Weierstrass Institute for
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Convergence of Simulable Processes for Coagulation with Transport

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Outline

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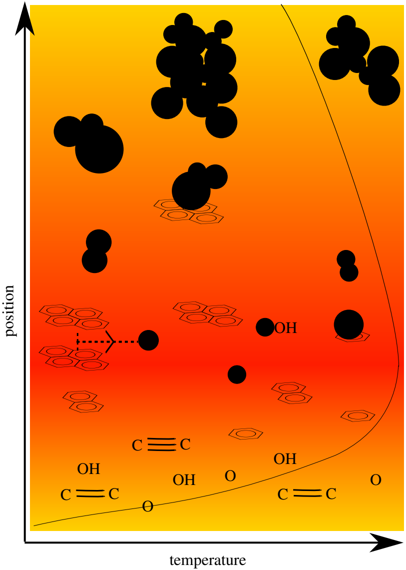
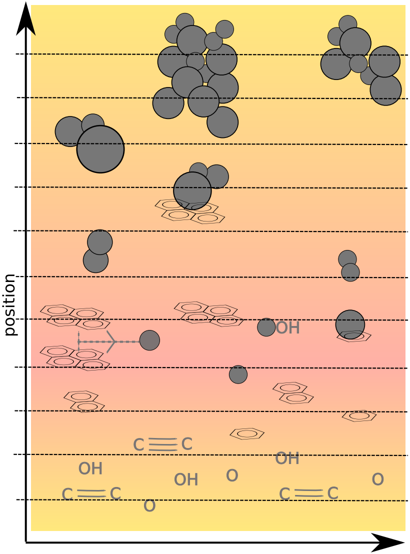
- Bounded region of laminar flow.
- Particle type space $\mathcal{X}$.
- Particles of type x_0 incepted with intensity $I \geq 0$.
- Pairs of particles collide and coagulate according to $K \geq 0$.
- Particles drift at velocity $u > 0$.
- Particles simply flow out of the domain from its end.



Delocalised Model

- Avoid simulating random walks and detecting collisions!
- Require a model for coagulation probabilities.
- Look for simulable dynamics.
- Follow Gas DSMC approach:
  - discretise space into cells,
  - delocaliase coagulation within each cell.
- We consider just one cell of size Δx in 1-d.



Existing Results

- Infinite homogeneous box, no flow:
  - Boltzmann setting: Wagner 92
  - Coagulation: Jeon 98, Norris 99
  - Famous review by Aldous 99
  - More general interactions: Eibeck & Wagner 03, Kolokoltsov book 10
- Diffusion in infinite domain: via jump process Guiaş 01
- Diffusion in infinite domain: via SDE Deaconu & Fournier 02
- Hammond, Rezakhanlou & co-workers 06-10

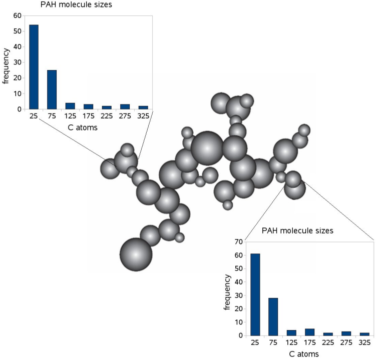
Physical Applications

Example Applications:

- Particle synthesis,
- Pollutant formation,
- Precipitation/crystallisation in clouds.

Mathematical Consequences:

- Bounded domain,
- Inflow & outflow,
- Outflow is dependent on rest of process,
- Convergence of approximations not covered in literature.



- Need a sequence of Markov Chains to study convergence; index n.
- Replace continuum with finite computable number of particles.
- Spatial cell is $[0, 1]$, i.e. $\Delta x = 1$.
- Scaling factor n: Inverse of concentration represented by one computational particle.
- Coagulation x and y at rate $K(x, y)/n$.
- Formation of new particles of type $x_0 \in \mathcal{X}$ at rate nI throughout the cell.
- Constant velocity $u > 0$ for all particles.
- Particles absorbed at 1.

