1.6 Quasi-Variational Inequalities and Optimal Control

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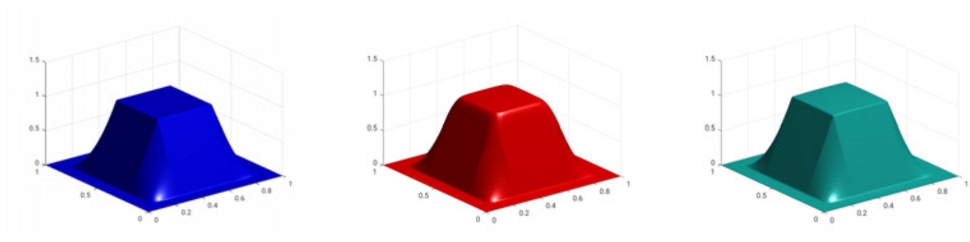
Introduction

A plethora of real-world applications involving nonlinear and non-smooth structures lead to a class of mathematical models called *quasi-variational inequalities* (QVIs). These are highly complex mathematical objects that have shown great versatility in their ability to be used to describe many phenomena in the applied and physical sciences such as game theory, solid mechanics, elasto-plasticity, superconductivity, and thermoforming. Our interests lie in studying the properties of QVIs, modeling of specific physical phenomena via QVIs, and using them as a base to study many aspects of what one calls *optimal control problems* where the aim is to find a control or action that most closely achieves some objective given that the underlying model is described by a QVI.

A large focus of our studies in QVIs is an application to thermoforming, which is the process where shapes (such as pots of yogurt or panels in cars) are mass-reproduced by heating up a plastic membrane to a high temperature and forcing it onto a mould shape (which is the desired shape to be reproduced), enabling it to take on the desired shape. Figure 1 shows the results of simulations based on a QVI model that describes such a thermoforming process. Indeed, Figure 1 (a) shows the (initial) mould shape that is to be reproduced, while Figure 1 (b) is the shape taken on by the membrane as a result of the thermoforming process. One sees that it is an accurate fit to the mould shape. In Figure 1 (c), we see the quasi-variational effect of the model in action: The mould shape has changed as a result of contact with the membrane.



(a) Mould shape (b) Membrane (c) Final mould shape

Fig. 1: *Computational results of thermoforming process*

Mathematical description

The defining feature of QVIs distinguishing them from variational inequalities (VIs) is that the constraint set in which the solution of the QVI is sought itself depends on the solution. Thus, QVIs are generalizations of VIs. The latter are simpler because the above-mentioned constraint sets are known *a priori* and are independent of the solution. This fundamental difference leads to considerable difficulties in the study of QVIs, requiring novel tools and methodologies.

The types of problems we are interested in are as follows. Given a constraint set $\mathbf{K}$ in some function space, VIs (of elliptic type) have the form

$$\text{Find } y \in \mathbf{K} : \langle Ay - f, y - v \rangle \leq 0 \quad \forall v \in \mathbf{K},$$

whereas QVIs take the form: given a set-valued map $\mathbf{K}(\cdot)$, consider

$$\text{Find } y \in \mathbf{K}(y) : \langle Ay - f, y - v \rangle \leq 0 \quad \forall v \in \mathbf{K}(y). \tag{1}$$

Note that the constraint set depends on the solution. We are primarily interested in inequalities of obstacle type, i.e.,

$$\mathbf{K}(y) := \{v \in V : v \leq \Phi(y)\},$$

where V is a reflexive Banach space which is also a vector lattice possessing an ordering $\leq$, $\Phi\colon V \to V$ is a given map, $f \in V^*$ is given data, and $A\colon V \to V^*$ is a linear elliptic operator. In the context of thermoforming, the inequality (1) could be the implicit obstacle problem where the membrane u lies below the mould shape $\Phi(u)$ and Φ could be the solution map of a partial differential equation (PDE) describing the relationship between the membrane and mould taking into account physical modeling assumptions. For full details, we refer the reader to [1, §6].

A fundamental pecularity of QVIs is that solutions of (1) are typically non-unique and in some cases they can be ordered with the existence of a smallest and largest solution.

We conducted extensive research on the following topics:

1. the analysis of the resulting QVIs of the above form: existence of solutions and properties [1, 2, 3],
2. directional differentiability of the solution map $\mathbf{Q}\colon V^* \rightrightarrows V$ that takes f into y and the solution maps related to smallest and largest solutions [1, 3],
3. optimal control problems with QVI constraints and stationarity conditions [2, 3].

The first two items relate directly to (1) whereas the third involves optimal control problems, which in this context looks like

$$\min_{\substack{u \in U_{ad} \\ y \in \mathbf{Q}(u)}} \frac{1}{2}\|y - y_d\|_H^2 + \frac{\nu}{2}\|u\|_H^2, \tag{2}$$

where H is a Hilbert space with $V \subset H$ and $U_{ad} \subset H$ is an admissible set of controls.

