HARD-SPHERE GASES: DETERMINISTIC DYNAMICS WITH RANDOM INITIAL DATA

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SAPIENZA UNIVERSITÀ DI ROMA

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1. The large scale limit problem
    - Basic tools from statistical mechanics
    - A warm-up problem
2. Hard sphere dynamics
    - Hard sphere gas: law of large numbers
    - Hard sphere gas: CLT and large deviations
3. Long time results
    - Fluctuating hydrodynamics

Material.

* H. Spohn.
Large scale dynamics of interacting particles.
Texts and Monographs in Physics, Springer, Heidelberg, 1991.

* C. Cercignani, R. Illner, M. Pulvirenti.
The Mathematical Theory of Dilute Gases.
Applied Math. Sci. **106**, Springer-Verlag, New York, 1994.

* T. Bodineau, I. Gallagher, L. Saint-Raymond, S. Simonella.
Dynamics of dilute gases: a statistical approach.
Proceedings of the International Congress of Mathematicians 2022, **2**, 750-795, Ed. D. Beliaev and S. Smirnov, EMS Press, 2023.

1. The large scale limit

deterministic dynamics

$$+$$

random initial data

deterministic dynamics

$+$

random initial data

motivation : *macroscopic theory of matter based on first principles*

MICRO

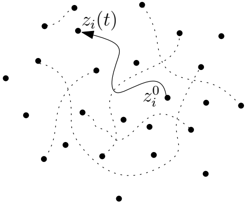
N particles (e.g. $N \sim 10^{23}$)

label $i = 1, 2, \cdots, N$

t : time

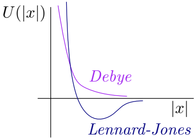
$z_i = (x_i, v_i) \in \mathbb{T}^d \times \mathbb{R}^d$: position, velocity of particle i

U : molecular potential (central)



deterministic dynamics: Newton's laws

$$\begin{cases} \frac{d}{dt}x_i = v_i \\ \frac{d}{dt}v_i = -\sum_{j \neq i} \nabla U\left(x_i - x_j\right) \qquad , \quad i = 1, \cdots, N \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \end{cases}$$



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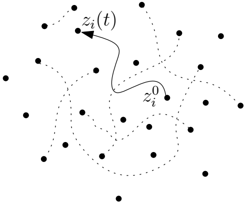

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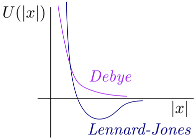
Flow : $(z_i^0)_i \xrightarrow[T_t]{} (z_i(t))_i$

Phase space : $(\mathbb{T}^d \times \mathbb{R}^d)^N$



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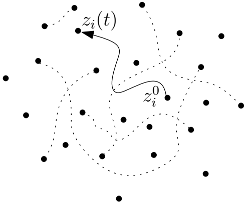

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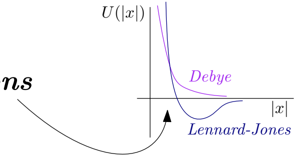
collisions



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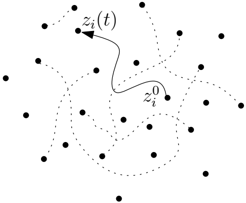

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- $T_{t_1+t_2} = T_{t_1} T_{t_2}$
- $dz_1 \cdots dz_N$ invariant
- $(x_i(-t), -v_i(-t))_i$ solution
- conservation of mass, momentum, energy

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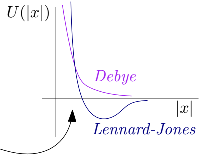

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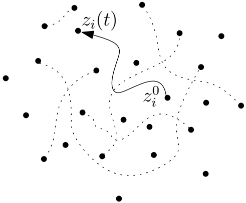

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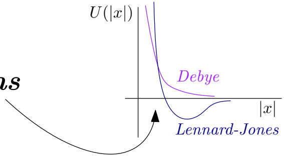

(no boundaries; no "$N = \infty$")

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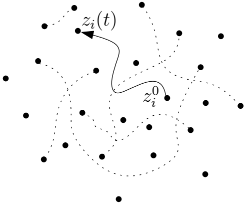

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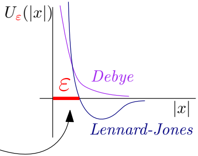
collisions

scale $\varepsilon > 0$



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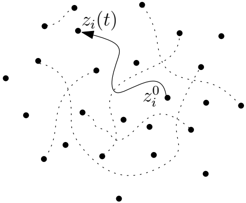

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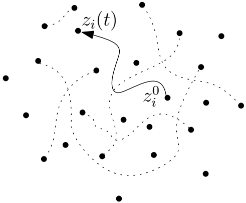

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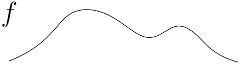
$N \longrightarrow \infty$

$\varepsilon \longrightarrow 0$

MACRO

f



$$\partial_t f \quad = \quad \mathcal{Q}(f)$$

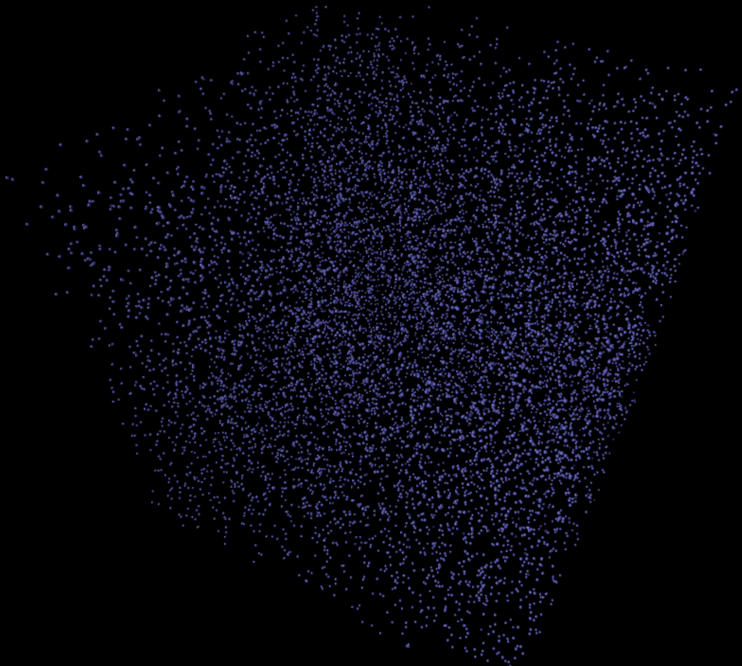
hydrodynamics

Euler, Navier-Stokes, heat...

