Powerlaws in “soft Bolztmann” equations via Landau Currents

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Plan:

1.- Divergence form of Boltzmann equation: Landau Currents.

2.- “KZ spectrum” for Boltzmann equation with soft potentials.

3.- The linearized equation around a KZ spectrum: deduction

The Boltzmann equation...

$$\frac{\partial}{\partial t} f(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f(t,x,\cdot))(v),$$

$x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, $t > 0$, where, $Q(f) = Q(f,f)$ and, for all $v \in \mathbb{R}^3$,

$$Q(f,g)(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} (f(v')g(v'_*) - f(v)g(v_*))\, W\,(v - v_*, \sigma)\, dv_* d\sigma$$

$$v' = v - (v - v_*) \cdot \sigma\, \sigma, \qquad v'_* = v_* - (v - v_*) \cdot \sigma\, \sigma$$

$$W\,(v - v_*, \sigma) = B(|v - v_*|, \cos\theta); \;\; \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle,$$

$$B(r, \cos\theta) = b(\cos\theta) r^{\gamma}.$$

The collision integral Q may also be written:

$$
\begin{aligned}
Q(f,g)(v) = \int_{\mathbb{R}^9} & \Gamma(v, v_*; v', v'_*)\delta(v + v_* - v' - v'_*)\times \\
& \times \delta(\omega + \omega_* - \omega' - \omega'_*)(f'g'_* - fg_*)dv_*dv'dv'_*
\end{aligned}
$$

$$
\Gamma(v, v_*; v', v'_*) = \frac{2}{|v - v_*|} W\left(v - v_*, \sigma\right)
$$

$$
\omega = |v|^2, \omega_* = |v_*|^2, \cdots
$$

Suitable conditions on γ and the function b are given in the statements.

Mass and energy currents

L. Landau, Phys. Z. Sowjet,'36 formally showed how to write

$$Q(f) = -\nabla_v \cdot J(f), \;\; J(f) : \text{Landau mass current.}$$

Similarly, $$Q(f)|v|^2 = -\nabla_v \cdot \Gamma(f)(v), \;\; \Gamma(f) : \text{Landau energy current.}$$

(See C. Villani's paper in the references for a more detailed discussion).

<u>We shall only consider mass current.</u>

For all $\phi \in C_c^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} Q(f,g)(v)\phi(v)dv =$$

$$= \int_{\mathbb{R}^6 \times \mathbb{S}^2} f(v)g(v_*) \underbrace{(\phi(v') - \phi(v))} B(|v - v_*|, \cos\theta) dv dv_* d\sigma$$

Then,

$$\int_{\mathbb{R}^3} Q(f,g)(v)\phi(v)dv =$$

$$= -\int_{\mathbb{R}^6\times\mathbb{S}^2} f(v)g(v_*)\left(\int_0^{(v-v_*)\cdot\sigma} \frac{d}{ds}\phi(v-s\sigma)ds\right) B(|v-v_*|,\cos\theta)dvdv_*d\sigma$$

$$= -\int_{\mathbb{R}^6\times\mathbb{S}^2} f(v)g(v_*)\left(\int_0^{(v-v_*)\cdot\sigma} \sigma\cdot\nabla\phi(v-s\sigma)ds\right) B(|v-v_*|,\cos\theta)dvdv_*d\sigma$$

$$= -\int_{\mathbb{R}^7\times\mathbb{S}^2} f(v)g(v_*)\mathbb{1}_{0<s<(v-v_*)\cdot\sigma}\sigma\cdot\nabla\phi(v-s\sigma)B(|v-v_*|,\cos\theta)dsdvdv_*d\sigma$$

$$= -\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^4\times\mathbb{S}^2} f(w+s\sigma)g(w_*+s\sigma)\mathbb{1}_{0<s<(w-w_*)\cdot n}B(|w-w_*|,\cos\theta)\times\right.$$

$$\left.\times\,\sigma dsdw_*d\sigma\right)\cdot\nabla\phi(w)dw$$

and so,

$$Q(f,g) = -\nabla \cdot J(f,g)$$
$$J(f,g)(v) = \int_{\mathbb{R}^4\times\mathbb{S}^2} f(v+s\sigma)g(v_*+s\sigma)\mathbb{1}_{0<s<(v-v_*)\cdot\sigma} B(|v-v_*|,\cos\theta)\sigma\, ds\, d\sigma\, dv_*$$

