Estimation of generation and propagation of chaos via cumulant hierarchies

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jointly with

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WIAS 2025

Introduction and motivation

Recently, significant advance has been made towards the *rigorous derivation of wave kinetic theory*, related to wave turbulence, for the nonlinear Schrödinger equation (NLS):

- (*continuum NLS, subkinetic times*) T Buckmaster, P Germain, Z Hani, J Shatah, *Invent. math.* 225, 787–855 (2021)
- (*continuum NLS, kinetic times*) Y Deng and Z Hani [arXiv:2104.11204 and arXiv:2110.04565]
- (*NLS with stochastic forcing*) A Dymov, S Kuksin, A Maiocchi, S Vladuts [arXiv:2104.11967 and arXiv:2110.13873]
- Also recent contributions of other groups to closely related models: Staffilani & Tran, Ampatzoglou & Collot & Germain, etc.

In addition, we have techniques from *previous results*:

- (*Controlling oscillatory integrals in discrete NLS*) JL and H Spohn *Invent. Math.* 183 (2011) 79–188, and earlier work by Erdős, Salmhofer, Yau, etc
- (*Wick polynomials*) JL, M Marcozzi, A Nota, e.g., *J. Math. Phys.* 57 (2016) 083301
- (*Propagation of regularity by Grönwall-type inequalities*) Chong, Lafleche, Saffirio [arXiv:2103.10946], and earlier work by Pickl, Schlein, etc

The aim of this talk (joint with S Pirnes and A Vuoksenmaa):

How far can we go wish these ideas to control the cumulant hierarchy and understand the origins and accuracy of kinetic equations?

Spoiler: The story is now essentially complete for the stochastic Kac model (mean field model with fast "mixing"), but an important step is still missing from DNLS and weakly interacting bosons and fermions

Part I

Cumulant hierarchy and Wick polynomials

[JL and M. Marcozzi, *J. Math. Phys.* **57** (2016) 083301 (27pp)]

What is "chaos" in propagation of chaos of kinetic theory?

Chaos in kinetic theory

In most of the mathematical examples of derivation of a kinetic theory, starting from the rarefied gas Boltzmann equation, "chaos" refers to (approximate) **statistical independence** of some of the evolving quantities

To fix some terminology for this talk:

Propagation of chaos
: means that if the above "evolving quantities" are (sufficiently) independent in the beginning, they remain so at least up to kinetic time scales

Generation of chaos
: means that even if the above "evolving quantities" are not independent in the beginning, they will become so later, at least in approximation and on some time-scale which need not be connected with kinetic theory

Why study cumulants?

Observation: If y, z are independent random variables, we have

$$\mathbb{E}[y^n z^m] = \mathbb{E}[y^n]\mathbb{E}[z^m] \neq 0$$

whereas the corresponding cumulant is zero if $n, m \neq 0$

Consider a random lattice field $\psi(x)$, $x \in \mathbb{Z}^d$, which is (very) strongly mixing under lattice translations:

Assume that the fields in well separated regions become asymptotically independent as the separation grows.

- Then $\kappa[\psi(x), \psi(x+y_1), \ldots, \psi(x+y_{n-1})] \to 0$ as $|y_i| \to \infty$
- Not true for corresponding moments, e.g., $\mathbb{E}[|\psi(x)|^2 |\psi(x+y)|^2]$
- NB: $\kappa[\psi(x_1), \psi(x_2), \ldots, \psi(x_n)] = 0$ if *for any j* the random variable $\psi(x_j)$ is independent from the rest

Wick polynomials

Generating functions

$$g_t(\eta) := \ln g_{\mathrm{mom},t}(\eta), \quad g_{\mathrm{mom},t}(\eta) := \mathbb{E}[\mathrm{e}^{\eta\cdot\psi_t}]$$

Then with $\partial_\eta^J := \prod_{i\in J}\partial_{\eta_i}$, $y^J = \prod_{i\in J} y_i$,

$$\kappa[\psi_t(x)_J] = \partial_\eta^J g_t(0)\,, \quad \mathbb{E}[\psi_t(x)^J] = \partial_\eta^J g_{\mathrm{mom},t}(0)$$

$$G_{\mathrm{w}}(\psi_t,\eta) := \frac{\mathrm{e}^{\eta\cdot\psi_t}}{\mathbb{E}[\mathrm{e}^{\eta\cdot\psi_t}]} = \mathrm{e}^{\eta\cdot\psi_t - g_t(\eta)}$$

$$\Rightarrow \quad \partial_t\kappa[\psi_t(x)_J] = \partial_\eta^J\partial_t g_t(\eta)\big|_{\eta=0} = \partial_\eta^J\mathbb{E}[\eta\cdot\partial_t\psi_t\, G_{\mathrm{w}}(\psi_t,\eta)]\big|_{\eta=0}$$
$$= \sum_{\ell\in J}\mathbb{E}[\partial_t\psi_t(x_\ell)\,\partial_\eta^{J\setminus\ell}G_{\mathrm{w}}(\psi_t,0)]$$

