf{VM}} > 150\})$



$\mathbb{P}(\{\sigma_{\mathsf{VM}} > 200\})$

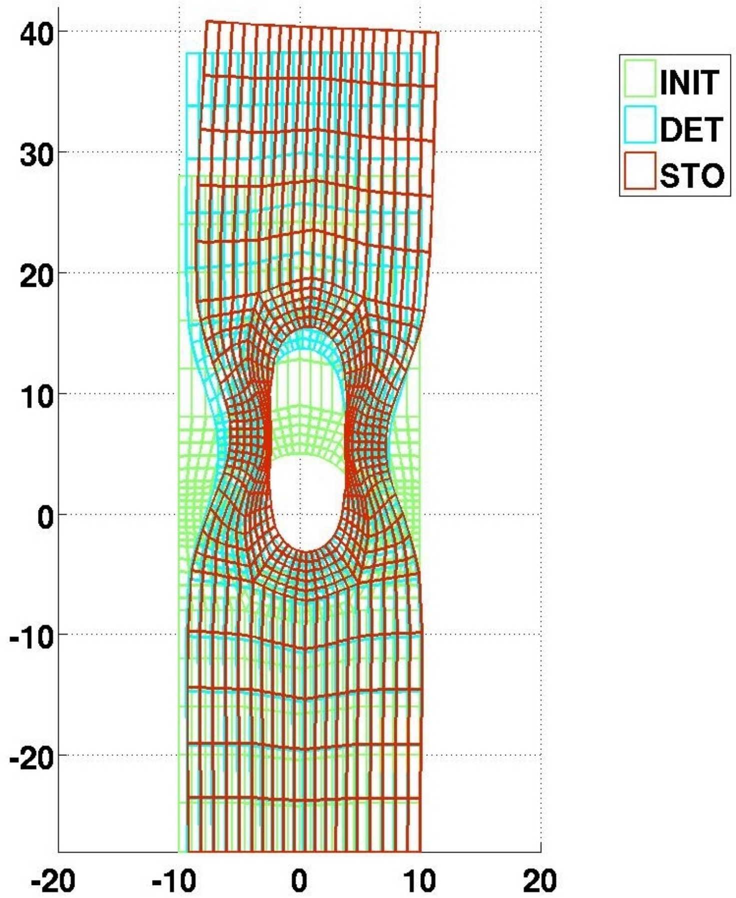


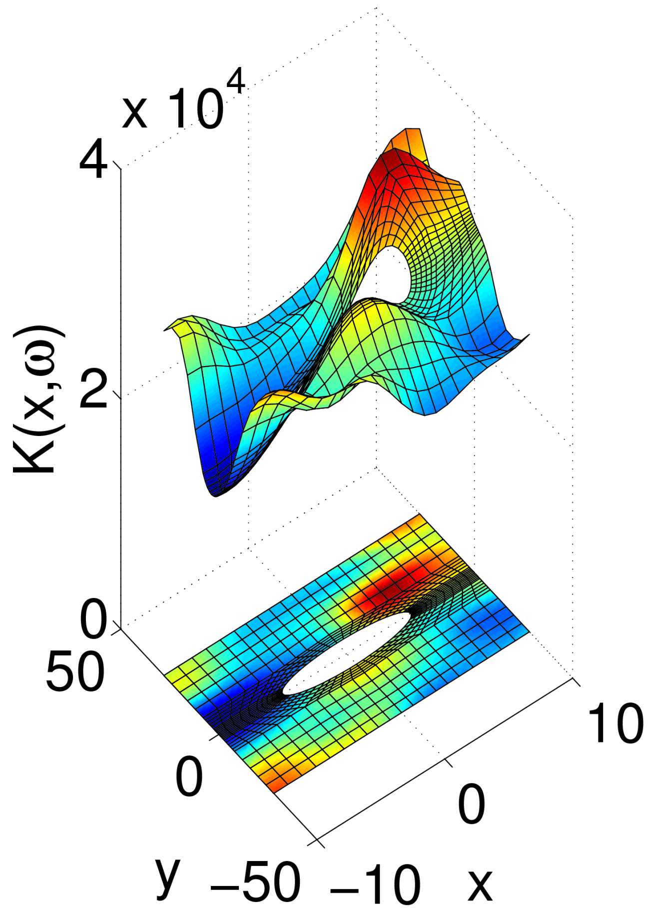
$\mathbb{P}(\{\sigma_{\mathsf{VM}} > 250\})$



