Hamilton-Jacobi PDEs in the space of probability measures, the metric nature explored

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Hamilton-Jacobi PDE in the space of probability measure is a new class of PDE. We give two examples. The first example concerns a statistical mechanics application where the equation is derived by probabilists studying Gibbs-Non-Gibbs transitions in the Netherlands community. Through a method by the speaker and Tom Kurtz, the equation's uniqueness will rigorously give the large deviation principle. However, this uniqueness problem is still open. This problem has a hidden spatial translation invariance. The second example concerns the variational formulation of a compressible Euler equation in continuum mechanics. The probability measure is just the density profile of infinite particles. The Hamilton-Jacobi equation characterizes the canonical transformations and is expected to give useful information on the large time dynamics of the Euler-Lagrange equation (Aubry-Mather theory for instance). We have now a well-podesness theory for this equation. This example has a hidden particle permutation invariance in the density profile representation

In the rigorous part of this talk, I focus on explaining how the well-posedness in the mechanical application is solved. An important observation is that the space of probability measure is best viewed as an infinite dimensional quotient space in this context. The quotient structure comes from particle permutation symmetry. To treat a PDE in such an infinitely folded space, we devise techniques based on the metric space analysis and the Wasserstein spaces. A key step is the use of geometric tangent cone concept in characterizing the velocity variables. Admissible velocity fields are more than function valued, they are relaxed to belong to a subspace of the space of Markov transition kernels. This augmentation of the tangent space is essential to allow us distinguish curves with mass condensation property in the physical space density profile variable with other curves without such feature in the Hamiltonian formulation.

The infinitely folded space structure here also has an intrinsic connection with the space of random variables. In studying the mean-field game theory, P.L. Lions setup some Hamilton-Jacobi equation in the space of random variables. A connection between the measure formulation here and the random-variable formulation there are conceptually possible but many questions are still left open.