Lecture 1

A. Note

Summer School: Probability & geometry on configuration spaces

July 2023

Lorentz Gas dynamics: particle systems & scaling limits with some perspectives on the role of long-range interactions

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1) The Paradigm of Kinetic Theory of Gases

- systems constituted by a very large number of (identical) components whose microscopic behaviour is based on the fundamental laws of mechanics (Newton Equations)

- Huge number of particles $\Longrightarrow$ behaviour of the system is too complicated to follow at MICROSCOPIC LEVEL and it is impossible to analyze

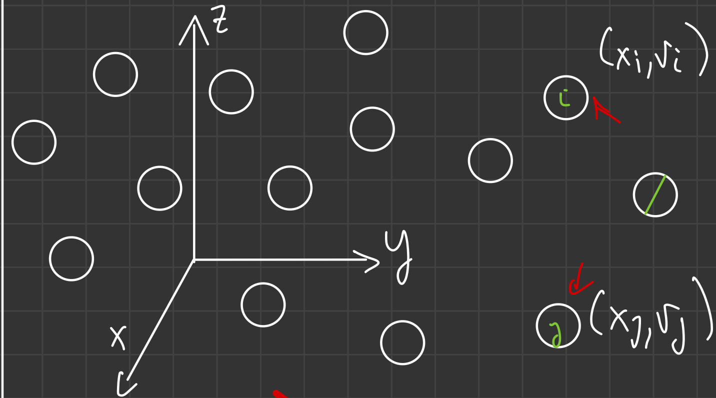
Ex: gases, planetary rings, plasmas, galaxies ...

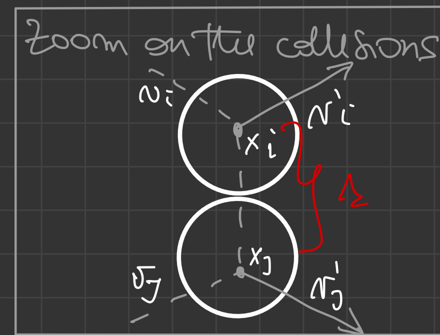
- Instead: look at the collective behaviour of the system on scales much longer than the ones characterizing the micro dynamics.

- On such MACROSCOPIC SCALES the system is "much" simpler to analyze and it is described by integro-differential equations for which the analysis is more feasible

- The problem of deriving these equations from the micro dynamics through suitable scaling limits is one of the central problem in non-equilibrium statistical mechanic

MICROSCOPIC DESCRIPTION





N particles
$\varepsilon > 0$ diameter

$z_i = (x_i, v_i) \in \mathbb{R}^3 \times \mathbb{R}^3$
$i = 1, \dots, N$

$N \simeq 10^{23}$

Kinetic limit

MESOSCOPIC DESCRIPTION

$$f : \underset{x}{\mathbb{R}^3} \times \underset{v}{\mathbb{R}^3} \times \underset{t}{\mathbb{R}_+} \longrightarrow \mathbb{R}_+ \quad , \quad f = f(x, v, t)$$

$$\underbrace{(\partial_t + v \cdot \nabla_x)}_{\text{FREE TRANSPORT}} f = \underbrace{Q(f, f)}_{\text{COLL. OP.}}$$

MICROSCOPIC DESCRIPTION

Newtonian dynamics for $(x_i(t), v_i(t))$ $i = 1, \dots, N$

$$(\text{N.eq})\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} F(x_i(t) - x_j(t)) \text{ "}+ F_{ext}\text{"} \end{cases}$$

interacting force; external force

$+$ in. cdt. $(x_i(0), v_i(t)) = (x_i, v_i)$

A state of the N-particle system is denoted

$$\underline{Z}_N = (\underline{X}_N, \underline{V}_N) = (x_1, \dots, x_N, v_1, \dots, v_N) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$$

phase space

·) due to the fact that the particles are <u>identical</u>

$$\mathcal{S} := \mathbb{R}^{3N} \times \mathbb{R}^{3N} / S_N \quad \rightarrow \text{permutation group.}$$

·) $m_i = 1 \quad \forall i = 1, \dots N$

·) $F_{ij} = F(x_i - x_j) = -\nabla_x \Phi(x_i - x_j)$

Φ interaction potential, $\Phi : \mathbb{R}^3 \to \mathbb{R}_+$

$\Phi(x) = \Phi(|x|)$

<u>Rk</u>: $\Phi \in C^2_b(\mathbb{R}^3) \Rightarrow \exists!$ of solution to (N. Eq.).

equivalently $\exists!$ $\quad Z_N \longrightarrow S^t(Z_N)$

Hamiltonian

(H) $$H(\underline{Z}_N) = \frac{1}{2}\sum_{i=1}^{N} v_i^2 + U(\underline{X}_N)$$

Kin. Pot.

$$U(\underline{X}_N) = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \Phi(x_i - x_j)$$

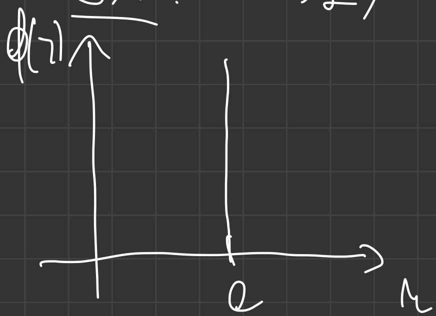
permutation invariance $$H(\underline{Z}_N) = H(\sigma(\underline{Z}_N)) \quad \forall \sigma \in S_N$$

Rk: (H.eq.)
$$\begin{cases} \dot{x}_i(t) = \nabla_{v_i} H(\underline{x}_N, \underline{v}_N) \\ \dot{v}_i(t) = -\nabla_{x_i} H(\underline{x}_N, \underline{v}_N). \end{cases} \quad \forall i = 1, \dots N$$

(equiv. to (N.eq))

Ex:

1) Hard-sphere pot.



$$\phi(r) = \begin{cases} 0 & r \geq a \\ +\infty & r < a \end{cases}$$

(a = hard-core diameter)

2) Coulomb pot.

$$\phi(r) = c\,\frac{Q}{r}, \quad c \in \mathbb{R}_+$$

Newtonian pot.

$$\phi(r) = -\frac{cM}{r}, \quad c \in \mathbb{R}_+$$

3) Inverse power law potentials

$$\phi(r) = \frac{1}{r^{s-1}} \qquad s > 2$$

STATISTICAL DESCRIPTION ?

introduce a prob. measure with density $f_0^N(\underline{z}_N)$ at time $t=0$

$$\left(f_0^N(\underline{X}_N, \underline{V}_N)\, \underbrace{d\underline{X}_N\, d\underline{V}_N}_{\text{Leb. meas.}} \right)$$

f_0^N satisfies

i) $f_0^N(\underline{X}_N, \underline{V}_N) \geq 0 \quad \forall (\underline{X}_N, \underline{V}_N) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$

ii) $$\iint_{\mathbb{R}^{3N}\,\mathbb{R}^{3N}} f_0^N(\underline{X}_N, \underline{V}_N)\, d\underline{X}_N\, d\underline{V}_N = 1$$

iii) f_0^N symmetric in the exchange of particles, Using Liouville Th. we obtain

$$\left[f^N(t, \underline{Z_N}) = f_0^N\left(S^{-t}(\underline{Z_N})\right)\right]$$

time evol. of the parts. mean

$f^N(t)$ is obtained by solving the Liouville Eq.

$$\frac{d}{dt} f^N(t) = \partial_t f^N(t) + \underline{V_N} \cdot \nabla_{\underline{X_N}} f^N(t) - (\nabla_{\underline{X_N}} U) \cdot \nabla_{\underline{V_N}} f^N(t) = 0$$

$$\left(\partial_t f^N + \{ f^N, H \} = 0 \right)$$

$$\sum_{i=1}^{N} v_i \cdot \nabla_{x_i}$$

$$\sum_{i=1}^{N} \nabla_{x_i} \Phi \cdot \nabla_{v_i}$$

$$(\text{L.E}_{\text{pr}})\quad \partial_t f^N(t) + V_N \cdot \nabla_{X_N} f^N(t) - \nabla_{X_N} U \cdot \nabla_{V_N} f^N(t) = 0$$

—

Kinetic Limit

(N, ϕ . . .

— (B.G. limit)

$N \to +\infty$, $\varepsilon \to 0$, $N\varepsilon^2 \to \lambda^{-1}$, $\lambda \in \mathbb{R}$
($N\varepsilon^2 = O(1)$), m.f.p.

($N\varepsilon^3 \to 0$ volume fraction)

$\sim$ ∘ $\backsim$ ∘

$N \simeq 10^{20}$, $\varepsilon \cong 10^{-8}$ cm

$N\varepsilon^3 = 10^{20} \cdot 10^{-24} = 10^{-4}$ cm, $N\varepsilon^2 = 10^{20} \cdot 10^{-16} = 10^4$ cm^2 $= 1\,m^2$

Other kinetic scalings:

- ∘) mean-field limit → <u>Vlasov Eq.</u> (collisionless plasma)
- ∘) weak-coupling limit → <u>Landau Eq.</u> (dense gas with weak collisions)
- ∘) High-density limit → <u>Lenard-Balescu Eq.</u>

<u>Ref</u>:

-) H. Spohn, Kinetic eq. from Hamiltonian dynamics Markovian limits, Rev. Mod. Phys. 1980

-) M. Pulvirenti, S. Simonella; Propagation of chaos and effective equations in kinetic theory, 2016

-) A. Nota, J. Velázquez, R. Winter; Interacting particle systems with long-range interactions: scaling limits & kinetic equations, Atti Acc. Lincei, 2021

Mesoscopic Description

1872.

$$\text{(B.Eq.)}\begin{cases} \partial_t f + v\cdot\nabla_x f = Q(f,f). \\ f = f(x,v,t) \quad : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+ \longrightarrow \mathbb{R}_+ \\ Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} dv_* \, d\omega \, \underbrace{B(v-v_*,\omega)}_{\geq 0} \left[f'f'_* - ff_*\right] \end{cases}$$

$$f' = f(v'), \quad f'_* = f(v'_*), \quad f_* = f(v_*), \quad f = f(v).$$

(v, v_*) incoming vel. $\longrightarrow$ (v', v'_*) outgoing vel.

$$\longrightarrow \quad (C) \begin{cases} v' = v - (\omega\cdot(v-v_*))\omega \\ v'_* = v_* + (\omega\cdot(v-v_*))\omega \end{cases}$$

(Rk: 1975 rigorous validation of the (B.Eq.) [O. Lanford]
[limitation: short time intervals])

PROPERTIES

P1) $Q(f,f)$ quadratic $\qquad (y = \varphi^2 \ldots)$

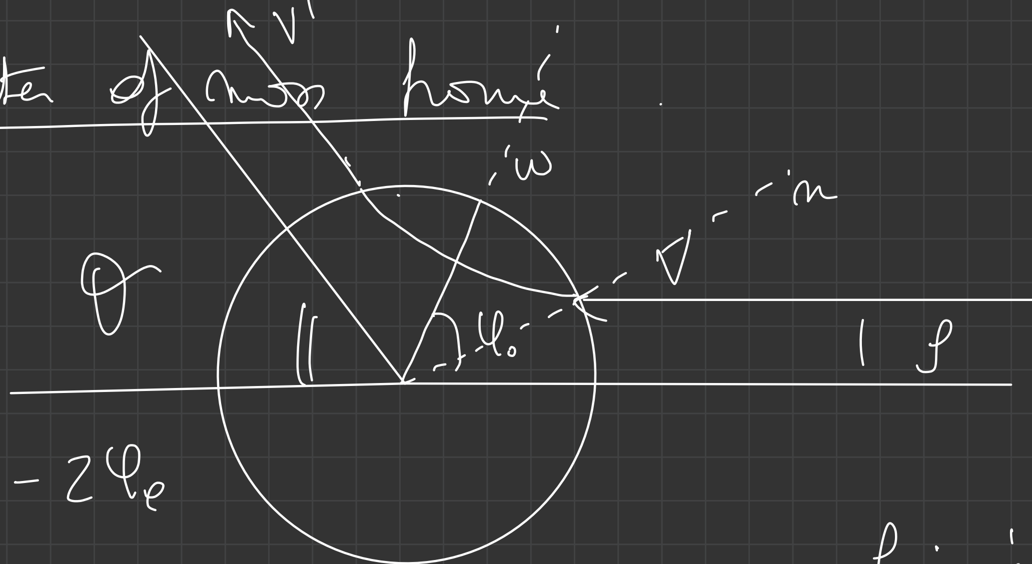
P2) x, t are parameters
(collisions are localized in space & time).