Example: Plate with hole

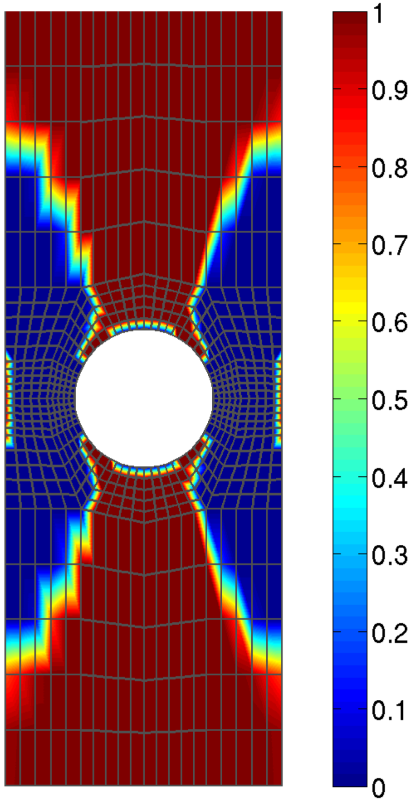
Large strain elasto-plasticity: uncertain bulk modulus



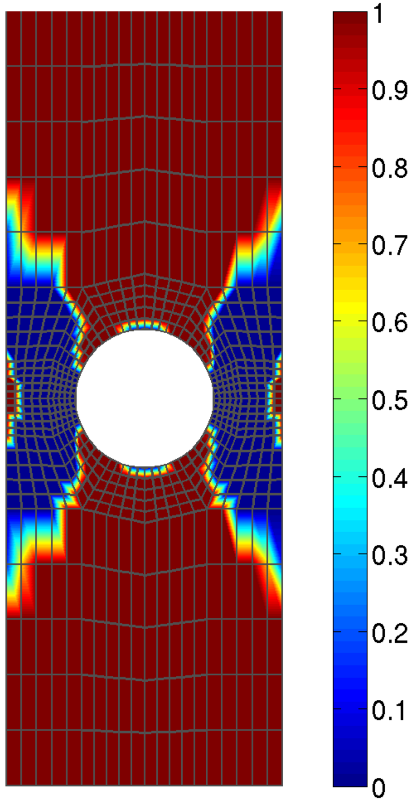


Results plate with hole II

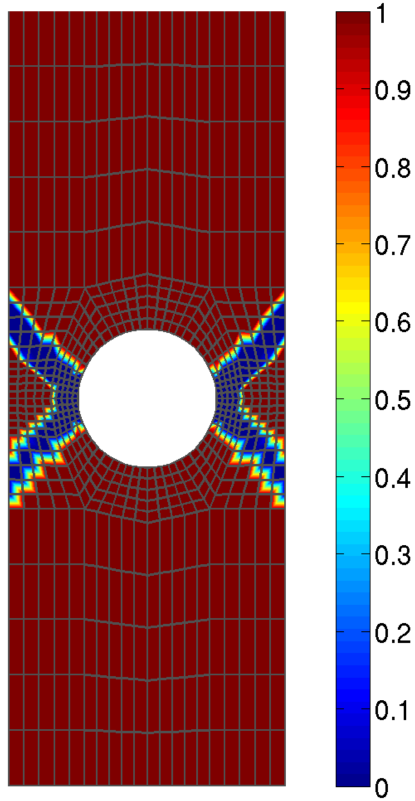
$\mathbb{P}(\{\sigma_{\mathsf{VM}} < 150\})$



