Large deviations for sparse random graphs and coagulation processes

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based on joint works with Wolfgang König, Heide Langhammer and Robert Patterson (WIAS Berlin)

Spatial coagulation processes

Particles with position and mass (total mass $\propto N$, $\mathcal{S}$ Polish space)

$$\left(X_i^{(N)}(t), M_i^{(N)}(t)\right) \in \mathcal{S} \times \mathbb{N}$$

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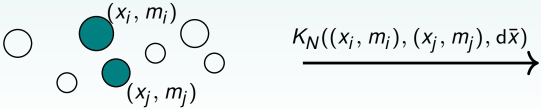

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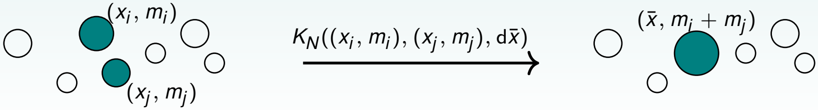

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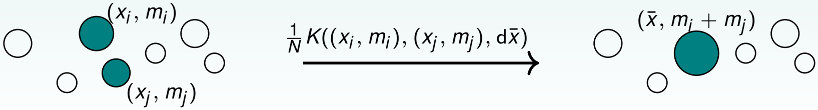

Empirical measure of the system:

$$\mu_t^{(N)} := \frac{1}{N} \sum_i \delta_{\left(X_i^{(N)}(t), M_i^{(N)}(t)\right)} \qquad \in \mathcal{M}(\mathcal{S} \times \mathbb{N}).$$

Phase transition: **gelation**

One particle (or more) of mass $\sim N$ in a **finite time**.

LLN: from microscopic to macroscopic description

Under certain assumptions on K (usually $K \leq Cm\,n$) the **hydrodynamic limit** is known:

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- No spatial component [NORRIS 1999], [FOURNIER, GIET 2004].
- Cluster coagulation process [NORRIS (2000), A., IYER, MAGNANINI (2023)].
- Particles also diffusing in $\mathbb{R}^d$ [HAMMOND, REZAKHANLOU (2007)].

Remark: uniqueness of solutions of the limiting equation is not always ensured (i.e. not always implies LLN).

One very special case: $K(m,n) = m\,n$

Initial condition with N particles of mass 1: $\mu_0^N = \delta_1$.

Collection of sizes of particles at any time t	$\longleftrightarrow$	Sizes of connected components of Erdős-Rènyi graph $\mathcal{G}(N, 1 - e^{-\frac{t}{N}})$

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Initial condition with N particles with general sizes: $\mu_0^N = \frac{1}{N} \sum_i \delta_{M_i^N(0)}$.

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μ_0^N : types of the vertices

$\kappa_N(m, n) = 1 - e^{-\frac{t}{N} m\, n}$
probability of an edge
between a vertex of type m and one of type n

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Can we get an LDP for the components of the inhomogeneous random graph?

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Sparse graph: $\#\{\text{ edges }\} \propto \#\{\text{ vertices }\}$

$$\mathcal{G}(N, \tfrac{\lambda}{N})$$

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- BORDENAVE, CAPUTO (2015): uniform graph with a given degree, $\mathcal{G}(n, m)$, $\mathcal{G}(n, \frac{\lambda}{n})$;
- BALDASSO, OLIVEIRA, PEREIRA, REIS (2022): $\mathcal{G}(n, m)$, $\mathcal{G}(n, \frac{\lambda}{n})$ with independent marks;
- RAMANAN, YASODHARAN (2024): entropy form for the *rate function*.

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- PUHALSKII (2005): large components of $\mathcal{G}(n, \frac{\lambda}{n})$ via large deviations of the exploration;
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Probability of rare events are on the same scale $\sim e^{-N[\dots]+o(N)}$:

- microscopic components
- macroscopic components

LDP for the sparse inhomogeneous random graph

- $\mathcal{S}$ a compact metric space: the **type space**;
- $\mu \in \mathcal{M}(\mathcal{S})$ a probability on $\mathcal{S}$;
- $\mathbf{x}^N = (x_1, \dots, x_N) \in \mathcal{S}^N$ vector of vertices' type

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \to \mu;$$

- a continuous symmetric kernel $\kappa : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$.

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The **sparse inhomogeneous random graph** $\mathcal{G}(N, \mathbf{x}^N, \kappa)$ is such that there is an edge between vertices i and j with probability that depends on their types:

$$i \sim j \text{ with probability } \frac{\kappa(x_i, x_j)}{N} \wedge 1.$$

Counting microscopic and macroscopic components ($\mathcal{S}$ finite)

$\mathcal{C}_i$ connected component of $\mathcal{G}(N, \mathbf{x}^N, \kappa) \to \text{types}(\mathcal{C}_i) \in \mathbb{N}^{\mathcal{S}}$.

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where $(\mathcal{C}_j)_j$ are the connected components of $\mathcal{G}(N, \mathbf{x}^N, \kappa)$.

