Different types of percolation for models of interacting cycles and loops on a graph

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Phase transitions in spatial particle systems

Berlin, July 31 - August 2, 2023

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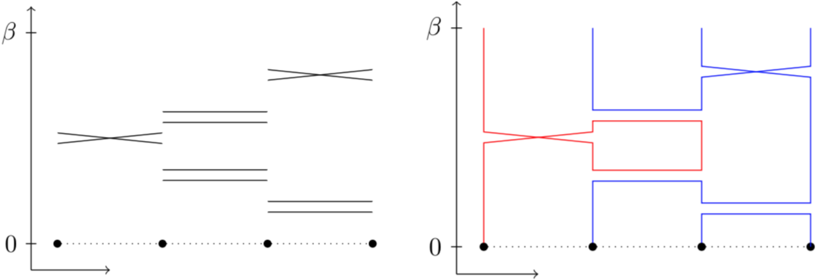
On many graphs (e.g. $\mathbb{Z}^2$) there is a value $p_c \in (0, 1)$ such that:

1) for $p < p_c$ connected components of P are finite a.s.

2) for $p > p_c$ there are a.s. infinite connected components of P.

The random loop model: intuition

We could place more complicated structures on edges of the graph:



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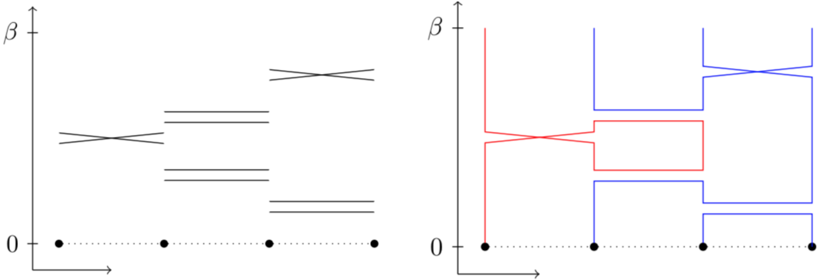
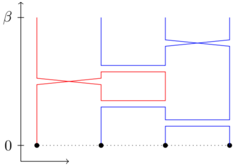


Each edge has a "time" interval $[0, \beta)$ on which we place *links* of two types, crosses (intensity u) and bars (intensity $1-u$).
$u = 1$: random transpositions $\rightsquigarrow$ random permutation.

Definition of the model

- Graph $G = (V, E)$ and Circle $[0, \beta)_{per}$.
- iid Poisson point processes $(X_e^{\chi})_{e \in E}$, $(X_e^{||})_{e \in E}$ on $[0, \beta)$, intensities u and $1 - u$, respectively. Law is $\rho(u, \beta)$.
- Relation ('**connectedness**') on $V \times [0, \beta)$: (v, t) is connected to (v', t') (wlog $t < t'$) if they are in the same loop.



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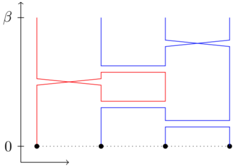


Now we have two types of percolation on the graph:

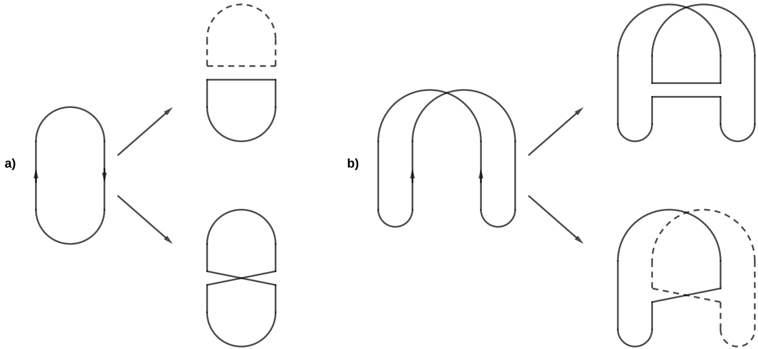
1) Percolation of edges with at least one link (*link percolation*)

2) Percolation of the connectedness condition (*loop percolation*).

A natural question is whether these two percolations coincide.