P3) collisions are elastic. < momentum / kin. energy are conserved

$$\begin{cases} v' + v'_* = v + v_* \\ (v')^2 + (v'_*)^2 = (v)^2 + (v_*)^2 \end{cases} \qquad \Vert$$

Center of mass frame

$$V = v - v_*$$
$$V' = v' - v'_*$$



θ

$$\theta = \pi - 2\varphi_0$$

ω : scattering vector

ρ : impact parameter

$$\theta \to \rho \qquad \underline{\theta}(\rho, V) = \pi - 2\int_{r_{min}}^{\infty} \frac{\rho}{r^2} \frac{dr}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{4\phi}{|V|}}}$$

$$B(v - v_*, \omega) = B(|v - v_*|, \omega)$$

$$B(|V|, \omega) = |V| \underbrace{\Sigma(\omega, |V|)} = |V| \frac{\rho}{\sin\theta} \cdot \frac{d\rho}{d\theta}$$

Ex: $B(|V|, \omega) = |V \cdot \omega|$ hard-sphere interactions.

Key RK: (different ϕ $\Rightarrow$ different B)

Validation: $\left[\underbrace{f_j^N(Z_N, t)}_{\text{particle marginal}} \xrightarrow[\text{BG limit}]{} \underbrace{f(t)^{\otimes j}}_{f \text{ is the unique sol. of B.Eq}} \right]$

Lecture 2

Summer School

Berlin 2023

A. Noto

Recap:

18.07.2023

(B.Eq.) $$(\partial_t + v\cdot\nabla_x) f = Q(f,f) \quad , \quad f = f(t,x,v)$$ one-particle prob. density

$$Q(f,f)(v) = \iint_{\mathbb{R}^3 \, S^2} dv_* \, d\omega \, B(v - v_*, \omega) \left[f' f'_* - f f_* \right]$$

$B \geq 0$: collision kernel

$f' = f(v')$, $f'_* = f(v'_*)$

PROPERTIES:

P1) Q is quadratic

P2) space & time are parameters in Q (collisions are localized in x, t)

P3) Collisions are elastic
$$\begin{cases} v + v_* = v' + v'_* & \text{momentum} \\ v^2 + (v_*)^2 = (v')^2 + (v'_*)^2 & \text{kin. en.} \end{cases}$$

P4) Collisions are reversible at MICROSCOPIC LEVEL

$$(v, v_*, \omega) \longrightarrow (v', v'_*, \omega)$$ is an involution $(\det J = 1)$

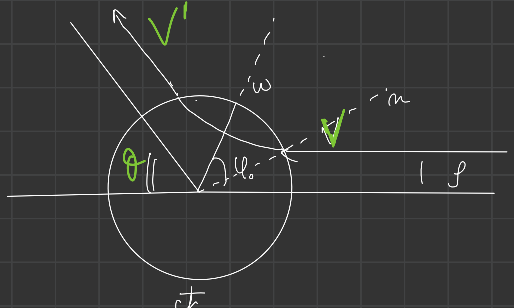
Structure of the coll. kernel B

∘) $0 \le B(v - v_*, \omega) = B\left(|v - v_*|, \omega \cdot \frac{(v - v_*)}{|v - v_*|}\right) = B(|v - v_*|, \cos\theta)$

(V = rel. velocity), $\omega \cdot \frac{(v-v_*)}{|v-v_*|}$ "= $\cos\theta$, θ scattering angle

∘) $\underbrace{B(\omega, V)}_{\text{coll. kernel}} = |V| \underbrace{\Sigma(\omega, V)}_{\text{scattering cross-sec}}$ ≃ one that describes the probability that a coll. takes place

$\Sigma(\omega, V)$? Solve the SCATTERING PROBLEM



∘) $\theta = \theta(\rho, V)$

∘) $\Sigma(\omega, V) = \frac{\rho}{\operatorname{sen}\theta} \left|\frac{d\rho}{d\theta}\right|$

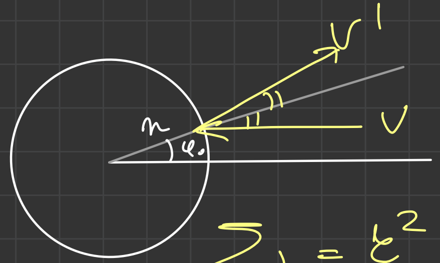
$\theta(\rho, V)$

$$\begin{cases} \theta = \pi - 2 \displaystyle\int_{z_{min}}^{+\infty} \frac{\frac{\rho}{z^2}\, dz}{\sqrt{1 - \frac{\rho^2}{z^2} - \frac{4\phi(z)}{|V|}}} \\ z_{min} \text{ s.t. } 1 = \frac{\rho^2}{z_{min}^2} + \frac{4\phi(z_{min})}{|V|} \end{cases}$$

Examples:

i) Φ hard-sphere interactions.

$\Rightarrow$ $$B(V, \omega) = |\omega \cdot V|$$



$$\Sigma = \frac{\varepsilon^2}{4}$$

ii) $\phi(r) = \frac{1}{r^{s-1}}$, $s>2$ in $d=3$ Inverse power law

$$B(|V|, \cos\theta) = \underbrace{|V|^{\gamma}}_{\text{kinetic cross-sec}} \underbrace{b_s(\cos\theta)}_{\text{angular cross-sec.}}$$

$$\gamma = \frac{s-5}{s-1}$$

- $\gamma > 0$ hard pot. , $\gamma < 0$ soft pot.
  $\gamma = 0$ Maxwellian molecules

[$\gamma = 1$ hard-sphere , $\gamma = -3$ Coulomb]

$b_s : [-1,1] \to [0,+\infty)$ implicitely def., locally smooth and has a singularity at $\theta=0$

- $\lim_{\theta\to 0} \theta^{1+\frac{2}{s-1}} (\text{sen}\,\theta)\, b_s(\cos\theta) = K_s$, $K_s > 0$

- $\lim_{s\to+\infty} b_s(\cos\theta) = \frac{1}{4}$

$\Downarrow$

$$B_s \xrightarrow[s\to\infty]{} B_{\text{hard-sphere}} = |v - v_*| \cos\varphi_0 \, \mathbb{1}_{\{\cos\varphi_0 \geq 0\}}$$

[See: J.W. Jang, B. Kepka, A. Nota, J. Velázquez, Vanishing angular singularity limit to the Hard-sphere B.Eq., JSP 2023.]

P5) Collisions involve only UNCORRELATED PARTICLES
(Boltzmann molecular chaos assumption)

Properties of the B. Eq.

$$\rho(x,t) := \int_{\mathbb{R}^3} f(x,v,t)\,dv$$ MASS DENSITY;

$$W(x,t) := \frac{1}{\rho(x,t)} \int_{\mathbb{R}^3} f(x,v,t)\, v\, dv$$ BULK VELOCITY;

$$\mathcal{E}(x,t) := \frac{1}{\rho(x,t)} \int_{\mathbb{R}} (v-w)^2 f(x,v,t)\,dv$$ ENERGY $\left[T = \frac{2}{3}\mathcal{E}\right]$

H- Theorem

$$H(f(t)) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x,v,t)\left(\log f(x,v,t)\right) dx\,dv$$

ENTROPY FUNCTIONAL

$$\frac{d}{dt} H(f)(t) = -\int_{\mathbb{R}^3} \underbrace{D(f(t,x,v))}_{\text{entropy dissipation}}\,dx \leq 0$$

Trend to equilibrium

- local eq. : $M_\ell(x,v) = \frac{\rho(x,v)}{(2\pi T(x))^{3/2}} e^{-\frac{|v-w(x)|^2}{2T(x)}}$

- global eq. : $M_\ell(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|w|^2}{2T}}$, $\rho \geq 0$, $w \in \mathbb{R}^3$, $T > 0$

II principle of Therm. : any non-eq. state of an isolated gas evolves towards the equilibrium represent.

Heuristic derivation of the B. Eq. (for hard-sphere int.)

N hard-spheres of radius $\varepsilon/2$

$\underline{z}_N = (\underline{x}_N, \underline{v}_N) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$, particles cannot overlap

$|x_i - x_j| > \varepsilon$

Phase space: $\Gamma_N = \Lambda_\varepsilon \times \mathbb{R}^{3N}$

$$\Lambda_\varepsilon = \{ \underline{x}_N : \underbrace{|x_i - x_j| > \varepsilon}_{\text{hard-core cond.}} , \; i \neq j, \; i,j = 1,\dots,N \}$$

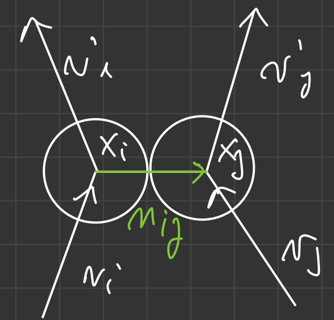
Dynamics: free flow until the first time in which 2 particles arrive at distance ε

$$\left(|x_j(t) - x_i(t)| \geq \varepsilon \quad \forall\, i,j = 1,\dots,N, \; i \neq j \right)$$

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = 0 \end{cases}$$

Then an instantaneous elastic collision happens:

bd. cdt.
$$\begin{cases} x_i(t^+) = x_i(t^-); & v_i(t^+) = v_i(t^-) - ((v_i(t^-) - v_j(t^-)) \cdot n) n \\ x_j(t^+) = x_j(t^-); & v_j(t^+) = v_j(t^-) + ((v_i(t^-) - v_j(t^-)) \cdot n) n \end{cases}$$



where $n = n_{ij} := \dfrac{x_i - x_j}{|x_i - x_j|}$

and after, again free flow.

$$\underline{Z_N} \to S^t(\underline{Z_N}) = (x_1 + v_1 t, v_1, x_2 + v_2 t, v_2, \ldots, x_N + v_N t, v_N)$$

RK: the hard-sphere flow $Z_N \to S^t(Z_N)$ is time rev.

RK: $S^t(Z_N)$ not globally well-def $\forall Z_N(0)$ due to possible pathological configurations

→ MULTIPLE COLL

→ GRAZING COLL. (i.e. $n\cdot(v_i - v_j) = 0$)

→ "CLUSTERING"

but such pathologies happen only for a set of initial configuration of vanishing measure

($\Rightarrow S^t$ a.e. def. w.r.t. Lebesgue)

- Prob. measure with density $f_0^N(\underline{Z_N})\, d\underline{Z_N}$ on Γ_N
- j-particle marginals ⟵ to have a reduced description

$$f_j^N(z_1, \dots, z_j, t) := \int_{\mathbb{R}^{3(N-j)} \times \mathbb{R}^{3(N-j)}} dz_{j+1} \cdots dz_N \, f^N(\underline{Z_j}, z_{j+1}, \dots, z_N, t)$$

$$f_j^N(z_1, \dots, z_j, t) = 0 \quad \forall j > N$$

1-particle marg.: $f_1^N(\underbrace{x_1, v_1}_{z_1}, t) = \int_{\mathbb{R}^{3(N-1)} \times \mathbb{R}^{3(N-1)}} dz_2 \dots dz_N \, f^N(z_1, z_2, \dots, z_N, t)$

<u>Goal</u>: Evolution Eq. for f_1^N?