Notation

- Individual particle and position an element of $\mathcal{X}' = \mathcal{X} \times [0,1]$.
- Fock state space for the particle systems $E = \bigcup_{k=0}^{\infty} \mathcal{X}'^k$.
- Let $\psi : \mathcal{X}' \to \mathbb{R}$ and define $\psi^{\oplus} : E \to \mathbb{R}$ by $\psi^{\oplus}(x_1, \ldots x_k) = \sum_{j=1}^{k} \psi(x_j)$.
- $X^n(t)$ is the E-valued process.
- $N(X^n(t))$ is the number of particles.
- $X^n(t,i) \in \mathcal{X}'$ is the type and location of the i-th particle.

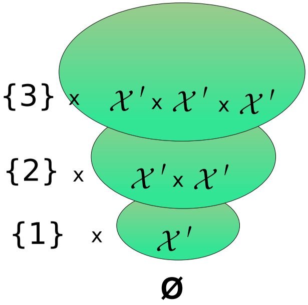


Figure: The disjoint union E.

Let $X \in E$, $X = (X(1), \ldots, X(N(X)))$, then the generators A_n satisfy

$$A_n \psi^{\oplus}(X) = nI \int_{[0,1]} \psi(x_0, y)\mathrm{d}y + u\nabla\psi^{\oplus}(X) +$$

$$\frac{1}{2} \sum_{\substack{j_1, j_2 = 1 \\ j_1 \neq j_2}}^{N(X)} \left[\psi\left(X(j_1) + X(j_2)\right) - \psi\left(X(j_1)\right) - \psi\left(X(j_2)\right)\right] \frac{K\left(X(j_1), X(j_2)\right)}{n}.$$

- inceptions of x_0 at times R_i^n,
- advection with velocity u,
- coagulations at times U_i^n,
- exits from 1 at times S_i^n,
- T_i^n will be time of any kind of jump.
- $R^n(t)$, $S^n(t)$,$U^n(t)$ and $T^n(t) \in \mathbb{N}$ are defined as the respective jump counting processes hence $R^n(t) + S^n(t) + U^n(t) = T^n(t)$.

Current Work

- Overall goal is proof of convergence of the simulable particle systems to a solution of the PBE.
- Immediate goal is relative compactness of approximating sequence via:
  - construction of approximating sequence,
  - martingales that converge to 0,
  - compact containment,
  - control of Modified Variation,
- Deviations and confidence intervals also interesting and studied by Kolokoltsov for the classical cases mentioned previously.
- Uniqueness of limit point is an additional question.

We now have piecewise deterministic Markov processes (Davis 1993) defined by jumps and jump rates.

Theorem

For all $t \geq 0$ *and* $k \in \mathbb{N}$ *there exists* $A_2(t,k)$ *which is* $\mathcal{O}(t^k)$ *uniformly in* n *such that:*

$$\mathbb{E}\left[\left(\frac{T^n(t)}{n}\right)^k\right] \leq A_2(t,k). \tag{1}$$

Proof.

Coagulation and exit events each remove one particle, hence $U^n(t) + S^n(t) \leq R^n(t)$, thus $T^n(t) \leq 2R^n(t)$ and $R^n(t) \sim \mathrm{Poi}(nIt)$. □

A further, important result of Davis (1993) is that the following processes are shown by direct calculation to be Martingales:

Theorem

Let $\psi \in C_B^{0,1}(\mathcal{X}') = C_B^{0,1}(\mathcal{X} \times [0,1])$ *then for all* n *the following process is a Martingale:*

$$M_n^{\psi}(t) = \frac{1}{n}\psi^{\oplus}\left(X^n(t)\right) - \frac{1}{n}\psi^{\oplus}\left(X^n(0)\right) - \int_0^t \frac{1}{n} A_n \psi^{\oplus}\left(X^n(s)\right) \mathrm{d}s + \frac{1}{n}\sum_{i=1}^{S^n(t)} \psi(Z_i^n, 1)$$

Proof.