(U coded in coefficients)

+ fluctuations

(revealing the coefficients)



⚠

observed macroscopic behaviour corresponds to “typical” configurations

⚠

not easy

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not easy ↓

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A *state* is a probability measure on the N particle phase space.

Initial measure : $\mathcal{W}_0$.

Evolution : $\mathcal{W}(t) = \mathcal{W}_0 \circ T_{-t}$.

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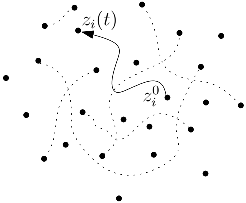

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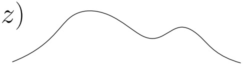
MACRO

$f = f(t, z)$



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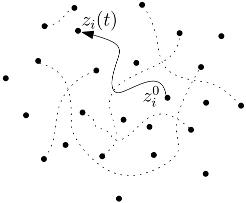

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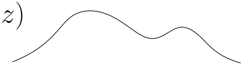

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(U coded in coefficients)

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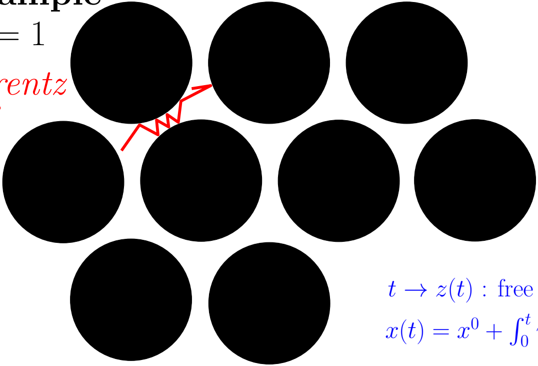
(Hilbert 1900)

Example

$N = 1$

Lorentz gas



Sinai billiard

$z = (x, v) \in \mathbb{R}^2 \times \mathbb{S}^1$

$t \to z(t)$: free flow + elastic reflection

$x(t) = x^0 + \int_0^t v(s)ds$ (finite horizon)

Initial data: uniform distribution over the basic cell and over $\mathbb{S}^1$

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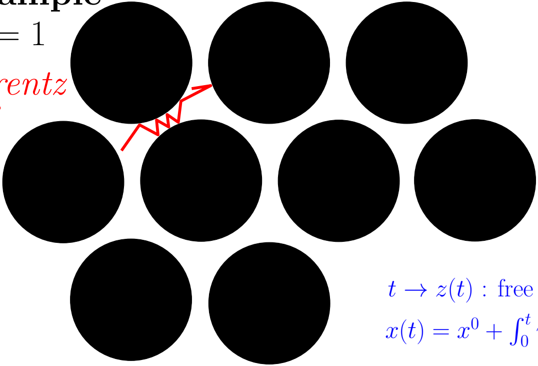

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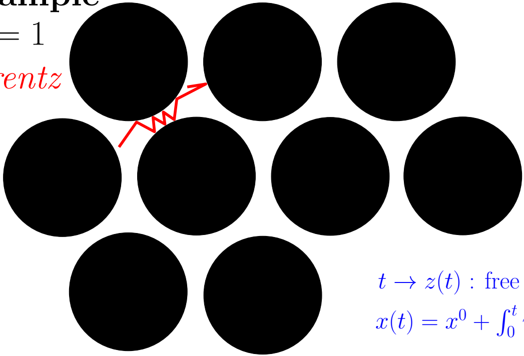

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D positive diffusion matrix, $b(t)$ standard Brownian motion.

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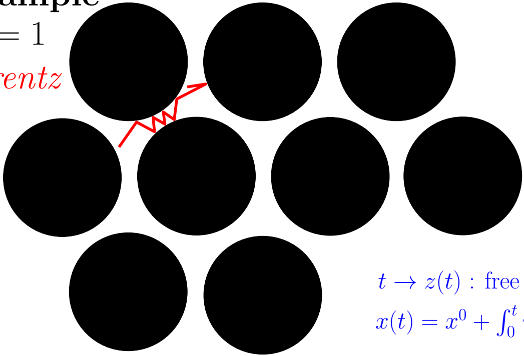

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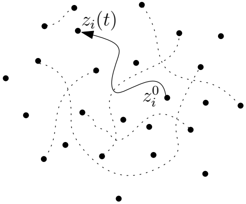
"I will not discuss derivations which include some external randomness in the dynamics by the Varadhan school. I will also not discuss the derivation of a diffusion equation for non-interacting particles moving among Sinai billiards. Those cases show what we could do if only our mathematics was better." [J. Lebowitz]

MICRO N particles



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$N \longrightarrow \infty$

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Kinetic Limit

$N\varepsilon^d \ll 1$ and/or $U_\varepsilon \ll 1$

+ random initial data

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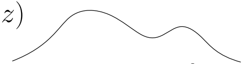
KIN

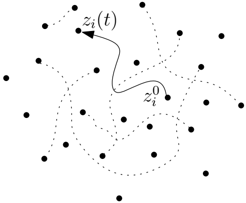
$f = f(t, z)$



$$f(t) \underset{t \to \infty}{\longrightarrow} e^{-\frac{v^2}{2}}$$

chaos propagation

MICRO N particles



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KIN

$$(\partial_t + v \cdot \nabla_x) f + \underbrace{F(x) \cdot \nabla_v f}_{\text{long-range mean-field}} = \underbrace{\mathcal{Q}(f,f)}_{\textit{collisions}}$$

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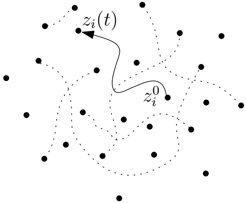
N particles



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$$(\partial_t + v \cdot \nabla_x) f \quad + \quad \underbrace{F(x) \cdot \nabla_v f}_{\text{long-range mean-field}} \quad = \quad \underbrace{\mathcal{Q}(f,f)}_{\textit{collisions}}$$

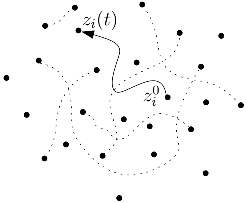
$\mathcal{Q}_B(f,f)(z) = \int B(v - v_*, \omega)\, [f' f'_* - f f_*]$

Boltzmann (gas dynamics)

$N\varepsilon^d \ll 1, \; U_\varepsilon(x) = U(x/\varepsilon)$

MICRO

N particles



label $i = 1, 2, \cdots, N$

t : time

$z_i = (x_i, v_i) \in \mathbb{T}^d \times \mathbb{R}^d$: position, velocity of particle i

