The Schrödinger problem from Brownian to interacting particles – An Introduction –

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In 1931 E. Schrödinger addressed the following question: suppose that

> *...wir hätten das Teilchen zur Zeit t_0 bei x_0, zur Zeit t_1 bei x_1 angetroffen [...] Ein Hilfsbeobachter hat die Lage des Teilchens zur Zeit t beobachtet, jedoch ohne uns sein Ergebnis mitzuteilen. Die Frage lautet dann: welche Wahrscheinlichkeitsschlüsse können wir aus unseren zwei Beobachtungen auf die Zwischenbeobachtung unseres Helfers ziehen?*

whence, in 1932, the following thought experiment:

> *Imaginez que vous observez un système de particules en diffusion, qui soient en équilibre thermodynamique. Admettons qu'à un instant donné t_0 vous les ayez trouvées en répartition à peu près uniforme et qu'à $t_1 > t_0$ vous ayez trouvé un écart spontané et considérable par rapport à cette uniformité. On vous demande de quelle manière cet écart s'est produit. Quelle est la manière la plus probable ?*

What is the Schrödinger problem?

Setting: N i.i.d. diffusive particles $X_t^1, \dots, X_t^N$ evolving according to

$$\mathrm{d}X_t^i = -\nabla U(X_t^i)\mathrm{d}t + \mathrm{d}B_t^i, \quad X_0^i \sim \mathfrak{m} \qquad i = 1, \dots, N$$

$$\text{empirical distribution} \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}, \quad t \in [0, T].$$

If $\mu_0^N \approx \mu$, then $\mu_T^N \approx \mathsf{P}_T\mu$. So $\mu_T^N \approx \nu$ is unlikely for $\nu \neq \mathsf{P}_T\mu$.

What is the most likely evolution between μ and ν?

If we look at the probability of the joint event $(\mu_0^N \approx \mu, \, \mu_T^N \approx \nu)$, then

$$\mathrm{Prob}[\mu_0^N \approx \mu, \, \mu_T^N \approx \nu] \asymp \exp\big(-N\mathcal{C}_T(\mu,\nu)\big)$$

$$\mathcal{C}_T(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \mathcal{H}(\pi \,|\, R_{0T}) \quad \text{where } R_{0T} \text{ joint law of } (X_0, X_T)$$

$$= \inf_{Q \,:\, Q_0 = \mu, \, Q_T = \nu} \mathcal{H}(Q \,|\, R) \quad \text{where } R \text{ path measure law of } X$$

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Structure of the minimizer

The problem $\inf_{\pi \in \Pi(\mu,\nu)} \mathcal{H}(\pi \mid R_{0T})$ has a unique minimizer π^T if μ, ν have finite entropy.

- existence: direct method of calculus of variations
- uniqueness: strict convexity of the entropy

How does π^T look like? The Euler equation reads as $\int \log\left(\frac{\mathrm{d}\pi^T}{\mathrm{d}R_{0T}}\right)\mathrm{d}\sigma = 0$ for every σ such that $p^1_*\sigma = p^2_*\sigma = 0$. This forces

$$\log\left(\frac{\mathrm{d}\pi^T}{\mathrm{d}R_{0T}}\right)(x,y) = a^T(x) + b^T(y).$$

Thus for $f^T := \exp(a^T), g^T := \exp(b^T)$ we have

$$\pi^T = f^T \otimes g^T R_{0T},$$

namely (f^T, g^T) is a solution to the Schrödinger system

$$\begin{cases} \mu = f^T \, \mathrm{P}_T(g^T) \, \mathfrak{m} \\ \nu = \mathrm{P}_T(f^T) \, g^T \, \mathfrak{m} \end{cases}$$

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The dynamics of entropic interpolations

The entropic interpolation/Schrödinger bridge between μ and ν is then defined by

$$\rho_t^T := \mathsf{P}_t(f^T)\mathsf{P}_{T-t}(g^T), \quad t \in [0, T],$$

Its dynamics can be described in a threefold way. The first two have to deal with a control problem interpretation of SP, coming from the observation that

$$\begin{cases} \varphi_t^T := \log \mathsf{P}_t f^T \\ \partial_t \varphi_t^T = \frac{1}{2}|\nabla \varphi_t^T|^2 + \frac{1}{2}\Delta \varphi_t^T \\ -\partial_t \rho_t^T + \mathrm{div}(\nabla \varphi_t^T \rho_t^T) = \frac{1}{2}\Delta \rho_t^T \end{cases} \qquad \begin{cases} \psi_t^T := \log \mathsf{P}_{T-t} g^T \\ -\partial_t \psi_t^T = \frac{1}{2}|\nabla \psi_t^T|^2 + \frac{1}{2}\Delta \psi_t^T \\ \partial_t \rho_t^T + \mathrm{div}(\nabla \psi_t^T \rho_t^T) = \frac{1}{2}\Delta \rho_t^T \end{cases}$$

whence, after Mikami '04, Léonard '14, Gentil-Léonard-Ripani '17, Gigli-T. '18,

$$\begin{aligned} \mathcal{C}_T(\mu,\nu) &= \mathcal{H}(\mu\,|\,\mathrm{m}) + \min_{FFP} \iint_0^T \frac{|v_t|^2}{2}\rho_t \mathrm{d}t\mathrm{d}\mathrm{m} \\ &= \mathcal{H}(\nu\,|\,\mathrm{m}) + \min_{BFP} \iint_0^T \frac{|v_t|^2}{2}\rho_t \mathrm{d}t\mathrm{d}\mathrm{m} \end{aligned}$$