Davis (1993) Theorem 31.3. □

The domain of the generator is in some sense restricted to ψ such that $\psi(\cdot, 1) \equiv 0$.

Notation

- Particle leaving at U_i^n is $Z_i^n \in \mathcal{X}$.
- Particle (with position) incepted at R_i^n is $Y_i^n \in \mathcal{X}'$.
- Let $\psi \in C_B(\mathcal{X}') = C_B(\mathcal{X} \times [0,1])$ and define $[\psi]$ by $[\psi](x_1, x_2) = \psi(x_1 + x_2) - \psi(x_1) - \psi(x_2)$ (one coagulation).
- A 'self coagulation' is given by $[[\psi]](x) = [\psi](x, x)$.
- Expected coagulation effects are given by

$$\mathcal{K}_n(\psi)(X) = \frac{1}{2n} \sum_{i_1, i_2 = 1}^{N(X)} [\psi] \left(X(i_1), X(i_2)\right) K.$$

- Expected (& unwanted) self-coagulation effects are given by

$$\widetilde{\mathcal{K}}_n(\psi)(X) = \frac{1}{2n} \sum_{i=1}^{N(X)} [[\psi]] \left(X(i)\right) K.$$

Expanding the generator gives a representation for the martingale that is easier to use for calculations:

$$M_n^{\psi}(t) = \frac{1}{n}\sum_{i=1}^{N(X^n(t))} \psi\left(X^n(t,i)\right) - \frac{1}{n}\sum_{i=1}^{N(X^n(0))} \psi\left(X^n(0,i)\right)$$
$$+ \frac{1}{n}\sum_{i=1}^{S^n(t)} \psi(Z_i^n, 1) - \int_0^t \frac{1}{n}\sum_{i=1}^{N(X^n(s))} u\nabla\psi\left(X^n(s,i)\right) \mathrm{d}s$$
$$- t\int_{[0,1]} \psi(x_0, y) I \mathrm{d}y - \int_0^t \frac{1}{n}\mathcal{K}_n\psi\left(X^n, s\right) \mathrm{d}s + \int_0^t \frac{1}{n}\widetilde{\mathcal{K}}_n\psi\left(X^n, s\right) \mathrm{d}s$$

Theorem

For all $t \geq 0$ *and* $\psi \in C_B^{0,1}(\mathcal{X}')$ *there exists* $A_4(t, \psi)$ *independent of* n *such that*

$$\mathbb{E}\left[M_n^\psi(t)^2\right] \leq \frac{A_4(t, \psi)}{n}.$$

Proof.

Not proved here. □

Theorem

For all $t \geq 0$ and $\psi \in C_B^{0,1}(\mathcal{X}')$ there exists $A_5(t,\psi)$, independent of n such that

$$\mathbb{P}\left(\sup_{s\leq t}\left|M_n^{\psi}(s)\right| \geq \epsilon\right) \leq \frac{A_5(t,\psi)}{\epsilon^2 n}.$$

Proof.

Doob's inequality □

This result has two important applications:

- Demonstates properties of weak limit points.
- As a technical tool in the remainder of the talk.

Necessary now to move to a weak point of view ...

The processes can be viewed as measures:

$$\mu_t^n := \frac{1}{n} \sum_{i=1}^{N(X^n(t))} \delta_{X^n(t,i)},$$

which are elements of the space of finite measures $\mathcal{M}(\mathcal{X}')$, which is given the topology generated by pairings with $\psi \in C_B^{0,1}(\mathcal{X}')$.