$\mathbb{P}(\{\sigma_{\mathsf{VM}} < 200\})$



$\mathbb{P}(\{\sigma_{\mathsf{VM}} < 250\})$



General approaches to computation

Alternative Formulations / Approaches

- Moments: Derive equations for the moments of the quantities of interest (QoI) $\boldsymbol{\Psi}$. Usually Perturbation.
- Probablity distributions / densities: Derive equations for the probability densities of $u(x, \omega)$, e.g. Master-Equation, Fokker-Planck.
- Direct Integration: Compute desired QoI $\boldsymbol{\Psi}$ via direct integration over Ω—high dimensional (e.g. Monte Carlo, quasi Monte Carlo, Smolyak ($=$ sparse grids)).
- Direct Approximation: Compute an approximation to $u(x, \omega)$, use this to compute everything else (traditional response surface methods, surrogate models, stochastic Galerkin, stochastic collocation)

Random Variables vs Measures

The Fokker-Planck equation computes measures (densities).

The last two — direct — approaches deal directly with random variables.

Measures live — geometrically speaking — in the positive cone on the unit ball in the Banach space of bounded measures.

Extreme points of this convex set are Dirac-δ's.

The random variables in the direct approach live in vector spaces; upon discretisation, computation via linear algebra.

Computational requirements

- How to represent a stochastic process for computation, both simulation or otherwise?
- Best would be as some combination of countably many independent random variables (RVs).
- How to compute the required integrals or expectations numerically?
- Best would be to have probability measure as a product measure $\mathbb{P} = \mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_\ell$, then integrals can be computed as iterated one-dimensional integrals via Fubini's theorem,

$$\int_\Omega \Psi(\omega)\, \mathbb{P}(\mathrm{d}\omega) = \int_{\Omega_1} \ldots \int_{\Omega_\ell} \Psi(\omega_1, \ldots, \omega_\ell)\, \mathbb{P}_1(\mathrm{d}\omega_1) \ldots \mathbb{P}_\ell(\mathrm{d}\omega_\ell)$$

Stochastic discretisation of fields

- Connected with the decompositions of the covariance:
  kernel: $c_\kappa(x,y) := \mathbb{E}\left(\kappa(x,\cdot) \otimes \kappa(y,\cdot)\right)$
  operator: $C_\kappa : \phi \to \psi(x) = \int_{\mathcal{G}} c_\kappa(x,y)\phi(y)\,\mathrm{d}y$
- Best known is the spectral or eigen decomposition of $C_\kappa \phi_m = \lambda_m \phi_m$, leading to singular value decompostion (SVD) of int.op. assoc. with κ, a.k.a. the Karhunen-Loève expansion:

$$\kappa(x,\omega) = \bar{\kappa}(x) + \sum_m \sqrt{\lambda_m}\, \phi_m(x)\xi^{(m)}(\omega).$$

- Uncorrelated RVs $\xi^{(m)}(\omega)$ can be expanded in polynomials of independent Gaussian RVs $\theta_\ell(\omega) \Rightarrow$ PCE in Hermite polynomials H_α:

$$\xi^{(m)}(\omega) = \sum_\alpha \varkappa_m^{(\alpha)} H_\alpha(\theta_1(\omega), \ldots, \theta_\ell(\omega), \ldots)$$

- Integration then over independent Gaussian measures $\mathbb{P}_\ell = \Gamma_\ell$

Spectral representation

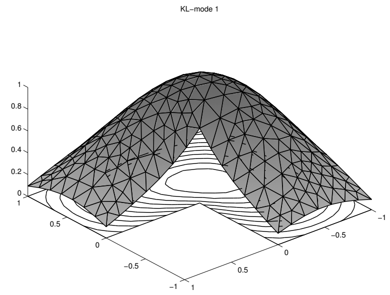
Although the Karhunen-Loève expansion relies on the spectral decomposition of C_κ, the name "spectral representation" is usually reserved for the special case where $c_\kappa(x, y) = c_\kappa(x + h, y + h)$ is invariant under translations (then $c_\kappa(x, y) = c(x - y)$); i.e. C_κ commutes with the translation operator, and the KLE eigenvalue equation becomes (e.g. with $\mathcal{G} = \mathbb{R}^d$) a convolution:

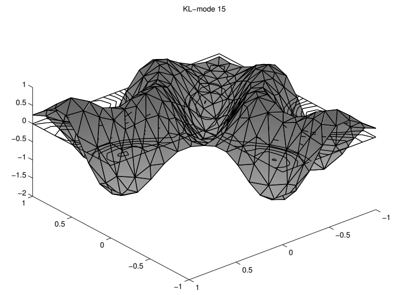
$$\int_{\mathcal{G}} c_\kappa(x, y)\phi(y)\,\mathrm{d}y = \int_{\mathcal{G}} c(x - y)\phi(y)\,\mathrm{d}y = \lambda\,\phi(x).$$

- Recall that commuting operators have same spectral resolution, and translation operator T_h satisfies $T_h \mathrm{e}^{\mathrm{i}k\cdot x} = \mathrm{e}^{\mathrm{i}k\cdot(x+h)} = \mathrm{e}^{\mathrm{i}k\cdot h}\,\mathrm{e}^{\mathrm{i}k\cdot x}$
- or recall that convolution equations are solved via Fourier transform

In any case, we have found the eigenfunctions $\mathrm{e}^{\mathrm{i}k\cdot x}$, eigenvalues are $\hat{c}(k)$ (FT of c), and as $c(h) = c(-h) \Rightarrow \hat{c}(k) = \hat{c}(-k)$ (same eigenvalue), $(\mathrm{e}^{\mathrm{i}k\cdot x}, \mathrm{e}^{-\mathrm{i}k\cdot x})$ combine to $(\cos(k \cdot x), \sin(k \cdot x))$.

Karhunen-Loève Expansion I

mode 1: 

mode 15: 

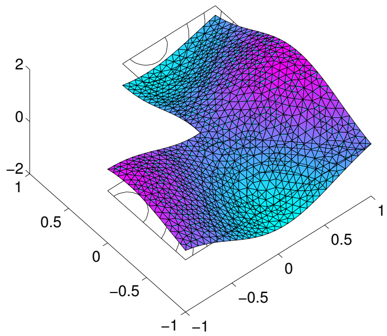
KLE: Other names: Proper Orthogonal Decomposition (POD), Singular Value Decomposition (SVD), Principal Component Analalysis (PCA): spectrum of $\{\lambda_j^2\} \subset \mathbb{R}_+$ and orthogonal KLE eigenfunctions $\phi_m(x)$:

$$\int_{\mathcal{G}} c_\kappa(x,y)\phi_m(y)\,dy = \lambda_m\,\phi_m(x) \quad \text{with} \quad \int_{\mathcal{G}} \phi_m(x)\phi_k(x)\,dx = \delta_{mk}.$$

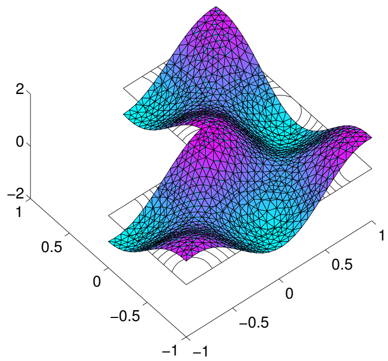
$\Rightarrow$ Mercer's representation of c_κ:

$$c_\kappa(x,y) = \sum_{m=1}^{\infty} \lambda_m\,\phi_m(x)\phi_m(y)$$

Karhunen-Loève Expansion II



mode 5



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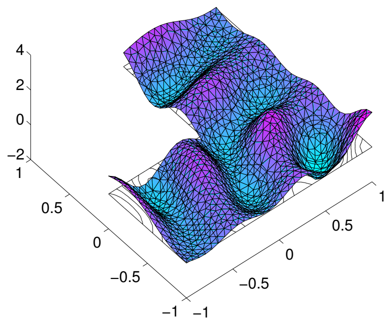


mode 25

Representation of κ (convergence in — basically L_2):

