Decay rate of Logarithmic Signature

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Algebraic and Geometric aspects of Signatures and Rough Analysis

Logarithmic signature

- The signature of X on $[0,1]$ is

$$S\left(X\right)_{0,1} = 1 + \mathbf{X}_{0,1}^{1} + \mathbf{X}_{0,1}^{2} + \cdots \in T\left(\left(\mathbb{R}^{d}\right)\right)$$

where

$$\mathbf{X}_{0,1}^{n} = \int_{0}^{1} \cdots \int_{0}^{t_2} \mathrm{d}X_{t_1} \otimes \cdots \otimes \mathrm{d}X_{t_n}.$$

- Tensor logarithm of $a \in T\left(\left(\mathbb{R}^{d}\right)\right)$ is defined as

$$\log\left(1+a\right) = \sum_{n=1}^{\infty} \left(-1\right)^{n-1} \frac{a^{\otimes n}}{n}.$$

- The log signature of X is defined as

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Log signature representation

- Consider differential equation

$$\mathrm{d}Y_t = Y_t \mathbf{M}\left(\mathrm{d}X_t\right), \; Y_0 = y_0 \tag{1}$$

  where

  - $\left(X_t\right)_{t \geq 0}$ is a $\mathbb{R}^d$-valued bounded variation path;
  - $\mathbf{M} : \mathbb{R}^d \to M_{m \times m}\left(\mathbb{R}\right)$

- Define linear map $\mathbf{M} : T\left(\left(\mathbb{R}^d\right)\right) \to M_{m \times m}$,

$$\mathbf{M}\left(v_1 \otimes \cdots \otimes v_n\right) = \mathbf{M}\left(v_1\right) \cdots \mathbf{M}\left(v_n\right).$$

- Assuming $\log S\left(X\right)_{0,1}$ decays quickly,

$$Y_t = \exp\left(\mathbf{M}\left(\log\left(S\left(X\right)_{0,1}\right)\right)\right) Y_0.$$

  - exp is matrix exponential.

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$$\mathrm{d}\begin{pmatrix} Y_t^1 & Y_t^2 \end{pmatrix} = \lambda \begin{pmatrix} Y_t^1 & Y_t^2 \end{pmatrix} \begin{pmatrix} \mathrm{d}X_t^1 & \mathrm{d}X_t^2 \\ \mathrm{d}X_t^2 & -\mathrm{d}X_t^1 \end{pmatrix}.$$

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$$\log S(X)_{0,T} = \int_0^T \mathrm{d}X_{t_1} + \frac{1}{2}\int_0^T \int_0^{t_2} [\mathrm{d}X_{t_1}, \mathrm{d}X_{t_2}] + \cdots.$$

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Log signature properties

- Terms in $S(X)_{0,1}$ not algebraically independent, e.g. $\exists$ linear map P

$$\frac{\left(\mathbf{X}^1_{0,1}\right)^{\otimes 2}}{2!} = P\left[\mathbf{X}^2_{0,1}\right],$$

  - Taking log eliminate algebraic dependencies.

- For $a, b \in T\left(\left(\mathbb{R}^d\right)\right)$, define

$$[a, b] = a \otimes b - b \otimes a$$

$$[V, W] = \operatorname{span}\{[a, b] : a \in V, b \in W\}.$$

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$$\log\left(S(X)_{0,1}\right) \in \mathbb{R}^d \times \left[\mathbb{R}^d, \mathbb{R}^d\right] \times \cdots$$

  - Not every element of $\mathbb{R}^d \times \left[\mathbb{R}^d, \mathbb{R}^d\right] \times \cdots$ is a log signature!

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Radius of convergence

A conjecture of Lyons-Sidorova

Let

$$\log S(X)_{0,1} = \underbrace{LS_1(X)}_{\in \mathbb{R}^d} + \underbrace{LS_2(X)}_{\in \left(\mathbb{R}^d\right)^{\otimes 2}} + \cdots .$$

Then $\log S(X)_{0,1}$ has *finite radius of convergence (ROC)* ie $\exists \lambda > 0$,

$$\sum_{n=0}^{\infty} \lambda^n \| LS_n(X) \| = \infty$$

- for all finite variation X except when X is a straight line.
- $\| \cdot \|$ can be projective norm/Hilbert Schmidt/Injective norm.

A class of counter examples

- If X is a straight line, then

$$\log\left(S\left(\overleftarrow{a}\star X\star a\right)_{0,1}\right)$$

has infinite ROC for any bounded variation path a.

  - $\overleftarrow{a}_t = a_{1-t}$.
  - $\star$ is concatenation of paths (joining paths).

- **Proof:**

$$\log\left(S\left(\overleftarrow{a}\star X\star a\right)_{0,1}\right) = \underbrace{S\left(\overleftarrow{a}\right)_{0,1}}_{\infty\text{ ROC}}\otimes\underbrace{\log\left(S\left(X\right)_{0,1}\right)}_{\infty\text{ ROC}}\otimes\underbrace{S\left(a\right)_{0,1}}_{\infty\text{ ROC}}.$$

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Modified conjecture

Modified Lyons-Sidorova Conjecture

If X has bounded variation, $\log S(X)_{0,1}$ has finite ROC except if X is **conjugate to** a straight line L.

Previous works

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$$\log S(X)_{0,1} = \underbrace{LS_1(X)}_{\in \mathbb{R}^d} + \underbrace{LS_2(X)}_{\in (\mathbb{R}^d)^{\otimes 2}} + \cdots$$
then $\exists C > 0$
$$\| LS_n(X) \| \leq C^n \qquad \forall n$$
    - due to Lyons-Sidorova for bounded variation X
    - due to Chevyrev-Lyons for rough paths X.
- Lyons and Sidorova proved special cases:
    - One coordinate of X is monotonic, OR
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- Friz-Lyons-Seigal proved if $\log S(X)_{0,1}$ cannot be a Lie polynomial except if X is tree-like equivalent to a straight line.