One therefore has $\mu^n \in \mathbb{D}\left(\mathbb{R}^+, \mathcal{M}(\mathcal{X}')\right)$ and

$$\frac{1}{n}\psi^{\oplus}\left(X^n(t)\right) = \langle \psi, \mu_t^n \rangle = \int_{\mathcal{X}'} \psi(x,y)\mu_t^n(\mathrm{d}x, \mathrm{d}y).$$

Weak Limit Equation

Note first that if $\psi(\cdot, 1) = 0$ then

$$M_n^{\psi}(t) = \langle \psi, \mu_t^n \rangle - \langle \psi, \mu_0^n \rangle - \int_0^t \langle u \nabla \psi, \mu_s^n \rangle \, \mathrm{d}s - t \int_{[0,1]} \psi(x_0, y) I \mathrm{d}y$$
$$- \frac{1}{2} \int_0^t \langle [\psi] K, \mu_s^n \otimes \mu_s^n \rangle \, \mathrm{d}s + \frac{1}{2n} \int_0^t \langle [[\psi]] K, \mu_s^n \rangle \, \mathrm{d}s.$$

Suppose that $\mu_t^n \xrightarrow{w} \mu_t$ for all t (convergence in Skorohod space is sufficient) then

$$0 = \langle \psi, \mu_t \rangle - \langle \psi, \mu_0 \rangle - \int_0^t \langle u \nabla \psi, \mu_s \rangle \, \mathrm{d}s - t \int_{[0,1]} \psi(x_0, y) I \mathrm{d}y$$
$$- \frac{1}{2} \int_0^t \langle [\psi] K, \mu_s \otimes \mu_s \rangle \, \mathrm{d}s,$$

which is a weak form of the PBE.

Are there any limit points?

Theorem

Let (E, r) be a complete and separable metric space and let $\{X_n\}$ be a sequence of processes with sample paths in $\mathbb{D}([0, \infty), (E, r))$. Then $\{X_n\}$ is relatively compact if and only if the following two conditions hold:

a) For every $\eta > 0$ and rational $t \geq 0$ there exists a compact set $\Gamma_{\eta,t} \subset E$ such that

$$\liminf_n \mathbb{P}\left(X_n(t) \in \Gamma^{\eta}_{\eta,t}\right) \geq 1 - \eta.$$

b) For every $\eta > 0$ and $T > 0$ there exists $\delta(\eta, T) > 0$ such that

$$\limsup_n \mathbb{P}\left(w'\left(X_n, \delta(\eta, T), T\right) \geq \eta\right) \leq \eta.$$

Proof.

This is Ethier & Kurtz (1986) Chap. 3 Coroll. 7.4. □

Definition

The modified variation of a càdlàg function f from $\mathbb{R}_0^+$ to a metric space (E, r) is defined by

$$w'(f, \delta, T) = \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} r(f(s), f(t)),$$

where the t_i define partitions of $[0, T]$ with minimum spacing at least δ.

- Modulus of continuity, that can ignore a few awkard points.
- Random if f is random.

Theorem

Let $\{X_n\}$ *be a sequence of processes with sample paths in* $\mathbb{D}([0,\infty),\mathcal{M}(\mathcal{X}'))$*.*
Then $\{X_n\}$ *is relatively compact if and only if the following two conditions hold:*
a) For every $\eta > 0$ *and rational* $t \geq 0$ *and* $\psi \in C_B^{0,1}(\mathcal{X}')$ *there exists a compact set* $\Gamma_{\eta,t}^{\psi} \subset \mathbb{R}$ *such that*

$$\liminf_n \mathbb{P}\left(\langle \psi, X_n(t)\rangle \in \Gamma_{\eta,t}^{\psi,\eta}\right) \geq 1-\eta.$$

b) For every $\eta > 0$, $T > 0$ *and* $\psi \in C_B^{0,1}(\mathcal{X}')$ *there exists* $\delta^{\psi} > 0$ *such that*

$$\limsup_n \mathbb{P}\left(w'\left(\langle \psi, X_n(\cdot)\rangle, \delta^{\psi}, T\right) \geq \eta\right) \leq \eta.$$

Proof.