For $s>0$ and $\sigma\in\mathbb{S}^2$, if $z=s\sigma$, then $|z|=s$ and $dz=s^2dsd\sigma=|z|^2dsd\sigma$,

$$J(f,g)(v) = \int_{\mathbb{R}^6} f(v+z)g(v_*+z)\mathbb{1}_{0<|z|^2<(v-v_*)\cdot z} B(|v-v_*|,\cos\theta)\frac{z}{|z|}\frac{dz}{|z|^2}dv_*$$

$$\boxed{\begin{aligned} J(f,g)(v) &= \int_{\mathbb{R}^3\times\mathbb{R}^3} A(v-v_1, v_1-v_2)f(v_1)g(v_2)dv_1dv_2 \\ A(\xi,\eta) &= \mathbb{1}_{|\xi|^2+\xi\cdot\eta<0} B\left(|\eta|, -\frac{\xi\cdot\eta}{|\xi||\eta|}\right)\frac{\xi}{|\xi|^3} \end{aligned}}$$

Notice: If $|\xi|>|\eta| \Longrightarrow |\xi|^2>|\eta\cdot\xi| \Longrightarrow A(\xi,\eta)=0$, (supp $A(\cdot,\eta)\subset\overline{B(0,|\eta|)}$)

Some properties of J.

Symmetry.

If $\tau_c(z) = z + c$ for $c \in \mathbb{R}^3$ and $R \in O_3(\mathbb{R})$ It is not difficult to check:

$$J(f \circ \tau_c, g \circ \tau_c) = J(f, g) \circ \tau_c, \ \forall c \in \mathbb{R}^3$$
$$J(f \circ R, g \circ R) = R^T J(f, g) \circ R, \ \forall R \in O_3(\mathbb{R})$$

$\longrightarrow$ Lemma (cf. F. Golse Cours Polytechnique, e. g.) If f is radial, there exists a radial distribution $j(f)$, real valued, such that $J(f)(v) = vj(f)(|v|)$.

Scaling. Set $S_\lambda z = \lambda z$ for each $\lambda > 0$. Then

$$J\left(f \circ S_\lambda, g \circ S_\lambda\right) = \lambda^{-\gamma-4} J(f, g) \circ S_\lambda.$$

Uniqueness of mass currents

Lemma Let $A(\cdot,\eta) \in \mathscr{D}'(\mathbb{R}^3)^3$ for each $\eta \in \mathbb{R}^3$ such that

$$\text{supp}(A(\cdot,\eta)) \subset \overline{B(0,|\eta|)}, \qquad \xi \times A(\xi,\eta) = 0$$
$$\nabla_\xi \cdot A(\xi,\eta) = 0, \qquad A(\cdot,\eta) \in L^1(B)^3$$

for some ball B such that $0 \in B$. Then $A(\cdot,\eta) = 0$ for all $\eta \in \mathbb{R}^3$.

Proof For all $\eta \in \mathbb{R}^3$ let $a(\cdot,\eta) \in \mathscr{D}'(\mathbb{R}^3 \setminus \{0\})$ be the distribution such that for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, $a(\xi,\eta)$ is the real number such that $A(\xi,\eta) = a(\xi,\eta)\xi$.

Since $\nabla_\xi \cdot (a(\xi,\eta)\xi) = 0$ it follows that $a(\cdot,\eta)$ is homogeneous of degree -3 on $\mathbb{R}^3 \setminus \{0\}$ (by Euler's Theorem).

Since $\text{supp}(a(\cdot,\eta)) \subset \overline{B(0,|\eta|)}$: $\text{supp}(a(\cdot,\eta)) \subset \{0\}$ and the $\text{supp}(A(\cdot,\eta)) \subset \{0\}$.

Since $A(\cdot,\eta) \in L^1(B)^3$ the conclusion follows.

Maxwellians. For all $\rho > 0, \theta > 0, u \in \mathbb{R}^3$,

$$\mathcal{M}_{(\rho,\theta,u)}(v) = \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}}$$

Equilibria of the Boltzmann equation:

Satisfy $Q(\mathcal{M}_{(\rho,\theta,u)}) = 0$

Maximize entropy $\int_{\mathbb{R}^3} f \log f dx$ under mass momentum and energy constraints.

As Landau' 36 obtained for Coulomb interactions:

Theorem Suppose that the collision kernel B is such that, $|B(z,\sigma)| \leq C|z|^\gamma$ with $\gamma > -3$. Then, for all $\rho > 0, \theta > 0, u \in \mathbb{R}^3$, $J(\mathcal{M}_{(\rho,\theta,u)}) = 0$.