$\partial_\eta^J G_{\mathrm{w}}(\psi_t,0) = {:}\psi_t(x)^J{:}$ are called Wick polynomials

- WP have been mainly used for Gaussian fields. They were introduced in quantum field theory where the unperturbed measure concerns Gaussian (free) fields

- Gaussian case has significant simplifications: If $C_{j'j} = \kappa[y_{j'}, y_j]$ denotes the *covariance matrix*,

$$G_{\mathrm{w}}(y, \eta) = \exp[\eta \cdot (y - \langle y \rangle) - \eta \cdot C\eta/2] \,.$$

  $\Rightarrow$ Cumulants of order ≥ 3 are *zero*,
  & Wick polynomials are *Hermite polynomials*

- The resulting orthogonality properties are used in the Wiener chaos expansion and Malliavin calculus

Wick polynomials and moments-to-cumulants formula

Truncated moments-to-cumulants formula

$$\mathbb{E}\left[y^{J'} :y^J:\right] = \sum_{\pi \in \mathcal{P}(J' \cup J)} \prod_{A \in \pi} \left(\kappa[y_A] \mathbb{1}_{\{A \not\subset J\}}\right) \tag{1}$$

- $:y^J:$ are μ-a.s. unique polynomials of order $|J|$ such that (1) holds for every J'

Multi-truncated moments-to-cumulants formula

Suppose $L \geq 1$ is given and consider a collection of $L+1$ index sequences J', J_ℓ, $\ell = 1, \ldots, L$. Then with $I = J' \cup (\cup_{\ell=1}^L J_\ell)$

$$\mathbb{E}\left[y^{J'} \prod_{\ell=1}^L :y^{J_\ell}:\right] = \sum_{\pi \in \mathcal{P}(I)} \prod_{A \in \pi} \left(\kappa[y_A] \mathbb{1}_{\{A \not\subset J_\ell, \forall \ell\}}\right)$$

Suppose that the evolution equation of the random variables $y_j(t)$ can be written in a form

$$\partial_t y_j(t) = \sum_I M_j^I(t) :y(t)^I:$$

Then the cumulants satisfy

$$\partial_t \kappa[y(t)_{I'}] = \sum_{\ell \in I'} \sum_I M_\ell^I(t) \mathbb{E}\big[:y(t)^I: :y(t)^{I' \setminus \ell}: \big]$$

where the *truncated moments-to-cumulants formula* implies

$$\mathbb{E}\big[:y(t)^I: :y(t)^{I' \setminus \ell}: \big] = \sum_{\pi \in \mathcal{P}(I \cup (I' \setminus \ell))} \prod_{A \in \pi} \Big(\kappa[y(t)_A] \mathbb{1}_{\{A \cap I \neq \emptyset,\ A \cap (I' \setminus \ell) \neq \emptyset\}} \Big)$$

$\Rightarrow$ evolution hierarchy for cumulants

Part II

Stochastic Kac model and its cumulant hierarchy

[JL, A Vuoksenmaa, arXiv:2407.17068]

Kac model

It was pointed out to us by C. Mouhot that there is a better controlled test case, simpler than NLS: **the stochastic Kac model with random velocity exchange**

- Toy model introduced by M. Kac in 1956 for deriving a Boltzmann equation
- N-particle system, where only velocities of the particles are tracked, and collisions between particles take place stochastically
- Originally velocities taken to be 3-dimensional ($3N$-dimensional phase space). Later analysis has also focused on the 1-dimensional case (N-dimensional phase space)

- System of N particles. Configurations are vectors $(v_i)_{i=1}^N$
- State is updated as follows: Randomly pick a pair of particles with indices (i,j) to collide. Pick a random collision angle $\theta \in 2\pi\mathbb{T}$ with some weight (say, uniform). Update velocities by *rotating*

$$v_i \mapsto \cos(\theta)v_i + \sin(\theta)v_j$$
$$v_j \mapsto \cos(\theta)v_j - \sin(\theta)v_i$$