Vague topology: Kallenberg (2001). Weak topology: Dawson (1993). □

Theorem

For every $T > 0$ and $\psi \in C_B^{0,1}(\mathcal{X}')$ there exists $\gamma^\psi(T) < \infty$ such that

$$\lim_n \mathbb{P}\left(\sup_{t \leq T} |\langle \psi, \mu_t^n \rangle| \leq \gamma^\psi(T) \right) = 1.$$

- Sufficient, not necessary.
- Growth rate not optimal.
- Assume $\mu_0^n = 0$.
- Recall for $\psi \in C_B^{0,1}(\mathcal{X}')$

$$M_n^\psi(t) = \langle \psi, \mu_t^n \rangle - \int_0^t \langle u \nabla \psi, \mu_s^n \rangle \, \mathrm{d}s - t \int_{[0,1]} \psi(x_0, y) I \mathrm{d}y$$

$$- \frac{1}{2} \int_0^t \langle [\psi] K, \mu_s^n \otimes \mu_s^n \rangle \, \mathrm{d}s + \frac{1}{2n} \int_0^t \langle [[\psi]] K, \mu_s^n \rangle \, \mathrm{d}s + \frac{1}{n} \sum_{i=1}^{S^n(t)} \psi(Z_i^n, 1).$$

$$
\begin{aligned}
|\langle\psi,\mu_t^n\rangle| \leq tI\int_{[0,1]}|\psi(x_0,y)|\,\mathrm{d}y + \int_0^t |\langle u\nabla\psi,\mu_s^n\rangle|\,\mathrm{d}s \\
+\frac{1}{2}\int_0^t |\langle[\psi]K,\mu_s^n\otimes\mu_s^n\rangle|\,\mathrm{d}s + \frac{1}{2n}\int_0^t |\langle[[\psi]]K,\mu_s^n\rangle|\,\mathrm{d}s \\
+\frac{1}{n}\sum_{i=1}^{S^n(t)}|\psi(Z_i^n,1)| + \left|M_n^\psi(t)\right|
\end{aligned}
$$

Defining $A_8(n,T)=\sup_{t\leq T}\mu_t^n(\mathcal{X}')$ one has

$$
\begin{aligned}
\sup_{t\leq T}|\langle\psi,\mu_t^n\rangle| \leq TI\,\|\psi\| + TA_8(n,T)\left(\|u\nabla\psi\| + \frac{3}{2n}\|\psi\|\,K\right) \\
+\frac{3}{2}T\,\|\psi\|\,KA_8(n,T)^2 + \frac{1}{n}S^n(T)\,\|\psi\| + \sup_{t\leq T}\left|M_n^\psi(t)\right|.
\end{aligned}
$$

$$\sup_{t\le T}|\langle\psi,\mu_t^n\rangle| \le TI\|\psi\| + TA_8(n,T)\left(\|u\nabla\psi\| + \frac{3}{2n}\|\psi\|K\right) + \frac{3}{2}T\|\psi\|KA_8(n,T)^2 + \frac{1}{n}S^n(T)\|\psi\| + \sup_{t\le T}\left|M_n^\psi(t)\right|.$$

- $TI\|\psi\|$ is constant.
- Already stated that

$$\mathbb{P}\left(\sup_{s\le t}\left|M_n^\psi(s)\right| > \epsilon\right) \le \frac{A_5(t,\psi)}{\epsilon^2 n}.$$

- $nA_8(n,T) \le R^n(T)$ since every particle must have been incepted.
- $S^n(T) \le R^n(T)$ since every particle must have been incepted before it can leave.

Containment Estimates

$$\sup_{t\leq T}|\langle\psi,\mu_t^n\rangle| \leq TI\|\psi\| + T\frac{R^n(T)}{n}\left(\|u\nabla\psi\| + \frac{3}{2n}\|\psi\|K\right) + \frac{3}{2}T\|\psi\|K\left(\frac{R^n(T)}{n}\right)^2 + \frac{R^n(T)}{n}\|\psi\| + \sup_{t\leq T}\left|M_n^\psi(t)\right|.$$

$R^n(T) \sim \mathrm{Poi}(nIT)$ thus $\mathbb{P}\left(\frac{R^n(T)}{n} > 2eIT\right) \leq 2\left(e2^{2e}\right)^{-nIT}$ and

Theorem

For every $T > 0$ and $\psi \in C_B^{0,1}(\mathcal{X}')$ there exists $\gamma^\psi(T) < \infty$ such that

$$\lim_n \mathbb{P}\left(\sup_{t\leq T}|\langle\psi,\mu_t^n\rangle| \leq \gamma^\psi(T)\right) = 1.$$

Proof.