The motivation for this is the following: often in a QVI model there is an external quantity that influences the state (i.e., the solution of the QVI). We wish to control this external quantity so that the state satisfies some predefined performance criteria. The task of finding and characterizing such a control that maximizes our criteria is the optimal control problem. The specific objective functional appearing in (2) is known as a *tracking-type* objective and it models the case where we wish the state to be as close as possible to a desired state y_d whilst trying to minimize the control cost.

More details

Directional differentiability

We studied in [3, 1] the directional differentiability associated to (1), in particular, the directional differentiability of the multi-valued (or set-valued) mapping $\mathbf{Q}$ taking the source term f into the set of solutions y. It is important to know if the map is directionally differentiable and to characterize the derivative as it enables us to find out the effect that changes in the source term have on the solution (which can be useful in applications such as thermoforming), and it is a necessary step for obtaining useful first-order characterizations of the optimal control problem (2). Furthermore, it is an interesting problem in mathematical analysis and a fundamental question that deserves to be addressed.

Showing directional differentiability is completely non-trivial due to the nonsmooth nature of the inequality (which in addition contains a nonsmooth obstacle mapping Φ) as well as the multiplicity of solutions to the QVI that can be expected in general. This means that one needs to take care with multivalued solution concepts as well as conduct a fine and careful analysis of the ensuing subproblems. For full details, we refer to the aforementioned papers.

Let V be a reflexive Banach space, $\Phi\colon V \to V$ a Hadamard differentiable operator, and $A\colon V \to V^*$ an elliptic operator. Theorem 3.2 of [3] essentially states that under some assumptions, given $f \in V^*$ and $y \in \mathbf{Q}(f)$, there exists $y^s \in \mathbf{Q}(f + sd)$ and $\alpha = \alpha(d)$ such that

$$\lim_{s \to 0^+} \frac{y^s - y - s\alpha}{s} = 0 \text{ in } V\,,$$

where α satisfies the QVI

$$\alpha \in \mathcal{K}_{\mathbf{K}(y)}(y,\alpha) : \langle A\alpha - d, v - \alpha\rangle \geq 0 \quad \forall v \in \mathcal{K}_{\mathbf{K}(y)}(y,\alpha),$$

and the constraint set appearing above is the *critical cone* defined by

$$\mathcal{K}_{\mathbf{K}(y)}(y,\alpha) := \{\varphi \in V : \varphi \leq \Phi'(y)(\alpha) \text{ q.e. on } \{y = \Phi(y)\} \text{ and } \langle Ay - f, \varphi - \Phi'(y)(\alpha)\rangle = 0\}.$$

This is a powerful result and it considerably improves our previous contribution [1] (which was the first result for the directional differentiability for QVIs in the infinite-dimensional setting) in which we needed further assumptions on the signs of the source and direction term. Let us also mention that directional differentiability of the minimal and maximal solution maps is studied in [4]; we refer the reader there for more details.

Optimal control

In [3, §4], we gave an existence result on the optimal points of the control problem (2), which we recall here:

$$\min_{\substack{u \in U_{ad} \\ y \in \mathbf{Q}(u)}} \frac{1}{2}\|y - y_d\|_H^2 + \frac{\nu}{2}\|u\|_H^2. \tag{2}$$

Furthermore, we also provided comprehensive first order characterizations of optimality in [3, §5]. Here, as mentioned above, U_{ad} is the so-called *admissible set of controls*, which is taken to be non-empty, closed, and convex.

These results were achieved by approximating the control problem (2) by

$$\min_{u \in U_{ad}} \frac{1}{2}\|y_\rho - y_d\|_H^2 + \frac{\nu}{2}\|u\|_H^2 \quad \text{where} \quad Ay_\rho + \frac{1}{\rho} m_\rho(y_\rho - \Phi(y_\rho)) = u, \tag{3}$$

deriving stationarity conditions for this problem (by standard constraint qualification) and then performing a delicate analysis in the passage to the limit in the parameter ρ under varying sets of assumptions. Here, for each $\rho > 0$, $m_\rho \colon V \to V^*$ is a C^1 map possessing certain properties that in some sense generalizes the positive part function $(\cdot)^+$. Thus, we have approximated the nonsmooth QVI by a sequence of more regular PDEs with penalization parameter ρ with the intention being to send $\rho \to 0$ and obtain results for the original problem.

The PDE in (3) is well posed in certain circumstances and solutions of (3) can be shown to converge to solutions of (2).

Stationarity systems are useful because they characterize optimal points and can often be easier to numerically solve than the original optimal control problem. There are a multitude of stationarity systems that can be derived depending on the hypotheses and structure of the problem.