Proof Without loss of generality consider $\mathcal{M}(v) = e^{-|v|^2}$. Since it is radial:

$J(e^{-|\cdot|^2})(v) = v\, j(|v|^2)$ and,

$$
\begin{aligned}
|j(|v|^2)||v| &\le \int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathcal{A}(v-v_1, v_1-v_2)|\, e^{-|v_1|^2} e^{-|v_2|^2} dv_1 dv_2 \\
&\le C \int_{\mathbb{R}^3} \frac{e^{-|v_1|^2}}{|v-v_1|^2} \underbrace{\int_{\mathbb{R}^3} |v_1-v_2|^\gamma e^{-|v_2|^2} dv_2}_{\equiv h(v_1)\in C_b(\mathbb{R}^3):}\, dv_1
\end{aligned}
$$

and then $j \in L^1(0,\infty)$. Define now:

$$
I(|v|^2) = \int_{|v|^2}^{\infty} j(r)dr : \;\; I \in L^\infty(1,\infty).
$$

$$
J\left(e^{-|\cdot|^2}\right)(v) = v\, j(|v|^2) = \frac{1}{2}\nabla_v I(|v|^2)
$$

and then $$\Delta_v I(|v|^2) = 2\nabla\cdot J\left(e^{-|\cdot|^2}\right)(v) = 2Q\left(e^{-|\cdot|^2}\right)(v) = 0.$$

The function $v \to I(|v|^2)$ is harmonic on $\mathbb{R}^3$ and bounded, it is then constant by Liouville's Theorem and $J(e^{-|\cdot|^2}) = \nabla I = 0$.

Power laws. Denote $p_\kappa(v) = |v|^{-\kappa}$.

Lemma If f is radial and homogeneous of degree θ and if $j(f)$ exists in $\mathscr{D}'(\mathbb{R}^3 \setminus \{0\})$ such that $J(f)(v) = vj(f)(|v|)$, then $j(f)$ is homogeneous of degree $\gamma + 3 + 2\theta$.

Proof For all $\lambda > 0$, $f \circ S_\lambda = \lambda^\theta f$ and so,

$$\underbrace{\lambda^{-\gamma-4} J(f) \circ S_\lambda = J(f \circ S_\lambda)}_{\text{Previous "Properties of } J\text{"}} = J(\lambda^\theta f) = \lambda^{2\theta} J(f).$$

Therefore $J(f) \circ S_\lambda = \lambda^{\gamma+4+2\theta} J(f)$ and $j(f) \circ S_\lambda = \lambda^{\gamma+3+2\theta} j(f)$.

Theorem If $\kappa \in \mathbb{R}$ is such that $J(p_\kappa) \in \mathscr{D}'(\mathbb{R}^3 \setminus \{0\})^3$ then,

$$\nabla \cdot (J(p_\kappa)) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^3 \setminus \{0\}) \Longleftrightarrow \kappa = \frac{\gamma}{2} + 3.$$

(and then $\kappa = \frac{\gamma}{2} + 3 \Longleftrightarrow Q(p_\kappa) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^3 \setminus \{0\})$).

Proof

Since $J\left(p_\kappa\right)$ is radial, then $J\left(p_\kappa(v)\right)=vj\left(p_\kappa\right)$ and $j\left(p_\kappa\right)\in\mathscr{D}'(\mathbb{R}^3\setminus\{0\})$ is radial.

By the previous Lemma, since p_k is homogeneous of degree $-\kappa$,

$$j\left(p_\kappa\right)\circ S_\lambda=\lambda^{\gamma+3-2\kappa}j\left(p_\kappa\right)$$

On the other hand, by Euler's identity

$$\nabla\cdot\left(J\left(p_\kappa\right)\right)=0\,\text{in}\,\mathscr{D}'(\mathbb{R}^3\setminus\{0\})\Longleftrightarrow j(p_k)\,\text{is homogeneous of degree }-3$$

then

$$\nabla\cdot\left(J\left(p_\kappa\right)\right)=0\Longleftrightarrow j\left(p_\kappa\right)\circ S_\lambda=\lambda^{-3}j\left(p_\kappa\right)$$

Hence

$$\nabla\cdot\left(J\left(p_\kappa\right)\right)=0\Longleftrightarrow\gamma+3-2\kappa=-3\Longleftrightarrow\kappa=3+\frac{\gamma}{2}$$

Theorem. Suppose $b \in L^\infty$. Then $J(p_\kappa) \in \mathscr{D}'(\mathbb{R}^3 \setminus \{0\})^3$ for $\kappa = 3 + \frac{\gamma}{2}, \gamma \in (-3, 0)$.