U : molecular potential (central)

$$\begin{cases} \frac{d}{dt} x_i = v_i \\ \frac{d}{dt} v_i = -\sum_{j \neq i} \nabla U_\varepsilon (x_i - x_j) \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \leftarrow \text{“i.i.d.} \sim f^0(x,v)\text{”} \end{cases},$$

deterministic dynamics: Newton's laws

$$\text{Flow : } (z_i^0)_i \xrightarrow[T_t]{} (z_i(t))_i$$

+ random initial data

$(z_i(t))_i$ random variables

$N \longrightarrow \infty$

$\varepsilon \longrightarrow 0$

Kinetic Limit

$N\varepsilon^d \ll 1$ and/or $U_\varepsilon \ll 1$

KIN

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Landau (plasma physics, Balescu-Lenard)

$N\varepsilon^d \sim 1, \; U_\varepsilon(x) = \sqrt{\varepsilon}\, U(x/\varepsilon)$

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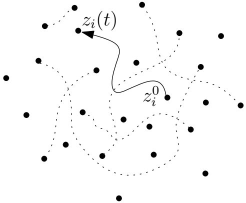

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(other: waves, quantum)

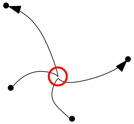
Kinetic Limit (in any fixed time interval $[0, t], t > 0$)

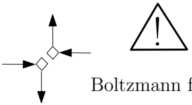
* *reversible → irreversible (time's arrow)*

* *memory (unbounded order) → Markov property*

* *N-particle interaction → nonlinearity (even small)*

dynamical instability





Boltzmann failure

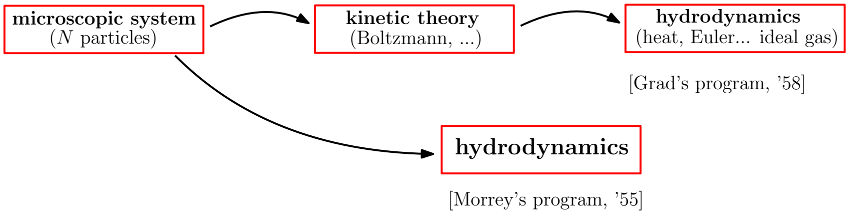
Kinetic Limit (in any fixed time interval $[0, t], t > 0$)

* *reversible* $\to$ *irreversible (time's arrow)*

* *memory (unbounded order)* $\to$ *Markov property*

* N*-particle interaction* $\to$ *nonlinearity (even small)*

Long times ($t \sim t_\varepsilon \to \infty$ gently)



1.1. Basic tools from statistical mechanics

Probability measures

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Grand canonical phase space : $\Omega = \bigcup_{n\geq 0} \Omega_n$, $\quad \Omega_n = (\mathbb{T}^d \times \mathbb{R}^d)^n$.

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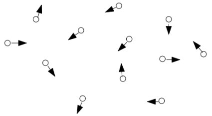
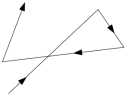
Low-density limit : $\varepsilon \to 0$, $\mathbb{E}_\varepsilon[N] \to \infty$, $\mathbb{E}_\varepsilon[N]\varepsilon^{d-1} \to 1$ $(\mathbb{E}_\varepsilon[N]\varepsilon^d \sim \varepsilon)$.





$$U_\varepsilon(x) = U\left(\frac{x}{\varepsilon}\right)$$

Correlation functions

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Correlation functions : $\{\rho_j^\varepsilon\}_{j\geq 0}$

$$\rho_j^\varepsilon : \Omega_j \to \mathbb{R}^+$$

$$\rho_j^\varepsilon(z_1, \cdots, z_j) = \sum_{k\geq 0} \frac{1}{k!} \int_{\Omega_k} W_{j+k}^\varepsilon(z_1, \cdots, z_{j+k}) dz_{j+1} \cdots dz_{j+k}$$

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$$W_j^\varepsilon(z_1, \cdots, z_j) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \int_{\Omega_k} \rho_{j+k}^\varepsilon(z_1, \cdots, z_{j+k}) dz_{j+1} \cdots dz_{j+k}$$

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$$\begin{cases} \rho_1^\varepsilon(z_1) = \mathbb{E}_\varepsilon[\sum_{i=1}^N \delta_{z_i^\varepsilon}(z_1)] \\ \\ \rho_2^\varepsilon(z_1,z_2) = \mathbb{E}_\varepsilon[\sum_{i_1\neq i_2} \delta_{z_{i_1}^\varepsilon}(z_1)\,\delta_{z_{i_2}^\varepsilon}(z_2)] \\ \\ \ldots \end{cases}$$

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$$\int_{\Omega_j} \rho_j^\varepsilon = \mathbb{E}_\varepsilon[N(N-1)\cdots(N-j+1)]\,.$$

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Rescaled correlation functions : $F_j^\varepsilon = \varepsilon^{j(d-1)} \rho_j^\varepsilon$ (BG limit : $\int_{\Omega_1} F_1^\varepsilon \to 1$).

Molecular chaos

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Assume that $F_j^\varepsilon \leq h^{\otimes j}$ for some suitable $h \in L^1(\Omega_1)$.

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We say that the state is chaotic (in the sense of molecular chaos) if there exists a smooth function $f : \Omega_1 \to \mathbb{R}^+$ such that, weakly as measures, for all $j \ge 1$,

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The particle state approximates f.

Empirical measure : $\pi^\varepsilon_{\text{emp}}(\varphi) = \varepsilon^{d-1} \sum_{i=1}^N \varphi(z_i^\varepsilon)$

$\pi^\varepsilon_{\text{emp}}(\varphi) \longrightarrow \int_{\Omega_1} f\varphi \quad \forall \varphi \in C_b^0(\Omega_1)$ in probability.

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However, this is **not** *possible...*

Quantitative chaos

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Moment generating function :

$$\mathbb{E}_\varepsilon \left[\exp\left(\pi^\varepsilon_{\text{emp}}(\varphi) \right) \right] = 1 + \sum_{j\geq 1} \frac{1}{j!} \int_{\Omega_j} \rho^\varepsilon_j \left(e^{\varepsilon^{d-1}\varphi} - 1 \right)^{\otimes j}$$

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Inverse formula: $F^\varepsilon_j(Z_j) = \sum_{k=1}^j \sum_{\sigma\in\mathscr{P}^k_j} \prod_{i=1}^k f^\varepsilon_{|\sigma_i|}\left(Z_{\sigma_i}\right)$

$Z_j = (z_1,\cdots,z_j),\quad Z_A = (z_\ell)_{\ell\in A},\quad \sigma = (\sigma_1,\cdots,\sigma_k),$

$\mathscr{P}^k_j$ the set of partitions of $\{1,\cdots,j\}$ in k parts.

