Algebraic and Geometric Aspects
of Signatures and Rough Analysis:
Open Problems

The following questions were presented during the open problems session of the workshop on 'Algebraic and Geometric Aspects of Signatures and Rough Analysis', WIAS Berlin, Feb 4-6.

1. **Problem:** How to describe algebraically and geometrically the strict subgroup of the Free Lie Group which is given as the image of the set of bounded variation paths under the signature?

   References: Partial answers: Le Donne and Züst (2021) Boedihardjo et al. (2016)

   Submitted by: Felix Medwed / Rosa Preiß

2. **Problem':** How to capture the analogous subgroup for rougher paths?

   References: Partial answers: Boedihardjo et al. (2016)

   Submitted by: Felix Medwed

3. **Problem:** Let I be an ideal in the commutative ring $T_{\sqcup}^{\leq k}(\mathbb{R}^d)$ (the coordinate ring of the free k-step nilpotent Lie group) and let J be the *halfshuffle* ideal generated by I in $T_{\sqcup}(\mathbb{R}^d)$. Question: is there an algorithm to determine the preimage $I_{\prec}$ of J under the embedding $T_{\sqcup}^{\leq k}(\mathbb{R}^d) \to T_{\sqcup}(\mathbb{R}^d)$?

   Submitted by: Felix Lotter

4. **Problem**: Let B be d-dimensional Brownian motion, $d \geq 9$. Let

$$\tau = \inf \{t : |B_t| = 1\}. \tag{1}$$

   Does the expected signature of B up to time τ have finite radius of convergence?

   Submitted by: Horatio Boedihardjo

5. **Problem:** Let $\mathsf{area}(x, y) := x \succ y - y \succ x$ be the antisymmetrization of the halfshuffle. Do Hall bracketings of area starting from the letters generate the shuffle algebra?

   References: Conjecture by Terry Lyons. Salvi et al. (2023) Diehl et al. (2021)

   Submitted by: Jeremy Reizenstein / Joscha Diehl / Rosa Preiß

6. **Problem:** Conjecture: There exists no Lie ideal L of the free Lie algebra such that the log signatures of non-tree-like-equivalent paths are still distinct mod L.

   References: Conjecture by R.P. 2022. Cf. Boedihardjo et al. (2016) Hambly and Lyons (2010)

   Submitted by: Rosa Preiß

7. **Problem:** Let M_p denote the adjoint map $\langle S(p(X)), x \rangle = \langle S(X), M_p x \rangle$ where $p : \mathbb{R}^d \to \mathbb{R}^e$ is a polynomial map. M_p is a homomorphism of Zinbiel algebras. How can we describe $\ker M_p$ for a general polynomial map p?

   References: Colmenarejo and Preiß (2020)

   Submitted by: Felix Lotter / Rosa Preiß

8. **Problem**: Let X be a bounded variation, tree-reduced paths. Let $\mathbf{X}_{0,1}^n$ be the n-th term of its signature. Is it true

$$\lim_{n \to \infty} \|n! \mathbf{X}_{0,1}^n\|^{\frac{1}{n}} = \|X\|_{1-var}? \tag{2}$$

   Submitted by: Horatio Boedihardjo

9. **Problem:** If X is the shortest path which has a given truncated signature, is it always true that then X is arcwise smooth?

   References: Friz and Victoir (2010), Not all sub-Riemannian length minimizers are smooth: Chitour et al. (2025)

   Submitted by: Rosa Preiß

10. **Problem** Let $k \geq 3$ and consider the signature varieties given by polynomial paths $\mathcal{P}_{d,k,m}$ and piecewise linear paths $\mathcal{L}_{d,k,m}$ Améndola et al. (2019). Prove that the stabilizing constant that makes m reach the universal variety $\mathcal{U}_{d,k}$ is given by $M = \lceil \lambda_{d,k}/d \rceil$. Moreover, if $m < M$ then the signature varieties are rationally identifiable. Pfeffer et al. (2019) have verified this for $m \leq 7$ and algebraic identifiability for $m \leq 30$.

    References: Améndola et al. (2019); Pfeffer et al. (2019)

    Submitted by: Carlos Améndola

11. **Problem**: Let $X : [0,1] \to \mathbb{C}^n$ be bounded variation path. What condition does X have to satisfy so that

$$S(X)_{0,1} = 1?$$

    Submitted by: Horatio Boedihardjo

12. **Problem** The function $(\mathbb{R}^2)^{\otimes 4} \to \mathbb{R} : X \mapsto 2(X_{1122} + X_{2211}) - (X_{1212} + X_{2121} + X_{1221} + X_{2112})$ is the unique SL_2-invariant linear function on fourth signatures of paths in $\mathbb{R}^2$ that only sees the degree 1 and 3 parts of the log-signature (it lies in the Thrall module $W_{3,1}(\mathbb{R}^2)$. Is there a geometric interpretation in terms of the path?

    Submitted by: Tim Seynnaeve

References

Améndola, C., Friz, P., and Sturmfels, B. (2019). Varieties of signature tensors. In *Forum of Mathematics, Sigma*, volume 7, page e10. Cambridge University Press.

Boedihardjo, H., Geng, X., Lyons, T., and Yang, D. (2016). The signature of a rough path: Uniqueness. *Advances in Mathematics*, 293:720–737. `doi:10.1016/j.aim.2016.02.011`.

Chitour, Y., Jean, F., Monti, R., Rifford, L., Sacchelli, L., Sigalotti, M., and Socionovo, A. (2025). Not all sub-riemannian minimizing geodesics are smooth.

Colmenarejo, L. and Preiß, R. (2020). Signatures of paths transformed by polynomial maps. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 61(4):695–717. `doi:10.1007/s13366-020-00493-9`.

Diehl, J., Lyons, T., Preiß, R., and Reizenstein, J. (2021). Areas of areas generate the shuffle algebra. Version 2, `arXiv:2002.02338v2 [math.RA]`.

Friz, P. K. and Victoir, N. B. (2010). *Multidimensional Stochastic Processes as Rough paths: Theory and Applications*, volume 120 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. `doi:10.1017/CBO9780511845079`.

Hambly, B. and Lyons, T. (2010). Uniqueness for the signature of a path of bounded variation and the reduced path group. *Annals of Mathematics Second Series*, 171(1):109–167. `doi:10.4007/annals.2010.171.109`.

Le Donne, E. and Züst, R. (2021). Space of signatures as inverse limits of carnot groups*. *ESAIM: COCV*, 27:37.

Pfeffer, M., Seigal, A., and Sturmfels, B. (2019). Learning paths from signature tensors. *SIAM Journal on Matrix Analysis and Applications*, 40(2):394–416.

Salvi, C., Diehl, J., Lyons, T., Preiss, R., and Reizenstein, J. (2023). A structure theorem for streamed information. *Journal of Algebra*, 634:911–938.