Just proved. □

Comments on Containment Proof

- $\gamma^{\psi}(T) \sim \mathcal{O}(T^3)$, which is not optimal.

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- Conjecture $\gamma^{\psi}(T) \sim \mathcal{O}(\sqrt{T})$ possible by exploiting outflow.
- Zero initial condition simplifies the calculation, more general version of the result is:

Theorem

For every $T > 0$ and $f \in C_B^{0,1}(\mathcal{X}')$ there exists $\gamma^{\psi}(T, \eta) < \infty$ such that

$$\liminf_n \mathbb{P}\left(\sup_{t \leq T} |\langle \psi, \mu_t^n \rangle| \leq \gamma^{\psi}(T, \eta)\right) \geq 1 - \eta.$$

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- Zero initial condition simplifies the calculation, more general version of the result is:

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$$\liminf_n \mathbb{P}\left(\sup_{t\leq T} |\langle \psi, \mu_t^n \rangle| \leq \gamma^\psi(T,\eta)\right) \geq 1 - \eta.$$

- Can replace $\mathbb{P}\left(\frac{R^n(T)}{n} > 2eIT\right) \leq 2\left(e2^{2e}\right)^{-nIT}$ with $\mathbb{P}\left(\frac{R^n(T)}{n} > 2IT\right) \leq \sqrt{\frac{1}{nIT}}$ to get smaller leading constant in $\gamma^\psi(T)$.

Definition

The modified variation of a function f from $\mathbb{R}_0^+$ to a metric space (E, r) is defined by

$$w'(f, \delta, T) = \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} r(f(s), f(t)),$$

where the t_i define partitions of $[0, T]$ with minimum spacing at least δ.

We will use the following partition: $t_0 = 0, t_1 = \delta, t_2 = 2\delta, \ldots, t_k = k\delta, t_{k+1} = T$ where $k = \lfloor T/\delta \rfloor - 1$ and consider a 'majorant' variation

$$\bar{w}(f, \delta, T) = \max_i \sup_{s,t \in [t_{i-1}, t_i)} r(f(s), f(t)) \geq w'(f, \delta, T)$$

defined on this particular partition, which has spacing between δ and 2δ.

In the case of non-zero initial conditions which lead to fixed jumps: adjust the partition points to include the fixed jumps.

Theorem

For every $T > 0$ and $\psi \in C_B^{0,1}(\mathcal{X}')$ there exists $\delta^\psi(T,\eta) < \infty$ such that

$$\lim_n \mathbb{P}\left(w'\left(\langle \psi, \mu_t^n \rangle, \delta^\psi(T,\eta), T\right) \geq \eta\right) = 0.$$

- Sufficient, not necessary.
- Assume $\mu_0^n = 0$.
- Recall for $\psi \in C_B^{0,1}(\mathcal{X}')$

$$M_n^\psi(t) = \langle \psi, \mu_t^n \rangle - \int_0^t \langle u\nabla\psi, \mu_s^n \rangle \, \mathrm{d}s - t \int_{[0,1]} \psi(x_0, y) I \mathrm{d}y$$

$$-\frac{1}{2}\int_0^t \langle [\psi]K, \mu_s^n \otimes \mu_s^n \rangle \, \mathrm{d}s + \frac{1}{2n}\int_0^t \langle [[\psi]]K, \mu_s^n \rangle \, \mathrm{d}s + \frac{1}{n}\sum_{i=1}^{S^n(t)} \psi(Z_i^n, 1).$$