Proof. $$J(p_\kappa) = \int_{\mathbb{R}^6} \mathcal{A}(v - v_1, v_1 - v_2)|v_1|^{-\kappa}|v_2|^{-\kappa} dv_2 dv_1 \qquad (1)$$

$$|\mathcal{A}(\xi, \eta)| \le ||b||_\infty \frac{|\eta|^\gamma}{|\xi|^2} \mathbb{1}_{|\xi|^2 + \xi\cdot\eta < 0} \le ||b||_\infty \frac{|\eta|^\gamma}{|\xi|^2} \mathbb{1}_{|\xi| \le |\eta|}$$

$$\le ||b||_\infty \frac{\mathbb{1}_{|\xi|\le|\eta|}}{|\eta|^{|\gamma|-\varepsilon}|\xi|^{2+\varepsilon}}, \quad \text{for } 0 < 2\varepsilon < |\gamma| < 3.$$

$$\Rightarrow |J(p_\kappa)| \le ||b||_\infty \int_{\mathbb{R}^3} \frac{1}{|v - v_1|^{2+\varepsilon}|v_1|^{3-\frac{|\gamma|}{2}}} \underbrace{\left(\int_{\mathbb{R}^3} \frac{dv_2}{|v_1 - v_2|^{|\gamma|-\varepsilon}|v_2|^{3-\frac{|\gamma|}{2}}} \right)}_{=\tilde{h}(|v_1|)<\infty} dv_1$$

Since $\tilde{h}(|v_1| < \infty$ and $\tilde{h}(\lambda v_1) = \lambda^{\varepsilon - \frac{|\gamma|}{2}} \tilde{h}(v_1)$: $\tilde{h}(v_1) = c|v_1|^{\varepsilon - \frac{|\gamma|}{2}}$ and, for $v \neq 0$,

$$|J(p_\kappa)| \le ||b||_\infty \int_{\mathbb{R}^3} \frac{\tilde{h}(|v_1| dv_1}{|v - v_1|^{2+\varepsilon}|v_1|^{3-\frac{|\gamma|}{2}}} \le c||b||_\infty \int_{\mathbb{R}^3} \frac{dv_1}{|v - v_1|^{2+\varepsilon}|v_1|^{3-\varepsilon}} < \infty.$$

For hard spheres, $(\gamma = 1)$ the integral in (1) diverges.

Extension of $Q(p_\kappa) \in \mathscr{D}'(\mathbb{R}^3 \setminus \{0\})$ to $\mathscr{D}'(\mathbb{R}^3)$ for $\kappa = \frac{\gamma}{2} + 3$:

Since $j(p_\kappa)$ is homogeneous of degree -3, by Euler's relation:

$$\nabla \cdot J\,(p_\kappa) = \nabla \cdot (vj(p_\kappa)) = 0, \text{ in } \mathscr{D}'(\mathbb{R}^3 \setminus \{0\})$$

The locally integrable function $J\,(p_\kappa)$ satisfies then $\text{supp}(\nabla \cdot J\,(p_\kappa)) \subset \{0\}$.

Since (each component of) $J\,(p_\kappa)$ is a homogeneous function of degree -2, the distribution $\nabla \cdot J\,(p_\kappa)$ is a finite linear combination of $D^\alpha(\delta_0)$ with $\alpha \in \mathbb{N}^3$.
Since $\nabla \cdot J\,(p_\kappa)$ is homogeneous of degree -3, for some $c_0 \in \mathbb{R}$:

$$Q(p_\kappa) = \nabla \cdot J\,(p_\kappa) = c_0\delta_0$$

The real number c is called the "residue of $J\,(p_\kappa)$ at 0". This residue is equivalently defined by the formula:

$$\langle J\,(p_\kappa)\,, \phi\rangle = \frac{c_0}{|\mathbb{S}^2|}\int_{\mathbb{R}^3} \phi(x)\frac{dx}{|x|^3}, \text{ in } \mathscr{D}'(\mathbb{R}^3).$$

Previous approach.

In references of the 70's, several Soviet authors (Kats, Kantorovich, Karats, Kontorovich, Kadomtsev...), studied power law solutions for Boltzmann equations. Prompted by previous works (60's) from Zakharov, Sagdeev, Filonenko and Kats on waves interactions (later wave turbulence):

For radial functions f they obtain $Q(f)(v) = \widetilde{Q}(f)(|v|)$ after integration of angular variables.

Clever use of symmetries and scaling shows $\widetilde{Q}(p_\kappa)(r) = 0, \ \forall r > 0$ for two values of κ. One value is $\kappa = 3 + \frac{\gamma}{2}$. The collision integral is not convergent in the other.

p_κ is then called a <u>KZ spectrum</u>.

Remark For $\kappa = 3 + \frac{\gamma}{2}$,

$$\widetilde{Q}(p_\kappa)(r) = 0 \qquad \forall r > 0,$$

$$Q(p_\kappa)(v) = c_0\delta_0, \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

Flux of Mass Given the mass current $J(p_k)$, if S is a closed surface enclosing a volume V containing the origin, the outward flux of mass through S is

$$\int_S J(p_k) \cdot n dS$$

where n is the outward normal vector to S. Since,

$$\int_S J(p_\kappa) \cdot n dS = \int_V \nabla \cdot J(p_\kappa)(v) dv = c_0$$

p_κ has a non zero flux of mass for any such surface S. The sign of c_0 gives the direction of the flux.

