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First- and second-order optimality conditions in the sparse optimal control of Cahn–Hilliard systems

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Abstract

This paper deals with the sparse distributed control of viscous and nonviscous Cahn–Hilliard systems. We report on results concerning first-order necessary and second-order sufficient optimality conditions that have recently established by the authors. The analysis covers both the cases when the nonlinear double well potential governing the evolution is of either regular or logarithmic type. A major difficulty originates from the sparsity-enhancing term in the cost functional which typically is nondifferentiable.

1 Introduction

Our contribution for the RISM conference is concerned with the optimal control of the initial-boundary value problem for a Cahn–Hilliard system (cf. [4]) with no-flux boundary conditions. Phase field systems of this type govern the evolution of diffusive phase transition processes with conserved order parameter. Let us describe the problem: $\Omega \subset \mathbb{R}^3$ denotes a bounded and connected domain with smooth boundary $\Gamma = \partial\Omega$ and unit outward normal $\boldsymbol{n}$, and $T > 0$ stands for some final time. Set

$$Q_t := \Omega \times (0,t), \quad \Sigma_t := \Gamma \times (0,t), \quad \text{for } t \in (0,T], \quad \text{and} \quad Q := Q_T, \quad \Sigma := \Sigma_T,$$

and consider the initial-boundary value system

$$\partial_t\varphi - \Delta\mu = 0 \qquad \text{a.e. in } Q, \tag{1.1}$$

$$\tau\partial_t\varphi - \Delta\varphi + f'(\varphi) = \mu + w \qquad \text{a.e. in } Q, \tag{1.2}$$

$$\gamma\partial_t w + w = u \qquad \text{a.e. in } Q, \tag{1.3}$$

$$\partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\varphi = 0 \qquad \text{a.e. on } \Sigma, \tag{1.4}$$

$$\varphi(0) = \varphi_0, \quad w(0) = w_0 \qquad \text{a.e. in } \Omega, \tag{1.5}$$

in which u denotes the control acting on (1.2) through the equation (1.3). The variables are φ, μ, w, while τ and γ denote positive coefficients. In particular, if $\tau > 0$ we refer to (1.1)–(1.5) as a viscous Cahn–Hilliard system. Please note that

- the equations (1.1)–(1.2) rule the evolution of the state variables φ and μ that are monitored through the input variable w;
- in turn, w is determined by the action of the control u via the linear control equation (1.3);
- φ plays the role of an *order parameter* that may attain its values in the physical interval $[-1,+1]$, while μ is the associated *chemical potential*;

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- equation (1.3) models how the "forcing" w is generated by the external control u. Much more general integro-differential equations, and even partial differential equations – modeling the relation between an L^2-control u and a smooth forcing w – could be considered;
- φ_0 and w_0 in (1.5) are given initial data;
- the derivative f' of a double-well potential f defines the local part of the thermodynamic force driving the evolution of the system;
- smooth or singular potentials of logarithmic type are covered by the analysis.

We mention that the typical behavior of the physically relevant logarithmic potential is given by

$$f_{\log}(r) = \begin{cases} c_1\big((1+r)\,\ln(1+r) + (1-r)\,\ln(1-r)\big) - c_2 r^2 & \text{if } r \in (-1,1) \\ 2c_1 \ln(2) - c_2 & \text{if } r \in \{-1,1\} \\ +\infty & \text{if } r \notin [-1,1] \end{cases}, \tag{1.6}$$

where the constants c_1, c_2 are nonnegative and such that $f_{\log}$ is nonconvex. For $f = f_{\log}$ it turns out that the term $f'(\varphi)$ in (1.2) becomes singular as $\varphi \searrow -1$ and $\varphi \nearrow 1$, which forces the order parameter φ to attain its values in the interval $(-1,1)$.

We are concerned with the optimal control problem:

(CP) Minimize the cost functional

$$\begin{aligned} \mathcal{J}(\varphi,u) &:= \frac{b_1}{2}\iint_Q |\varphi-\varphi_Q|^2 + \frac{b_2}{2}\int_\Omega |\varphi(T)-\varphi_\Omega|^2 + \frac{b_3}{2}\iint_Q |u|^2 + \kappa\, G(u)\,, \\ &=: J(\varphi,u) + \kappa\, G(u) \end{aligned} \tag{1.7}$$

subject to $u \in \mathcal{U}_{\rm ad}$, with φ first component of the solution (φ,μ,w) to the system (1.1)–(1.5), and with

$$\mathcal{U}_{\rm ad} = \{u \in \mathcal{U} = L^\infty(Q) : \ \underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t) \text{ for a.a. } (x,t) \text{ in } Q\}, \tag{1.8}$$

where the bounds $\underline{u}, \overline{u} \in \mathcal{U}$ satisfy $\underline{u} \le \overline{u}$ almost everywhere in Q.