<u>Without Collisions</u>:

$$\frac{d}{dt} f_1^N(x_1 + v_1 t, v_1, t) = 0$$

$$\parallel$$

$$(\partial_t + v_1 \cdot \nabla_{x_1}) f_1^N$$

With collisions: $(\partial_t + v_1 \cdot \nabla_{x_1}) f_1^N = \text{Coll} \; (= G - L)$

$L \, dx_1 dv_1 dt$ = loss of particles from the cell $\boxed{dx_1 dv_1}$ of the phase space in the time interval $(t, t+dt)$

$G \, dx_1 dv_1 dt$ = gain of particles entering the cell $\boxed{dx_1 dv_1}$ of the phase space in the time interval $(t, t+dt)$

$$\ldots\ldots \Longrightarrow (\text{Coll})(f_2^N) = \underbrace{(N-1)\varepsilon^2}_{\text{"}O(1)\text{"}} \int_{\mathbb{R}^3}\int_{S^2} dv_2\, dn\, f_2^N(x_1, v_1, \underbrace{x_1 + n\varepsilon}_{\to x_1}, v_2)\, \underbrace{(v_2 - v_1 \cdot n)}_{\text{"}B\text{"}}$$

$\Longrightarrow$ THE EQUATION (*) $(\partial_t + v_1 \cdot \nabla_{x_1}) f_1^N = Q(f_2^N, f_2^N)$ IS NOT CLOSED!

Hp: (S) $\quad f_2^N(x_1, v_1, x_2, v_2) = f_1^N(x_1, v_1) f_1^N(x_2, v_2)$ PROPAGATION OF CHAOS !!

(S) holds asymptotically (in the B.G. limit)

$\Longrightarrow$ from (*) we obtain the Boltzmann Eq. (B.Eq).

Some references :

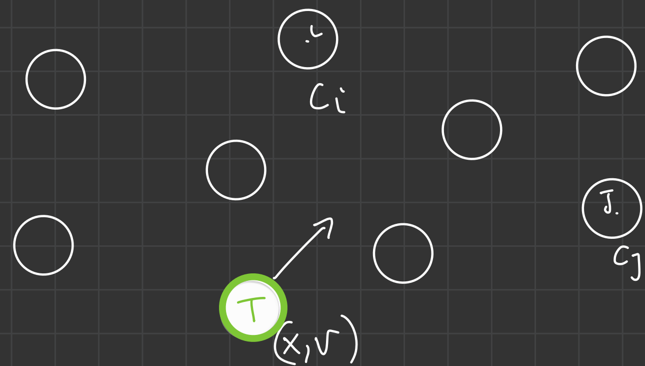
1) C. Cercignani, R. Illner, M. Pulvirenti
The Mathematical Theory of Dilute Gases, Springer 1994.

2) C. Cercignani, The Boltzmann Equation and its applications, Springer 1988.

3) O.E. Lanford, Time evolution of large classical systems. In "Dynamical Systems, theory and applications", Lecture Notes in Physics, Springer 1975

From a many body problem to an effective single-particle system: the Lorentz Gas

(H. Lorentz 1905)



fixed conf. of randomly distr. obstacles (infinitely heavy)
$\{c_1, \dots, c_j, \dots\}$ centers

MICROSCOPIC DESCR.

$$(\text{N.Eq})\quad \begin{cases} \dot{x} = v \\ \dot{v} = -\sum_i \nabla_x \Phi(x - c_i) \end{cases}$$

$+ (x(0), v(0)) = (x, v)$ in. cdt.

Kinetic limit $\longrightarrow$ (Markovian approx.)

MESOSCOPIC DESCR.

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}(f)$$

$$\mathcal{L} f(v) = \int_S d\omega \, B(\omega, v) [f' - f]$$

$f = f(x, v, t)$ prob. dens.

$f' = f(v')$, $f = f(v$

Rk:

MICRO → **MACRO**

N. eq. (N particles) → Euler / Navier Stokes

kinetic scaling → MESOSCOPIC → (fast relax. $\lambda^{-1} \to +\infty$) → MACRO

B. eq. (Kinetic eq.)

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linear scaling

MICRO —— hydrodynamical limit ——→ **MACRO**

N. eq. for free Lorentz particle

(*) 1980 Bunimovich Sinai

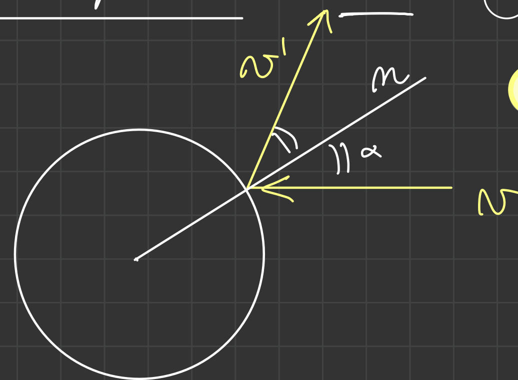
$$\partial_t \rho = D \Delta \rho$$

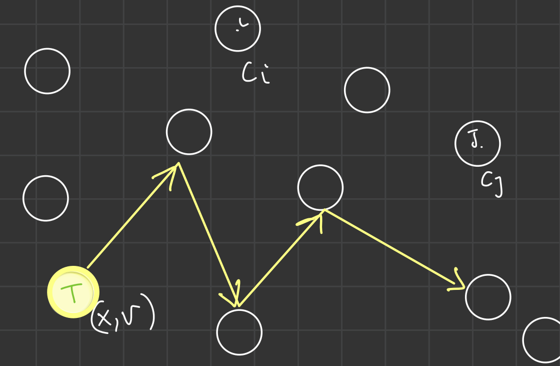
kinetic limit → MESOSCOPIC → diffusion limit → MACRO

L. B. eq.

Consider:

- .) $d = 2$
- .) hard-sphere int.





$$\begin{cases} p = \operatorname{sen} \alpha \in [-1,1] \\ v' = v - 2(n \cdot v) n \\ dp = \left|\frac{dp}{d\alpha}\right| d\alpha = |n \cdot v| \, dn \end{cases}$$

- .) $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}_+$

$$\partial_t f + v \cdot \nabla_x f = \int_{S^1} dn \, |n \cdot v| \left(f(v') - f(v) \right)$$

$$\left(= \int_{-1}^{1} dp \left(f(v') - f(v) \right) \right)$$

The Linear Boltzmann Eq. (for hard-sphere Int.)

$K = (G) = \int_{-1}^{1} d\rho\, f(v')$

$\mathcal{L} = \int_{-1}^{1} d\rho\, f(v)$

Let $f_0 \in C_b(\mathbb{R}^2 \times \mathbb{R}^2)$

Since $\mathcal{L} f(v) = c(K - I) f(v)$ (set $c = 1$)

$$(\partial_t + v \cdot \nabla_x) f + I f = K f \quad \text{(L.B. eq.)}$$

Source

$$f(x,v,t) = e^{-t} f_0(x - vt, v) + \int_0^t ds\, e^{-(t-s)} (Kf)(x - v(t-s), v, s)$$

Duhamel

$S(t-s)(Kf)(x,v,s)$

Set $f(x,v,t) := S(t) f_0(x,v) = e^{-t} f_0(x - vt, v)$

evolution sgp.

By iterating $\rightsquigarrow$ formal series expansion.

$$f(x,v,t) = S(t) f_0(x,v) + \sum_{m>0} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m$$

$$S(t-t_1)\, K\, S(t_1-t_2) \dots K\, S(t_m)\, f_0$$

Using the explicit form of $S(t) \Longrightarrow$

$$f(x,v,t) = e^{-t} f_0(x-vt,v) + \sum_{m>0} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m$$

$$\int_{-1}^{1} dv_1 \int_{-1}^{1} dv_2 \dots \int_{-1}^{1} dv_m \; f_0\big(x - v(t-t_1) - v_1(t_1-t_2) \dots - v_m t_m, v_m\big) \qquad (B.8d)$$

where: • $\{t_i\}_{i=0}^{m}$ s.t. $0 \le t_m < t_{m-1} \dots t_1 < t_0 = t$

COLLISION TIMES

$(t_i - t_{i+1}) \sim \mathcal{E}xp(1)$

• $\{v_i\}_{i=0}^{m}$ sequence of velocities

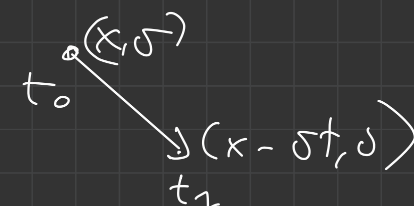
$v \to v_1 \to v_2 \to \dots \to v_m$; $v_m = v_{m-1} - 2(n \cdot v_{m-1})n$

Stochastic Trajectory:

let $(x, v) \in \mathbb{R}^2 \times \mathbb{S}^1$ $(|v| = 1)$

- $s \in (t_1, t_0)$

$$\begin{cases} x(-s) = x - vs \\ v(-s) = v \end{cases}$$

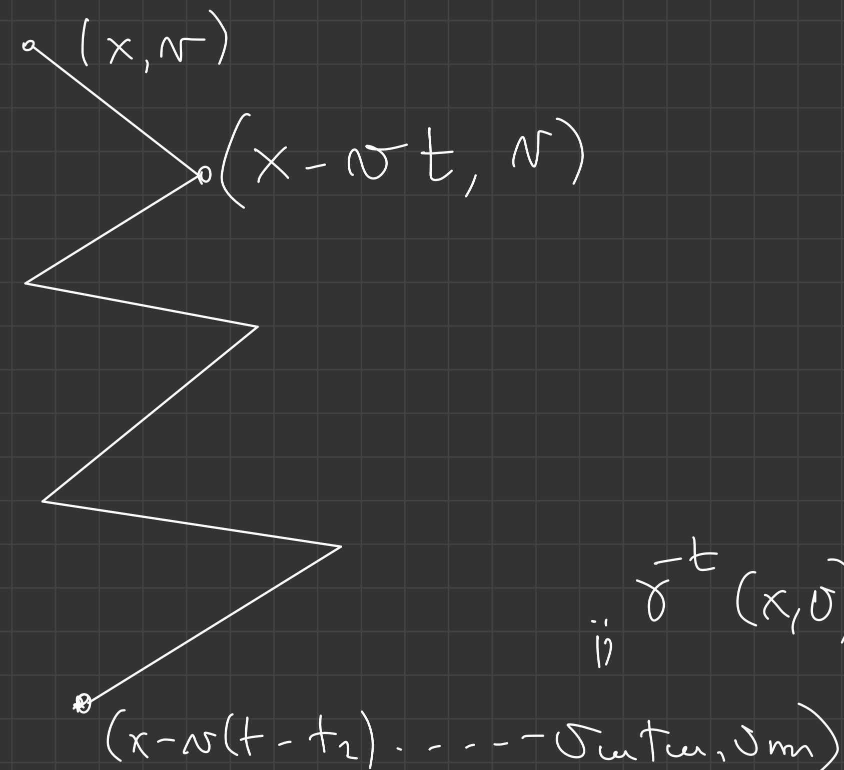


- $\forall s \in [t_{i+1}, t_i)$

$$\begin{cases} x(-s) = x - \sum_{k=0}^{i-2} v_k (t_k - t_{k+1}) - (t_i - (t - s))v_i \end{cases} ; \; v(-s) = v_i$$

