# Bilinear Coagulation via Random Graphs

D. Heydecker, joint work with Robert Patterson

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# Motivation: Interaction Clusters in Kac Dynamics

# Starting Point: the Boltzmann-Grad Limit

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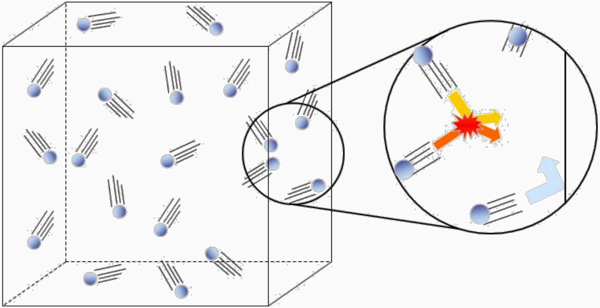

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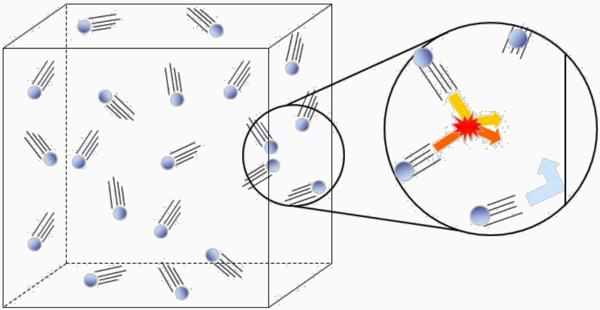

- Ballistic dynamics, hard core exclusion of radius $r_N \sim N^{-1/2}$

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    - Bogolyubov's *interaction clusters* partition $N$ particles.
    - Related to short-time proof of the Boltzmann equation by Simonella, Gallagher, Bodineau: Two particles are in the same interaction cluster if the associated collision trees are joined by a recollision.

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    - When particles collide, energy and momentum are conserved.

# From Interaction Clusters to Bilinearity and Random Graphs

# Key Computation 1

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$$K(x,y,S) = \sum_{v\in x, w\in y} (a + b|v-w|^2) = \sum_{v\in x, w\in y} (a + b|v|^2 - 2v\cdot w + |w|^2)$$
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- $\pi$ is **additive**: $\pi(z) = \pi(x) + \pi(y)$ for $K(x, y, \cdot)$-a.e. $z$, $\pi(x') = \pi(x)$ for $J^N(x, \cdot)$-a.e. $x'$.

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- "Any fact about $\mu_t^N$ which depends only on the conserved quantities is the same for the graph model".

# Coagulation Equations: General Framework

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  for $f \in C_c(S)$.

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- Gelation time

$$t_{\mathrm{gel}} := \inf\{t : g_t \neq 0\}.$$

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4. $g_t$ is differentiable except at $t_{\rm gel}$, right-differentiable at the gelation time $t_{\rm gel}$ and $g'_{t_{\rm gel}+} > 0$. Moreover, for some convex combination $\theta_i \geq 0, \sum_{i=1}^n \theta_i = 1$, it holds that

$$\frac{\sum_{i=1}^n \theta_i (g'_{t_{\rm gel}+})_i}{(g'_{t_{\rm gel}})_0} > \sum_{i=1}^n \frac{\theta_i \langle \pi_i, \mu_0 \rangle}{\langle \pi_0, \mu_0 \rangle}.$$

# Main Result: Convergence of the Particle System

Under mild assumptions on the initial data $X_0^N = (x_1(0), \ldots, x_{l_N(0)}(0))$ at time 0, corresponding to an initial measure $\mu_0$,

- The empirical measure $\mu_t^N := N^{-1} \sum \delta_{x_i^N(t)} \to \mu_t$, where $\mu_t$ is the unique solution to the Flory equation starting at $\mu_0$, uniformly in $t \geq 0$, in probability and

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- In particular, $t_{\mathrm{gel}}$ is the first time a macroscopic cluster appears, and this cluster contains all of the conserved quantities escaping to infinity.

# Key Elements in the Proof

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3. Convergence of the gel $g_t^N \to g_t$

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- Conclude with diagonal argument and tuning $\xi_N$.

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- The connection between blowup and gelation was established by Norris (2000,2001) assuming that $K$ is *approximately multiplicative*, meaning that

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- Construct $\nu_0^\epsilon$ by shifting $\nu_0$ by $(\epsilon, \epsilon, \ldots, \epsilon)$ and cutting out large values.

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- Characterisations of both blowup and gelation times are now continuous in the initial data, so we can take limits of $t^\epsilon_{\mathrm{expl}} \to t_{\mathrm{expl}}$ and $t^\epsilon_{\mathrm{gel}} \to t_{\mathrm{gel}}$.

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- As a result, $\mu_t = \hat{\mu}_t$, which has finite second moments thanks to the behaviour **before** gelation!

# Back to Interaction clusters

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- **So: verified, for toy model, that macroscopic interaction clusters occur before $t_{\mathrm{mf}}$!**