Difficulties of the model

We certainly need link percolation in order to have loop percolation, but the converse relation is not so clear. Indeed, the model does not enjoy monotonicity properties, i.e. adding a link may split a loop into two smaller loops, pictorially like this:



The connection with quantum spin systems

The loop model is closely related to the following hamiltonian on G

$$H = -\sum_{\{x,y\}\in E} \sigma_x^1\sigma_y^1 + (2u-1)\sigma_x^2\sigma_y^2 + \sigma_x^3\sigma_y^3, \quad \text{on } \otimes_{x\in V}\mathbb{C}^2 \equiv \mathbb{C}^{2^{|V|}}$$

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$$\mathbb{P}_{\theta,u,\beta}(A) = \mathbb{E}_{\rho(u,\beta)}[\mathbb{1}_A \theta^L] / \mathbb{E}_{\rho(u,\beta)}[\theta^L].$$

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We have the following connection

$$\langle \sigma_x^1 \sigma_y^1 \rangle_{\beta,u} = tr(\sigma_x^1 \sigma_y^1 e^{-\beta H}) / tr(e^{-\beta H}) = \mathbb{P}_{2,u,\beta}((x,0) \leftrightarrow (y,0)).$$

Loop percolation

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Question:

Are there intermediate β where link percolation occurs but loop percolation does not?
Let us stick to $\theta = 1$ (independent links on each edge) for now.

Answer on K_n, $\theta = 1$: no

Take the complete graph on n vertices and realisation w of links on $[0, \beta/(n-1)) \times E$.

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Theorem (Schramm '05 and Björnberg et al '19)

1) $\mathbf{u} = \mathbf{1}$. As $n \to \infty$ the law of $\mathcal{X}_w$ converges weakly to the Poisson Dirichlet distribution PD(1). (Schramm '05)

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So in particular, loop and link percolation coincide. This is true for many (probably all) graphs of diverging degree.

A special case: trees. Summarising a long story

Vertices have fixed degree but the graph has no cycles (so it is much easier). On a regular tree, $\theta > 0$, Björnberg and Ueltschi ‘17, ‘18

$$\frac{\beta_c(u)}{\theta} = \frac{1}{d} + \frac{1 - \theta u(1-u) - \theta^2(1-u)^2/6}{d^2} + o(d^{-2})$$

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Betz et al '18 considered Galton-Watson trees, finding explicit sufficient conditions on the offspring distribution for infinite loops. Betz et al '21 even found an expansion for $\beta_c(u)$

$$\beta_c(u) = \sum_{k=0}^{n} \frac{\alpha_k(u)}{d^{k+1}} + O(d^{-n-2}),$$

where α_k's are degree $2k$ polynomials that can be computed recursively. This heavily uses the tree structure.

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Let G be a countably infinite connected graph of uniformly bounded degree. Let β_c^{per} be the critical β for link percolation and $\beta_c(u)$ the critical β for loop percolation then if $\beta_c^{per} \in (0, \infty)$ we have $\beta_c(u) > \beta_c^{per}$ for all $u < 1$.

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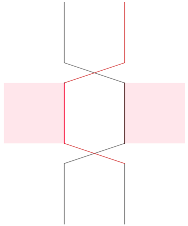
It is even possible to provide an explicit lower bound on the length of the interval $[\beta_c^{per}, \beta_c(u))$ (Klippel et al '23+).

Bounded degree graphs - proof idea

The proof idea for bounded degree graphs is quite beautiful and simple. The links on edges are independent and some link structures will “block” loops.



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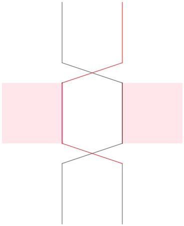

Two crosses in a row on the edge, and no crosses in the shaded red area. These events occur on the link percolation, thinning it and resulting in an effective decrease of the loop percolation parameter.

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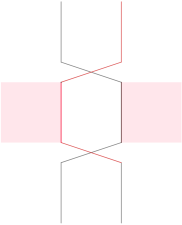

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This works because links are independent for $\theta = 1$, so link percolation $=$ Bernoulli percolation with parameter $1 - e^{-\beta}$. If $\theta \neq 1$ links are not independent making things difficult.

Loop models for classical spin systems

There are also loop representations for classical spin systems. Their loop percolation properties are simpler than even the $\theta = 1$ case above, but the difficulty of whether link percolation = loop percolation seems to lie somewhere between the $\theta = 1$ and $\theta \neq 1$ case.