Referring now to (1.7), we point out that the targets φ_Q, φ_Ω are known functions, while $b_1 \ge 0$, $b_2 \ge 0$, $b_3 > 0$ are constant coefficients, as well as $\kappa > 0$, which represents the sparsity parameter. The sparsity-enhancing functional $G : L^2(Q) \to \mathbb{R}$ is nonnegative, continuous and convex. Typically, G has a nondifferentiable form like, e.g.,

$$G(u) = \|u\|_{L^1(Q)} = \iint_Q |u|\,. \tag{1.9}$$

Since the seminal work [37], a vast body of literature has explored the well-posedness and asymptotic behavior of both viscous and nonviscous Cahn–Hilliard systems with Neumann or dynamic boundary conditions. While it is not possible to cite all relevant contributions here, we quote [2, 3, 11–16, 18, 22, 25, 35, 36, 44, 50] and focus now on the state system (1.1)–(1.5), which appears to have not been previously studied in this specific form, where the control influences the chemical potential through the term w. Notice that the typical regularity to be expected for an L^2-control u is $w \in H^1(0,T;L^2(\Omega))$, which in three-dimensional cases with a logarithmic potential is typically needed to derive a *separation property* for the state variable φ from (1.2).

There is also extensive research on the optimal control of Cahn–Hilliard-type systems across various contexts. Without aiming to provide an exhaustive list, we highlight some relevant works. General contributions include [32, 46, 62, 63], while studies in the framework of diffusive tumor growth models can be found in [17, 26–28, 33, 34, 39, 42, 58]. Problems involving dynamical boundary conditions have been analyzed in [9, 10, 20–24, 26, 43], whereas convective Cahn–Hilliard systems have been addressed in [22, 23, 43, 51, 60, 61]. Furthermore, several works have investigated Cahn–Hilliard systems coupled with other models. In this context, we mention the Cahn–Hilliard–Navier–Stokes models (see [38,45,47,48,56,59]), as well as the Cahn–Hilliard–Oono system (see [18,41]), the Cahn–Hilliard–Darcy system (see [1, 55]), the Cahn–Hilliard–Brinkman system (see [34]) and the Cahn–Hilliard system with curvature effects (see [19]).

None of the previously cited works address the aspect of *sparsity*, specifically the possibility that a locally optimal control may vanish over subregions of positive measure within the space-time cylinder Q, governed by the sparsity parameter κ. The geometry of these subregions depends on the specific choice of the convex functional G, which may vary across different contexts. Sparsity properties can be derived from the variational inequality appearing in the first-order necessary optimality conditions and from the structure of the subdifferential ∂G. Here, we focus on sparsity, specifically the case of full sparsity, which is associated with the $L^1(Q)$-norm functional G introduced in (1.9). Other types of sparsity, such as directional sparsity with respect to time or with respect to space (see, e.g., [52]) are not treated here.

Sparsity in the optimal control theory for partial differential equations has become an actively investigated aspect. The use of sparsity-enhancing functionals originates from inverse problems and image processing. Several studies have focused on first-order optimality conditions for sparse optimal controls in single elliptic and parabolic equations. In [6, 7], both first- and second-order optimality conditions have been analyzed in the context of sparsity for the (semilinear) FitzHugh–Nagumo system. More recently, the sparsity of optimal controls for reaction-diffusion systems of the Cahn–Hilliard type has been investigated in [29, 40, 52]. Additionally, measure control of the Navier–Stokes system has been studied in [5]. However, the analysis of second-order sufficient optimality for sparse controls for the Cahn–Hilliard and viscous Cahn–Hilliard equations is a relatively recent research.

Second-order sufficient optimality conditions typically rely on a coercivity condition that is required to hold for the smooth part J of $\mathcal{J}$ within a specific *critical cone*. The nonsmooth part G contributes to sufficiency through its convexity. For the strength of sufficient conditions it is crucial that the critical cone be as small as possible. In [7], Casas, Ryll and Tröltzsch introduced a technique that allows for the selection of a particularly small and advantageous critical cone.