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Yet another (symmetric) variational representation

Beside optimal control formulations, SP also admits a 'noised' optimal transport interpretation, arising from the consideration that

$$v_t^T := \frac{1}{2}\nabla(\psi_t^T - \varphi_t^T), \qquad \partial_t \rho_t^T + \mathrm{div}(v_t^T \rho_t^T) = 0$$

After Chen-Georgiou-Pavon '16 and Gigli-T. '18, leveraging on ideas by Nelson '67,

$$\mathcal{C}_T(\mu,\nu) = \frac{1}{2}\Big(\mathcal{H}(\mu\,|\,\mathfrak{m}) + \mathcal{H}(\nu\,|\,\mathfrak{m})\Big) + \min_{CE} \iint_0^T \Big(\frac{|v_t|^2}{2} + \frac{1}{8}|\nabla \log \rho_t|^2\Big)\rho_t \mathrm{d}t\mathrm{d}\mathfrak{m}$$

which has to be compared with the celebrated Benamou-Brenier formula

$$\frac{1}{2}W_2^2(\mu,\nu)^2 = \min_{CE} \iint_0^1 \frac{|v_t|^2}{2}\rho_t \mathrm{d}t\mathrm{d}\mathfrak{m}.$$

A strictly related quantity (which will play an important role in the following) is

$$\mathcal{E}_T(\mu,\nu) := \int \Big(\frac{|v_t^T|^2}{2} - \frac{1}{8}|\nabla \log \rho_t^T|^2\Big)\rho_t^T \mathrm{d}\mathfrak{m} = -\int \langle \nabla\psi_t^T, \nabla\varphi_t^T\rangle \rho_t^T \mathrm{d}\mathfrak{m}$$

i.e. the energy of the system (=constant in $t \in [0,T]$).

Why the Schrödinger problem?

Small-time regime: $\rightsquigarrow$ optimal transport

$$\inf_{\pi \in \Pi(\mu,\nu)} \mathcal{H}(\pi \,|\, R_{0T}) \approx \inf_{\pi \in \Pi(\mu,\nu)} \frac{1}{2} \int |x-y|^2 \pi(\mathrm{d}x\mathrm{d}y) + T\mathcal{H}(\pi \,|\, \mu \otimes \nu)$$

$$T\mathcal{C}_T(\mu,\nu) = \frac{T}{2}\Big(\mathcal{H}(\mu \,|\, \mathfrak{m}) + \mathcal{H}(\nu \,|\, \mathfrak{m})\Big) + \min_{CE} \iint_0^1 \Big(\frac{|v_t|^2}{2} + \frac{T^2}{8}|\nabla \log \rho_t|^2\Big)\rho_t \mathrm{d}t\mathrm{d}\mathfrak{m}$$

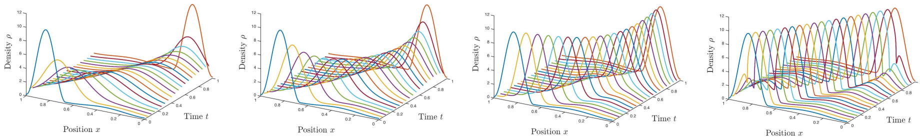

Figure: SB with $\sqrt{T} = 0.5$, $\sqrt{T} = 0.3$, $\sqrt{T} = 0.15$; displacement interpolation

Long-time regime: $\rightsquigarrow$ turnpike property/quantified convergence to equilibrium

Literature review

Convergence of the cost in the small-time regime

- 0th order: $v_\varepsilon = v_0 + o(1)$
  Mikami '04, Mikami-Thieullen '08, Léonard '12, Carlier-Duval-Peyré-Schmitzer '17
- 1st order: $v_\varepsilon = v_0 + \frac{d}{2}\varepsilon \log(1/\varepsilon) + \varepsilon \frac{\mathcal{H}(\mu^- \mid \mathcal{L}^d) + \mathcal{H}(\mu^+ \mid \mathcal{L}^d)}{2} + o(\varepsilon)$
  Adams-Dirr-Peletier-Zimmer '11, Erbar-Maas-Renger '15, Pal '19, Carlier-Pegon-T. '22
- 2nd order: $v_\varepsilon = v_0 + \frac{d}{2}\varepsilon \log(1/\varepsilon) + \varepsilon \frac{\mathcal{H}(\mu^- \mid \mathcal{L}^d) + \mathcal{H}(\mu^+ \mid \mathcal{L}^d)}{2} + \frac{\varepsilon^2}{4} \int_0^1 \mathcal{I}(\mu_t^0) \mathrm{d}t + o(\varepsilon^2)$
  Conforti-T. '19, Chizat-Roussillon-Léger-Vialard-Peyré '20