After the update, the velocity vector is given by

$$R_{i,j}(\theta)v$$

where the diagonal and off-diagonal elements of the matrix $R_{i,j}(\theta)$ are

$$[R_{i,j}(\theta)]_{k,k} = \mathbb{1}_{\{i,j\neq k\}}1 + \mathbb{1}_{\{i,j=k\}}\cos(\theta)$$
$$[R_{i,j}(\theta)]_{k_1,k_2} = \mathbb{1}_{\{i=k_1,j=k_2\}}\sin(\theta) + \mathbb{1}_{\{i=k_2,j=k_1\}}(-\sin(\theta))$$

- Let $S^{N-1} := S^{N-1}(\sqrt{N}) = \{v \in \mathbb{R}^N : \|v\|_{\ell^2_N} = \sqrt{N}\}$
- If $v \in S^{N-1}$, then also $R_{i,j}(\theta)v \in S^{N-1}$
- *Markov transition operator* $Q = Q_N$ given by

$$Q\phi(v) = \frac{1}{N(N-1)} \sum_{i,j;i\neq j} \int_{-\pi}^{\pi} \phi(R_{i,j}(\theta)v) \frac{\mathrm{d}\theta}{2\pi}$$

- Start from a probability density f_0 on S^{N-1} with respect to the uniform probability measure μ_N on S^{N-1}. Evolve the density according to Kac's model, and denote the density at time t by f_t and the full measure by $F(t)$

- The collision times are Poisson, and let us scale time so that the collision rate is equal to N. Then the evolution of the density f_t is determined by the semigroup

$$f_t = S_t f_0 = \mathrm{e}^{tN(Q_N - I)} f_0$$

- Writing the time evolution in terms of the generator, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = N(Q_N - I) f_t$$

Known results

- Kac 1956: Showed that if the initial densities f_0^N are chaotic (comparing to a product state), also f_t^N are chaotic. Furthermore, the first marginal of f_t^N converges to f_t as $N \to \infty$, where f_t solves

$$\begin{cases} \partial_t f_t(v) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}} (f_t(v') f_t(w') - f_t(v) f_t(w))\, \mathrm{d}w \mathrm{d}\theta \\ f_0(v) = (\text{limit of marginal of } f_0^N) \end{cases}$$

  Here

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = R(\theta) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

- Evolution equation similar to the spatially homogeneous Boltzmann equation

Known results

- For each N, $f_t^N \mu_N \to \mu_N$, so the density equilibrates. Can this be quantified?
- Janvresse [*Ann. Probab.* 2001] proved that the operator Q_N has a *spectral gap*. This shows that

$$\|f_t^N - 1\|_{L^2(S^{N-1},\mu_N)} \leq \mathrm{e}^{-ct}\|f_0^N - 1\|_{L^2} \to 0$$

with the rate c uniform in N. Unfortunately, the prefactor depends on the L^2 size of the initial data which is often $\propto C_0^N$

$\Rightarrow$ Observable relaxation time estimated to be $\propto N$

- Carlen, Carvalho, Loss [*Acta Math.* 2003] (cf. also Maslen [*Math. Z.* 2003]) provided the spectral gaps of Q_N explicitly,

$$\Delta_N = \frac{1}{2}\frac{N+2}{N-1}$$

$$\Rightarrow \quad \|f_t^N - 1\|_{L^2} \leq \mathrm{e}^{-\frac{t}{2}}\|f_0^N - 1\|_{L^2}$$

- Carlen et. al [Kinetic & related models 2009] studied the *entropy production* of the Kac model. Showed that there is no universal lower bound for the entropy production (bad news for fast equilibriation).
- Einav [Kinetic & related models 2011] showed that entropy production behaves almost as badly as $\frac{1}{N}$.
- Mischler, Mouhot [*Invent. math.* 2013] prove several results concerning quantitative uniform in time propagation of chaos, propagation of entropic chaos, and quantitative estimates (independent of the number N of particles) on relaxation times

All of the previous results assume a permutation invariant (initial) state and so do we.

Evolution of observables

Instead the full probability density f_t^N, we may instead consider the correlation structure of the time-evolved random variables:

Let $\phi : S^{N-1} \to \mathbb{R}$ be an observable. Since Q_N preserves μ_N, the evolution is self-adjoint

$$\begin{aligned}\mathbb{E}_{F(t)}[\phi] &= \int_{S^{N-1}} \phi(v) f(t,v) \mu_N(\mathrm{d}v) \\ &= \int_{S^{N-1}} (\mathrm{e}^{tN(Q-I)}\phi(v)) f_0(v) \mu_N(\mathrm{d}v)\end{aligned}$$

- We can then consider evolution of, for instance, moments $\phi(v) := v^I$ for any index sequence I
- The moments will satisfy a “closed” hierarchy of evolution equations (the rate of change of moment of order n will only depend on moments of order $\leq n$)

Energy observables

Because of the conservation law $\sum_{i=1}^{N} v_i^2 = N$ a.s., it is easier to analyse the local energy variables.