In particular, for $s = t_m$

$$\begin{cases} x(-t) = x - \sum_{k=0}^{m-1} v_k (t_k - t_{k-1}) - t_m v_m \\ v(-t) = v_m \end{cases}$$

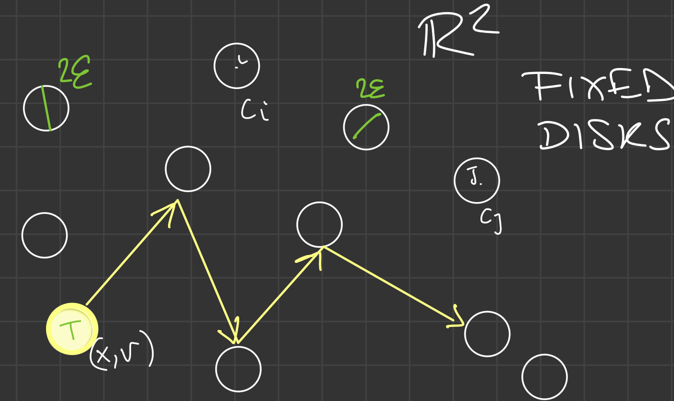


$$\text{ii}\quad \gamma^{-t}(x, v)$$

The particle model (Lorentz Gas in $\mathbb{R}^2$)

$\{c_j\}_j$ centers $\sim$ Poiss(μ), $\mu > 0$

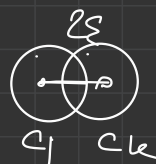
$\varepsilon > 0$ radius



$$\mathbb{P}(\#(\{c_j\} \cap A) = N) = \frac{e^{-\mu|A|}(\mu|A|)^N}{N!} dc_1 \ldots dc_N, \quad \forall A \subseteq \mathbb{R}^2 \text{ bounded}$$

obstacles may overlap: for some $j \neq k$ $\underbrace{|c_j - c_k|}_{d} < 2\varepsilon$

(we allow for configurations $\{c_j\}$ s.t.



)

Dynamics (billiard flow)

Let $(x, v) \in \mathbb{R}^2 \times S^1$, $\underline{c}_N = (c_1, \dots, c_N)$, $|v| = 1$.

$\Longrightarrow$ $T^t_{\varepsilon, \underline{c}_N}(x, v) := (x_\varepsilon(t), v_\varepsilon(t)) \in \mathbb{R}^2 \times S^1$ for $t > 0$ is defined by:

i) if $|x_\varepsilon(t) - c_j| > \varepsilon$ $\forall j \in J \subseteq \mathbb{N}$

then

$$(N.eq) \begin{cases} \dot{x}_\varepsilon(t) = v_\varepsilon(t) \\ \dot{v}_\varepsilon(t) = 0 \end{cases}$$

ii) if $|x_\varepsilon(t) - c_j| = \varepsilon$ for some $j \in J \subseteq \mathbb{N}$ we use the boundary cdt.

$$\begin{cases} x_\varepsilon(t^+) = x_\varepsilon(t^-) ; \\ v_\varepsilon(t^+) = v_\varepsilon(t^-) - (v_\varepsilon(t^-) \cdot n)\, \underline{n} \end{cases}, \qquad n = \frac{x_\varepsilon(t^-) - c_j}{\varepsilon}$$

Kinetic Scaling : (low-density or B.G. limit)

The intensity $\mu \longrightarrow \mu_\varepsilon := \varepsilon^{-(d-1)}\mu$, $\mu > 0$, $d > 2$

$\left(\mu_\varepsilon := \varepsilon^{-1}\mu \quad \text{in } d=2\right)$

Rk: $\mu_\varepsilon = \langle N \rangle_{1_A}$, $A \subseteq \mathbb{R}^2$

$$\mu_\varepsilon \cdot \varepsilon^2 = \langle N \rangle_{1_A} \cdot \varepsilon^2 = \varepsilon^{-1} \cdot \varepsilon^2 \mu = \varepsilon\mu \xrightarrow[\varepsilon \to 0]{} 0 \qquad \text{DILUTE}$$

$$\mu_\varepsilon \cdot \varepsilon = \langle N \rangle_{1_A} \cdot \varepsilon = \varepsilon^{-1}\varepsilon\mu = \mu = O(1)$$

Notation : $\mathbb{P} \to \mathbb{P}_\varepsilon$, $\mathbb{E}[\cdot] \mapsto \mathbb{E}_\varepsilon[\cdot\cdot] := \mathbb{E}_\varepsilon\left[\ldots \mathbb{1}_{\{\min_i |x - c_i| > \varepsilon\}}\right]$

Given $(x,v) \in \mathbb{R}^2 \times \mathbb{S}^1$, given $\underline{c}_N$, f_0 "smooth"

consider $T^{-t}_{\varepsilon,\underline{c}_N}(x,v)$ and define

$$(\square) \quad f_\varepsilon(x,v,t) := \mathbb{E}_\varepsilon\left[f_0\left(T^{-t}_{\varepsilon,\underline{c}_N}(x,v)\right)\right]$$

Goal: prove that $f_\varepsilon \longrightarrow f$ where f solves
as $\varepsilon \to 0$
the Linear
Boltz. Eq.
(L.B.Eq.).

Theorem: (Gallavotti 1972)

Let $f_0 \in L^1 \cap W^{1,\infty}(\mathbb{R}^2 \times S^1)$, let $T>0$ and let $f_\varepsilon(x,v,t)$ be def. as in (I)

Then, for any $t \in [0,T]$ we have

$$\lim_{\varepsilon \to 0} \| f_\varepsilon(t) - f(t) \|_{L^1} = 0 \qquad (\text{unif. in } t)$$

when f solves

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f = 2\mu \int_{-1}^{1} d\rho \left(f(v') - f(v) \right) \\ f(x,v,0) = f_0(x,v) \end{cases}$$

RK: qualitative result! For a quantitative result that controls the error in this Markovian approximation ["Error" $\sim c\,\varepsilon^{1/2} t^2$]

see for instance:

(∘) Basile, Nota, Pezzotti, Pulvirenti (Comm. Math. Phys. 2015)

(∘) Nota (Springer Proceedings in Mathematics & Statistics 2015)

(∘) Lods, Nota, Winter (Journal Stat. Phys. 2019)

If we consider also the effect of an external field?

(∘) Nota, Saffirio, Simonella (Ann. Inst. H. Poincaré, Prob. & Statist. (B), 2022)

Strategy: constructive approach based on a suitable change of variables which leads to a Markovian approximation (for the Lorentz process) described by a linear Boltzmann Eq.

Technical difficulties: some of the random configurations lead to trajectories that "remember" too much preventing the Markovianity of the limit.

Lecture 3

Summer School

Berlin 2023

A. Note

The mesoscopic description of a Lorentz Gas is given by a linear kinetic equation

MICRO

MESO

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\sum \nabla_x \Phi(x - c_i) \end{cases}^{(N.\,eq.)} \xrightarrow{\text{kinetic limit}} (\partial_t + v\cdot\nabla_x) f = \mathcal{L}(f)$$

$\sum \nabla_x \Phi(x-c_i)$ " $F(x; C_N)$ "

(Markovian approx.)

$$\text{•) Boltz. eq.}: \quad \mathcal{L}f(v) = \int_{S^1} dw \, B(n\cdot w) \left[f(v'(w)) - f(v) \right] \quad \text{(low-density)}$$

h.s. $= |n\cdot w|$

$$\text{•) Landau eq.}: \quad \mathcal{L}f(v) = K \Delta_{v_\perp} f(v) \quad \text{(weak-coupling)}$$

Theorem: (Gallavotti 1972)

Let $f_0 \in L^1 \cap W^{1,\infty}(\mathbb{R}^2 \times \mathbb{S}^1)$, let $T>0$ and let $f_\varepsilon(x,v,t)$ be def. as in (I)

$f_\varepsilon(x,v,t) = \mathbb{E}_\varepsilon[f_0(T^{-t})]$

Then, for any $t \in [0,T]$ we have

$$\lim_{\varepsilon \to 0} \| f_\varepsilon(t) - f(t) \|_{L^1(\mathbb{R}^2 \times \mathbb{S}^1)} = 0 \qquad \text{(unif. in } t\text{)}$$

where f solves

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f = 2\mu \int_{-1}^{1} d\rho \, (f(v') - f(v)) \\ f(x,v,0) = f_0(x,v) \end{cases}$$

RK: qualitative result! For a quantitative result

Proof (main steps)

(1) $$f_\varepsilon(x,v,t) = e^{-\mu_\varepsilon |B_t(x)\setminus B_\varepsilon(x)|} \sum_{N\geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{(B_t(x)\setminus B_\varepsilon(x))^N} d\underline{c}_N \; f_0\left(T^{-t}_{\varepsilon,\underline{c}_N}(x,v)\right)$$

Set $B_t^\varepsilon(x) := B_t(x)\setminus B_\varepsilon(x)$

→ distinguish the obstacles $\underline{c}_N = (c_1, \dots, c_N)$ into

- → INTERNAL OBS. : if $\inf_{s\in[0,t]} |x_\varepsilon(-s) - c_i| = \varepsilon$
- → EXTERNAL OBS. : if $\inf_{s\in[0,t]} |x_\varepsilon(-s) - c_i| > \varepsilon$

and we decompose the configuration $\underline{c}_N = \underline{b}_m \cup \tilde{\underline{b}}_p$

$\underline{b_n} = (b_1, \ldots, b_n)$ internal obst.

$\tilde{b}_p = (\tilde{b}_1, \ldots, \tilde{b}_p)$ extend obst. $(p = N - n)$

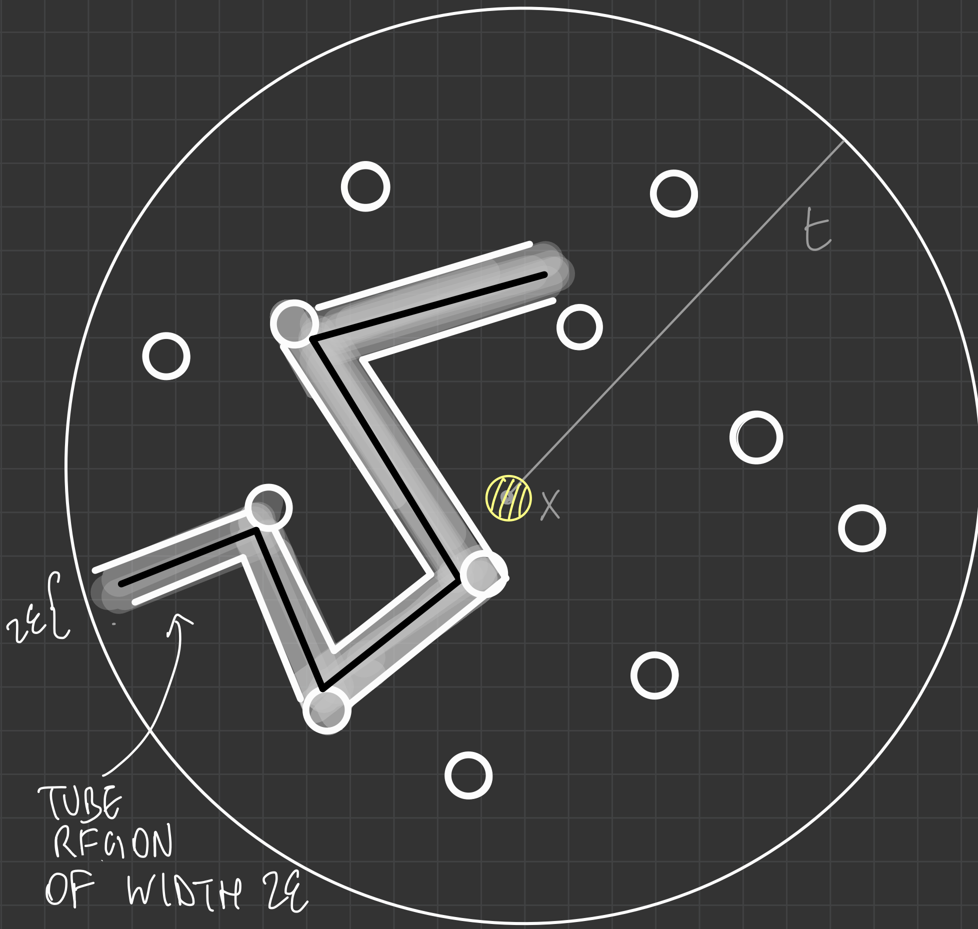
N obstacles $\begin{cases} n \text{ internal} \\ p = N-n \text{ extend} \end{cases} \Rightarrow \binom{N}{n}$ different ways