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$\forall\, \lambda \in \mathcal{M}(\mathbb{N}^{\mathcal{S}})$ $\quad c(\lambda)(r) := \sum_{k \in \mathbb{N}^{\mathcal{S}}} k_r \, \lambda_k \quad \forall r \in \mathcal{S}$ **Integrated type configuration** of λ

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The large deviation principle

Theorem (A., König, Langhammer, Patterson (2023))

The pair of measures $(\mathrm{Mi}_N, \mathrm{Ma}_N)$ *satisfies a large deviations principle with speed N and* **explicitly given rate function**

$$I(\lambda, \alpha) = I_{\mathrm{Mi}}(\lambda) + I_{\mathrm{Ma}}(\alpha) + I_{\mathrm{Me}}(\mu - c(\lambda) - c(\alpha)).$$

$$\mathbb{P}((\mathrm{Mi}_N, \mathrm{Ma}_N) \approx (\lambda, \alpha)) = e^{-N\, I(\lambda,\alpha) + o(N)}$$

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I_{\mathrm{Ma}}(\alpha) &= \int_{\mathcal{M}(\mathcal{S})\setminus\{0\}} \alpha(\mathrm{d}y)\left(\left\langle y, \log\frac{y}{(1-\mathrm{e}^{-\kappa y})\mu}\right\rangle + \frac{1}{2}\langle y, \kappa(\mu - y)\rangle\right), \\
I_{\mathrm{Me}}(\nu) &= \left\langle \nu, \log\frac{\nu}{(\kappa\nu)\mu}\right\rangle + \frac{1}{2}\langle \nu, \kappa\mu\rangle.
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- Spotting the phase transition: look for minimizers of

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No minimizer with $c(\lambda) = \mu \quad \rightarrow \quad \exists$ giant component!

Main ingredients for the proof

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Let $\quad L_k^{(N)} := \#\{j : \text{types}(\mathcal{C}_j) = k\} \quad \forall\, k \in \mathbb{N}^{\mathcal{S}}$

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Lemma

For any $N \in \mathbb{N}$ and for any $\ell = (\ell_k)_k$ we have that

$$\mathbb{P}\left(L_k^{(N)} = \ell_k, \forall k\right) = \textit{Explicit terms}(\ell, N) \times \prod_{k \leq N\mu_N} \frac{p_N(k)^{\ell_k}}{\ell_k!},$$

where for any $k \in \mathbb{N}^{\mathcal{S}}$:

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We fix $R < \infty$ and $\epsilon > 0$.

Main ingredients for the proof

$\mathcal{S}$ finite

Let $\quad L_k^{(N)} := \#\{j\colon \text{types}(\mathcal{C}_j) = k\} \quad \forall\, k \in \mathbb{N}^{\mathcal{S}}$

Lemma

For any $N \in \mathbb{N}$ and for any $\ell = (\ell_k)_k$ we have that

$$\mathbb{P}\left(L_k^{(N)} = \ell_k, \forall k\right) = \textit{Explicit terms}(\ell, N) \times \prod_{k \le N\mu_N} \frac{p_N(k)^{\ell_k}}{\ell_k!},$$

where for any $k \in \mathbb{N}^{\mathcal{S}}$:

$$p_N(k) = \mathbb{P}\big(\mathcal{G}(|k|, \mathbf{x}, \tfrac{1}{N}\kappa) \textit{ is connected}\big), \qquad N \in \mathbb{N}.$$

We fix $R < \infty$ and $\epsilon > 0$.

For any pair (λ, α)

$$\mathbb{P}\Big((\mathrm{Mi}_N, \mathrm{Ma}_N) \simeq (\lambda, \alpha)\Big)$$

$$\simeq \text{Explicit terms}(\ell, N) \times \prod_{|k| \le R} \frac{p_N(k)^{\ell_k}}{\ell_k!} \prod_{|k| \ge \epsilon N} \frac{p_N(k)^{\ell_k}}{\ell_k!} \prod_{\text{otherwise}} \frac{p_N(k)^{\ell_k}}{\ell_k!}$$

LDP for spatial coagulation process

[A., König, Langhammer, Patterson (2024)]

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Poisson monodisperse initial condition: for $\mu \in \mathcal{M}_1(\mathcal{S})$

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We are interested in describing, under $\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\cdot)$,

$$\mu_T^{(N)} \in \mathcal{M}(\mathcal{S} \times \mathbb{N}).$$

$$(X_i(T), M_i(T)) \in \mathcal{S} \times \mathbb{N}$$

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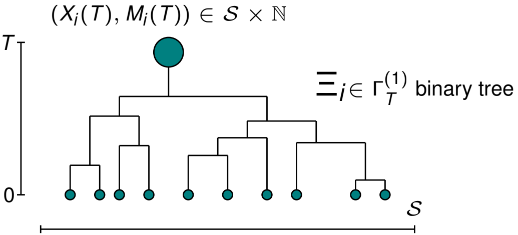
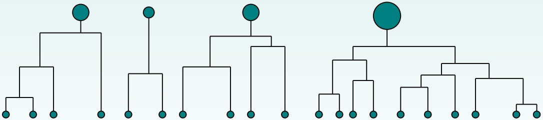


Description via binary trees



$\Gamma_T^{(1)}$ set of binary trees embedded in $[0, T]$: a subset of $\mathbb{D}([0, T], \mathcal{M}(\mathcal{S} \times \mathbb{N}))$.