Energies as random variables

For each particle i, its energy $e_i := v_i^2$ is a random variable whose cumulant hierarchy is closed, similarly to v_i before. At equilibrium and as $N \to \infty$, these should converge to i.i.d. χ^2 -random variables

How to measure "chaoticity" of a cumulant hierarchy? 21

Assumption (chaos bounds)

Let $c \geq 0$, $0 < \alpha < 1$, $N \geq 2$, and $n^* \geq 1$ be given. We say that $B > 0$ is a constant for the *chaos bound* of type α, c up to order n^* if for every $n \in [n^*]$ the joint cumulants are bounded as

$$|\kappa_0^{n,N}[e_r]| \leq B^{n-1}(n-1)! N^{c(n-1)-\alpha(\mathrm{len}(r)-1)}$$

- Here, r is a sequence of n labels ($|r| = n$)
- $\mathrm{len}(r) \leq n$ denotes the number of *different* energy labels in r
- Such a constant B can be found but B might need be very large for large N. The idea here is to find c, α such that B is "order one"
- We say $c = 0$ corresponds to a *chaotic state*
- In fact, for any symmetric state and α we can always satisfy the bounds with $c = 1$ and $B = 4$: namely,

$$|\kappa_0^{n,N}[e_r]| \leq 4^{n-1}(n-1)! N^{n-\mathrm{len}(r)}, \quad n \leq N/2 + 1$$

These bounds are saturated by symmetrizing deterministic data where one particle has all the energy

Let $c \geq 0$ and $\alpha \in (0,1)$. Let $n^* \in \mathbb{N}$ be a maximal order of cumulants. Then there exists a $N_0 = N_0(n^*, \alpha, c) \geq 2$ such that

Theorem (Generation of chaotic bounds)

Consider some fixed $N \geq N_0$ and some permutation invariant initial data F_0^N on $S^{N-1}(\sqrt{N})$. Denote the corresponding joint cumulants at order $n \in [n^*]$ and at time $t \geq 0$ by $\kappa_t^n[e_r]$, $|r| = n$. Assume that the initial data satisfies *chaos bound* of type α, c up to order n^* with a constant $B \geq 1$.

Then there exists a constant C, **depending only on** B, such that for all $n \leq n^*$, the time-evolved cumulants satisfy

$$|\kappa_t^{n,N}[e_r]| \leq \frac{1}{(N-1)^{\alpha(\mathrm{len}(r)-1)}} C^{n^2} n! \left(N^{c(n-1)} \mathrm{e}^{-\frac{1}{4}t} + 1 \right)$$

Convergence to equilibrium

Let c, α, n^*, N_0 be given as above. Similarly, assume $N \geq N_0$ and take some symmetric initial data F_0^N, whose cumulants are $\kappa_t^n[e_r]$.

Theorem (Equilibration)

Denote the stationary cumulants (corresponding to the uniform probability distribution on $S^{N-1}(\sqrt{N})$) by $\bar{\kappa}^n[e_r]$. Assume that the initial data satisfies *chaos bound* of type α, c up to order n^* with a constant B.

Then there exists a constant C, depending only on B, such that the time-evolved energy cumulants satisfy the following bound for all $|r| = n \leq n^*$ and $t \geq 0$

$$\left|\kappa_t^{n,N}[e_r] - \bar{\kappa}^{n,N}[e_r]\right| \leq \frac{1}{(N-1)^{\alpha(\mathrm{len}(r)-1)}} n! C^{n^2} N^{c(n-1)} \mathrm{e}^{-\frac{t}{4}} \tag{2}$$

- If the initial state is chaotic ($c = 0$), equilibration in time $O(1)$
- If the initial state is not chaotic ($c > 0$), need a time $\propto \ln N$ to equilibrate

Accuracy of kinetic theory

1. Take symmetrized initial data as above.
2. Wait until time t_0 such that the state is already chaotic: Choose $t_0 = 0$ if $c = 0$, and $t_0 = 4c(n^* - 1)\ln N$, otherwise
3. At time t_0, compute the first marginal $f_0(v)$ of $F^N_{t_0}$ and solve the Boltzmann–Kac equation (yielding $\nu_t := f_t(v)\mathrm{d}v$)

$$\partial_t f_t(v) = 2\int_{-\pi}^{\pi}\int_{\mathbb{R}} (f_t(v')f_t(w') - f_t(v)f_t(w))\,\mathrm{d}w\frac{\mathrm{d}\theta}{2\pi}$$

4. Construct product measures $\tilde{F}^N_T := \otimes_{i=1}^N \nu_T$ on $\mathbb{R}^N$

How close are the cumulants of $\tilde{F}^N_T$ to the "real" ones, of $F^N_{t_0+T}$?