(2)

$$\Rightarrow f_\varepsilon(x,v,t) = e^{-\mu_\varepsilon |B_t^\varepsilon(x)|} \sum_{n \geq 0} \frac{\mu_\varepsilon^n}{n!} \int_{(B_t^\varepsilon(x))^n} d\underline{b}_n \, 1\!\!1_{\{\underline{b}_n \text{ int}\}} \cdot \sum_{p \geq 0} \frac{\mu_\varepsilon^p}{p!} \int_{(B_t^\varepsilon(x))^p} d\tilde{b}_p \, 1\!\!1_{\{\tilde{b}_p \text{ ext}\}} \, f_0\left(T^{-t}_{\varepsilon, \underline{b}_n}(x,v)\right)$$

$$T^{-t}_{\varepsilon, \underline{c}_N}(x,v) = T^{-t}_{\varepsilon, b_n}(x,v)$$

TUBE



$$\mathcal{T}_t(x, r; \underline{b}_n) = \{ y \in \mathcal{B}^{\varepsilon}_t(x) : \exists\, s \in [0, t] \text{ s.t. } |y - x_\varepsilon(-s)| \le \varepsilon \}$$

$$\mathcal{T}_t(x, r; \underline{b}_n) = \mathcal{T}_t(\underline{b}_n)$$

Integrating over the external obstacles:

$$(3)\quad f_\varepsilon(x,v,t) = \sum_{n\geq 0} \frac{\mu_\varepsilon^n}{n!} \int_{(B_t^\varepsilon(x))^n} d\underline{b}_n \left(e^{-\mu_\varepsilon |\mathcal{Z}_t(b_n)|} \right) \mathbb{1}_{\{\underline{b}_n \text{ internal}\}} f_0(T^{-t}_{\underline{b}_n}(x,v))$$

Goal: Remove from f_ε all trajectories that involve "bad events" $\rightsquigarrow$ multiple collisions.

Define

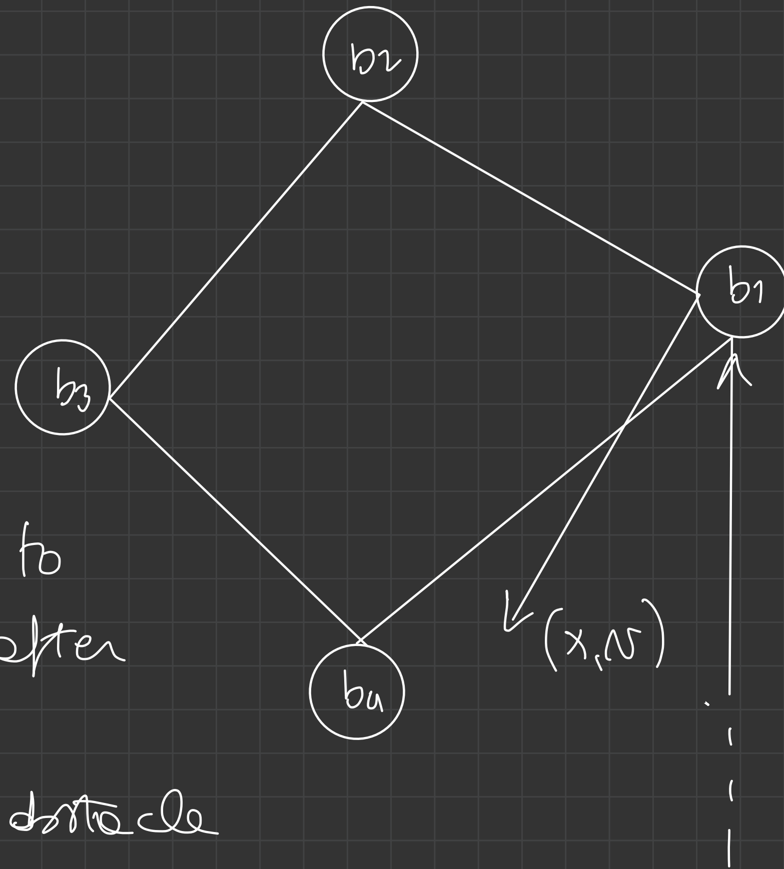
$$(4)\quad \tilde{f}_\varepsilon(x,v,t) = \sum_{n\geq 0} \frac{\mu_\varepsilon^n}{n!} \int_{(B_t^\varepsilon(x))^n} d\underline{b}_n \, e^{-\mu_\varepsilon |\mathcal{Z}_t(\underline{b}_n)|} \mathbb{1}_{\{\underline{b}_n \in \Delta_n(t)\}} f_0(T^{-t}_{\underline{b}_n}(x,v))$$

(set of obstacles hit only once by the tagged particle) ← $\Delta_n(t)$

$$\mathbb{1}_{\{\underline{b}_n \in \Delta_n(t)\}} = \mathbb{1}_{\{\underline{b}_n \text{ internal}\}} \mathbb{1}_{\{\text{exactly } n \text{ coll.}\}}$$

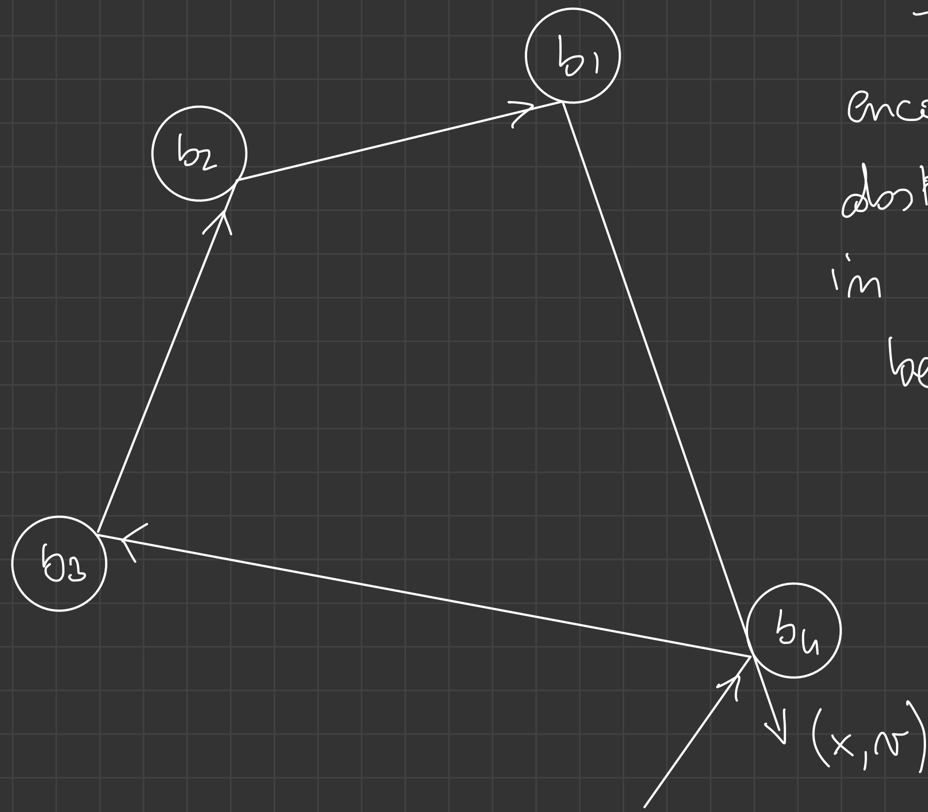
Pathological conf:

·) Recollisions
(backward)



the trajectory returns to a collision point after having collided with a different obstacle

◦) Interferences (backward)



the trajectory encounters a new obstacle in a point in space which has been already visited

$$\mathbb{E}[f_\varphi(T^{-1}(x,v))] = \mathbb{E}(f_\theta(T^{t}(x,v))]$$

$$\mathbb{1}_{\{\underline{b}_n \in P\}} = \mathbb{1}_{\{\underline{b}_n \in R \cup I\}} = \mathbb{1}_{\{\underline{b}_n \in R\}} + \mathbb{1}_{\{\underline{b}_n \in I\}} - \mathbb{1}_{\{\underline{b}_n \in R \cap I\}}$$

$$\mathbb{1}_{\{\underline{b}_n \in P\}} = \mathbb{1}_P, \qquad \mathbb{1}_{\{\underline{b}_n \in R \cap I\}} = \mathbb{1}_{\{rec\}} \mathbb{1}_{\{int\}}$$

Proposition ("Markovian part"):

Let f_ε be def. as in (3). Then, $\tilde{f}_\varepsilon$ def as in (4) satisfies:

$$0 \le \bar{f}_\varepsilon(t) \le \tilde{f}_\varepsilon(t) \le f_\varepsilon(t)$$

where

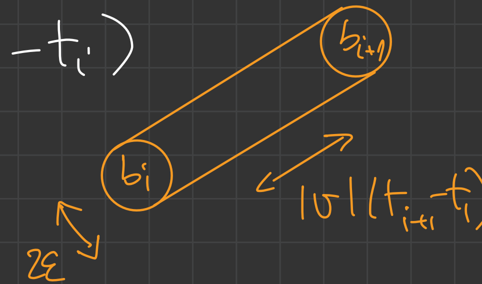
$$\bar{f}_\varepsilon(x,v,t) = e^{-2\mu t} \sum_{n=0}^{\infty} (2\mu)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \int_{-1}^{1} d\rho_1 \dots \int_{-1}^{1} d\rho_n \left(1 - \mathbb{1}_P\right) f_0\left(\gamma^{-t}(x,v)\right)$$

$(\mu_\varepsilon = \varepsilon^{-1}\mu)$

limit flow $\gamma^{-t}(x,v) = (x(-t), v(-t))$

Proof: $$|Z_t(\underline{b}_n)| \le \sum_{i=0}^{n} (2\varepsilon)\,|v|\,(t_{i+1}-t_i)$$

$$\le 2\varepsilon |v| t$$



$$(*)\quad e^{-\mu_\varepsilon |Z_t(\underline{b}_n)|} \ge e^{-2\mu_\varepsilon \cdot \varepsilon t |v|} = e^{-2\mu_\varepsilon \cdot \varepsilon t} = e^{-2\mu t}$$

$$\Rightarrow h_\varepsilon(x,v,t) = e^{-2\mu t} \sum_{n\ge 0} \frac{\mu_\varepsilon^n}{n!} \int_{(B^\varepsilon_t(x))^n} d\underline{b}_n \, \mathbb{1}_{\{\underline{b}_n \in \mathcal{D}_n(t)\}} f_0\left(T^{-t}_{\underline{b}_n,\varepsilon}(x,v)\right)$$

$$\Rightarrow 0 < h_\varepsilon(t) \le \tilde{f}_\varepsilon(t) \le \underline{f_\varepsilon(t)}$$

$\Rightarrow$ we need to prove that $h_\varepsilon = \bar{f}_\varepsilon$.