Convergence of the potentials Nutz-Wiesel '21

Convergence of the gradients of the potentials

- qualitative: Chiarini-Conforti-Greco-T. '22
- quantitative: Pooladian-Niles-Weed '22, Carlier-Pegon-T. '22,

The results

Theorem (Conforti-T. '19)

Let M be a Riemannian manifold satisfying CD(κ, N) with $\kappa \in \mathbb{R}$, $N < \infty$; $\mu, \nu \ll \mathfrak{m}$ with bounded densities and supports. Then

$$T\mathcal{C}_T(\mu,\nu) = \frac{1}{2}W_2^2(\mu,\nu) + \frac{T}{2}\Big(\mathcal{H}(\mu\,|\,\mathfrak{m}) + \mathcal{H}(\nu\,|\,\mathfrak{m})\Big) + \frac{T^2}{8}\int_0^1 \mathcal{I}(\mu_t^0)\,\mathrm{d}t + o(T^2).$$

Theorem (Conforti-T. '19)

Let M be a Riemannian manifold, $\mathfrak{m} = e^{-V}\mathrm{vol}$ with $\mathrm{Ric}_g + \mathrm{Hess}(V) \geq \kappa > 0$; $\mu, \nu \ll \mathfrak{m}$ with bounded densities and supports. Then $\mathcal{C}_T(\mu,\nu) \to \mathcal{H}(\mu\,|\,\mathfrak{m}) + \mathcal{H}(\nu\,|\,\mathfrak{m})$ as $T \to \infty$ with

$$|\mathcal{C}_T(\mu,\nu) - \mathcal{H}(\mu\,|\,\mathfrak{m}) - \mathcal{H}(\nu\,|\,\mathfrak{m})| \leq \frac{2}{e^{\kappa T/2} - 1}\Big(\mathcal{H}(\mu\,|\,\mathfrak{m}) + \mathcal{H}(\nu\,|\,\mathfrak{m})\Big).$$

Rmk: the rate $e^{-\kappa T/2}$ is sharp and can be checked on $(\mathbb{R}^d, \gamma_d)$.

Rmk: if $\kappa = 0$ (and $\mathfrak{m}(M) = \infty$), then $\mathcal{C}_T(\mu,\nu)$ may diverge as $T \to \infty$, but not faster than $\log T$, see Clerc-Conforti-Gentil '20.

A first step for both results

First step: the cost $T \mapsto \mathcal{C}_T(\mu,\nu)$ is $C^1((0,\infty))$ and there holds

$$\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{C}_T(\mu,\nu) = -\mathcal{E}_T(\mu,\nu)$$

Idea: rescale

$$\mathcal{C}_T(\mu,\nu) = \frac{1}{2}\Big(\mathcal{H}(\mu\,|\,\mathfrak{m}) + \mathcal{H}(\nu\,|\,\mathfrak{m})\Big) + \min_{CE}\iint_0^T \Big(\frac{|v_t|^2}{2} + \frac{1}{8}|\nabla\log\rho_t|^2\Big)\rho_t \mathrm{d}t\mathrm{d}\mathfrak{m}$$

into

$$T\mathcal{C}_T(\mu,\nu) = \frac{T}{2}\Big(\mathcal{H}(\mu\,|\,\mathfrak{m}) + \mathcal{H}(\nu\,|\,\mathfrak{m})\Big) + \min_{CE}\iint_0^1 \Big(\frac{|v_t|^2}{2} + \frac{T^2}{8}|\nabla\log\rho_t|^2\Big)\rho_t \mathrm{d}t\mathrm{d}\mathfrak{m}$$

and use the envelope theorem.

Key ingredient: for all $0 \leq T_1 < T_2 < \infty$ it holds

$$\iint_0^1 |\nabla\log\rho_t^{T_1}|^2\rho_t^{T_1}\mathrm{d}t\mathrm{d}\mathfrak{m} \geq \iint_0^1 |\nabla\log\rho_t^{T_2}|^2\rho_t^{T_2}\mathrm{d}t\mathrm{d}\mathfrak{m}.$$

Small-time asymptotics: proof

Upper bound: integrate $\frac{\mathrm{d}}{\mathrm{d}T}(T\mathcal{C}_T(\mu,\nu)) = \mathcal{C}_T(\mu,\nu) - T\mathcal{E}_T(\mu,\nu)$

$$T\mathcal{C}_T(\mu,\nu) - \frac{1}{2}W_2^2(\mu,\nu) = \frac{T}{2}(H_0+H_1) + \int_0^T \frac{\tau}{4}\iint_0^1 |\nabla \log \rho_t^\tau|^2 \rho_t^\tau \,\mathrm{d}t\mathrm{d}\mathfrak{m}\,\mathrm{d}\tau$$
$$\leq \frac{T}{2}(H_0+H_1) + \frac{T^2}{8}\iint_0^1 |\nabla \log \rho_t^0|^2 \rho_t^0 \,\mathrm{d}t\mathrm{d}\mathfrak{m}$$