This method was initially developed for a class of semilinear second-order parabolic problems with smooth nonlinearities. More recently, the works [30, 31, 53, 54] have demonstrated that this approach can be adapted to Allen–Cahn systems with dynamic boundary conditions, a broad class of tumor growth models, and Cahn–Hilliard systems of the form (1.1)–(1.5).

In this paper, we review some of these developments, with a particular focus on the results of [31] concerning the viscous case with $\tau > 0$. The adaptation to this setting is far from being trivial, as the Cahn–Hilliard structure leads to a fourth-order PDE for the order parameter φ (which follows directly from substituting μ from (1.2) into (1.1)). Consequently, several additional technical challenges arise, both in proving the Fréchet differentiability of the control-to-state operator and in analyzing the properties of the adjoint variables. Some of these difficulties stem from the singular behavior of the derivative $f'(\varphi)$ of the logarithmic nonlinearity (1.6) in (1.2). The nonviscous case $\tau = 0$ is examined in [30] for smooth potentials: however, dealing with a logarithmic nonlinearity in that case presents significant difficulties, making the analysis challenging even in the two-dimensional setting.

This note is organized as follows.

In the next section, we present the general assumptions and discuss the state system, formulating existence and continuous dependence results. We also recall the uniform separation property for the solution component φ.

In Section 3, we outline the properties of the control-to-state operator. By introducing the linearized and bilinearized systems, we demonstrate that the control-to-state operator is twice continuously Fréchet differentiable between suitable Banach spaces. Additionally, we revisit the local Lipschitz properties for the first and second derivatives.

In Section 4, we investigate the optimal control problem **(CP)** with sparsity and analyze the corresponding adjoint problem.

Finally, in Section 5, we derive the first-order necessary optimality conditions, examine the case of full sparsity, and establish the second-order sufficient optimality conditions for controls that are locally optimal in the $L^2(Q)$-sense.

2 Preliminaries and well-posedness

First, we fix some notation. For any Banach space X, we denote by $\|\cdot\|_X$, X^*, and $\langle\,\cdot\,,\,\cdot\,\rangle_X$, the corresponding norm, its dual space, and the related duality pairing between X^* and X. Standard notations are used for Lebesgue and Sobolev spaces. For convenience, we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \big\{v \in H^2(\Omega) :\ \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma\big\},$$

and denote by $(\,\cdot\,,\,\cdot\,)_H$ the natural inner product in H. As usual, H is identified with a subspace of the dual spaces V^* according to the identity

$$\langle u, v\rangle_V = (u,v)_H \quad \text{for every } u \in H \text{ and } v \in V.$$

We then have the Hilbert triplet $V \subset H \subset V^*$ with dense and compact embeddings. Here are the general assumptions for the data of the state system (1.1)–(1.5):

(A1) $f = f_1 + f_2$, where $f_1 : \mathbb{R} \to [0, +\infty]$ is convex and lower semicontinuous with $f_1(0) = 0$, and $f_2 : \mathbb{R} \to \mathbb{R}$ has a Lipschitz continuous first derivative f_2' on $\mathbb{R}$. Moreover, we require that $f_1 \in C^5(-1,1)$ and $f_2 \in C^5[-1,1]$, and we assume that

$$\lim_{r \searrow -1} f_1'(r) = -\infty\,, \quad \lim_{r \nearrow 1} f_1'(r) = +\infty\,. \tag{2.1}$$

(A2) $\tau > 0$ and $\gamma > 0$ are prescribed constants. Moreover, $w_0 \in L^\infty(\Omega)$, $\varphi_0 \in W$, and it holds that

$$-1 < \min_{x\in\overline{\Omega}} \varphi_0(x) \le \max_{x\in\overline{\Omega}} \varphi_0(x) < 1\,. \tag{2.2}$$

(A3) let $R > 0$ be a fixed constant such that

$$\mathcal{U}_{\mathrm{ad}} \subset \mathcal{U}_R := \{u \in L^\infty(Q) : \ \|u\|_{L^\infty(Q)} < R\}. \tag{2.3}$$

The condition **(A1)**, with (2.1) in particular, yields that the derivative f_1' is just defined in $(-1,1)$ and gives rise to a maximal monotone operator in $\mathbb{R}\times\mathbb{R}$. Note that **(A1)** is fulfilled if f is given by the logarithmic potential $f_{\log}$ in (1.6), where

$$f_1(r) = c_1\big((1+r)\,\ln(1+r) + (1-r)\,\ln(1-r)\big) \ \text{ if } r \in (-1,1)$$
$$\text{and } \ f_2(r) = -c_2 r^2 \ \text{ for } r \in \mathbb{R}.$$