Theorem (accuracy of the "Boltzmann–Kac" hierarchy)

$$|\kappa^{n,N}_{t_0+T}[e_r] - \tilde{\kappa}^{n,N}_T[e_r]| \le 2(N-1)^{-\alpha} C^{n^2} n! = O(N^{-\alpha})$$

Part III

Outline of the tools and proofs

[JL, A Vuoksenmaa, arXiv:2407.17068]

Kac model

Let $F_0^N \in \mathcal{P}_{\mathsf{sym}}(S^{N-1}(\sqrt{N}))$. Consider $F_t^N := S_t^N(F_0^N)$, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi, F_t^N \rangle = \langle N(Q_N - I)\phi, F_t^N \rangle, \quad \forall \phi \in C_b(S^{N-1})$$

$$(Q_N\phi)(v) = \frac{1}{N(N-1)} \sum_{i,j=1}^{N} \mathbb{1}_{\{i\neq j\}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \phi(R_{i,j}(\theta)v)$$

Here we focus on the case where

- $N \geq 2$ *is fixed but sufficiently large* and
- $t \geq 0$ is arbitrary (varied independently of N)

Energy variables and their chaos

We study the chaos and equilibration of the *kinetic energies*. Random variables on $(S^{N-1}(\sqrt{N}), F_t^N)$, defined by $e_i(v) = v_i^2$.

These satisfy the conservation law $\sum_{i=1}^{N} e_i = N$

Study their time-dependent joint cumulants:
$\kappa_t^N(e_I)$ of $\{e_i\}_{i=1}^N$. $I = (i_j)_{j=1}^n$, $i_j \in [N]$,

Goal is to show that energies will become chaotic when $t \to \infty$
$\Leftrightarrow$ mixed cumulants become small ($\ll_N 1$).

Partition classifiers

The measure F_t^N is symmetric.

$\Rightarrow$ Instead of sequences I of labels, the joint cumulants *of order n* can be indexed using *partition classifiers* in $\mathcal{C}_n$:

Multi-index $r \in \mathcal{C}_n \subset \mathbb{N}_0^n$ if and only if

1. $\sum_{\ell=1}^{n} r_\ell = n$, and
2. $r_\ell \geq r_{\ell+1}$

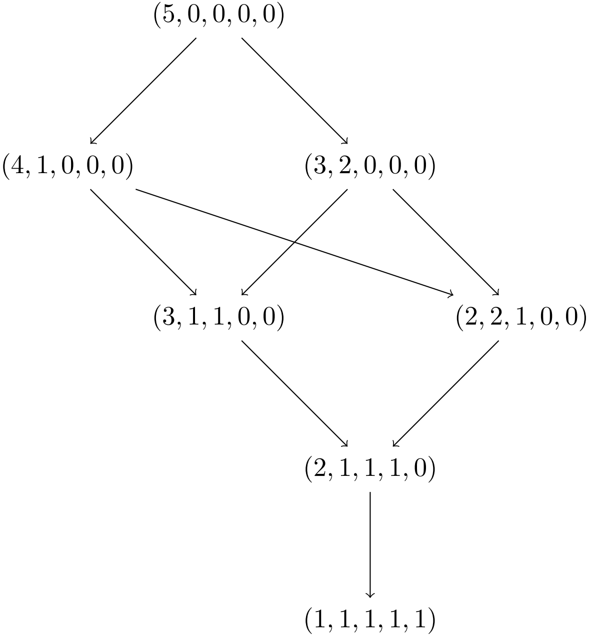
$$\kappa_t^N(e_1, e_1, e_1, e_1, e_1)$$

$$\kappa_t^N(e_1, e_1, e_1, e_1, e_2) \qquad \kappa_t^N(e_1, e_1, e_1, e_2, e_2)$$

$$\kappa_t^N(e_1, e_1, e_1, e_2, e_3) \qquad \kappa_t^N(e_1, e_1, e_2, e_2, e_3)$$