In this case it may be proved that $c_0 > 0$, and the flux is then outgoing.

Of course, since $J(\mathcal{M}\text{axwellian}) = 0$, the mass flux of the Maxwellians is zero.

In 70's Soviet literature, similar results for non zero energy flux power laws, with arguments as described before.

Linearization around p_κ, $\kappa = 3 + \frac{\gamma}{2}$; $\gamma \in (-3, 0)$.

To linearize the Boltzmann collision operator around a Maxwellian $\mathcal{M}$:
The perturbation is written as $f = \mathcal{M} + \sqrt{\mathcal{M}}\, G$ and the linear operator reads

$$L_{\mathcal{M}}(G)(v) = -\nu(v)G(v) + K(G)(v), \;\; v \in \mathbb{R}^N$$

for some function $\nu > 0$ and K a non negative self adjoint operator in a weighted L^2 space. (For soft potentials R. S. Caflisch, Comm. Maths. Phys. '80.)

We consider rather a perturbation of p_k of the form:

$$f(t, v) = p_\kappa(1 + G(t, v))$$

in the Boltzmann equation written as:

$$\frac{\partial f}{\partial t}(t, v) = \int_{\mathbb{R}^9} \Gamma(v, v_*; v', v'_*)\delta(v + v_* - v' - v'_*)\times$$
$$\times\, \delta(\omega + \omega_* - \omega' - \omega'_*)(f' f'_* - f f_*) dv_* dv' dv'_*$$

$$\Longleftrightarrow p_\kappa \frac{\partial G}{\partial t} = c_0\delta_0 + T(G) + Q(p_\kappa G)$$

$$\frac{\partial G}{\partial t} = \cancel{c_0 p_\kappa^{-1}}\delta_0 + L(G) + \cancel{p_\kappa^{-1} Q(p_\kappa G)}$$

$$\Rightarrow L(G)(v) = N(|v|)G(t,v) + \int_{\mathbb{R}^3} U(v,v')G(t,v')dv'$$

$$U(v,v') = \frac{2}{p_\kappa}\int_{\mathbb{R}^6} dv_* dv'_* \Gamma(v,v_*;v',v'_*)\delta(v+v_*-v'-v'_*)\times$$
$$\times\,\delta(\omega+\omega_*-\omega'-\omega'_*)p'_{\kappa*}p'_\kappa -$$
$$-\frac{p'_\kappa}{p_\kappa}\int_{\mathbb{R}^6} dv_* dv'_* \Gamma(v,v_*;v',v'_*)\delta(v+v_*-v'-v'_*)\times$$
$$\times\,\delta(\omega+\omega_*-\omega'-\omega'_*)p'_{\kappa*}.$$

$$N(|v|) = p_\kappa^{-1}\int_{\mathbb{R}^9} dv_* dv' dv'_* \Gamma(v,v_*;v',v'_*)\delta(v+v_*-v'-v'_*)\times$$
$$\times\,\delta(\omega+\omega_*-\omega'-\omega'_*)p_{\kappa*}.$$

General case: expand $U(v, v')$, $G(t, v)$ in Legendre polynomials/spherical harmonics:

$$U(v, v') = \sum_{\ell=0}^{\infty} U_\ell(|v|, |v'|) P_\ell(\cos\theta_{v,v'})$$

$$G(t, v) = \sum_{\ell,\nu} G_{\ell,\nu}(t, |v|) Y_\ell^\nu(\hat{v}), \;\; \hat{v} = \frac{v}{|v|}$$

deduce, with $r = |v|$, for $\ell \in \mathbb{N}$, $\nu \in \{-\ell, -\ell+1, \cdots, \ell-1, \ell\}$,

$$\frac{\partial G_{\ell,\nu}}{\partial t}(t, r) = -N_\ell(r) G_{\ell,\nu}(t, r) + \frac{4\pi}{2\ell+1} \int_0^\infty U_\ell(r, r') G_{\ell,\nu}(t, r') r^2 dr'.$$

Simplest case: $\ell = 0$, $P_0 \equiv 1$ and $\Gamma(v, v_*, v', v'_*) = |v - v'|^m$.

Then $U(v, v') \equiv U_0(r, r')$ and denote $G_{0,0}(t, r) \equiv A(t, r)$.