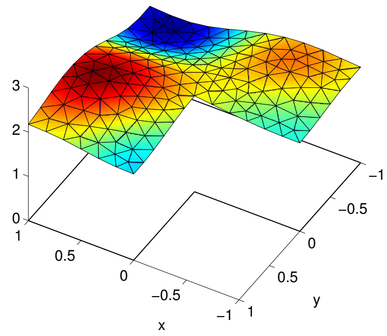
$$\kappa(x,\omega) = \bar{\kappa}(x) + \sum_{m=1}^{\infty} \lambda_m \, \phi_m(x)\xi_m(\omega) =: \sum_{m=0}^{\infty} \lambda_m \, \phi_m(x)\xi_m(\omega)$$

with centred, normalised, uncorrelated random variables $\xi_m(\omega)$:

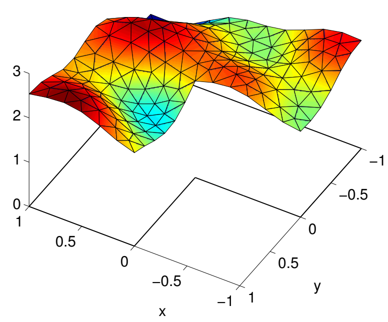
$$\mathbb{E}\left(\xi_m\right) = 0, \quad \mathbb{E}\left(\xi_m \xi_k\right) =: \langle \xi_m, \xi_k \rangle_{L_2(\Omega)} = \delta_{mk}.$$

Karhunen-Loève Expansion III

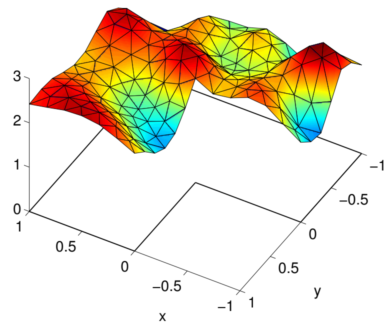
Realisation with:



6 modes



15 modes



40 modes

Truncate after M largest eigenvalues

$\Rightarrow$ optimal—in variance—expansion in M RVs.

First Summary

- Motivation, Probabiliy, aleatoric and epistemic Uncertainty
- Formulation as a well-posed problem
- RVs and Random Fields
- Karhunen-Loève Expansion — special case "spectral representation"
- Still open:
  - How to discretise RVs ?
  - How to actually compute $u(\omega)$?
  - How to perform integration ?

Polynomial Chaos Expansion in Gaussians

Each ξ_m from KLE may be expanded in plynomial chaos expansion (PCE) $\xi_m(\omega) = \sum_\alpha \xi_m^{(\alpha)} H_\alpha(\boldsymbol{\theta}(\omega))$, with orthogonal polynomials of independent Gaussian RVs $\{\theta_m(\omega)\}_{m=1}^\infty =: \boldsymbol{\theta}(\omega)$:

$$H_\alpha(\boldsymbol{\theta}(\omega)) = \prod_{\jmath=1}^{\infty} h_{\alpha_\jmath}(\theta_\jmath(\omega)),$$

where $h_\ell(\vartheta)$ are the usual Hermite polynomials, and

$$\mathcal{J} := \{\alpha \,|\, \alpha = (\alpha_1, \ldots, \alpha_\jmath, \ldots),\ \alpha_\jmath \in \mathbb{N}_0,\ |\alpha| := \sum_{\jmath=1}^{\infty} \alpha_\jmath < \infty\}$$

are multi-indices, where only finitely many of the $\alpha_\jmath$ are non-zero. Here $\langle H_\alpha, H_\beta \rangle_{L_2(\Omega)} = \mathbb{E}\,(H_\alpha H_\beta) = \alpha!\,\delta_{\alpha\beta}$, where $\alpha! := \prod_{\jmath=1}^{\infty}(\alpha_\jmath!)$.