$$\kappa_t^N(e_1, e_1, e_2, e_3, e_4)$$

$$\kappa_t^N(e_1, e_2, e_3, e_4, e_5)$$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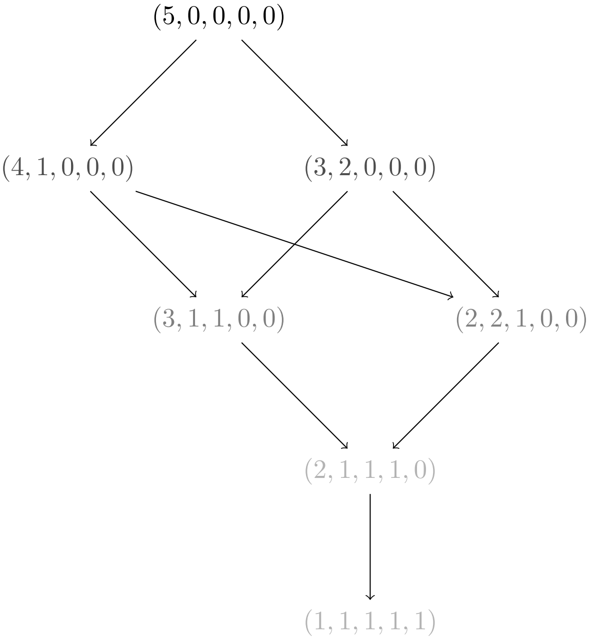
(4, 1, 0, 0, 0)
(3, 2, 0, 0, 0)
(3, 1, 1, 0, 0)
(2, 2, 1, 0, 0)
(2, 1, 1, 1, 0)
(1, 1, 1, 1, 1)


(5, 0, 0, 0, 0)
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(3, 2, 0, 0, 0)
(3, 1, 1, 0, 0)
(2, 2, 1, 0, 0)
(2, 1, 1, 1, 0)
(1, 1, 1, 1, 1)

Time-evolution of the joint cumulants

The time-evolution of the joint cumulants is given by $(n \geq 2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\kappa_t^{n,N})(r) = (L_{n,N}\kappa_t^{n,N})(r) + (\mathcal{N}_<^{n,N}[\kappa_t^{[\cdot],N}])(r)$$

The linear part splits into $L_{n,N} = L_n + R_{n,N}$.

The operator L_n couples to *more mixed cumulants* and generates exponentially decaying semigroup in the correct norm, $R_{n,N}$ is a perturbation.

The nonlinear term includes only cumulants of strictly lower order.

α-modulated chaos spaces

Let $\alpha \in (0, 1)$.

- Consider the vector of all joint energy cumulants of order n:

$$\kappa^{n,N} = (\kappa^{n,N}(e_r))_{r \in \mathcal{C}_n}$$

- And the space $X_\alpha = (\mathbb{R}^{\mathcal{C}_n}, \| \cdot \|_{\alpha,n,N})$, with the norm

$$\|\kappa^{n,N}\|_\alpha = \|\kappa^{n,N}\|_{\alpha,n,N} = \sup_{r \in \mathcal{C}_n} (N-1)^{\alpha(\mathrm{len}(r)-1)} |\kappa^{n,N}(e_r)|$$

- Now the task is to analyse $\|\kappa_t^{n,N}\|_\alpha$

Results

Theorem (generation of chaos)

Assume that we have a sequence of initial data $(F_0^N)_{N\geq 2}$. Assume the initial data has cumulants that satisfy

$$\|\kappa_0^{n,N}\|_\alpha \leq B^{n^2} N^{n-1}.$$

Then there exists a constant C, depending only on B, such that for every $n^* \in \mathbb{N}_1$, we have a threshold $N_0 = N_0(n^*, \alpha)$, such that for all $N \geq N_0$ and $n \leq n^*$, the time-evolved cumulants satisfy

$$\|\kappa_t^{n,N}\|_\alpha \leq C^{n^2} \left(N^{n-1} \mathrm{e}^{-\frac{1}{4}t} + 1\right). \tag{3}$$

Proof outline (generation of chaos)

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa_t^{n,N}(e_r) = \sum_{s\in\mathcal{C}_n} C_N(r,s)\kappa_t^{n,N}(e_s) + \mathcal{N}_{<n,N}[\kappa_t^{[\cdot],N}](e_r)$$

- Prove inductively on the order of cumulants n that the *totally nonrepeated energy cumulants* $\kappa_t^{n,N}(e_1,\ldots,e_n)$ are controlled nicely.
- For the rest of the cumulants prove inductively on the order n, that
  - The linear part minus the totally nonrepeated cumulants leads to exponential decay in the X_α-norm. $\Leftarrow$ Identify an explicitly dissipative term and show that the rest is its perturbation
  - With Duhamel's formula, the bounds propagate in the norm when the source term (nonlinear expression of lower order cumulants + nonrepeated cumulants) are taken into account.