Example

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"Maximally chaotic state" (MCS) :

$$\frac{1}{n!} W_n^\varepsilon(z_1, \cdots, z_n) := \frac{1}{\mathcal{Z}_\varepsilon} \frac{\varepsilon^{-n(d-1)}}{n!} f^{\otimes n}(z_1, \cdots, z_n)\, \psi_n^\varepsilon(x_1, \cdots, x_n)\,, \qquad n \geq 0$$

$\mathcal{Z}_\varepsilon$: partition function

$f : \Omega_1 \to \mathbb{R}^+$: smooth macroscopic density

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Exercises

Set $U = U_{\text{h.c.}}(x) = \begin{cases} \infty & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$.

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Remark : For $f(x, v) = M_\beta(v)$, Maxwellian with inverse temperature β, we recover the Gibbs state.

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Set $U = U_{\text{h.c.}}(x) = \begin{cases} \infty & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$.

(Q1) Show that the second cumulant $f_2^\varepsilon \to 0$ in L^1, in the BG limit.

(Q2) Is the convergence true in L^∞? Find a pointwise estimate for f_2^ε.

(Q3) Find an L^1 estimate on f_j^ε, $j \geq 2$.

Definition. The *fluctuation field* ζ^ε is

$$\zeta^\varepsilon(\varphi) = \frac{1}{\varepsilon}\left(\pi^\varepsilon_{\mathrm{emp}}(\varphi) - \mathbb{E}_\varepsilon\left[\pi^\varepsilon_{\mathrm{emp}}(\varphi)\right]\right)$$

Exercises

(Q1') Compute the covariance of the fluctuation field in terms of the first cumulants $f_1^\varepsilon, f_2^\varepsilon$

(Q2') Deduce that, for the hard-core MCS, ζ^ε converges to white noise.

(Q3') For which other potentials is this still true?

Dynamics

Dynamics

Newton's flow on Ω : $T_t^\varepsilon \quad (T_t^{(n,\varepsilon)})_n$

Potential : $U_\varepsilon(x) = U\left(\frac{x}{\varepsilon}\right)$

Initial state : $\mathscr{W}_0^\varepsilon$

Time-evolved state : $\mathscr{W}^\varepsilon(t) = \mathscr{W}_0^\varepsilon \circ T_{-t}^\varepsilon$

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$$\left(\partial_t + \sum_i v_i \cdot \nabla_{x_i} - \sum_{i\neq j} \nabla U_\varepsilon(x_i - x_j) \cdot \nabla_{v_i}\right) \rho_j^\varepsilon = \sum_i \int \nabla U_\varepsilon(x_i - x_{j+1}) \cdot \nabla_{v_i} \rho_{j+1}^\varepsilon dz_{j+1}$$

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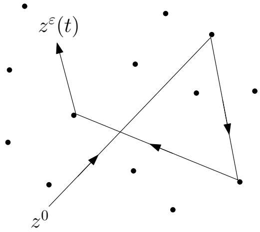
Cumulant Hierarchy : $\cdots$ [Ernst - Cohen '81]

1.2. A warm-up problem

$N = 1$

Lorentz gas at low density



$z = (x, v) \in \mathbb{T}^d \times \mathbb{S}^{d-1}\,, \quad d \geq 2$

$t \to z^\varepsilon(t)$: free flow + elastic reflection

Hard core scatterers, radius $\varepsilon > 0$

randomly distributed

Density of scatterers : $\varepsilon^{-(d-1)}$

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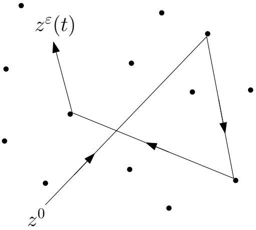

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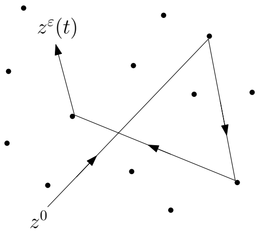

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Theorem. [Gallavotti '69, Spohn '78] $z^\varepsilon(t) \Rightarrow z(t)$

$z(t)$ Markov jump process with forward equation

$$(\partial_t + v \cdot \nabla_x)\, f = r(x) \int_{\mathbb{S}^{d-1}} [v \cdot \omega]_+ \{f(x, v - 2[v \cdot \omega]\omega) - f(x, v)\} d\omega$$

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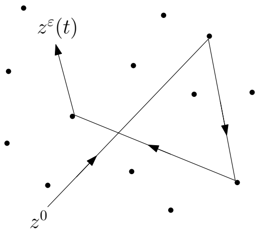

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$P\,(dt_1 d\omega_1 \cdots dt_k d\omega_k) = \frac{1}{Z} \prod_i r(x(t_i))[v(t_i^+) \cdot \omega_i]_+ dt_1 d\omega_1 \cdots dt_k d\omega_k$

on path space $\Lambda_t = \cup_{k \geq 0} \Lambda_{t,k}, \Lambda_{t,k} = \{(t_1, \omega_1, \cdots, t_k, \omega_k), 0 < t_1 < \cdots t_k < t, \omega_i \in \mathbb{S}^{d-1}\}$.

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$\Rightarrow$ for ε small, each scatterer is hit at most once in $\mathscr{A}$.

Step 2 : *parametrization*

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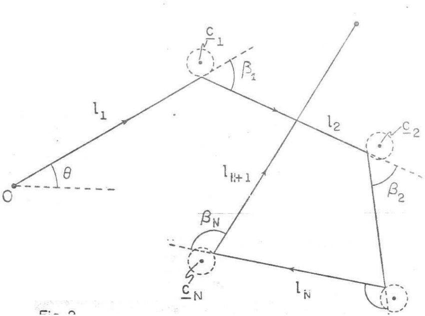

Lorentz path

Gallavotti '69: "The proof is based on several simple changes of variables..."

$$c_1, c_2, \cdots \longrightarrow t_1, \omega_1, t_2, \omega_2, \cdots \qquad \big((x^0, v^0)\ \text{fixed}\big)$$

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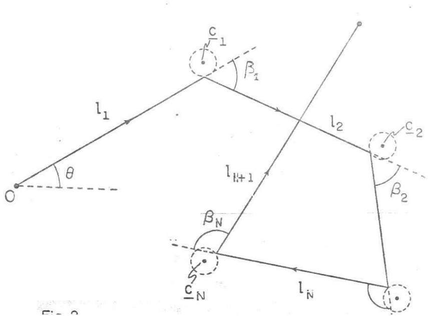

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Lemma. $dc_i = \varepsilon^{d-1} [v^\varepsilon(t_i^-) \cdot \omega_i]_- \, dt_i \, d\omega_i$

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For simplicity : $d = 2$, $\frac{W_n^\varepsilon}{n!} = \frac{1}{\mathcal{Z}_\varepsilon} e^{-\varepsilon^{-1}} \frac{\varepsilon^{-n}}{n!}$, $\mathcal{Z}_\varepsilon \simeq 1$, $n \geq 0$.