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Ideas from Lyons-Sidorova

- Define homomorphism

$$\mathbf{M} : T\left(\left(\mathbb{R}^2\right)\right) \to \mathfrak{sl}_2\left(\mathbb{R}\right) \text{ (2×2 zero-trace matrices)}$$

by

$$\mathbf{M}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} c_1 v_1 & c_2 v_2 \\ -c_2 v_2 & -c_1 v_1 \end{pmatrix}$$

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Lyons-Sidorova's argument

- $$\log S(X)_{0,1} \text{ has infinite ROC}$$
$$\Longrightarrow \mathbf{M}\left(\log S(X)_{0,1}\right) \in \mathfrak{sl}_2(\mathbb{R})$$

- $$\Longrightarrow \mathbf{M}\left(S(X)_{0,1}\right) = \exp\left(\mathbf{M}\left(\log S(X)_{0,1}\right)\right) \in \exp\left(\mathfrak{sl}_2(\mathbb{R})\right).$$
$$\Longrightarrow \int_0^T e^{2i\lambda y_t} \mathrm{d}e^{2(2k+1)i\pi x_t} = 0 \qquad \forall k \in \mathbb{Z}, \lambda \in \mathbb{R}.$$

- Final step uses $Y_t = \mathbf{M}\left(S(X)_{0,t}\right)$ satisfies
$$\mathrm{d}Y_t = \mathbf{M}(\mathrm{d}X_t)\, Y_t, \qquad Y_0 = I.$$

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A main result

Theorem (B.-Geng-Wang 25)

Let $X : t \to (x_t, y_t)$ be a bounded variation path such that $X_0 = (0,0)$ and $X_1 = (1,0)$. If $\log S(X)_{0,1}$ has infinite ROC, then

1. for all $k \in \mathbb{Z}, \lambda \in \mathbb{R}$

$$\int_0^1 e^{2k\pi \mathrm{i} x_s + \lambda \mathrm{i} y_s} \mathrm{d}y_s = 0;$$

2. for all $p, q \in \mathbb{Z} \backslash \{0\}$ with $p + q \neq 0$,

$$\int_{0<s<t<1} \left(e^{2\pi \mathrm{i}(px_s + qx_t)} - e^{2\pi \mathrm{i}(px_t + qx_s)} \right) \mathrm{d}y_s \mathrm{d}y_t = 0;$$

3. for all $m, p_1, \ldots, p_m \in \mathbb{N}$, the coefficient of $e_1 \otimes \cdots \otimes e_m$ in the log signature of

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Baker-Campbell-Hausdorff formula

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$$e^{v} \otimes e^{w} = e^{\sum_n D_n H},$$

where

$$D_n H \text{ is degree } n \text{ in the letter } v.$$

- $$D_n H = v + \frac{1}{2}[v, w] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (\mathrm{ad}_w)^{2n} v,$$

where

$$\mathrm{ad}_w(v) = [w, v],$$

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m.$$

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Variation of constants (Agrachev et al. 1989)

- Let $\{e_1, e_2\}$ be standard basis in $\mathbb{R}^2$.
- Let $t \to X_t = x_t e_1 + y_t e_2$ be $(x_0, y_0) = (0, 0)$, $(x_1, y_1) = (1, 0)$.
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- Define

$$D_n g = \text{projection of } g \text{ to terms with } n\ e_2\text{s}.$$

- By BCH formula + representation for signature

$$\Longrightarrow D_1 \log \left(S\left(X\right)_{0,1} \right) = \sum_{m=0}^{\infty} \frac{B_m}{m!} \mathrm{ad}_{e_1}^m \left(\int_0^1 e^{x_t \mathrm{ad}_{e_1}} \left(e_2\right) \mathrm{d} y_t \right),$$

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Projection to matrices

- Let A, D be 2×2 matrices

$$[A, D] = D.$$

- Define

$$F_\lambda(\mathbf{e}_1) = \lambda A, \qquad F_\lambda(\mathbf{e}_2) = D$$
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- $\log S(X)_{0,1}$ has infinite ROC
- $\implies \frac{\lambda}{e^\lambda - 1}\int_0^1 e^{\lambda x_t} \mathrm{d}y_t D$ well-defined for all λ.
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Higher order iterated integral conditions

$\mathfrak{sl}_3(\mathbb{C})$ has followings as basis

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

If $[A, B] = AB - BA$, then $[H_{ij}, H_{kl}] = 0$, $[H_{ij}, E_{kl}] \in \text{span}(E_{kl})$

$$[E_{ij}, E_{kl}] = \begin{cases} E_{il}, & \text{if } j = k, i \neq l \\ H_{il}, & \text{if } j = k, i = l \\ -E_{kj} & \text{if } j \neq k, i = l \\ 0, & \text{otherwise.} \end{cases}$$

Higher order iterated integral conditions

- Higher iterated integral conditions require
  - $\mathfrak{sl}_n(\mathbb{C})$ and **root space decomposition**.

- $\mathfrak{sl}_n(\mathbb{C})$ is set of zero-trace $n \times n$ matrices.

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{g}^{\alpha_1} \oplus \cdots \oplus \mathfrak{g}^{\alpha_n},$$

where

- $\forall H_1, H_2 \in \mathfrak{h}$,

$$\mathrm{ad}_{H_1}(H_2) = 0;$$

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A weak version of LS conjecture

Theorem

Let $X : [0,1] \to \mathbb{R}^d$ be a bounded variation path. If $\log S(X)_{s,t}$ has infinite ROC for all $s < t$, then X is a straight line.

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References

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