Functions of Simpler RVs

What kind of simpler RVs ?
What kind of functions? — Usually polynomials or other algebras.

- Gaussian RVs —classical Wiener Chaos
- Poissonian RVs —discrete Poisson Chaos
- other RVs, e.g. uniform, exponential, Gamma, Beta, etc.
  This is called generalised Polynomial Chaos (gPC).

Best is to use orthogonal polynomials w.r.t. relevant measure, i.e. Hermite polynomials for Gaussian RVs, Charlier polynomials for Poisson RVs, Legendre polynomials for uniform RVs, Laguerre polynomials for exponential RVs, etc. $\Rightarrow$ Askey scheme.

Why White Noise Analysis?

Comes from directly constructing Ω as (a subset of) $\mathcal{S}'(\mathcal{G})$ (tempered distributions) with a Gaussian or Poissonian measure $\mathbb{P}$
$\Rightarrow$ Gaussian or Poissonian white noise.

Elements from $\mathcal{S}(\mathcal{G})$ (rapidly falling test functions) are then naturally Gaussian or Poissonian RVs.

Let $\mathfrak{F} = \mathfrak{F}(\{\xi_\jmath(\omega)\}_{\jmath=1,\dots,\infty})$ be the σ-algebra generated by $\xi_\jmath(\omega)$.
Want to approximate $L_2(\Omega, \mathfrak{F}, \mathbb{P}) \subseteq L_2(\Omega, \mathbb{P})$.

Density results: Polynomial algebra, algebra of exponentials, and algebra of trigonometric polynomials of Gaussian RVs is dense in $L_2(\Omega, \mathfrak{F}, \mathbb{P})$, polynomial algebra of Poissonian RVs is dense in $L_2(\Omega, \mathfrak{F}, \mathbb{P})$.

Choices

Stochastic discretisation can be performed at different solution stages:

- In $\boldsymbol{a}(w,u)$ on $\mathscr{W} = \mathcal{W} \otimes \mathcal{S}$, replace κ with its Karhunen-Loève expansion, giving $\boldsymbol{a}_{KLE}(w,u)$; truncate at L terms, giving $\boldsymbol{a}_L(w,u)$.
  Q: How does $\boldsymbol{a}_{KLE}(w,u)$ approximate $\boldsymbol{a}(w,u)$,
  how does $\boldsymbol{a}_L(w,u)$ approximate $\boldsymbol{a}_{KLE}(w,u)$?
  Is $u_{KLE} = u$, how does u_L converge to u_{KLE} or u?

  Then discretise $\mathscr{W}$ to $\mathscr{W}_{N,M} = \mathcal{W}_N \otimes \mathcal{S}_M$ by choosing a N-dimensional subspace $\mathcal{W}_N \subset \mathcal{W}$ and M-dimensional subspace $\mathcal{S}_M \subset \mathcal{S}$.

- Or first discretise $\mathscr{W}$ to $\mathscr{W}_{N,M}$, and then in $\boldsymbol{a}(w,u)$ on $\mathscr{W}_{N,M}$ replace κ by truncated KLE. Simpler, as $\mathscr{W}_{N,M}$ is finite dimensional.

Computational path

Principal Approach:

1. Discretise / approximate deterministic model (e.g. via finite elements, [your favourite method]), and approximate stochastic model (processes, fields) in finitely many independent random variables (RVs), $\Rightarrow$ stochastic discretisation.
2. Special case: Low variance $\Rightarrow$ perturbation.
3. Very special case: All linear, Gaussian $\Rightarrow$ analytic solution.
4. Direct:Compute QoI via integration over Ω—high dimensional (e.g. Monte Carlo, Quasi Monte Carlo, Smolyak (= sparse grids)).
5. Proxy: construction of approximate solution (functional / spectral approx., response surface) as function of known RVs $\Rightarrow$ e.g. polynomial chaos expansion (PCE).