Lower bound

$$T\mathcal{C}_T(\mu,\nu) - \frac{1}{2}W_2^2(\mu,\nu) = \frac{T}{2}(H_0+H_1) + \frac{T^2}{8}\iint_0^1 |\nabla \log \rho_t^T|^2 \rho_t^T \,\mathrm{d}t\mathrm{d}\mathfrak{m}$$
$$+ \frac{1}{2}\iint_0^1 |v_t^T|^2 \rho_t^T \,\mathrm{d}t\mathrm{d}\mathfrak{m} - \frac{1}{2}\iint_0^1 |v_t^0|^2 \rho_t^0 \,\mathrm{d}t\mathrm{d}\mathfrak{m}$$
$$\geq \frac{T}{2}(H_0+H_1) + \frac{T^2}{8}\iint_0^1 |\nabla \log \rho_t^T|^2 \rho_t^T \,\mathrm{d}t\mathrm{d}\mathfrak{m}$$

- $T \mapsto \iint_0^1 |\nabla \log \rho_t^T|^2 \rho_t^T \,\mathrm{d}t\mathrm{d}\mathfrak{m}$ is continuous at $T=0$,
  $T \mapsto T\iint_0^1 |\nabla \log \rho_t^T|^2 \rho_t^T \,\mathrm{d}t\mathrm{d}\mathfrak{m}$ is cont. differentiable at $T=0$;
- refined convergence properties of ρ_t^T to ρ_t^0 [Gigli-T. '18]

Long-time asymptotics: proof

- Observe that $\mathcal{C}_T(\mu,\nu) \to \mathcal{H}(\mu\,|\,\mathfrak{m}) + \mathcal{H}(\nu\,|\,\mathfrak{m})$ as $T \to \infty$, e.g. proving that

$$\Gamma - \lim_{T\to\infty} \mathcal{H}(\cdot\,|\,R_{0T}) = \mathcal{H}(\cdot\,|\,\mathfrak{m}\otimes\mathfrak{m})$$

  on $\Pi(\mu,\nu)$ w.r.t. the narrow topology.

- Integrate $\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{C}_T(\mu,\nu) = -\mathcal{E}_T(\mu,\nu)$

$$|\mathcal{C}_T(\mu,\nu) - \mathcal{H}(\mu\,|\,\mathfrak{m}) - \mathcal{H}(\nu\,|\,\mathfrak{m})| = \left|\int_T^\infty \mathcal{E}_\tau(\mu,\nu)\mathrm{d}\tau\right| \le \int_T^\infty |\mathcal{E}_\tau(\mu,\nu)|\mathrm{d}\tau$$

- 'Energy-Transport' inequality (Conforti-T. '19, Backhoff-Conforti-Gentil-Léonard '19)

$$|\mathcal{E}_T(\mu,\nu)| \le \frac{\kappa}{e^{\kappa T/2}-1}\sqrt{(\mathcal{C}_T(\mu,\nu)-\mathcal{H}(\mu\,|\,\mathfrak{m}))(\mathcal{C}_T(\mu,\nu)-\mathcal{H}(\nu\,|\,\mathfrak{m}))}$$

- 'Entropic' Talagrand inequality (Conforti '17, Conforti-T. '19)

$$\mathcal{C}_T(\mu,\nu) \le \frac{1}{1-e^{-\kappa Tt}}\mathcal{H}(\mu\,|\,\mathfrak{m}) + \frac{1}{1-e^{-\kappa T(1-t)}}\mathcal{H}(\nu\,|\,\mathfrak{m})$$

The role of $\mathrm{Ric}_g + \mathrm{Hess}(V) \geq \kappa$

- (Bakry-Émery) From $\Gamma_2 \geq \kappa\Gamma$ it follows that $h_f(t) := \int \varphi_t^T \rho_t^T \,\mathrm{d}\mathfrak{m}$ is such that

$$h_f'(t) = \frac{1}{2}\int |\nabla\varphi_t^T|^2 \rho_t^T \,\mathrm{d}\mathfrak{m}, \quad h_f''(t) = \frac{1}{2}\int \Gamma_2(\varphi_t^T)\rho_t^T \,\mathrm{d}\mathfrak{m},$$

whence $h_f''(t) \geq -\kappa h_f'(t)$. This implies the 'Energy-Transport' inequality as well as the corrector estimate

$$\|\nabla \log \rho_t^T - \dot{\rho}_t^T\|_{L^2(\rho_t^T)} = 4\|\nabla\varphi_t^T\|_{L^2(\rho_t^T)} \lesssim e^{-\kappa(T-t)}.$$