Let us now describe the cascade of stationarity systems that we derived, in increasing order of the number of assumptions needed.

In the general vector lattice setting, we showed in [3, Theorem 5.5] the existence of multipliers $(p^*, \xi^*, \lambda^*) \in V \times V^* \times V^*$ satisfying what we call the *weak C-stationarity* system

$$y^* + (\mathrm{Id} - \Phi'(y^*))^* \lambda^* + A^* p^* = y_d, \tag{4a}$$

$$Ay^* - u^* + \xi^* = 0, \tag{4b}$$

$$\xi^* \geq 0 \text{ in } V^*, \quad y^* \leq \Phi(y^*), \quad \langle \xi^*, y^* - \Phi(y^*) \rangle = 0, \tag{4c}$$

$$u^* \in U_{ad} : (\nu u^* - p^*, u^* - v)_H \leq 0 \quad \forall v \in U_{ad}, \tag{4d}$$

$$\langle \lambda^*, p^* \rangle \geq 0. \tag{4e}$$

This is a system that lies in between the traditional notions of weak stationarity and C-stationarity, hence its name.

In case V is a Sobolev space over some domain $\Omega \subset \mathbb{R}^n$, we have at our disposal a specific family m_ρ that possesses enough regularity allowing us to improve the above system to an $\mathcal{E}$*-almost C-stationarity* system [3, Theorem 5.11] by additionally giving us the conditions

$$\langle \xi^*, (p^*)^+ \rangle = \langle \xi^*, (p^*)^- \rangle = 0, \tag{5a}$$

$$\langle \lambda^*, y^* - \Phi(y^*) \rangle = 0, \tag{5b}$$

$$\forall \tau > 0, \exists E^\tau \subset \mathcal{I} \text{ with } |\mathcal{I} \setminus E^\tau| \leq \tau : \langle \lambda^*, v \rangle = 0 \quad \forall v \in V : v = 0 \text{ a.e. on } \Omega \setminus E^\tau. \tag{5c}$$

The final condition arises from an application of Egorov's theorem. Under an extra assumption of continuity of $(\mathrm{Id} - \Phi) \colon V \to L^\infty(\Omega)$, we can strengthen the condition (5c) to

$$\langle \lambda^*, v \rangle = 0 \quad \forall v \in V : v = 0 \text{ a.e. on } \{y^* = \Phi(y^*)\}.$$

This is a fully *C-stationarity* system (with no need for the $\mathcal{E}$-almost damping).

By making further assumptions on the admissible set U_{ad}, we showed in [3, Theorem 5.16] that (y^*, u^*) is a *strong stationarity* point, i.e., the multipliers $(p^*, \xi^*, \lambda^*) \in V \times V^* \times V^*$ indeed satisfy

$$\begin{aligned}
y^* + (\mathrm{Id} - \Phi'(y^*)^*)\lambda^* + A^* p^* &= y_d, \\
Ay^* - u^* + \xi^* &= 0, \\
\xi^* \geq 0 \text{ in } V^*, \quad y^* \leq \Phi(y^*), \quad \langle \xi^*, y^* - \Phi(y^*) \rangle &= 0, \\
u^* \in U_{ad} : (\nu u^* - p^*, u^* - v) \leq 0 &\quad \forall v \in U_{ad}, \\
p^* \geq 0 &\quad \text{q.e. on } \mathcal{B}(y^*) \text{ and } p^* = 0 \text{ q.e. on } \mathcal{A}_s(y^*), \\
\langle \lambda^*, v \rangle \geq 0 &\quad \forall v \in V : v \geq 0 \text{ q.e. on } \mathcal{B}(y^*), \text{ and} \\
&\quad v = 0 \text{ q.e. on } \mathcal{A}_s(y^*).
\end{aligned}$$

Here, note that $\mathcal{A}_s(y^*) := \{\xi^* > 0\}$ is the *strongly active* set, and $\mathcal{B}(y^*) := \{y^* = \Phi(y^*)\} \cap \{\xi^* = 0\}$ is the *biactive* set. This is the strongest form of stationarity available; note in particular the pointwise q.e. (quasi everywhere) sign conditions on the adjoint p^* as well as a finer characterization of the multiplier λ^* in comparison to the previous systems.

Conclusions and outlook

As mentioned above, directional differentiability results for the minimal and maximal solution mappings do appear in our work [4], however, only for signed source and direction terms. The removal of these restrictions, as well as the obtainment of stationarity systems for optimal control problems with minimal/maximal control-to-state maps, are highly delicate and subjects of ongoing work.

As a final remark, we highlight that this work has been conducted within projects funded via the DFG SPP 1962 Priority Programme in collaboration with Carlos N. Rautenberg (George Mason University). A joint work [5] with José-Francisco Rodrigues (Lisbon University) has also been finished.

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