After Mellin transform: $\widehat{A}(t, s) = \int_0^\infty A(t, x) x^{s-1} dx$, Proposition If (restriction condition on Γ)

$$\Gamma(v,v_*;v',v'_*) = |v-v'|^m \quad \left(\text{or } B(|v-v_*|,\cos\theta) = \frac{1}{2}|v-v_*|^{m+1}(1-\cos\theta)^{-\frac{m}{2}}\right):$$

$$\boxed{\frac{\partial}{\partial t}\widehat{A}(t,s+h) = W(m,s)\widehat{A}(t,s),\ \Re e(s) < -\frac{m+1}{2};\ \ h = \frac{3-m}{2}, m \in (-3,-1)}$$

1) W is meromorphic in the region $s \in \mathbb{C}; \Re es < -\frac{m+1}{2}$.

2) Poles at $a = \frac{m-19}{2}, b = -\frac{3+m}{2}$.

W has a set of zeros. In particular at some $\sigma_- \in (a,b), \sigma^+ \in (b, b+h)$.

W is analytic in $\Re e(s) \in (a,b)$.

3) $W(m,s) = -\gamma_m + W_0(m,s)$ for a constant $\gamma_m > 0$ and

for all bounded interval $I \subset \left(-\infty, -\frac{m+1}{2}\right)$, there exists $C > 0$ such that

$$|W_0(m,s)| \leq C|\Im ms|^{-\frac{2\Re es+15+m}{4}}, \quad \forall s \in \mathbb{C};\ \Re e(s) \in I,\ |\Im ms| \geq 1.$$

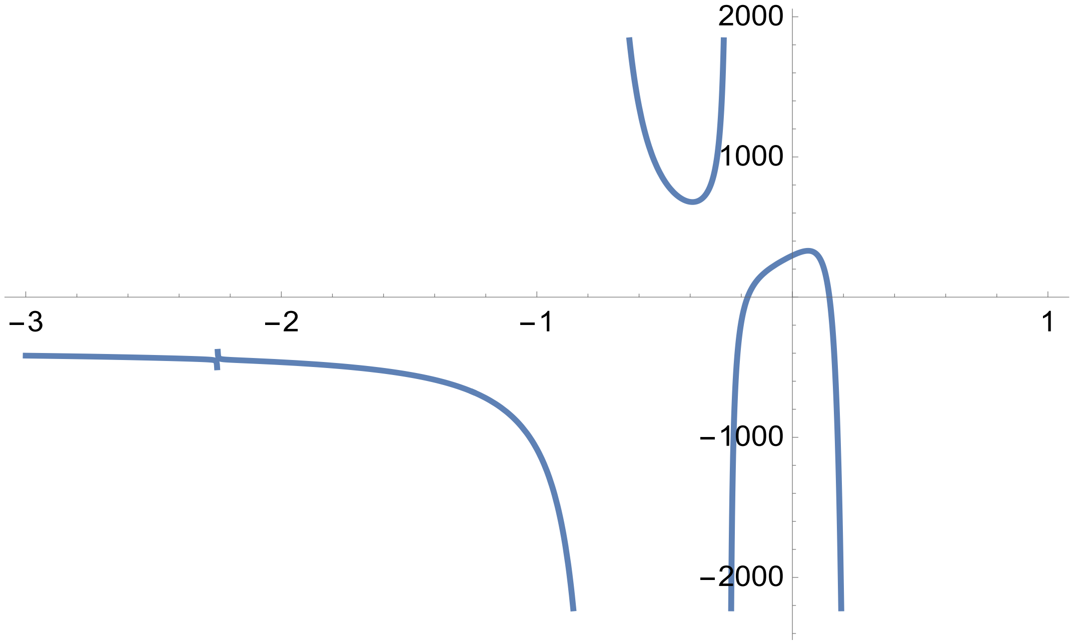
4) $w(u) = 0$ for $u \in (\sigma_-, b)$.

The winding $w(u)$ of the function $W(s)$ along the line $\Re e(s) = u$:

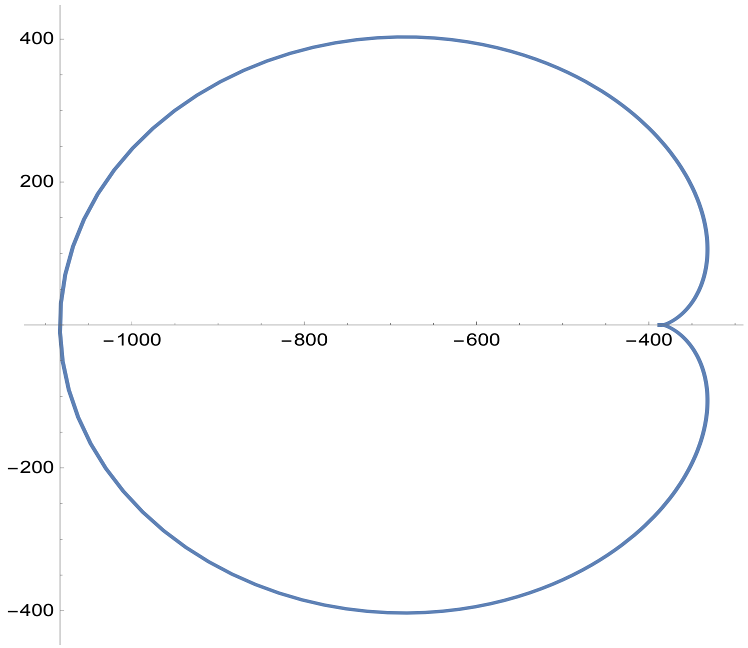
It is the complete increment divided by 2π of the continuously changing argument of the complex quantity $W(s)$ as s moves from $u - i\infty$ to $u + i\infty$ along the line $\Re e(s) = u$).

The equation for $\widehat{A}$ may then be solved using the arguments presented in the references [1, 6] (see also [3]).

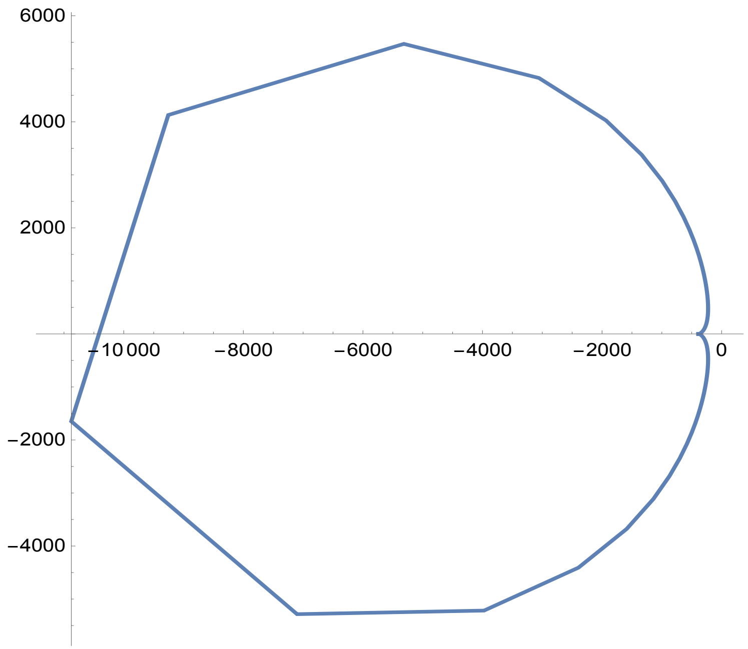
Some pictures of the function $W(s)$ for $s \in \mathbb{R}$ and $s \in \mathbb{C}$:



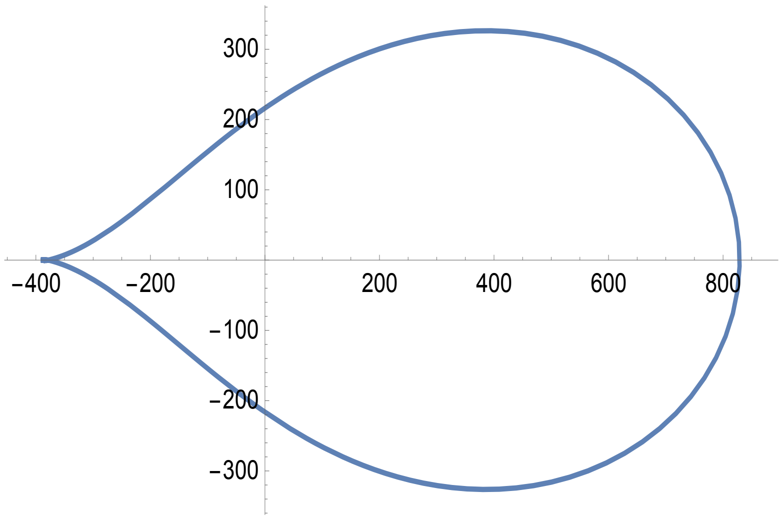
The function $W(-3/2, s)$, $s \in (-3, 1)$