- order the obstacles, i.e. b_i collided before b_j if $i < j$

- ρ_i : impact parameter, $t_i := \sup_{\tau > 0} \{ \inf_{0 \le t \le \tau} |x_\varepsilon(t) - b_i| > \varepsilon \}$
  ↑ hitting time

- CHANGE OF VARIABLES : $b_1, \dots, b_n \xrightarrow{(C.V.)} (\rho_1, t_1), \dots, (\rho_n, t_n)$
  with
  $0 \le t_n < t_{n-1} < \dots < t_1 < t_0 = t$

- conversely, fixed the $\{t_i\}_i$, $\{\rho_i\}_i$ we construct the centers of the obstacles $\{b_i\}$, $b_i = b_i(\rho_i, t_i)$ and the flow $\gamma^{-s}(x, v) = (x(-s), v(-s))$ $\forall s \in [0, t)$

Rk: ◦) $\gamma^{-t}(x,v) \neq T^{-t}_{\varepsilon, b_n}(x,v)$!!!

◦) $\mathbb{1}_{\{\underline{b}_n \in \Delta_n(t)\}} \implies \gamma^{-t}(x,v) = T^{-t}_{\varepsilon, b_n}(x,v)$

(outside the pathol. conf.)

$\implies$ (C.V.) is a one to one map.

In the new variables

$$\mathbb{1}_{\{\underline{b}_n \in \Delta_n(t)\}} = (1 - \mathbb{1}_P) = (1 - \mathbb{1}_R)(1 - \mathbb{1}_I)$$

and

$$\frac{db_1 \dots db_n}{n!} = dt_1 \dots dt_n \, dp_1 \dots dp_n$$

$\implies$ using (C.V.) in h_ε we arrive at

$$h_\varepsilon(x,v,t) = e^{-2\mu t} \sum_{n\geq 0} (2\varepsilon)^n \mu_\varepsilon^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{-1}^{1} dv_1 \cdots \int_{-1}^{1} dv_n \,(1-\mathbb{1}_R)(1-\mathbb{1}_I)\, f_0(\gamma^{-t}(x,v))$$

$$\Rightarrow \quad h_\varepsilon \equiv \bar{f}_\varepsilon \left(= f_\varepsilon^{\text{Markovian}} \right)$$

and

$$0 \leq \bar{f}_\varepsilon(t) \leq \tilde{f}_\varepsilon(t) \leq f_\varepsilon(t)$$

$$\left[\text{Rk:} \quad f_\varepsilon(t) = f_\varepsilon^{\text{Markovian}} + f_\varepsilon^{\text{Non Markovian}} = \bar{f}_\varepsilon + \text{"error"} \right]$$

Since

$$\boxed{\mathbb{E}_\varepsilon[(1-\mathbb{1}_P)] \xrightarrow[\varepsilon\to 0]{} 1}$$

$$\left(\text{equiv. } \mathbb{E}_\varepsilon[\mathbb{1}_P] \xrightarrow[\varepsilon\to 0]{} 0\right)$$

STEP (I): Compare $\overline{f}_\varepsilon$ with f sol. to (B_eq.)

1) we have pointwise conv. : $\overline{f}_\varepsilon(x,v,t) \underset{\varepsilon\to 0}{\longrightarrow} f(x,v,t)$

2) the generic term of the series defining $\overline{f}_\varepsilon$ is dominated by

$$\|f_0\|_\infty e^{-2\mu t} (2\mu)^n \frac{t^n}{n!}$$

(term of a conv. series!)

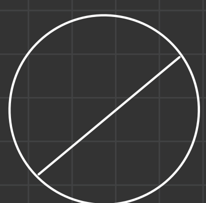
$$1) + 2) \underset{\text{dominated convergence}}{\Longrightarrow} \overline{f}_\varepsilon(t) \longrightarrow f(t) \text{ in } L^1(\mathbb{R}^2 \times S^1)$$

STEP (II): compare f_ε with f sol to (B.eq.)

(B.E. sol.)

$$f(x,v,t) = e^{-2\mu t} \sum_{n\geq 0} \mu^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{-1}^{1} dv_1 \cdots \int_{-1}^{1} dv_n \, f_0\left(\gamma_n^{-t}(x,v)\right)$$

$$\begin{cases} x(-t) = x - v(t-t_1) - v_1(t_1-t_2) - \cdots - v_n t_n \\ v(-t) = v_n \end{cases}$$



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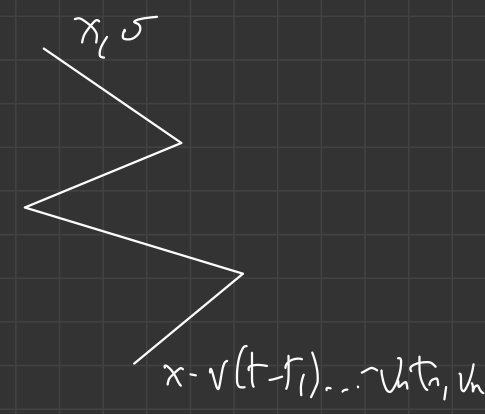


$$\| f_\varepsilon(t) - f(t) \|_{L^1} \le \| f_\varepsilon(t) - \bar{f}_\varepsilon(t) \|_{L^1} + \underbrace{\| \bar{f}_\varepsilon(t) - f(t) \|_{L^1}}_{\text{STEP (I)}}$$

1) Monotonicity (of the constructive argument)

2) Conservation of mass :

$$\| f_0 \|_{L^1} = \| f_\varepsilon \|_{L^1} = \| f \|_{L^1}$$

$$\Rightarrow \lim_{\varepsilon \to 0} \| f_\varepsilon(t) - f(t) \|_{L^1} = 0$$

✓

Adopt. : $\underline{\Phi \in C^2_b}$ (short range)

WHAT HAPPENS FOR LONG-RANGE INTERACTION POTENTIALS?

$$\left[\ \Phi(|x|) \simeq |x|^{-s} \text{ for } |x| \text{ large}\ \right]$$

Key remarks: 1) For short-range potentials the mixing properties of the random field ⟹ statistical independence of the trajectories in the limit!

2) Very slow decay of correlations of the random field for long-range

Literature:

1) Nota, Simonella, Velázquez: On the theory of Lorentz Gases with long-range interactions. ←
Rev. Math. Phys., 2018

2) Nota, Velázquez, Winter: Interacting particle systems with long-range interactions: scaling limits and kinetic equations, Atti Acc. Naz. Lincei Rend. Lincei Mat. Appl. 2021

3) Nota, Velázquez, Winter: Interacting particle systems with long-range interactions: approximation by tagged particles in random fields, Atti Acc. Naz. Lincei Rend. Mat. Appl. 2022

Outline

Task 1: Existence of the limit stochastic force field (generalized Holtsmark field) generated by the distrib. of sources yielding a potential $\Phi(x) \simeq |x|^{-s}$ and identification of conditions for translation invariance. [Chandrasekhar 1943, Holtsmark 1919]

Task 2: Estimate the diffusive timescale and identify conditions for the vanishing of correlations to obtain the correct Markovian approximation

Generalized Holtsmark fields

Setting

- $\Omega = \left\{ \{c_n\}_{n\in\mathbb{N}},\ \#\{c_n\}_{n\in\mathbb{N}} \cap K < \infty \right\}$

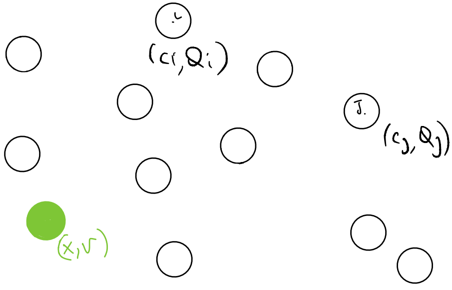
  ν : uniform Poisson meas. with $\lambda = 1$

- $I = \{Q_1, Q_2, Q_3, \ldots, Q_L\},\ Q_j \in \mathbb{R}$

  μ : probability meas. in the set I

- $\Omega \otimes I$ set of charged scatterer conf.

  $\omega = \left\{ (c_n, Q_n) \right\}_{n\in\mathbb{N}} \in \Omega \otimes I$



$$(\nu\otimes\mu)\left(\cap_{j=1}^{L}\left[\mathfrak{U}_{U,n_j,j}\right]\right) = \frac{\exp\left(-|U|\right)\prod_{j=1}^{L}\left[\mu\left(Q_j\right)|U|\right]^{n_j}}{\prod_{j=1}^{L}\left(n_j\right)!}$$

/ $U \subset \mathbb{R}^3$ bounded

Class of potentials

$s > 1/2$

$$\mathcal{C}_s := \left\{ \Phi \in C^2\left(\mathbb{R}^3 \setminus \{0\}; \mathbb{R}\right) \text{ s.t. } \Phi(x) = \Phi(|x|) \text{ and } \exists A \neq 0,\ r > \max(s, 2) \right.$$

$$\left. \text{s.t., for } |x| \geq 1,\ \left|\Phi(x) - \frac{A}{|x|^s}\right| + |x|\left|\nabla\left(\Phi(x) - \frac{A}{|x|^s}\right)\right| \leq \frac{C}{|x|^r} \right\}$$

Definition: Let be $\Phi \in \mathcal{C}_s$ and $\omega \in \Omega \otimes I$. The random field $\{F(x) : x \in \mathbb{R}^3\}$ is a generalized Holtsmark field if $\exists\, U \subset \mathbb{R}^3$ open with $0 \in U$ s.t.

$$F(x)\,\omega = F_U(x)\,\omega = \lim_{R \to \infty} F_U^{(R)}(x)\omega$$

$$\text{where} \quad F_U^{(R)}(x) := -\sum_{c_n \in RU} Q_{j_n} \nabla \Phi(x - c_n) \qquad (\star)$$

$\forall\, x \in \mathbb{R}^3$ and the *convergence is in law*.

Theorem [N., Simonella, Velázquez 2018]

Let be $\Phi \in \mathcal{C}_s$. Then the limit field F exists for potentials $\Phi \in \mathcal{C}_s$ and defines a random field if:

1. $s > 2$ **or** $\sum_{j=1}^{L} Q_j \mu(Q_j) = 0$ ('neutrality') and $s > 1/2$

   or $F_U^{(R)} = F_U^{(R),0}$, $\int_{|y|<1} |\nabla \Phi(y)|\, dy < \infty$ and $s > 1/2$

   (Translation Invariance)

2. $1 < s \leq 2$ and $\int_{U \setminus \{|y| < \frac{1}{2}\}} \nabla\left(\frac{1}{|y|^s}\right) dy = 0$ (Dependence on the geometry)

3. $s = 1$ and $\int_U \nabla\left(\frac{1}{|y|}\right) dy = 0$ (Translation invariance is lost)

Theorem [N., Simonella, Velázquez 2018]

Let be $\Phi \in \mathcal{C}_s$. Then the limit field F exists for potentials $\Phi \in \mathcal{C}_s$ and defines a random field if:

1. $s > 2$ **or** $\sum_{j=1}^{L} Q_j \mu(Q_j) = 0$ ('neutrality') and $s > 1/2$

   or $F_U^{(R)} = F_U^{(R),0}, \quad \int_{|y|<1} |\nabla\Phi(y)|\, dy < \infty$ and $s > 1/2$

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Comments:

- For $s \leq 2$ $\sum_{c_n} Q_{j_n} \nabla\Phi(x - c_n)$ only conditionally convergent.
- In absence of neutrality we need a stringent assumption on U (cf. (2), (3)) (geometrical condition on the cloud scatterer distribution).
- Small displacements of the domain U can yield limit random force fields with a non-zero component in one particular direction (cf. (2))

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3. $s = 1$ and $\int_U \nabla\left(\frac{1}{|y|}\right) dy = 0$ (Translation invariance is lost)

Comments:

- In $\mathbb{R}^2$ the critical value is $s = 1$. The Theorem can be adapted for $s > 0$.