$$\left|\left\langle f,\mu^n_{t_2}\right\rangle - \left\langle f,\mu^n_{t_1}\right\rangle\right| \le \int_{t_1}^{t_2} \left|\left\langle u\nabla f,\mu^n_s\right\rangle\right| \mathrm{d}s + (t_2-t_1) I \int_{[0,1]} |f(x_0,y)|\,\mathrm{d}y$$
$$+ \frac{1}{2}\int_{t_1}^{t_2} \left|\left\langle [f]K,\mu^n_s\otimes\mu^n_s\right\rangle\right| \mathrm{d}s + \frac{1}{2n}\int_{t_1}^{t_2} \left|\left\langle [[f]]K,\mu^n_s\right\rangle\right| \mathrm{d}s$$
$$+ \frac{1}{n}\sum_{i=S^n(t_1)}^{S^n(t_2)} |f(Z^n_i,1)| + \left|M^f_n(t_2) - M^f_n(t_1)\right|$$

Recalling $A_8(n,t) = \sup_{s\le t}\mu^n_s(\mathcal{X}')$ one has, for $r < s \le t$

$$|\langle\psi,\mu^n_s\rangle - \langle\psi,\mu^n_r\rangle| \le (s-r)I\,\|\psi\| + (s-r)\frac{3}{2}\,\|\psi\|\,KA_8(n,t)^2$$
$$+ (s-r)A_8(n,t)\left(\|u\nabla\psi\| + \frac{3}{2n}\,\|\psi\|\,K\right)$$
$$+ \frac{1}{n}\left(S^n(s) - S^n(r)\right)\|\psi\| + 2\sup_{s\le t}\left|M^\psi_n(s)\right|.$$

Outline of Proof 2

Focusing on one interval $[t_{i-1}, t_i)$ of the time partition and noting $t_i - t_{i-1} \leq 2\delta$

$$\sup_{r,s \in [t_{i-1}, t_i)} |\langle \psi, \mu_s^n \rangle - \langle \psi, \mu_r^n \rangle| \leq 2\delta I \|\psi\| + 3\delta \|\psi\| K A_8(n,T)^2 + 2\delta A_8(n,T) \left(\|u \nabla \psi\| + \frac{3}{2n} \|\psi\| K \right) + \frac{1}{n} (S^n(t_i) - S^n(t_{i-1})) \|\psi\| + 2 \sup_{s \leq T} \left| M_n^\psi(s) \right| .$$

- Same bound on $R^n(T)/n \geq A_8(n,T)$ as before provides the key.
- M_n^ψ vanishes as $n \to \infty$.
- $S^n(t_i) - S^n(t_{i-1})$ can be estimated as $\mathrm{Poi}(2\delta n I)$.

For every interval

$$\mathbb{P}\left(\sup_{r,s\in[t_{i-1},t_i)} |\langle\psi,\mu_s^n\rangle - \langle\psi,\mu_r^n\rangle| > \eta\right) \leq \mathcal{O}\left(\sqrt{\frac{1}{n}}\right).$$

Recall the partition: $0, \delta, 2\delta, 3\delta, \ldots, k\delta, T$ where $k = \lfloor T/\delta \rfloor - 1$ and ‘majorant’ variation.

For fixed $\delta \leq \delta^\psi = \delta^\psi(\eta, T)$

$$\begin{aligned}
&\mathbb{P}\left(\bar{w}\left(\langle\psi,\mu^n\rangle, \delta, T\right) \geq \eta\right) \\
&\leq \sum_{i=0}^{i=\lfloor T/\delta\rfloor-1} \mathbb{P}\left(\sup_{s,r\in[t_{i-1},t_i)} |\langle\psi,\mu_s^n\rangle, \langle\psi,\mu_r^n\rangle| \geq \eta\right) = \lfloor T/\delta\rfloor \mathcal{O}\left(\sqrt{\frac{1}{n}}\right).
\end{aligned}$$

Theorem

$$\limsup_n \mathbb{P}\left(w'\left(\langle f, \mu_t^n \rangle, \delta^f, T\right) \geq \eta\right) = 0,$$

Proof.

See above. □

Theorem

The μ^n are relatively compact in distribution on $\mathbb{D}_{\mathcal{M}(\mathcal{X}')}(\mathbb{R}_0^+)$.

Proof.