Sketch of solution

- For the solution make ansatz: $u(x, \omega) = \sum_m \sum_\alpha \mathrm{u}_m^{(\alpha)} H_\alpha(\boldsymbol{\theta}) N_m(x)$, where $N_m(x)$ are FEM functions. $u(x, \omega)$ represented by tensor $\mathrm{u}_m^{(\alpha)}$.
- Solution $\mathrm{u}_m^{(\alpha)}$ by inserting ansatz into SPDE, and applying

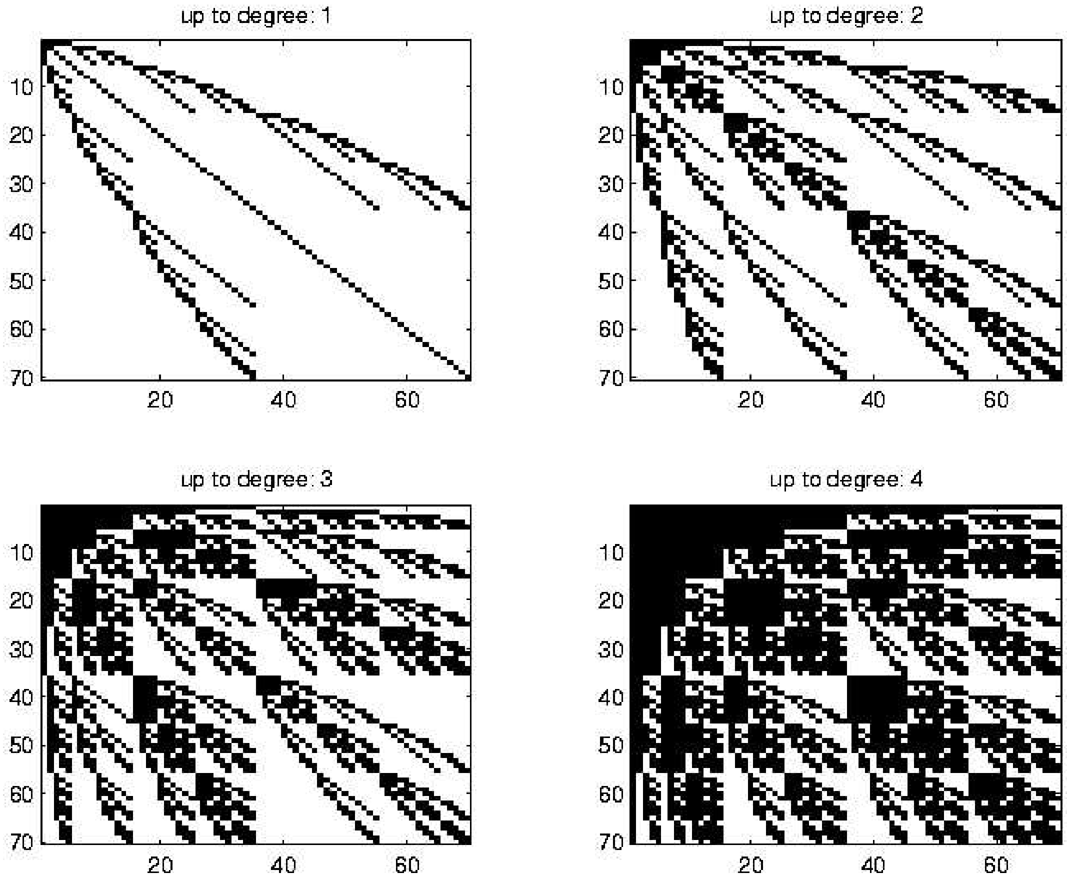
  Collocation $\Rightarrow$ Interpolation, i.e. solve SPDE on interpolation points ω_i — decoupled, non-intrusive solve.

  Projection: Simple as H_α are orthogonal. Compute projection inner product (integral) by quadrature, i.e. solve SPDE on quadrature points ω_n — decoupled, non-intrusive solve.