  The Coulombian case corresponds to a logarithmic potential.

  If $s \leq 1$ a nontrivial condition on the geometry of the finite clouds is required.

- The result holds also for time-dependent random fields.

  ([N., Velázquez, Winter 2019])

Theorem [N., Simonella, Velázquez 2018]

Let be $\Phi \in \mathcal{C}_s$. Then the limit field F exists for potentials $\Phi \in \mathcal{C}_s$ and defines a random field if:

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   (Translation Invariance)

2. $1 < s \leq 2$ and $\int_{U \setminus \{|y|<\frac{1}{2}\}} \nabla\left(\frac{1}{|y|^s}\right) dy = 0$ (Dependence on the geometry)
3. $s = 1$ and $\int_U \nabla\left(\frac{1}{|y|}\right) dy = 0$ (Translation invariance is lost)

Strategy:

Pointwise convergence of the $J-$point characteristic function of $F_U^{(R)}(x)$:

$$m^{(R)}\Big(\underbrace{\eta_1, \dots, \eta_J}_{\in\mathbb{R}^3}; \underbrace{y_1, \dots, y_J}_{\in\mathbb{R}^3}\Big) := \mathbb{E}\left[\exp\left(i \sum_{k=1}^{J} \eta_k \cdot F_U^{(R)}(y_k)\right)\right]$$

$$= \mathbb{E}\left[\prod_{k=1}^{J} \prod_{c_n \in RU} \exp\left(-iQ_{j_n}\eta_k \cdot \nabla\Phi(y_k - c_n)\right)\right], \quad \forall J \geq 1$$

The importance of electroneutrality for Coulombian potentials

In the case of Coulombian potentials $\Phi(x) = \frac{1}{|x|}$ the random force field $F_U(x)$ satisfies a system of stationary (Maxwell) differential equations.

For almost every $\omega = \{(c_n, Q_{j_n})\}_{n\in\mathbb{N}}$ the function $\psi(x) := F_U(x)\omega$ is a weak solution of

$$\operatorname{div}\psi = \sum_n Q_{j_n}\delta(\cdot - c_n) \quad , \quad \operatorname{curl}\psi = 0.$$

If the random force field $\left\{F(x) : x \in \mathbb{R}^3\right\}$ is translation invariant

and $\mathbb{E}[|F(x)|] < \infty$ for any point $x \in \mathbb{R}^3$ $\quad \Rightarrow \quad \sum_{j=1}^{L} Q_j \mu(Q_j) = 0.$

- electroneutrality is necessary in order to obtain the translation invariance!

Dynamics of the tagged particle and Kinetic Limit

The type of linear kinetic equation arising in the scaling limit strongly depends on the microscopic details of the interactions (dependence on the decay as well as on the singularities of the potential)!

Kinetic Limit

Consider $\{\Phi(x,\varepsilon);\ \varepsilon > 0\}$. $\quad\varepsilon$ tuning the mean free path.

- *mean free path* ℓ_ε: typical length that the tagged particle must travel to have a change in velocity comparable to $|v|$
- *typical distance* between scatterers $d = 1$. characteristic speed $O(1)$
- *collision length* λ_ε $(\to 0)$: characteristic distance for deflections $O(1)$

  If λ_ε exists then the *characteristic time* between collisions is $T_{BG} = \frac{1}{\lambda_\varepsilon^2}$

  Ex: $\Phi(x,\varepsilon) = \frac{\varepsilon^s}{|X|^s}$ $\quad \lambda_\varepsilon = \varepsilon.$

  $\Phi(x,\varepsilon) = \varepsilon G(x)$, G globally bounded $\Rightarrow$ No collision length

Kinetic limit : • $1 = d \ll \ell_\varepsilon$ as $\varepsilon \to 0$ (KL1)

- statistical independence of the deflections at distances $O(\ell_\varepsilon)$

$\uparrow$

$\left(\mathbb{E}[D(0)D(T)] \ll \sqrt{\mathbb{E}[D(0)^2]\,\mathbb{E}[D(T)^2]} \quad \text{as } \varepsilon \to 0\right)$

Dynamics of the test particle

- $(x(t), v(t))$: position and velocity of the tagged particle. (x_0, v_0) in. data.

$$\begin{cases} \frac{dx}{dt} = v \\ \frac{dv}{dt} = F(x,\varepsilon)\,\omega \end{cases}$$

- $T^t(x_0, v_0; \varepsilon; \omega)$: Hamiltonian flow. $f_0 \in \mathcal{M}_+\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$

Goal : $f_\varepsilon(\ell_\varepsilon t, \ell_\varepsilon x, v) = \mathbb{E}[f_0(T^{-\ell_\varepsilon t}(\ell_\varepsilon x, v; \varepsilon; \cdot))]$ as $\varepsilon \to 0$?

Tool : to control the deflections at distances larger than λ_ε split Φ as

$$\Phi(x,\varepsilon) = \Phi_B(x,\varepsilon) + \Phi_L(x,\varepsilon)$$

$$\Phi_B(x,\varepsilon) := \Phi(x,\varepsilon)\,\eta\left(\frac{|x|}{M\lambda_\varepsilon}\right) \qquad \Phi_L(x,\varepsilon) := \Phi(x,\varepsilon)\left[1-\eta\left(\frac{|x|}{M\lambda_\varepsilon}\right)\right], \quad M>0$$

big deflections within λ_ε — deflections at distances $\gg \lambda_\varepsilon$

$\eta \in C^\infty(\mathbb{R}^3)$ s.t. $0 \le \eta \le 1$, $\eta(|x|) = 1$ if $|x| \le 1$, $\eta(|x|) = 0$ if $|x| \ge 2$

Dynamics of the test particle

- $(x(t), v(t))$: position and velocity of the tagged particle. (x_0, v_0) in. data.

$$\begin{cases} \frac{dx}{dt} = v \\ \frac{dv}{dt} = F(x,\varepsilon)\,\omega \end{cases}$$

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Goal : $f_\varepsilon(\ell_\varepsilon t, \ell_\varepsilon x, v) = \mathbb{E}[f_0(T^{-\ell_\varepsilon t}(\ell_\varepsilon x, v; \varepsilon; \cdot))]$ as $\varepsilon \to 0$?

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$$\Phi(x,\varepsilon) = \Phi_B(x,\varepsilon) + \Phi_L(x,\varepsilon)$$

$$\underbrace{\Phi_B(x,\varepsilon) := \Phi(x,\varepsilon)\eta\Big(\frac{|x|}{M\lambda_\varepsilon}\Big)}_{\text{big deflections within } \lambda_\varepsilon} \qquad \underbrace{\Phi_L(x,\varepsilon) := \Phi(x,\varepsilon)\Big[1-\eta\Big(\frac{|x|}{M\lambda_\varepsilon}\Big)\Big]}_{\text{deflections at distances} \gg \lambda_\varepsilon}, \quad M > 0$$

At distances $O(\lambda_\varepsilon)$ the particle is deflected for an amount $O(1)$ by Φ_B.

Time scale T_L in which the deflections produced by Φ_L become relevant ?

Test Particle Deflections produced by Φ_L

Consider the dynamics in the Holtsmark field $F_L(x,\varepsilon)\omega$ for $t \in [0, T]$

$$T \text{ small } \Rightarrow x(t) \simeq v_0 t \quad \& \quad D_T(\varepsilon)\omega := \int_0^T F_L(v_0 t, \varepsilon)\omega \, dt \quad \text{(change of velocity)}$$

- Characteristic function: $\quad m_T^{(\varepsilon)}(\theta) = \mathbb{E}\left[\exp\left(i\theta \cdot D_T(\varepsilon)\omega\right)\right] \quad , \quad \theta \in \mathbb{R}^3$
- Characteristic time for the deflections:

$$\sigma(T;\varepsilon) := \sup_{|\theta|=1} \int_{\mathbb{R}^3} dy \left(\theta \cdot \int_0^T \nabla_x \Phi_L(vt - y, \varepsilon)\, dt\right)^2$$

- Landau time scale T_L: $\quad \sigma(T_L;\varepsilon) = 1$
- $1 = d \ll \ell_\varepsilon$ becomes $\ell_\varepsilon = \min\{T_{BG}, T_L\} \gg 1$ as $\varepsilon \to 0$

  $\Rightarrow$ *the time scale for the kinetic evolution is the shortest among* T_{BG}, T_L

Different cases:

$$T_L \gg T_{BG} \quad \text{or} \quad T_L \ll T_{BG} \quad \text{or} \quad \frac{T_L}{T_{BG}} \to C_* \in (0,\infty) \text{ as } \varepsilon \to 0$$

Test Particle Deflections produced by Φ_L

Consider the dynamics in the Holtsmark field $F_L(x, \varepsilon)\omega$ for $t \in [0, T]$

T small $\Rightarrow$ $x(t) \simeq v_0 t$ & $D_T(\varepsilon)\omega := \int_0^T F_L(v_0 t, \varepsilon)\omega \, dt$ (change of velocity)

- Characteristic function: $m_T^{(\varepsilon)}(\theta) = \mathbb{E}\left[\exp\left(i\theta \cdot D_T(\varepsilon)\omega\right)\right]$, $\theta \in \mathbb{R}^3$
- Characteristic time for the deflections:

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Different cases:

$T_L \gg T_{BG}$ or $T_L \ll T_{BG}$ or $\frac{T_L}{T_{BG}} \to C_* \in (0, \infty)$ as $\varepsilon \to 0$

Collisions vs. diffusion

Family of potentials: $\{\Phi(x,\varepsilon);\ \varepsilon > 0\}$, $\quad \Phi(\cdot,\varepsilon) \in \mathcal{C}_s, \quad s > 1/2$

$$\Phi(x,\varepsilon) = \Psi\left(\frac{|x|}{\varepsilon}\right), \qquad \Psi \in C^2\left(\mathbb{R}^3 \setminus \{0\}\right)$$

$$\Psi(y) \sim \frac{A}{|y|^s}, \quad \nabla\Psi(y) \sim -\frac{sAy}{|y|^{s+2}} \quad \text{as} \quad |y| \to \infty, \quad 0 \neq A \in \mathbb{R}.$$

Collision length: $\lambda_\varepsilon = \varepsilon$. $\quad$ Boltzmann-Grad time scale: $T_{BG} = \dfrac{1}{\varepsilon^2}$.

Collisions vs. diffusion

Family of potentials: $\{\Phi(x,\varepsilon)\,;\ \varepsilon>0\}$, $\Phi(\cdot,\varepsilon)\in\mathcal{C}_s$, $s>1/2$

$$\Phi(x,\varepsilon)=\Psi\left(\frac{|x|}{\varepsilon}\right),\qquad \Psi\in C^2\left(\mathbb{R}^3\setminus\{0\}\right)$$

$$\Psi(y)\sim\frac{A}{|y|^s},\quad \nabla\Psi(y)\sim-\frac{sAy}{|y|^{s+2}}\ \text{ as }\ |y|\to\infty,\quad 0\neq A\in\mathbb{R}.$$

Collision length: $\lambda_\varepsilon=\varepsilon$. Boltzmann-Grad time scale: $T_{BG}=\dfrac{1}{\varepsilon^2}$.