- (Lott-Sturm-Villani) The entropy $\mathcal{H}$ satisfies a 'distorted' κ-convexity inequality

$$\mathcal{H}(\mu_t^T \,|\, \mathfrak{m}) \leq \alpha_{\kappa,T}(1-t)\mathcal{H}(\mu \,|\, \mathfrak{m}) + \alpha_{\kappa,T}(t)\mathcal{H}(\nu \,|\, \mathfrak{m}) + \beta_{\kappa,T}(t)\mathcal{C}_T(\mu,\nu)$$

with $\alpha_{\kappa,T}(t) \to t$, $\beta_{\kappa,T}(t) \to -\kappa Tt(1-t)$ as $T \to 0$, so that in the limit

$$\mathcal{H}(\mu_t^0 \,|\, \mathfrak{m}) \leq (1-t)\mathcal{H}(\mu \,|\, \mathfrak{m}) + t\mathcal{H}(\nu \,|\, \mathfrak{m})\frac{\kappa}{2}t(1-t)W_2^2(\mu,\nu).$$

The 'Entropic' Talagrand inequality is a corollary.

What is the MFSP?

From i.i.d. diffusive particles

$$\mathrm{d}X_t^i = -\nabla U(X_t^i)\mathrm{d}t + \mathrm{d}B_t^i, \quad X_0^i \sim \mathfrak{m} \qquad t \in [0, T]$$

$$\Downarrow$$

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{j=1}^{N} \nabla W(X_t^i - X_t^j)\mathrm{d}t + \mathrm{d}B_t^i, \quad X_0^i \sim \mu \qquad t \in [0, T]$$

to interacting particles (Backhoff-Conforti-Gentil-Léonard '19).

Assumption: $W \in C^2(\mathbb{R}^d; \mathbb{R})$ symmetric with $\kappa \le \mathrm{Hess}(W) \le K$.

The empirical distributions satisfy an LDP with rate function

$$\tilde{\mathcal{C}}_T^{mf}(\mu, \nu) := \inf_{Q\,:\, Q_0 = \mu,\, Q_T = \nu} \mathcal{H}(Q \,|\, \Gamma(Q))$$

where $\Gamma(Q)$ is the law of the unique strong solution of the McKean-Vlasov SDE

$$\mathrm{d}X_t = -\nabla W * Q_t(X_t)\mathrm{d}t + \mathrm{d}B_t, \quad X_0 \sim \mu$$

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Control formulation and optimality conditions

By Girsanov theorem

$$\tilde{\mathcal{C}}_T^{mf}(\mu,\nu) = \inf \frac{1}{2}\mathbb{E}_Q\Big[\int_0^T |\alpha_t^Q|^2 \mathrm{d}t\Big]$$

where the infimum runs over all Q's such that $Q_0 = \mu$, $Q_T = \nu$ and the law of $X_t - \int_0^t (-\nabla W * P_s(X_s) + \alpha_s^Q)\mathrm{d}s$ under Q is the Wiener measure with starting distribution μ.

For the optimal (Q, α^Q) it holds $\alpha_t^Q = \nabla\psi_t^T$ and the optimality conditions for $\mu_t^T := (X_t)_* Q^T$ and ψ_t^T read as

$$\begin{cases} -\partial_t \psi_t^T = \frac{1}{2}|\nabla\psi_t^T|^2 + \frac{1}{2}\Delta\psi_t^T - \int \nabla W(x-y)\cdot(\nabla\psi_t^T(x) - \nabla\psi_t^T(y))\mathrm{d}\mu_t^T(y) \\ \partial_t\mu_t^T + \mathrm{div}(\nabla\psi_t^T - \nabla W * \mu_t^T)\mu_t^T) = \frac{1}{2}\Delta\mu_t^T \end{cases}$$

The optimality conditions for the 'backward' MFSP are instead

$$\begin{cases} \partial_t \varphi_t^T = \frac{1}{2}|\nabla\varphi_t^T|^2 + \frac{1}{2}\Delta\varphi_t^T - \int \nabla W(x-y)\cdot(\nabla\varphi_t^T(x) - \nabla\varphi_t^T(y))\mathrm{d}\mu_t^T(y) \\ -\partial_t\mu_t^T + \mathrm{div}(\nabla\varphi_t^T - \nabla W * \mu_t^T)\mu_t^T) = \frac{1}{2}\Delta\mu_t^T \end{cases}$$

Control formulation and optimality conditions

By Girsanov theorem

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Analogies with 'classical' SP

- The link between forward and backward controls is given by
$$\nabla\varphi_t^T + \nabla\psi_t^T = \nabla \log \mu_t^T + 2\nabla W * \mu_t^T .$$
- The energy is defined as
$$\mathcal{E}_T^{mf}(\mu,\nu) := -\frac{1}{2}\int \nabla\psi_t^T \cdot \nabla\varphi_t^T \, \mathrm{d}\mu_t^T$$
- There is a third variational representation à la Benamou-Brenier:
$$\mathcal{C}_T^{mf}(\mu,\nu) = \frac{\mathcal{F}(\mu)+\mathcal{F}(\nu)}{2} + \min_{CE} \iint_0^T \left(\frac{|v_t|^2}{2} + \frac{1}{8}|\nabla \log \rho_t + 2\nabla W * \rho_t|^2\right)\rho_t \mathrm{d}t\mathrm{d}\mathfrak{m}$$