A consequence of **(A2)** is that the mean value of φ_0,

$$m_0 := \frac{1}{|\Omega|}\int_\Omega \varphi_0\,, \ \text{belongs to the interior of the domain } (-1,1) \text{ of } f_1'. \tag{2.4}$$

In the sequel, the notation $\overline{v}$ is used the mean value of a generic function $v \in L^1(\Omega)$. More generally, noting that the constant function 1 is an element of V, we set

$$\overline{v} := \frac{1}{|\Omega|}\,\langle v, 1\rangle_V \quad \text{for every } v \in V^*. \tag{2.5}$$

On the other hand, the condition **(A3)** just fixes a bounded open subset of the control space $L^\infty(Q)$ that contains $\mathcal{U}_{\mathrm{ad}}$.

Let us point out our notion of solution: for u given in $L^2(0,T;H)$, the triple (φ,μ,w) is a solution to (1.1)–(1.5) if

$$\begin{aligned}
&\varphi \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W),\\
&-1 < \varphi(x,t) < 1 \quad \text{for a.e. } (x,t) \in Q,\\
&\mu \in L^2(0,T;W), \quad w \in H^1(0,T;H),\\
&(\varphi,\mu,w) \ \text{ solves (1.1)–(1.5).}
\end{aligned}$$

In particular, the variational formulations of equations (1.1)–(1.3) are

$$\int_\Omega \partial_t\varphi(t) v + \int_\Omega \nabla\mu(t)\cdot\nabla v = 0 \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{2.6}$$

$$\tau\int_\Omega \partial_t\varphi(t) v + \int_\Omega \nabla\varphi(t)\cdot\nabla v + \int_\Omega f'(\varphi(t)) v$$
$$= \int_\Omega (\mu(t) + w(t)) v \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{2.7}$$

$$\int_\Omega \gamma\,\partial_t w(t) z + \int_\Omega w(t) z = \int_\Omega u(t) z \quad \text{for a.e. } t \in (0,T) \text{ and every } z \in H, \tag{2.8}$$

with the initial conditions

$$\varphi(0) = \varphi_0 \quad \text{in } V, \quad w(0) = w_0 \quad \text{in } H. \tag{2.9}$$

Note that, by this definition and (2.7), (1.2) holds true and, by comparison of terms, it follows that $f'(\varphi) \in L^2(0,T;H)$. In addition, it is clear that (φ,μ,w) is a strong solution of (1.1)–(1.5). Owing to the linear equation (1.3) and the second initial condition in (2.9), it turns out that the component w of the solution can be explicitly written in terms of u by

$$w(x,t) = w_0(x)\exp(-t/\gamma) + \int_0^t \exp(-(t-s)/\gamma)u(x,s)ds, \quad \text{a.e. } (x,t) \in Q. \tag{2.10}$$

The existence of a solution, even more regular, can be proved along with the separation property expressed in (2.12), provided that $u \in \mathcal{U}_R$.

Theorem 2.1. *Assume* **(A1)–(A3)***. Then the state system* (1.1)–(1.5) *has for any* $u \in L^2(0,T;H)$ *a unique solution* (φ, μ, w) *with the regularity*

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)\,, \tag{2.11}$$
$$\varphi \in C^0(\overline{Q}) \quad \text{and} \quad -1 < \varphi < 1 \quad \text{in } Q\,, \tag{2.12}$$
$$\mu \in L^\infty(0,T;W) \cap L^2(0,T;H^3(\Omega)) \subset L^\infty(Q)\,, \tag{2.13}$$
$$w \in H^1(0,T;H)\,. \tag{2.14}$$

In addition, there is a constant $K_1 > 0$, which depends only on $\|u\|_{L^2(0,T;H)}$ and the data, such that

$$\begin{aligned}&\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)\cap C^0(\overline{Q})} \\ &+ \|\mu\|_{L^\infty(0,T;W)\cap L^2(0,T;H^3(\Omega))\cap L^\infty(Q)} + \|w\|_{H^1(0,T;H)} \le K_1\,,\end{aligned} \tag{2.15}$$

where (φ, μ, w) denotes the solution corresponding to u. Moreover, if $u \in \mathcal{U}_R$, then the solution component w satisfies

$$w \in W^{1,\infty}(0,T;L^\infty(\Omega)) \subset L^\infty(Q)\,, \tag{2.16}$$

and a uniform strict separation property is fulfilled: there are constants r_-, r_+, depending only on R and the data of the state system, such that

$$-1 < r_- \le \varphi(x,t) \le r_+ < 1 \quad \text{for every } (x,t) \in \overline{Q}, \tag{2.17}$$

where φ represents the first component of the solution (φ, μ, w) to the state system related to $u \in \mathcal{U}_R$.