Theorem [N., Simonella, Velázquez 2018]

$s>1$	$s=1$	$1/2<s<1$
$\limsup_{\varepsilon\to 0}\sigma(T_{BG};\varepsilon)\le\delta(M)$ $\lim_{M\to\infty}\delta(M)=0$ *	$T_L\sim\dfrac{1}{A^2\varepsilon^2\log\left(\frac{1}{\varepsilon}\right)}$	$T_L\sim\left(\dfrac{1}{W_sA^2\varepsilon^{2s}}\right)^{\frac{1}{3-2s}}$

Hence $T_L\ll T_{BG}$ as $\varepsilon\to 0$ if $s\le 1$

*Small deflections due to interactions at distances larger than $M\lambda_\varepsilon$ become irrelevant as $M\to\infty$ in the timescale T_{BG}

Collisions vs. diffusion

Theorem [N., Simonella, Velázquez 2018]

$s > 1$	$s = 1$	$1/2 < s < 1$
$\limsup_{\varepsilon \to 0} \sigma(T_{BG}; \varepsilon) \leq \delta(M)$ $\lim_{M \to \infty} \delta(M) = 0 \ *$	$T_L \sim \dfrac{1}{A^2 \varepsilon^2 \log\left(\frac{1}{\varepsilon}\right)}$	$T_L \sim \left(\dfrac{1}{W_s A^2 \varepsilon^{2s}}\right)^{\frac{1}{3-2s}}$

Hence $T_L \ll T_{BG}$ as $\varepsilon \to 0$ if $s \leq 1$

Remark: For the Ideal Rayleigh gas (not interacting background affected by the tagged particle) the diffusive time scales are the same.

Different type of diffusion coefficient !

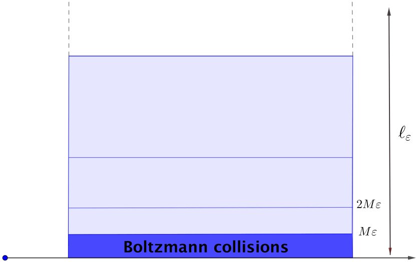
(Diffusion not restricted on the sphere of constant velocity: the energy of the tagged particle is no longer conserved in the collisions!)

([N., Velázquez, Winter 2021, 2022])

The Coulombian Logarithm

What is the effect of Φ_L?

Dyadic decomposition $A_k := [2^k M\varepsilon, 2^{k+1} M\varepsilon]$ between $M\varepsilon$ and the m.f.p. $\ell_\varepsilon \approx T_\varepsilon t$



$$\#\{A_k\} = \frac{\log\left(\frac{T_\varepsilon t}{M\varepsilon}\right)}{\log 2}$$

Deflection D_k due to particles in A_k : $\mathbb{E}[D_k] = 0, \quad \mathbb{E}[(D_k)^2] \simeq \varepsilon^2 T_\varepsilon t$

Variance of the total deflection $D := \sum_k D_k$:

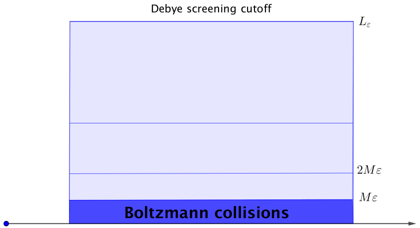
$$Var(D) = \sum_k Var(D_k) = \frac{\log\left(\frac{T_\varepsilon t}{M\varepsilon}\right)}{\log 2} \varepsilon^2 T_\varepsilon t \sim t$$

$\Rightarrow Var(D) \sim t \quad \Leftrightarrow \quad T_\varepsilon = \frac{C}{\varepsilon^2 |\log(\frac{1}{\varepsilon})|} := T_L$ with $C = 3$ (Landau timescale)

The Coulombian Logarithm in the interacting particle case

What is the effect of Φ_L ?

Dyadic decomposition $A_k := [2^k M\varepsilon, 2^{k+1} M\varepsilon]$ between $M\varepsilon$ and $L_\varepsilon = \frac{1}{\sqrt{\varepsilon}}$ (Debye screening length)



$$\#\{A_k\} = \frac{\log\left(\frac{t}{\varepsilon\sqrt{\varepsilon}}\right)}{\log 2}$$

Same computations performed for the Lorentz Gas

$\Rightarrow$ Total deflection $Var(D) \sim t \quad \Leftrightarrow \quad T_\varepsilon = \frac{\tilde{C}}{\varepsilon^2 |\log(\frac{1}{\varepsilon})|} =: T_L$

The difference is in the numerical factor $\tilde{C} = \frac{3}{2}$!

Correlations when $T_L \ll T_{BG}$ $\quad (s \leq 1)$

Deflection

$$D\left(x_0, v; \zeta T_L\right) = \int_0^{\zeta T_L} \nabla_x \Phi_L\left(x_0 + vt, \varepsilon\right) \omega\, dt \qquad x_0, v \in \mathbb{R}^3,\ |v| = 1, \quad \zeta > 0$$

Theorem [N., Simonella, Velázquez 2018]

- $s = 1$: $\quad \mathbb{E}\left[D(x_0, v; \zeta T_L) D(x_0 + v\zeta T_L, v; \zeta T_L)\right] = O(T_L \varepsilon^2) \to 0$ as $\varepsilon \to 0$

  and $\displaystyle \frac{1}{2}\int_{\mathbb{R}^3} \left(\theta \cdot \int_0^{\zeta T_L} dt\, \nabla_x \Phi_L\left(v\zeta T_L - y, \varepsilon\right)\right)^2 dy \underset{\varepsilon \to 0}{\to} \kappa \zeta \left|\theta_\perp\right|^2, \quad \kappa > 0.$

  $\theta_\perp = \theta - \theta \cdot \frac{v}{|v|} \frac{v}{|v|}$

- $s \in \left(\frac{1}{2}, 1\right)$: $\mathbb{E}\left[D(x_1, v_1; \zeta T_L) D(x_2, v_2; \zeta T_L)\right] \to \zeta^2 K\left(X, V\right) \neq 0$ as $\varepsilon \to 0$

  $$T_L X = \left(x_2 - x_1\right), \qquad T_L V = \left(v_2 - v_1\right)$$

  and $\displaystyle \frac{1}{2}\int_{\mathbb{R}^3} \left(\theta \cdot \int_0^{\zeta T_L} dt\, \nabla_x \Phi_L\left(v\zeta T_L - y, \varepsilon\right)\right)^2 dy \underset{\varepsilon \to 0}{\to} \kappa \zeta^{3-2s} \left|\theta_\perp\right|^2, \quad \kappa > 0.$

  (non vanishing correlations on macroscopic times!)

Kinetic equations

On the correct kinetic scale the tracer particle distribution

$$f_\varepsilon\left(\ell_\varepsilon t, \ell_\varepsilon x, v\right) = \mathbb{E}_\omega[f_0(T^{-\ell_\varepsilon t}\left(\ell_\varepsilon x, v; \varepsilon; \cdot\right))] \underset{\varepsilon \to 0}{\longrightarrow} f(t, x, v) \quad \text{solution to}$$

Claim:

$[s > 1]$ Linear Boltzmann equation

$$\left(\partial_t f + v \cdot \nabla_x f\right)(t, x, v) = \sum_{j=1}^{L} \mu(Q_j) \int_{S^2} B\left(v; \omega; Q_j\right) \left[f\left(t, x, |v|\,\omega\right) - f\left(t, x, v\right)\right] d\omega$$

$[s > 1]$ Linear Landau equation

$$\left(\partial_t f + v \nabla_x f\right)(t, x, v) = \kappa \Delta_{v_\perp} f\left(t, x, v\right), \qquad \kappa > 0$$

$\sum_{i,j=1}^{3} \partial_{v_i} A_{i,j}(v) \partial_{v_j} f$ with $A_{i,j}(v) = K\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)$

$\left[s \in \left(\frac{1}{2}, 1\right)\right]$ Stochastic differential equation with correlated noise

$$x(\tau + d\tau) - x(\tau) = v(\tau) d\tau$$

$$v(\tau + d\tau) - v(\tau) = D(x(\tau), v(\tau); d\tau), \quad D = O((d\tau)^\beta), \quad \beta \in (0, 1)$$

How sensitively the time scales T_{BG}, T_L and the kinetic equations depend on the specific details of the interaction ?

Consider different families of potentials $\Phi(x,\varepsilon) = \varepsilon G(|x|)$, $G \in \mathfrak{C}_s$, $s > 1/2$.

- $G \in C^2(\mathbb{R}^3 \setminus \{0\})$ and $G(x) \sim \frac{A}{|x|^s}$ as $|x| \to \infty$,

  $G(x) \sim \frac{B}{|x|^r}$ as $|x| \to 0$, $r \geq 0$.

- Collision length: $\lambda_\varepsilon = \varepsilon^{\frac{1}{r}}$. Boltzmann-Grad timescale: $T_{BG} = \varepsilon^{-\frac{2}{r}}$.

	$s > 1$	$s = 1$
$r > 1$	$\limsup_{\varepsilon \to 0} \sigma(T_{BG}; \varepsilon) \leq \delta(M)$ $\delta(M) \to 0, \quad M \to \infty$	$\limsup_{\varepsilon \to 0} \sigma(T_{BG}; \varepsilon) \leq \delta(M)$ $\delta(M) \to 0, \quad M \to \infty$
$r \leq 1$	$r = 1 \quad T_L \sim \frac{1}{B^2 \varepsilon^2 \lvert\log(\varepsilon)\rvert}$ $r < 1 \quad T_L \sim \frac{C}{\varepsilon^2}, \quad T_L \ll T_{BG}$	$T_L \sim \frac{C}{\varepsilon^2 \lvert\log(\varepsilon)\rvert}, \quad T_L \ll T_{BG}$

How sensitively the time scales T_{BG}, T_L and the kinetic equations depend on the specific details of the interaction ?

Consider different families of potentials $\Phi(x, \varepsilon) = \varepsilon G(|x|)$, $G \in \mathcal{C}_s$, $s > 1/2$.

- $G \in C^2\left(\mathbb{R}^3 \setminus \{0\}\right)$ and $G(x) \sim \frac{A}{|x|^s}$ as $|x| \to \infty$,

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- Collision length: $\lambda_\varepsilon = \varepsilon^{\frac{1}{r}}$. Boltzmann-Grad timescale: $T_{BG} = \varepsilon^{-\frac{2}{r}}$.

Kinetic description?

	$s > 1$	$s = 1$	$1/2 < s < 1$
$r > 1$	Boltzmann eq.	Boltzmann eq.	• $2s > 3 - r$ Boltzmann eq. • $2s < 3 - r$ Stochastic eq.
$r \leq 1$	Landau eq.	Landau eq.	Stochastic diff. eq. with correlations

Summary

- Conditions on the interactions to have a kinetic description:
  weak enough interaction to have $\ell_\varepsilon \gg d$. Then:
  - if the fastest process yielding particle deflections are binary collisions with single scatterers $\Rightarrow$ linear Boltzmann eq.
  - If the deflections due to the accumulation of a large no. of small interactions yield a relevant change in the direction of v before a binary collision takes place $\Rightarrow$ linear Landau eq.
    (The deflections over times of order T_L must be uncorrelated.)
  - Potentials for which this lack of correlations does not take place. Then macroscopic deflections must be taken into account.
- The proof of the independence of the deflections is the crucial step towards any rigorous derivation of the kinetic equations !

THANK YOU

FOR YOUR ATTENTION !