The Boltzmann entropy $\mathcal{H}$ is here replaced by
$$\mathcal{F}(\mu) := \mathcal{H}(\mu\,|\,\mathcal{L}^d) + \int W * \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^d}\mathrm{d}\mu - \mathcal{F}(\mu_\infty)$$
where μ_∞ is the (unique) equilibrium measure [McCann '97].
$$\mathcal{C}_T^{mf}(\mu,\nu) := \mathcal{F}(\mu) + \inf_{Q\,:\,Q_0=\mu,\,Q_T=\nu} \mathcal{H}(Q\,|\,\Gamma(Q))$$

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The results

Theorem (Conforti-T. '19)

Under the previous assumption, for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathcal{F}(\mu), \mathcal{F}(\nu) < \infty$ and $\int x\mu(\mathrm{d}x) = \int x\nu(\mathrm{d}x)$ it holds

$$T\mathcal{C}_T^{mf}(\mu,\nu) = \frac{1}{2}W_2^2(\mu,\nu) + \frac{T}{2}\Big(\mathcal{F}(\mu) + \mathcal{F}(\nu)\Big) + \frac{T^2}{8}\int_0^1 |\partial\mathcal{F}|^2(\mu_t^0)\,\mathrm{d}t + o(T^2),$$

where

$$|\partial\mathcal{F}|^2(\mu) = \|\nabla^{W_2}\mathcal{F}\|^2_{L^2(\mu)} = \int |\nabla \log\rho + 2\nabla W * \rho|^2\,\mathrm{d}\mu$$

if $\mu = \rho\mathfrak{m}$, $+\infty$ else.

Theorem (Conforti-T. '19)

If in addition $\kappa > 0$, then $\mathcal{C}_T^{mf}(\mu,\nu) \to \mathcal{F}(\mu) + \mathcal{F}(\nu)$ as $T \to \infty$ and the rate of convergence is at least $e^{-\kappa T/2}$: there exists a decreasing function B such that

$$|\mathcal{C}_T^{mf}(\mu,\nu) - \mathcal{F}(\mu) - \mathcal{F}(\nu)| \le e^{-\kappa T/2}B(\kappa)(\mathcal{F}(\mu) + \mathcal{F}(\nu)).$$

Long-time asymptotics: proof

First step: the cost $T \mapsto \mathcal{C}_T^{mf}(\mu,\nu)$ is $AC_{loc}([0,\infty))$ and there holds

$$\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{C}_T^{mf}(\mu,\nu) = -\mathcal{E}_T^{mf}(\mu,\nu)$$

[Consequence of Monsaingeon-T.-Vorotnikov '20.]

Then we proceed as in the 'classical' SP:

- Integrate $\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{C}_T^{mf}(\mu,\nu) = -\mathcal{E}_T^{mf}(\mu,\nu)$

$$|\mathcal{C}_T^{mf}(\mu,\nu) - \mathcal{F}(\mu) - \mathcal{F}(\nu)| = \left|\int_T^\infty \mathcal{E}_\tau^{mf}(\mu,\nu)\mathrm{d}\tau\right| \leq \int_T^\infty |\mathcal{E}_\tau^{mf}(\mu,\nu)|\mathrm{d}\tau$$

- 'Energy-Transport' inequality (Backhoff-Conforti-Gentil-Léonard '19)

$$|\mathcal{E}_T^{mf}(\mu,\nu)| \leq \frac{4\kappa}{e^{\kappa T/2}-1}\sqrt{(\mathcal{C}_T^{mf}(\mu,\nu)-\mathcal{F}(\mu))(\mathcal{C}_T^{mf}(\mu,\nu)-\mathcal{F}(\nu))}$$

- 'Entropic' Talagrand inequality (Backhoff-Conforti-Gentil-Léonard '19)

$$\mathcal{C}_T^{mf}(\mu,\nu) \leq \frac{1}{1-e^{-\kappa T t}}\mathcal{F}(\mu) + \frac{1}{1-e^{-\kappa T(1-t)}}\mathcal{F}(\nu)$$

The metric Schrödinger problem

$$\min_{CE} \iint_0^1 \Big(\frac{|v_t|^2}{2} + \frac{T^2}{8}|\nabla \log \rho_t|^2\Big)\rho_t \mathrm{d}t\mathrm{d}\mathfrak{m} \quad \Rightarrow \quad \inf_{\gamma\,:\,\gamma_0 = x,\ \gamma_1 = y} \frac{1}{2}\int_0^1 \Big(|\dot\gamma_t|^2 + T^2|\partial \mathrm{E}|^2(\gamma_t)\Big)\mathrm{d}t$$

Assumptions:

- $\mathrm{E} : X \to \mathbb{R} \cup \{+\infty\}$ is lsc. with dense domain; there exists $\lambda \in \mathbb{R}$ such that for any $x \in X$ there exists an EVI_λ-gradient flow of E starting from x.
- (X, d) is a Polish space with a Hausdorff topology σ on X such that d-bounded sets are sequentially σ-compact. Moreover, d and $|\partial \mathrm{E}|$ are σ-sequentially lsc.