Evolution of the energy cumulants

The energy cumulants evolve according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa_t^N(e_I) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_\eta^I \log \mathbb{E}_t[\mathsf{e}^{\eta\cdot e}]\right)|_{\eta=0}$$
$$= \left(\partial_\eta^I\left(\mathbb{E}_t[\mathsf{e}^{\eta\cdot e}]^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_t[\mathsf{e}^{\eta\cdot e}]\right)|_{\eta=0}\right)$$

The time-evolution of the mgf $\mathbb{E}_t[\mathsf{e}^{\eta\cdot e}]$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_t[\mathsf{e}^{\eta\cdot e}] = \frac{1}{N-1}\sum_{i\neq j}\int_{-\pi}^{\pi}\frac{\mathsf{d}\theta}{2\pi}\int \mathsf{e}^{\eta\cdot e}\left(\mathsf{e}^{\eta\cdot P_{i,j}^\theta(v)}-1\right)F_t^N(\mathsf{d}v)$$

Here $P_{i,j}^\theta(v)\in\mathbb{R}^N$, with $(P_{i,j}^\theta(v))_k = 0,$ for $k\neq i,j$, and

$$(P_{i,j}^\theta(v))_i = -\sin(\theta)^2 e_i + 2\cos(\theta)\sin(\theta)v_iv_j + \sin(\theta)^2 e_j$$
$$(P_{i,j}^\theta(v))_j = -\sin(\theta)^2 e_j - 2\cos(\theta)\sin(\theta)v_iv_j + \sin(\theta)^2 e_i$$

Evolution of the energy cumulants

$$\mathbb{E}_t[\mathrm{e}^{\eta\cdot e}]^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_t[\mathrm{e}^{\eta\cdot e}]$$
$$= \mathbb{E}_t[\mathrm{e}^{\eta\cdot e}]^{-1}\frac{1}{N-1}\sum_{i\neq j}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\int \mathrm{e}^{\eta\cdot e}\left(\mathrm{e}^{\eta\cdot P_{i,j}^{\theta}(v)}-1\right)F_t^N(\mathrm{d}v)$$

Recognizing that $\mathbb{E}_t[\mathrm{e}^{\eta\cdot e}]^{-1}\mathrm{e}^{\eta\cdot e} = G_w(\eta;e)$ is the *Wick polynomial generating function*, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa_t(e_I) = \frac{1}{N-1}\sum_{i\neq j}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\mathbb{E}_t[\partial_\eta^I\left(G_w(\eta;e)(\mathrm{e}^{\eta\cdot P_{i,j}^{\theta}(v)}-1)\right)|_{\eta=0}]$$

Evolution of the energy cumulants

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa_t(e_I) = \frac{1}{N-1}\sum_{i\neq j}\sum_{\emptyset\neq J\subseteq I}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\mathbb{E}_t[{:}e_{I\setminus J}{:}\,\partial_\eta^J\left(\mathrm{e}^{\eta\cdot P^\theta_{i,j}(v)}\right)|_{\eta=0}]$$

Those resulting $(P^\theta_{i,j}(v))^J$ terms which are odd in v are also odd in θ
$\Rightarrow$ *polynomial in* e

Since the expectations of products of Wick polynomials and monomials of random variables can be written in terms of cumulants as follows:

$$\mathbb{E}_t[{:}e_{I'}{:}\,e_{J'}] = \sum_{\pi\in\mathcal{P}(I'+J')}\left(\prod_{A\in\pi}\mathbb{1}_{\{A\cap J'\neq\emptyset\}}\kappa_t(e_A)\right)$$

we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa_t(e_I) = \sum_{|J|=|I|}C_N(I,J)\kappa_t(e_J) + \mathcal{N}_{<|I|}[\kappa_t](e_I)$$

Proof outline (linear part)

- In the vector form, we have

$$\kappa_t^{n,N} = \mathrm{e}^{tM_{n,N}}\kappa_0^{n,N} + \mathrm{e}^{tM_{n,N}}\int_0^t \mathrm{d}s\mathrm{e}^{-sM_{n,N}}\mathcal{N}_{<,n}[\kappa_s^{[\cdot],N}] \quad (4)$$