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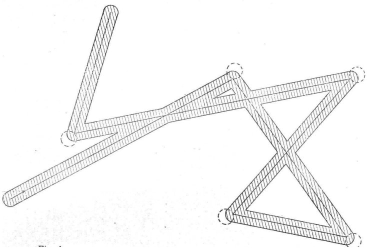

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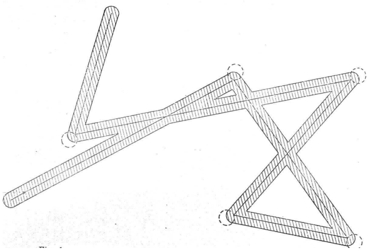

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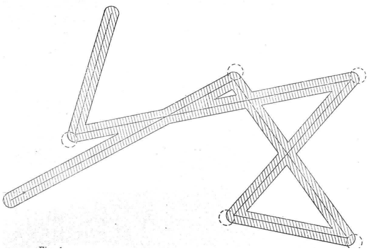

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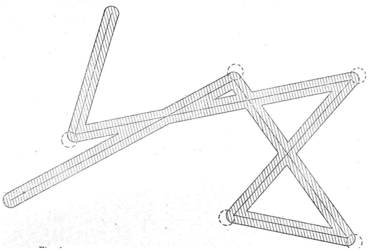

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□

Question.

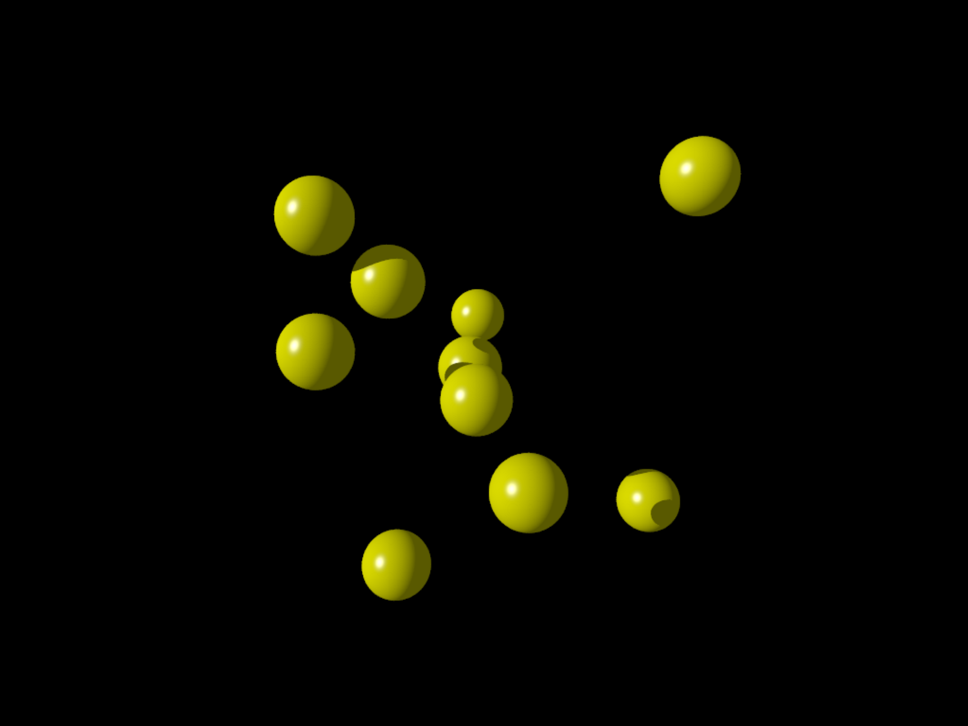
Replace $U = U_{\text{h.c.}}(x) = \begin{cases} \infty & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$

with $U = U_k(x) = \frac{1}{|x|^k}$, $k > 0$.

Is the theorem still valid?

2. Hard sphere dynamics

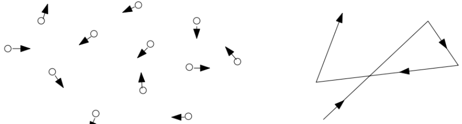
2.1. Hard sphere gas: law of large numbers



MICRO

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = 0 \end{cases} \qquad \Omega_n^{\varepsilon} := \left\{ |x_i - x_k| > \varepsilon \text{ for } i \neq k \right\} \subset \left(\mathbb{T}^d \times \mathbb{R}^d \right)^n$$

$$\begin{cases} v_i' = v_i - \omega[\omega \cdot (v_i - v_k)] \\ v_k' = v_k + \omega[\omega \cdot (v_i - v_k)] \end{cases} \quad \text{if} \quad x_k - x_i = \varepsilon\omega, \quad \omega \in S^{d-1}$$



scaling

N = number of spheres ; ε = sphere diameter

rate of coll. $\simeq \mathbb{E}_\varepsilon[N]\varepsilon^{d-1} \to 1$; 'volume' density $\simeq \mathbb{E}_\varepsilon[N]\varepsilon^d \sim \varepsilon$

$\varepsilon \to 0$: low density limit (Boltzmann-Grad limit)

Given n, ε, the hard-sphere flow $T_t^{(n,\varepsilon)} : \Omega_n^\varepsilon \to \Omega_n^\varepsilon$ is undefined for:

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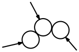
- multiple collisions : 

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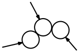
- multiple collisions : 
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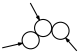
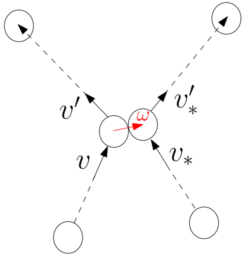
- multiple collisions : 
- accumulating collisions (infinite collisions in finite time)

Proposition. [Alexander '75, Vaserstein '79] The hard sphere flow $T_t^{(n,\varepsilon)}$ exists for all times t, almost everywhere in Ω_n^ε with respect to the Lebesgue measure. It admits the standard flow properties.