Examples:

- internal energy, i.e. $\mathrm{E}(\mu) := \int_M U(\rho)\mathrm{d}\mathfrak{m} + U'(\infty)\mu^\perp(M)$ if $\mu = \rho\mathfrak{m} + \mu^\perp$, on $(\mathscr{P}_2(M), W_2)$, M RCD$(0, N)$;
- (MFSP) on $(\mathscr{P}_2(\mathbb{R}^n), W_2)$, i.e. $\mathrm{E}(\mu) = \mathcal{H}(\mu\,|\,\mathcal{L}^n) + \int_{\mathbb{R}^n} W * \rho\,\mathrm{d}\mu$;
- internal energy on $(\mathscr{P}_2(\Omega), W_m)$, $\Omega \subset \mathbb{R}^n$ convex bounded, W_m induced by pseudo-Riemannian norm associated with (some) non-linear mobility m

$$\|\dot\mu\|_\mu^2 = \min_v \Big\{ \int_\Omega |v|^2 m(\mu) \,:\, \partial_t\mu_t + \mathrm{div}(vm(\mu)) = 0\Big\}.$$

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$$\|\dot\mu\|_\mu^2 = \min_v \left\{ \int_\Omega |v|^2 m(\mu) \;:\; \partial_t\mu_t + \mathrm{div}(vm(\mu)) = 0 \right\}.$$

The results

$$\mathscr{C}_\varepsilon(x,y) := \inf\left\{\frac{1}{2}\int_0^1\Big(|\dot\gamma_t|^2 + \varepsilon^2|\partial \mathsf{E}|^2(\gamma_t)\Big)\mathrm{d}t \,:\, \gamma\in AC([0,1],X),\ \gamma_0 = x,\ \gamma_1 = y\right\}.$$

Theorem (Monsaingeon-T.-Vorotnikov, '20)

- The map $\varepsilon \mapsto \mathscr{C}_\varepsilon(x,y)$ is $AC_{loc}([0,\infty))$, left- and right differentiable everywhere in $(0,\infty)$ with

$$\frac{\mathrm{d}^-}{\mathrm{d}\varepsilon}\mathscr{C}_\varepsilon(x,y) = 2\varepsilon \max_{\Lambda_\varepsilon(x,y)} \mathcal{I}, \qquad \frac{\mathrm{d}^+}{\mathrm{d}\varepsilon}\mathscr{C}_\varepsilon(x,y) = 2\varepsilon \min_{\Lambda_\varepsilon(x,y)} \mathcal{I},$$

  where $\Lambda_\varepsilon(x,y) := \mathrm{argmin}\mathscr{C}_\varepsilon(x,y)$ and $\mathcal{I}(\gamma) := \int_0^1 |\partial \mathsf{E}|^2(\gamma_t)\mathrm{d}t$.
- If there exists $\omega^0 \in \Lambda_0(x,y)$ with $\mathcal{I}(\omega^0) < \infty$, then $\varepsilon \mapsto \mathscr{C}_\varepsilon(x,y)$ is right differentiable also at $\varepsilon = 0$ with vanishing derivative and

$$\mathscr{C}_\varepsilon(x,y) - \mathscr{C}_0(x,y) = \varepsilon^2 \inf_{\Lambda_0(x,y)} \mathcal{I} + o(\varepsilon^2).$$

However, 'Energy-Transport' and 'Entropic' Talagrand inequalities are geometric, not purely metric!

The Schrödinger problem for lattice gases

Setting: indistinguishable interacting particles performing random walks on the macroscopic blow-up $\Lambda_N := N\Lambda \cap \mathbb{Z}^d$, $\Lambda \subset \mathbb{R}^d$.

In the diffusive limit $N \to \infty$, empirical distributions converge to the hydrodynamic limit equation

$$\partial_t \rho + \mathrm{div}(J(t,\rho)) = 0.$$

Macroscopic quantities: hydrodynamic current J, mobility χ, diffusion coefficient D, (for simplicity we assume no external field) linked via $J(t,\rho) = -D(\rho)\nabla\rho$ and Einstein's relation $D(\rho) = f''(\rho)\chi(\rho)$, f being the free energy.

Examples: simple exclusion process $\chi(\rho) = \rho(1-\rho)$, $J(t,\rho) = -\nabla\rho$; zero-range process $J(t,\rho) = -\chi'(\rho)\nabla\rho$.

The rate function quantifying the large deviations from the hydrodynamic limit is

$$\mathcal{I}_{[0,T]}(\rho,j) := \frac{1}{4}\int_0^T \int_\Lambda (j - J(t,\rho)) \cdot \chi(\rho)^{-1} \cdot (j - J(t,\rho))\mathrm{d}x\mathrm{d}t,$$

where ρ and j are linked through the continuity equation.