  Galerkin: Apply Galerkin weighting. Coupled equations, is it intrusive? When solved in a partitioned way, residua computed by quadrature, it is non-intrusive, needs only residua on quadrature points.

Galerkin matrix

Matlab spy picture of block-structure



Computational cost

We have to integrate over $\theta_1, \ldots, \theta_s$. For simplicity assume
$\mathscr{I} := \int_{[0,1]^s} f(\theta_1, \ldots, \theta_s) \approx \sum_{n=1}^N w_n f(\theta_{1,n}, \ldots, \theta_{s,n}) =: \mathscr{Q}$.

Q1: What is $\mathscr{E} = |\mathscr{I} - \mathscr{Q}|$ in relation to N and s?
Q2: How much does evaluation of $f(x_1, \ldots, x_s)$ cost?

A1: Deterministic quadrature rules can have very fast $\mathscr{E} \to 0$ as $N \to \infty$, these worst case bounds depend on regularity of f, but grow (often exponentially) as $s \to \infty$; e.g. for QMC: $\mathscr{E} = \mathcal{O}((\log N)^s)/N$.
Random(ised) quadratures (e.g. MC) can have $\mathscr{E}$ independent of s, e.g. for MC: std dev$(\mathscr{E}) = \sqrt{\text{var}(f)/N}$.
A2: One evaluation of $f(x_1, \ldots, x_s)$ costs at least $\mathcal{O}(s)$. For direct methods and spectral projection each evaluation is one PDE solve.
For Galerkin it is one residual evaluation.
QoI computation is cheap evaluation for all proxy methods.

Early references (incomplete)

Stochastic FEM: Belytschko, Liu; Kleiber; M., Bucher; Deodatis, Shinozuka; Der Kiureghian;, Kleiber, Hien; Ladevèze; Papadrakakis; Schuëller

Formulation of SPDEs: Babuška, Tempone, Nobile; Holden, Øksendal; Karniadakis, Xiu, Lucor; Lions; M., Keese; Rozanov; Roman, Sarkis; Schwab, Tudor; Zabaras

Spatial/temporal expansion of stochastic processes/ random fields: Adler; Grigoriu; Karhunen, Loève; Krée, Soize; Vanmarcke

White noise analysis/ polynomial chaos (PCE): Wiener; Cameron, Martin; Hida, Potthoff; Holden, Øksendal; Itô; Kondratiev; Malliavin; Galvis, Sarkis

Galerkin / collocation methods for SPDEs: Babuška, Tempone, Nobile; Benth, Gjerde; Cao; Eiermann, Ernst; Elman; Ghanem, Spanos; Galvis, Sarkis; Knio, Le Maître; Karniadakis, Xiu, Wan, Hesthaven, Lucor; M., Keese; Schwab, Tudor; Zabaras

Recent developments

- Try and keep a sparse (usually low-rank) tensor approximation troughout, from input fields to output solution.
- One possibility: Iterate (cheaply) on low-rank representation.
  $\Rightarrow$ Perturbed / truncated iterations.
- Build solution rank-one by rank-one, i.e. with already computed $u_R(x,\omega) = \sum_{r=1}^{R} w(x)_r\, \eta^{(r)}(\omega)$ alternate in $w(x)_{R+1}$ and $\eta^{(R+1)}(\omega)$:
  $$\min_{w_{R+1},\eta^{(R+1)}} \Phi(u_R(x,\omega) + w(x)_{R+1}\, \eta^{(R+1)}(\omega))$$
  $\Rightarrow$ successive rank-one updates (SR1U),
  proper generalised decomposition (PGD).
- This Galerkin procedure only solves "small" problems, good approximations often with small R.

Outlook

- Stochastic problems at very beginning (like FEM in the 1960's), when to choose which stochastic discretisation?
- Non-linear problems possible.
- Time dependend problems straight forward—Itō-integral via PCE
- Development of framework for stochastic coupling and parallelisation.
- Computational (low-rank) algorithms have to be further developed.
- Bayesian identification possible.