KIN

$$(v, v_*) \longrightarrow (v', v'_*)$$

$$v' = v - \omega[\omega \cdot (v - v_*)]$$
$$v'_* = v_* + \omega[\omega \cdot (v - v_*)]$$

$$v, v_* \sim \text{i.i.d.}$$

$f = f(t, x, v) \qquad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \qquad d \geq 2$

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f)$$

$$Q(f,f)(x,v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \Big[\omega \cdot (v - v_*)\Big]_+ \Big\{ f(x, v') f(x, v'_*) - f(x, v) f(x, v_*) \Big\} d\omega \, dv_*$$

$$f|_{t=0} = f_0$$

Properties : $\frac{d}{dt} \int (1, v, v^2) f(t, x, v) \, dx \, dv = 0, \qquad \frac{d}{dt} \int f \log f \, dx \, dv \leq 0.$

Theorem. [Lanford '75)] $(d = 3)$

Let the initial probability distribution of hard spheres $\left\{\frac{W_{0,n}^{\varepsilon}}{n!}\right\}$ over $\bigcup_n (\mathbb{T}^3 \times \mathbb{R}^3)^n$ ($W_{0,n}^{\varepsilon}$ supported in Ω_n^{ε}) satisfy the two assumptions:

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$$\pi^\varepsilon_{\mathrm{emp}}(t,\varphi)=\varepsilon^2\sum_{i=1}^N\varphi(z_i^\varepsilon(t))\longrightarrow\int_{\mathbb{T}^3\times\mathbb{R}^3}f(t)\varphi$$

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Note. Lanford used the hard-sphere BBGKY hierarchy (written by Cercignani in '72).

Remark. Topology encodes *irreversibility*.

$$\mathcal{B}_\varepsilon^{j\pm}(t) = \Big\{(z_1, \cdots, z_j) \in \Omega_j^\varepsilon, \qquad \exists s \in [0,t]\,, \exists i, j, \qquad |x_i - x_j \pm s(v_i - v_j)| \leq \varepsilon\Big\}$$

- $F_j^\varepsilon(0)$ **converges** on $\mathcal{B}_\varepsilon^{j+}(t)$ *collisions towards the future*
- $F_j^\varepsilon(t)$ does **not** converge on $\mathcal{B}_\varepsilon^{j-}(t)$ *collisions towards the past*

Elements of proof

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Proposition. The hard sphere MCS

$$\frac{W^\varepsilon_{0,n}}{n!} = \frac{1}{\mathcal{Z}_\varepsilon} \frac{\varepsilon^{-2n}}{n!} f_0^{\otimes n} \mathbf{1}_{\Omega^\varepsilon_n}, \qquad n \geq 0 \tag{HSMCS}$$

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For simplicity, we will assume the initial state is the HSMCS from now on.

Microscopic flow :

$$Z_N^0 = (z_1^0, \cdots, z_N^0) \to Z_N^\varepsilon(t) = (z_1^\varepsilon(t), \cdots, z_N^\varepsilon(t)) \,, \qquad t \in [0, T]$$

well defined (almost surely) for arbitrary T.

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(ii) A *dynamical cluster* is any maximal connected component of the neighbour relation (i).

(iii) A *cluster path* $\lambda = \lambda([0, T])$ is the trajectory in $[0, T]$ of dynamical cluster (ii).

$\mathscr{P}_A$ = partitions of A

$$\left(Z_N^0 \longrightarrow \left(Z_N^\varepsilon(t)\right)_{t\in[0,T]}\right) \implies \{\lambda_1, \lambda_2, \cdots\} \in \mathscr{P}_{\left(Z_N^\varepsilon(t)\right)_{t\in[0,T]}}$$

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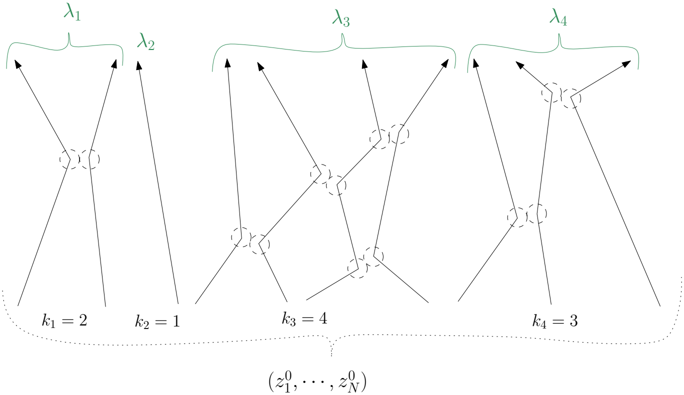

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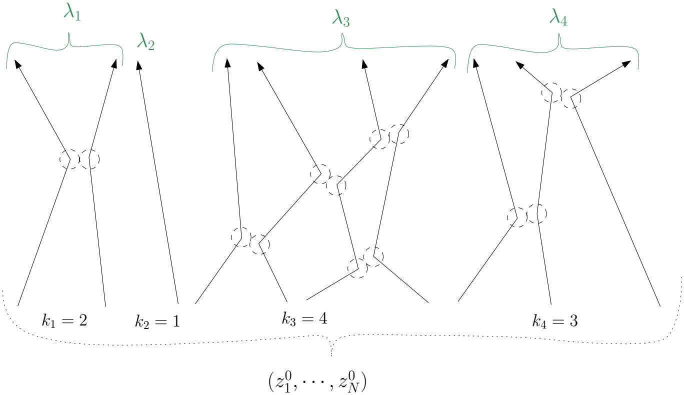
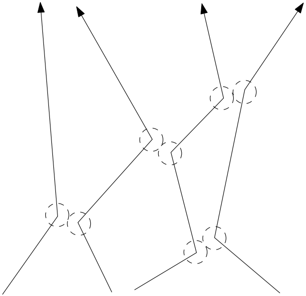


cluster path decomposition

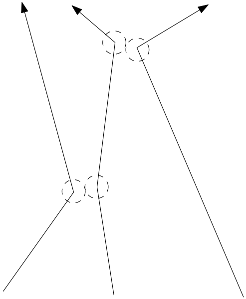
Empirical measure on cluster paths : $\pi^\varepsilon_{\text{emp}}(\lambda) = \varepsilon^2 \sum_i \delta_{\lambda_i([0,T])}(\lambda)$

<u>Step 1</u> : we can define a *bad set* as for the Lorentz gas.

