The fundamental gap conjecture

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At the end of his study [J. Statist. Phys., 83] on thermodynamic functions of a free boson gas, van den Berg conjectured that the difference between the two smallest eigenvalues

$$\Gamma^V(\Omega) := \lambda_2^V(\Omega) - \lambda_1^V(\Omega);$$

of the Schrödinger operator $-\Delta + V$ on a convex domain Ω in $\mathbb{R}^d$, $d \geq 1$, equipped with homogeneous Dirichlet boundary conditions satisfies

$$\Gamma^V(\Omega) \geq \Gamma(I_D) = \frac{3\pi^2}{D^2}, \tag{1}$$

where I_D is the interval $(-D/2, D/2)$ of length $D = \text{diameter}(\Omega)$. The term $\Gamma^V(\Omega)$ is called the *fundamental gap* and describes an important physical quantity: for example, in statistical mechanics, $\Gamma^V(\Omega)$ measures the energy needed to jump from the ground state to the next excited eigenstate, or computationally, it can control the rate of convergence of numerical methods to compute large eigenvalue problems [SIAM, 2011]. Thus, one is interested in (optimal) lower bounds on $\Gamma^V(\Omega)$. Since the late 80s, the *fundamental conjecture* (1) attracts consistently the attention of many researcher including M. S. Ashbaugh & R. Benguria [Proc. Amer. Math. Soc., 89], R. Schoen and S.-T. Yau [Camb. Press, 94.] (see also [Geneva, 86]), B. Andrews and J. Clutterbuck [J. Amer. Math. Soc., 11].

In this talk, I present new results on the fundamental gap conjecture (1) for the Schrödinger operator $-\Delta + V$ on a convex domain Ω equipped with *Robin boundary conditions*. In particular, we present a proof of this conjecture in dimension one, and mention results for the p-Laplacian.

The talk is based on the joint works [1, 2] with B. Andrews and J. Clutterbuck.

References

[1] Ben Andrews, Julie Clutterbuck, and Daniel Hauer. Non-concavity of the Robin ground state. *Camb. J. Math.*, 8(2):243–310, 2020.

[2] Ben Andrews, Julie Clutterbuck, and Daniel Hauer. The fundamental gap for a one-dimensional Schrödinger operator with Robin boundary conditions. *Proc. Amer. Math. Soc.*, 149(4): 1481–1493, 2021.