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The spin model of interest is the $O(N)$-spin model, the classical version of the hamiltonian we saw. For a collection $\boldsymbol{S} = (S_x)_{x \in V}$ of unit vectors (spins) in $\mathbb{R}^N$, $N > 1$ define the hamiltonian

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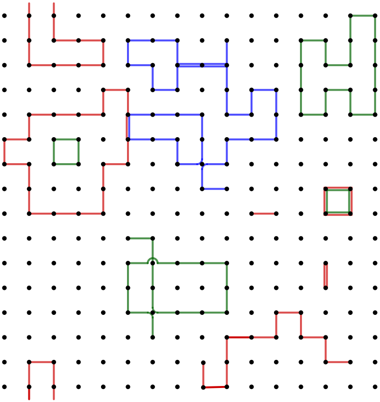
A function of these spins f has expectation

$$\mathbb{E}_{\beta,N}(f) = \frac{1}{Z_{\beta,N}} \int \mathrm{d}\boldsymbol{S}\, f(\boldsymbol{S})\, e^{-\beta H(\boldsymbol{S})}$$

The random path model



The loop model for this system has 3 ingredients:

- a collection of natural numbers $\boldsymbol{m} = (m_e)_{e \in E}$ where m_e is *the number of links on edge* e
- a colouring $\boldsymbol{c}$ of these links with colours in $\{1, \ldots, N\}$.
- a *pairing* π of links at each vertex to form loops and paths

The measure on loops

Different choices of measure for $w = (\boldsymbol{m}, \boldsymbol{c}, \pi)$ give different models of interest such as dimers, spatial random permutations, $O(N)$-loops.... the most important example is the $O(N)$-spin model.

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The measure is

$$\mu_{G,N,\beta}(w) = \frac{1}{Z_{G,N,\beta}} \prod_{e \in E} \frac{\beta^{m_e}}{m_e!} \prod_{x \in V} U_x(w).$$

with

$$U_x(w) = \frac{1}{2^{n_x(w)} \Gamma\left(n_x(w) + \frac{N}{2}\right)}$$

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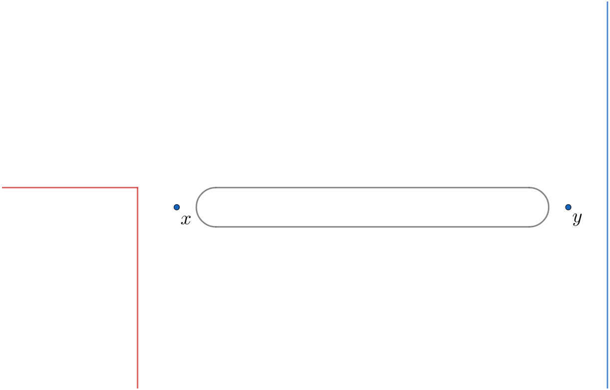
where $n_x(w) =$ is the local time of paths at x. Under this measure $Z_{\beta,N} =$ total weight of configurations with only loops.
Loop percolation occurs for large β (Fröhlich, Simon, and Spencer '76 for $O(N)$-spins).

The analogue to blocking structures - sausages

Again we have the question: do connected components of edges carrying links percolate at a different value of β to loops?
Just like the case $\theta = 1$, this model has a natural blocking structure that prevent loops from crossing an edge.



Loop vs link percolation for $O(N)$-spins

Theorem (Betz, Klippel, L. '23++)

Let $d \geq 3$ and G be any d-regular graph, then there is a $\delta > 0$ depending on G such that

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The main difficulty in the proof is that links are far from independent, so the blocking edges (apparently) do not thin the link clusters in a simple way.

Rough proof strategy blocking edges occur with some "density"

First let us suppose we are on a "good" set of random path configurations on G_n: For some $K > 0$ let $\mathcal{G}_\varepsilon(K)$ be the set of configurations such that $\leq \varepsilon|V_n|$ vertices have local time $> K$.

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It can be shown that for a set $A \subset E$ such that $\text{dist}(e, e') \geq 4$ for every distinct pair $e, e' \in A$ we have for any $w \in \mathcal{G}_\varepsilon(K)$

$$\mu_{G,N,\beta}(\text{every edge in } A \text{ is blocking} \big| w|_{N_2(A)^c}) \geq c^{|A|}$$

where $N_2(A)$ is the 2-neighbourhood of A, for some $c > 0$ that depends on K, G, β (but not on A or G_n).

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So blocking edges aren't independent, but if they are far enough apart their occurrence can be bounded below in a nice way.

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We can hence sum over all SAWs and find that our bound on blocking edges leads to a decrease in the connective constant of the link clusters. With some work, the result follows.

Thank you for listening.

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