A *good cluster path* is a path λ of size $|\lambda| = \ell$ with exactly $\ell - 1$ collisions.



bad



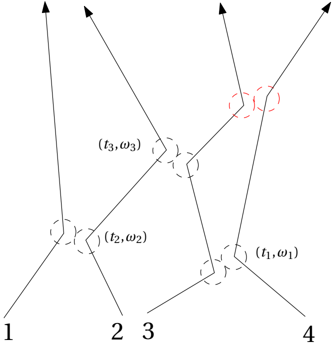
good

Step 2 : parametrization

$$\left(z_1^0,\cdots,z_\ell^0\right) \longrightarrow \lambda = \left(\mathcal{T}_\ell, x_{\rm cm}, V_\ell^0, t_1, \omega_1, \cdots, t_{\ell-1}, \omega_{\ell-1}\right)$$

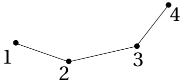
particles $\longrightarrow$ cluster paths

cluster path space : $\Lambda_T = \bigcup_{\ell\in\mathbb{N}} \{\mathcal{T}_\ell\} \times \mathbb{T}^3 \times \mathbb{R}^{3\ell} \times [0,T]^{\ell-1} \times \mathbb{S}^{2(\ell-1)}$



Ingredients describing the cluster path $\lambda = (z_1(t),\cdots,z_\ell(t))_{t\in[0,T]}$ of size ℓ :

- a tree graph $\mathcal{T}_\ell$ on ℓ vertices



- the center of mass at time zero $x_{\rm cm}$
- the collection of velocities V_ℓ^0
- impact times and angles $(t_i,\omega_i) \in [0,T] \times \mathbb{S}^2$

(ignoring recollisions)

$$dZ_\ell^0 = d\lambda\, \varepsilon^{2(\ell-1)} \prod_{e=\{\alpha,\beta\}\in E(\mathcal{T}_\ell)} \left[\omega_e \cdot \left(v_\alpha(t_e) - v_\beta(t_e)\right)\right]_+$$

Lemma. Let λ be a good cluster path. Then

$$\mathbb{E}_\varepsilon\left[\pi^\varepsilon_{\mathrm{emp}}(\lambda)\right] = \frac{1}{\tilde{\mathcal{Z}}_\varepsilon} \sum_{k\geq 0} \frac{\varepsilon^{-2k}}{k!} \int d\nu(\lambda_0)\cdots d\nu(\lambda_k)\, \delta_{\lambda_0}(\lambda) \prod_{\substack{i,j=0 \\ i\neq j}}^{k} \mathbf{1}_{\lambda_i \nsim \lambda_j}$$

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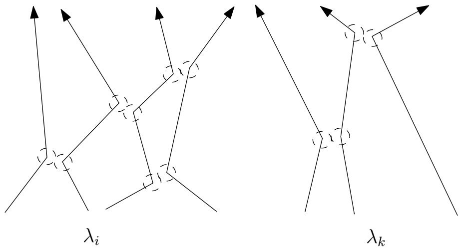
disconnection relation

$\lambda_i \nsim \lambda_k$



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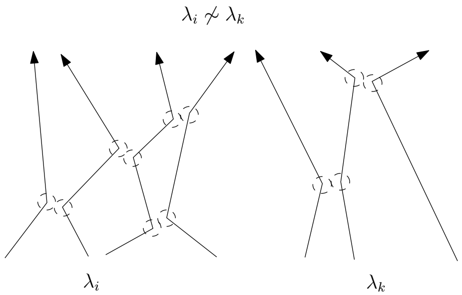

overlaps : expansion of constraints $\left(1-\mathbf{1}_{\lambda_i\sim\lambda_k}\right)$

Step 3 : *background*

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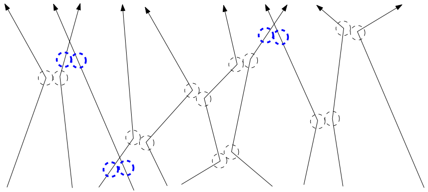

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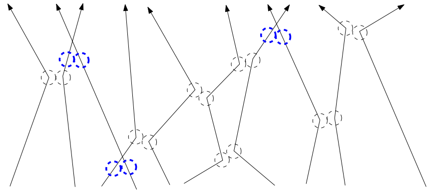
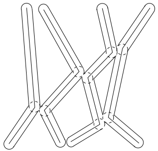


cluster tubes $\mathscr{T}^\varepsilon(\lambda)$



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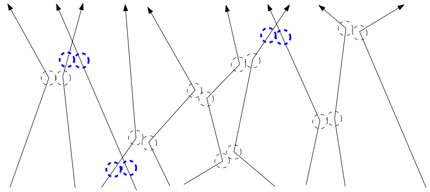

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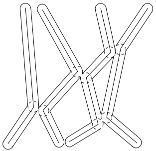

Question. How many effective terms in ψ^ε_k ?

H : smooth test function on the space of cluster paths.

(Rescaled) Cumulant gen. func. :

$$\mathscr{I}^{\varepsilon}(T,H) = \varepsilon^2 \log \mathbb{E}_{\varepsilon}\left[e^{\varepsilon^{-2}\pi^{\varepsilon}_{\text{emp}}(H)}\right] = \varepsilon^2 \log \mathbb{E}_{\varepsilon}\left[e^{\sum_i H(\lambda_i([0,T]))}\right] \qquad \text{(CGF)}$$

Remark : $\mathbb{E}_{\varepsilon}\left[\pi^{\varepsilon}_{\text{emp}}(\lambda)\right] = \partial_u \mathscr{I}^{\varepsilon}(T,uH)\Big|_{u=0}$.

Remind : $\mathscr{I}^{\varepsilon}(T,H) = \varepsilon^2 \sum_{j\geq 1} \frac{\varepsilon^{-2j}}{j!} \int_{\Omega_j} \tilde{f}^{\varepsilon}_j \left(e^H - 1\right)^{\otimes j}$ ($\tilde{f}^{\varepsilon}_j$ cumulant of $\pi^{\varepsilon}_{\text{emp}}(\lambda)$)

Proposition. [Bodineau - Gallagher - Saint-Raymond - S. '22]

There exists $T > 0$ and a constant C (depending only on $f_0, \|H\|_\infty$) such that

$$\|\tilde{f}^{\varepsilon}_j\|_1 \leq (CT)^j \; j^{j-2}\, \varepsilon^{2(j-1)}$$

for ε small enough and all $j \geq 1$, and the expansion of the CGF converges absolutely.

Further consequences.