SPLG and optimality conditions

The associated SP is thus given by (Chiarini-Conforti-T. '21)

$$\mathcal{C}_T(\mu,\nu) := \inf \left\{\mathcal{I}_{[0,T]}(\rho,j) \,:\, \partial_t\rho + \mathrm{div}(j) = 0,\ \rho_0 = \mu,\ \rho_T = \nu\right\}$$

By first variation, the optimal (ρ, j) is such that $\chi(\rho)^{-1}\cdot(j - J(t,\rho)) = \nabla\psi$ for some corrector ψ, so that $\mathcal{C}_T(\mu,\nu)$ can be rewritten in 'forward' form as

$$\inf\left\{\frac{1}{4}\int_0^T\int_\Lambda |\nabla\psi|^2\chi(\rho)\mathrm{d}x\mathrm{d}t\right\}$$

under the constraints

$$\partial_t\rho_t + \mathrm{div}(\chi(\rho_t)(\nabla\psi_t - f''(\rho_t)\nabla\rho_t)) = 0, \quad \rho_0 = \mu,\ \rho_T = \nu.$$

The optimality conditions for the optimal (ρ_t^T, ψ_t^T) thus read as

$$\begin{cases} -\partial_t\psi_t^T = \dfrac{1}{2}\chi'(\rho_t^T)|\nabla\psi_t^T|^2 + D(\rho_t^T)\Delta\psi_t^T \\ \partial_t\rho_t^T + \mathrm{div}(\chi(\rho_t^T)\nabla\psi_t^T) = \mathrm{div}(D(\rho_t^T)\nabla\rho_t^T) \end{cases}$$

SPLG and optimality conditions

$\mathcal{C}_T(\mu,\nu)$ can also be rewritten in 'backward' form as

$$\inf\left\{\frac{1}{4}\int_0^T\int_\Lambda |\nabla\varphi|^2\chi(\rho)\mathrm{d}x\mathrm{d}t\right\}$$

under the constraints

$$-\partial_t\rho_t + \mathrm{div}(\chi(\rho_t)(\nabla\varphi_t - f''(\rho_t)\nabla\rho_t)) = 0,\ \rho_0 = \mu,\ \rho_T = \nu.$$

The optimality conditions for the optimal (ρ_t^T, φ_t^T) then read as

$$\begin{cases}\partial_t\varphi_t^T = \frac{1}{2}\chi'(\rho_t^T)|\nabla\varphi_t^T|^2 + D(\rho_t^T)\Delta\varphi_t^T \\ -\partial_t\rho_t^T + \mathrm{div}(\chi(\rho_t^T)\nabla\varphi_t^T) = \mathrm{div}(D(\rho_t^T)\nabla\rho_t^T)\end{cases}$$

and the link between forward and backward controls is given by

$$\varphi_t^T(x) + \psi_t^T(x) = 2f'(\rho_t(x)) \qquad (t,x)\in[0,T]\times\Lambda.$$

For $\chi(\rho)=\rho$, we recover the optimality conditions for the 'classical' SP.

Further analogies

- In the 'classical' SP, $\mathcal{H}$ is convex along Schrödinger bridges provided $\kappa \geq 0$.
- Now $\mathcal{H}$ is replaced by the quasi-potential

$$\mathcal{F}(\rho) := \int_\Lambda f(\rho)\,\mathrm{d}x$$

and, if g denotes the primitive of $\chi' D$, we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(\rho_t) = {} & \frac{1}{2}\int (D(\rho_t)\chi(\rho_t) - g(\rho_t))\big((\Delta\varphi_t)^2 + (\Delta\psi_t)^2\big)\,\mathrm{d}x \\ & + \frac{1}{2}\int g(\rho_t)\big(\Gamma_2(\varphi_t) + \Gamma_2(\psi_t)\big)\mathrm{d}x \\ & - \frac{1}{4}\int \big(|\nabla\varphi_t|^2 + |\nabla\psi_t|^2\big)|\nabla\rho_t|^2 D(\rho_t)\chi''(\rho_t)\mathrm{d}x\end{aligned}$$

- Following Carrillo-Lisini-Savaré-Slepčev '10, if

$$\chi'' \leq 0, \qquad D\chi \geq (1 - 1/d)g$$

then $\mathcal{F}$ is convex along Schrödinger bridges.

- While SP and FSMP are related to Wasserstein geometry and Otto calculus, SPLG is related to geometry of Dolbeault-Nazaret-Savaré distances!

Thank you for your attention!

Link with quantum mechanics

The Schrödinger equation

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\Delta\psi + i\mathrm{b}(t,x)\cdot\nabla\psi - V(t,x)\psi = 0$$

is a wave equation, thus no particle notion should be encoded in it, but Born's statistical interpretation (stemming from the fact that $\psi\overline{\psi}$ is a probability density) suggests the opposite.

- first attempt: Brownian motion (as particle with position but no velocity)
- better attempt: the Schrödinger problem $\rightsquigarrow$ time symmetry and Euclidean analogue of Born's identity

$$\rho_t^T = \mathrm{P}_t f^T \mathrm{P}_{T-t} g^T$$

with $\mathrm{P}_t f^T$ (resp. $\mathrm{P}_{T-t} g^T$) solving a forward (resp. backward) 'heat' equation.