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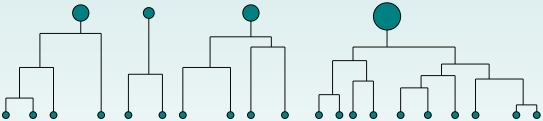

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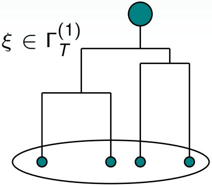
$\xi \in \Gamma_T^{(1)}$



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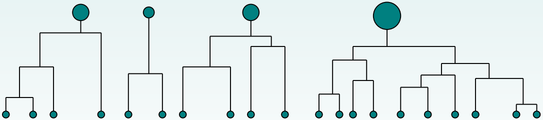

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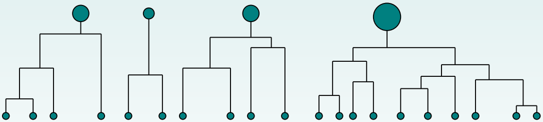

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$$N \int \xi_0 \mathcal{V}_N^{(T)}(\mathrm{d}\xi) \quad \in \mathcal{M}_{\mathbb{N}_0}(\mathcal{S})$$

$\Rightarrow$ initial distribution of points in $\mathcal{S}$

The empirical measure at time $t \leq T$

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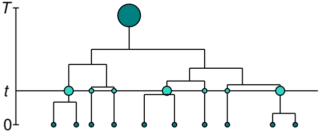
Projection at time t

For $t \in [0, T]$, take the function

$$\begin{aligned} \rho_t : \quad & \mathcal{M}(\Gamma_T^{(1)}) && \to && \mathcal{M}(\mathcal{S} \times \mathbb{N}) \\ & \nu^{(T)} && \mapsto && \rho_t(\nu^{(T)}) := \int \nu^{(T)}(\mathrm{d}\xi)\, \xi_t \end{aligned}$$



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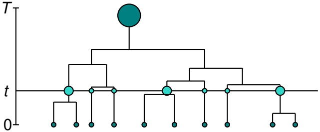

No gelation at time T: full mass of $\rho_t(\nu^{(T)})$

$$\langle m, \rho_t(\nu^{(T)}) \rangle \equiv 1, \qquad \forall t \in [0, T].$$

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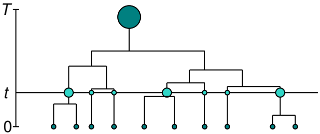

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For any fixed N

$$\mathbb{E}\left[\langle m, \rho_t(\mathcal{V}_N^{(T)}) \rangle \right] \equiv 1, \qquad \forall t \in [0, T]$$

Does the law of $\mathcal{V}_N^{(T)}$ satisfy a **large deviations principle**?

$$\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\mathcal{V}_N^{(T)} \in \mathrm{d}\nu) \approx e^{-NI(\nu)+o(N)}$$

Many-body system approach

Theorem [A., König, Langhammer, Patterson, 2024]

For $\nu \in \mathcal{M}(\Gamma_T^{(1)})$

$$\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\mathcal{V}_N^{(T)} \in \mathrm{d}\nu) = \exp\Big\{ - \frac{1}{2N} \sum_{i \neq j} R^{(T)}(\xi_i, \xi_j) \Big\} \, \mathrm{P}^{(N)}_{NM^{(T)}_{\mu,N}} \left(\frac{Y}{N} \in \mathrm{d}\nu \right)$$

where Y is a Poisson Point process on the space $\Gamma_T^{(1)}$ with intensity measure $NM^{(T)}_{\mu,N}$.

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$$R^{(T)}(\xi,\xi') = \int_0^T \mathrm{d}t \sum_{m,m'\in\mathbb{N}} \int_{\mathcal{S}} \xi_t(\mathrm{d}x,m) \int_{\mathcal{S}} \xi'_t(\mathrm{d}x',m')\, K\big((x,m),(x',m')\big).$$

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Interpretation:
a many bodies system with interaction R.

The intensity measure of Y

$Y = \sum_i \delta_{\Xi_i} \sim \mathrm{Poi}_{NM^{(T)}_{\mu,N}}$, where

$$M^{(T)}_{\mu,N}(\mathrm{d}\xi) = N^{|\xi_0|-1} \mathrm{ePoi}_\mu \otimes \mathbb{P}^{(N)}(\mathrm{d}\xi), \qquad \xi \in \Gamma^{(1)}_T.$$

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- For all $K \leq Hm \cdot n$, all $k \in \mathcal{M}_{\mathbb{N}_0}(\mathcal{S})$,

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Explicit expression for $\mathbb{Q}^{(T)}_k$ and $\tau^{(T)}_k$.

To be continued...

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Thank you!