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Theorem. *The emp. meas. on cluster paths in* $[0,t]$ *converges:*

$$\forall \delta > 0\,, \qquad \mathbb{P}_\varepsilon\left(\left|\pi^\varepsilon_{\mathrm{emp}}(H) - \int_{\Lambda_t} d\Upsilon(t,\lambda) H(\lambda)\right| > \delta\right) \xrightarrow[\varepsilon\to 0]{} 0\,, \qquad t \in [0,T)$$

$\forall H$ *smooth on* Λ_t*, where* $\Upsilon(t)$ *solves the Boltzmann-cluster equation:*

$$\begin{cases} \int d\Upsilon(t,\lambda) H(\lambda) = \int d\Upsilon(0,\lambda) H(\lambda) \\ \qquad + \frac{1}{2} \int d\tau\, d\omega\, d\Upsilon(\tau,\lambda)\, d\Upsilon(\tau,\lambda_*) \sum_{\substack{i\in\lambda \\ j\in\lambda_*}} \Big[(v_i(\tau) - v_j(\tau))\cdot\omega\Big]_+ \delta_{x_i(\tau)}(x_j(\tau)) \\ \qquad\qquad \times \Big\{ H\Big([\lambda \wedge \lambda_*]^{i,j,\tau,\omega}\Big) - H(\lambda) - H(\lambda_*)\Big\} \\ \Upsilon|_{t=0} = f_0(x_{\mathrm{cm}}, v_1^0)\,\delta_{\ell=1}\,, \quad \Upsilon|_{t<t_i,\ell>1} = 0 \end{cases}$$

where $[\lambda \wedge \lambda_*]^{i,j,\tau,\omega}$ *is the* merging *of the clusters due to a binary collision between particles* i *and* j *at time* τ *with scattering angle* ω*.*

Lemma.

Define

$$f(t,z) := \int d\Upsilon(t,\lambda)\,\ell\,\delta_z(z_1(t))$$

For times short enough, f solves the Boltzmann eq. with initial datum f_0.

Lemma.

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For times short enough, f solves the Boltzmann eq. with initial datum f_0.

Corollary.

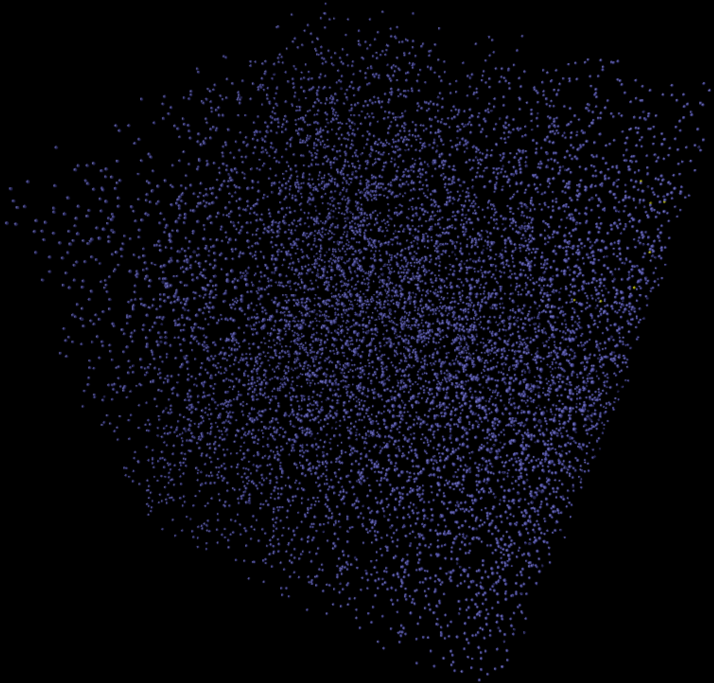
Let

$$R(t,x,v) := \int f(t,x,v_*)\, [\omega\cdot(v-v_*)]_+\, d\omega dv_*$$

be the free-flight time rate. For times short enough, the Boltzmann-cluster distribution is given by ($t > \max_i t_i$)

$$\Upsilon(t,\lambda) = \frac{1}{\ell!}\prod_{i=1}^{\ell} e^{-\int_0^t R(s,z_i(s))ds} f_0(z_i(0)) \prod_{e=\{\alpha,\beta\}\in E(\mathcal{T}_\ell)} \left[\omega_e\cdot\left(v_\alpha(t_e)-v_\beta(t_e)\right)\right]_+$$

$\Big($ for the 2D Lorentz gas $e^{-2t} f_0(z_1(0)) \prod_{j=1}^{k}\left[\omega_j\cdot v_1(t_j)\right]$ $\Big)$



Toy model (Kac):
- no space dependence
- $[\omega \cdot (v - v_*)]_+ \longrightarrow |\mathbb{S}^2|^{-1}$

$$n(t,k) = \int d\Upsilon(t,\lambda)\,\ell\,\delta_{\ell=k} = \frac{k^{k-2}}{(k-1)!}\,t^{k-1}\,e^{-kt} \simeq \frac{\left(ete^{-t}\right)^k}{\sqrt{2\pi}\,t k^{3/2}}$$

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Hard spheres:

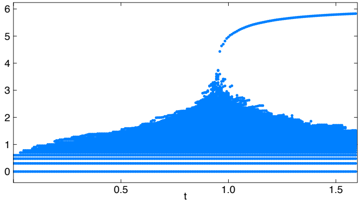


Fig. 5 Sizes of all clusters for a single realization of the system (logarithmic scale).

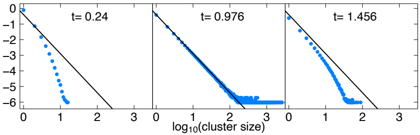


Fig. 6 Cluster size distribution before, at and after the gelation time (log-log scale). The solid lines show the power law with exponent $-\frac{5}{2}$.

3. Long time results

Long time results

1. **Dispersing cloud in $\mathbb{R}^d$.** [Illner - Pulvirenti '89, Denlinger '18]

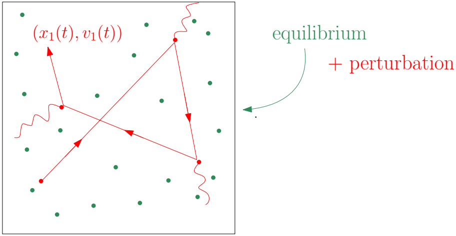
   Equation as in Theorem 0.

   [Deng - Hani - Ma preprint]

2. **Tracer particle.** [van Beijeren - Lanford - Lebowitz - Spohn '80, Bodineau - Gallagher - Saint-Raymond '16]

$$Q_{RB}(g)(x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[\omega\cdot(v-v_*)]_+ M(v_*)\left\{g(x,v') - g(x,v)\right\}$$

$$M(v) = (2\pi)^{-3/2}\exp\left(-v^2/2\right)$$



3. **Fluctuation field.** [Bodineau - Gallagher - Saint-Raymond - S. '24]