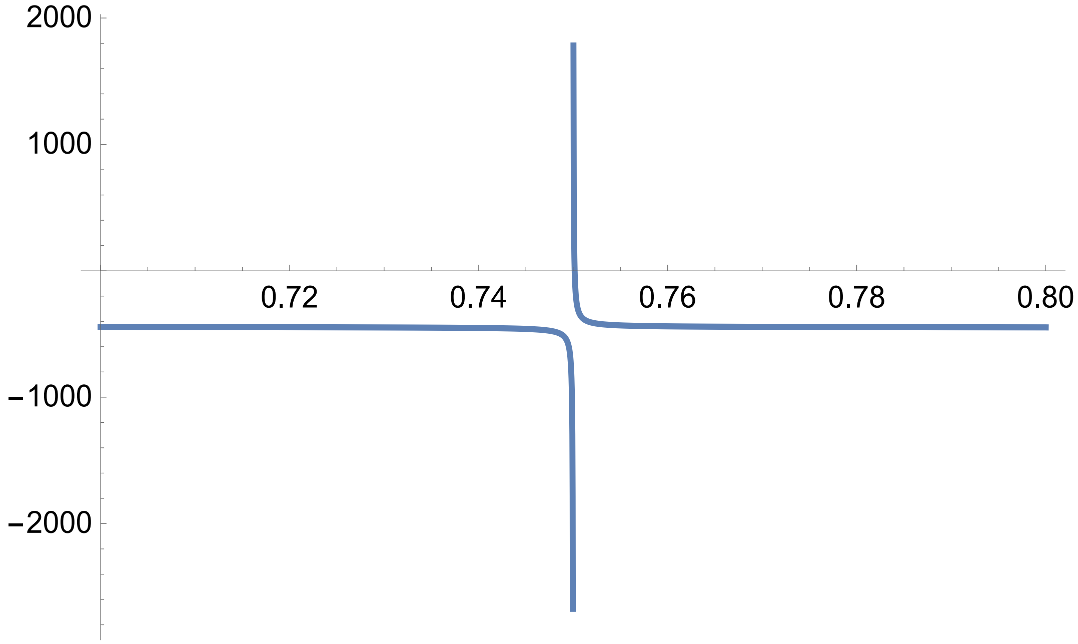
The function $(\Re e(W(-3/2, s)), \Im m(W(-3/2, s))$, $\Re e s = -1$, $\Im m(s) \in (-20, 20)$



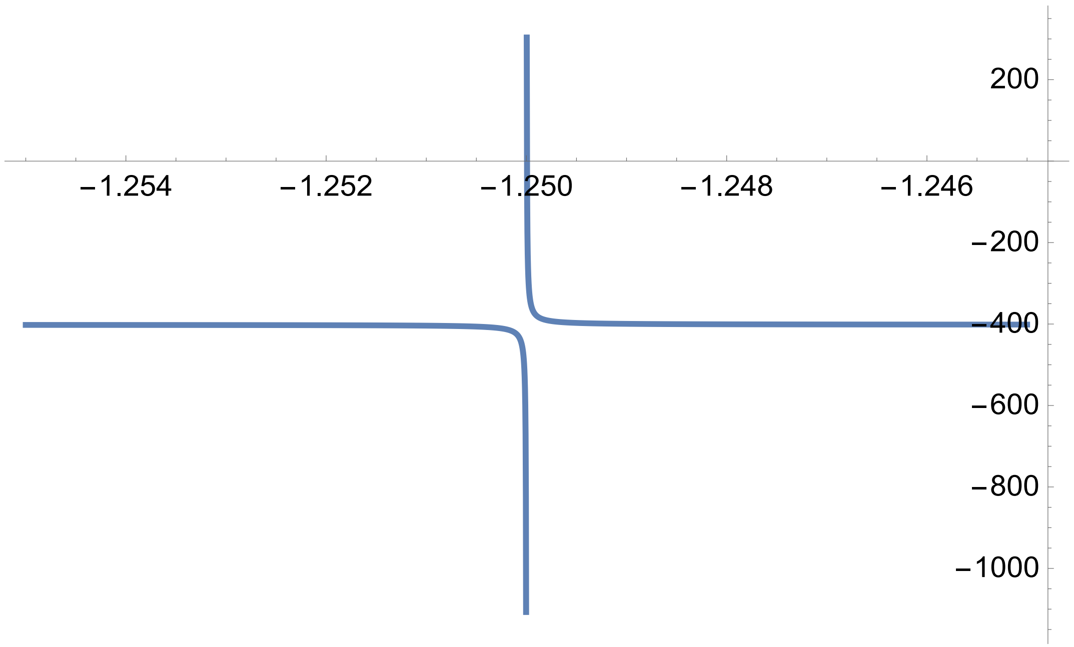
The function $(\Re e(W(-3/2,s)), \Im m(W(-3/2,s)), \Re es = -1.23,\ \Im m(s) \in (-20,20)$



The function $(\Re e(W(-3/2,s)), \Im m(W(-3/2,s)), \Re es = -1.25,\ \Im m(s) \in (-20,20)$



Zero and pole of $W(s)$ near -2.25



Zero and pole of $W(s)$ near -4.25

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