Most of this talk so far! □

Recall the Martingale

$$M_n^{\psi}(t) = \langle \psi, \mu_t^n \rangle - \langle \psi, \mu_0^n \rangle - \int_0^t \langle u \nabla \psi, \mu_s^n \rangle \,\mathrm{d}s + \frac{1}{n} \sum_{i=1}^{S^n(t)} \psi(Z_i^n, 1)$$
$$- t \int_{[0,1]} \psi(x_0, y) I \mathrm{d}y - \frac{1}{2} \int_0^t \langle [\psi] K, \mu_s^n \otimes \mu_s^n \rangle \,\mathrm{d}s + \frac{1}{2n} \int_0^t \langle [[\psi]] K, \mu_s^n \rangle \,\mathrm{d}s.$$

We only get a limit equation for pairings with ψ such that $\psi(\cdot, 1) \equiv 0$, in which case

$$\langle \psi, \mu_t \rangle = \langle \psi, \mu_0 \rangle + \int_0^t \langle u \nabla \psi, \mu_s \rangle \,\mathrm{d}s + t \int_{[0,1]} \psi(x_0, y) I \mathrm{d}y$$
$$+ \frac{1}{2} \int_0^t \langle [\psi] K, \mu_s \otimes \mu_s \rangle \,\mathrm{d}s,$$

which is effectively a restriction on the domain of the generator.

- Suppose a limit point $\mu_t(\mathrm{d}x, \mathrm{d}y)$ has a density $c_t(x, y)$
- Suppose the existence of regular time derivatives.

$$\partial_t \int_{\mathcal{X}\times[0,1]} \psi(x,y)c_t(x,y)\mathrm{d}x\mathrm{d}y = \int_{[0,1]} \psi(x_0,y)I\mathrm{d}y$$

$$+ \int_{\mathcal{X}\times[0,1]} u\left(\partial_y \psi(x,y)\right) c_t(x,y)\mathrm{d}x\mathrm{d}y$$

$$+ \frac{1}{2}\int_{(\mathcal{X}\times[0,1])^2} \left[\psi(x_1+x_2,y_1) - \psi(x_1,y_1) - \psi(x_2,y_2)\right] K$$

$$c_t(x_1,y_1)c_t(x_2,y_2)\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2.$$

Integrating by parts and using $\psi(\cdot, 1) \equiv 0$, the second term becomes

$$-\int_{\mathcal{X}} u\psi(x,0)c_t(x,0)\mathrm{d}x - \int_{\mathcal{X}\times[0,1]} u\psi(x,y)\partial_y c_t(x,y)\mathrm{d}x\mathrm{d}y.$$

Letting ψ approach a δ function at any interior point of $\mathcal{X} \times [0,1]$ we see:

$$\begin{aligned}\partial_t c_t(x,y) + u\partial_y c_t(x,y) &= I\mathbb{1}_{\{x_0\}}(x) \\ &\quad + \frac{1}{2}\int_{\substack{\mathcal{X}^2\times[0,1] \\ x_1+x_2=x}} K c_t(x_1,y) c_t(x_2,y_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_2 \\ &\quad - K c_t(x,y)\int_{\mathcal{X}\times[0,1]} c_t(x_2,y_2)\mathrm{d}x_2\mathrm{d}y_2.\end{aligned}$$

Letting $\psi(x,y)$ approach $\delta_{\{x_1\}}(x)\mathbb{1}_{\{0\}}(y)$:

$$u c_t(x,0) = I_{\mathrm{in}}(x)$$

where $I_{\mathrm{in}}(x)$ is the inception rate on the inflow boundary (assumed 0 above).

- No boundary condition at $y=1$ since $\psi(\cdot,1)\equiv 0$.
- First order equation should have one boundary condition.

Summary

What we know:

- Limit points satisfy a weak equation.
- The simulation algorithm has limit points.

Open questions:

- Is there a unique limit point?
- What can we say about the distribution of $\langle f, \mu_t^n \rangle$ for finite n?
- Can we refine the spatial grid and 're-localise' the coagulation?