- The linear part splits into two: $M_{n,N} = M_n + R_{n,N}$, where M_n looks at *more mixed cumulants* and $R_{n,N}$ is an $O(N^{\alpha-1})$ perturbation in $\|\cdot\|_\alpha$.
- If follows that for sufficiently large $N(n^*,\alpha)$

$$\|\mathrm{e}^{tM_{n,N}}\|_\alpha \leq 10\mathrm{e}^{-\frac{1}{4}t}, \quad (5)$$

Proof outline (equilibration)

Let $h_t^{n,N}(e_r) := \kappa_t^{n,N}(e_r) - \bar{\kappa}^{n,N}(e_r)$. Note that we have for all $n \le n^*$ and $N \ge N_0(n^*, \alpha)$

$$\|\bar{\kappa}^{n,N}\|_\alpha \le B^{n^2} \tag{6}$$

for some constant B.

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t^{n,N}(e_r) = \sum_{s \in \mathcal{C}_n} \widetilde{L}_N(s,r) h_t^{n,N}(e_s) + \widetilde{\mathcal{N}}_{<n,N}[h_t^{[\cdot],N}](e_s)$$

- Similar to the generation of chaos but now the source terms contain $(\bar{\kappa}^{m,N})_{m<n}$.

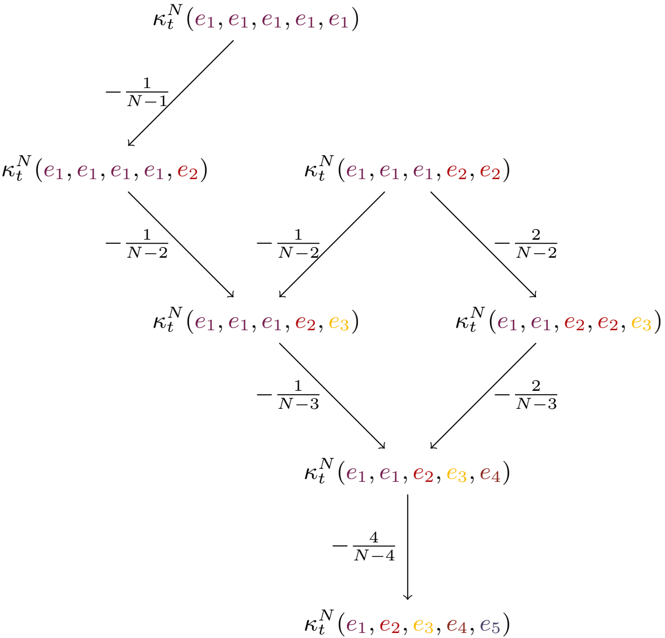
$$\kappa_t^N(e_1, e_1, e_1, e_1, e_1)$$

$$\kappa_t^N(e_1, e_1, e_1, e_1, e_2) \qquad \kappa_t^N(e_1, e_1, e_1, e_2, e_2)$$

$$\kappa_t^N(e_1, e_1, e_1, e_2, e_3) \qquad \kappa_t^N(e_1, e_1, e_2, e_2, e_3)$$

$$\kappa_t^N(e_1, e_1, e_2, e_3, e_4)$$

$$\kappa_t^N(e_1, e_2, e_3, e_4, e_5)$$


$\kappa_t^N(e_1, e_1, e_1, e_1, e_1)$
$-\frac{1}{N-1}$
$\kappa_t^N(e_1, e_1, e_1, e_1, e_2)$
$\kappa_t^N(e_1, e_1, e_1, e_2, e_2)$
$-\frac{1}{N-2}$
$-\frac{1}{N-2}$
$-\frac{2}{N-2}$
$\kappa_t^N(e_1, e_1, e_1, e_2, e_3)$
$\kappa_t^N(e_1, e_1, e_2, e_2, e_3)$
$-\frac{1}{N-3}$
$-\frac{2}{N-3}$
$\kappa_t^N(e_1, e_1, e_2, e_3, e_4)$
$-\frac{4}{N-4}$
$\kappa_t^N(e_1, e_2, e_3, e_4, e_5)$

Comments (DNLS)

Next step:

Complete the missing estimates of the proof...
(DNLS is still missing proper analysis of the semigroup generated by linear part)

Inhomogeneous initial data:

Does kinetic theory perform as well?
...with a Vlasov–Poisson term?

Proper setup and assumptions for the inhomogeneous case?

Properties of fluctuations around W_t?

Thank you for your attention!