This theorem is proved in [31, Section 2] by approximating f_1' by its Yosida regularization, recalling the possible use of a Faedo–Galerking scheme, then deriving uniform estimates and passing to the limit as the approximation parameter tends to 0. The separation property is shown in the last step of the proof, directly on the state system.

Note that a consequence of **(A1)** and (2.17) is that there exists a constant K_2, depending only on r_-, r_+, f_1, f_2, such that the component φ of the solution satisfies

$$\max_{0\le i\le 5}\Big(\max_{j=1,2}\|f_j^{(i)}(\varphi)\|_{C^0(\overline{Q})} + \|f^{(i)}(\varphi)\|_{C^0(\overline{Q})}\Big) \le K_2\,, \tag{2.18}$$

where $f^{(i)} = f_1^{(i)} + f_2^{(i)}$ for $i = 0, 1, \ldots, 5$.

Remark 2.2. The nonviscous case, that is, the same problem but with $\tau = 0$, has been thoroughly studied in [30]. In the approach presented in [30], assumptions **(A1)** and **(A2)** are modified as follows:

(A1)$_0$ $f = f_1 + f_2$, where $f_1 \in C^5(\mathbb{R})$ is a convex and nonnegative function with $f_1(0) = 0$ and $f_2 \in C^5(\mathbb{R})$ has a Lipschitz continuous first derivative f_2' on $\mathbb{R}$.

(A2)$_0$ $\gamma > 0$ is a constant. Moreover, $w_0 \in V$ and $\varphi_0 \in H^3(\Omega) \cap W$.

In this case, the logarithmic potential (1.6) cannot be considered. Instead, the admissible potentials must be defined over the entire real line. A typical example is the standard smooth double-well potential, given by

$$f(r) = \frac{1}{4}(r^2 - 1)^2, \quad r \in \mathbb{R}, \tag{2.19}$$

with the components $f_1(r) = r^4/4$ and $f_2(r) = (1 - 2r^2)/4$, $r \in \mathbb{R}$. The regularity of the solution in [30, Theorem 2.1] differs slightly due to the absence of viscosity. In the nonviscous case, we have

$$\varphi \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \cap C^0(\overline{Q}), \tag{2.20}$$

$$\mu \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W \cap H^3(\Omega)), \tag{2.21}$$

while (2.14) remains unchanged. Of course, also here φ is still bounded above and below by two constants, though these constants are not necessarily within the range $[-1,1]$. An estimate similar to (2.15) remains valid in this setting, considering the function spaces given in the conditions above.

A continuous dependence result can be shown. This results ensures the uniqueness of the solution provided by Theorem 2.1.

Theorem 2.3. *Let* **(A1)**–**(A3)** *be fulfilled. Then, if* $u_i \in L^2(0,T;H)$, $i = 1,2$, *are given and* (φ_i, μ_i, w_i), $i = 1,2$, *are corresponding solutions to* (1.1)–(1.5)*, there is a constant* K_3*, depending only on* τ, γ, T *and the Lipschitz constant of* f_2'*, such that*

$$\|\varphi_1 - \varphi_2\|_{C^0([0,T];H)\cap L^2(0,T;V)} + \|w_1 - w_2\|_{H^1(0,T;H)} \le K_3 \, \|u_1 - u_2\|_{L^2(0,T;H)}. \tag{2.22}$$

If, in addition, $u_i \in \mathcal{U}_R$, $i = 1,2$, *then the further estimate*

$$\begin{aligned} &\|\varphi_1 - \varphi_2\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)} + \|\mu_1 - \mu_2\|_{L^2(0,T;W)} \\ &\quad + \|w_1 - w_2\|_{H^1(0,T;H)} \le K_4 \, \|u_1 - u_2\|_{L^2(0,T;H)}, \end{aligned} \tag{2.23}$$

holds for a constant K_4 *that depends only on* K_2, τ, γ, Ω, *and* T.

For the proof of Theorem 2.3, realized via some suitable contracting estimates, we refer to [31, Section 2].

Remark 2.4. Regarding the nonviscous case analyzed in [30], we note that the estimate (2.23) is replaced by

$$\begin{aligned} &\|\varphi_1 - \varphi_2\|_{H^1(0,T;V^*)\cap C^0([0,T];V)\cap L^2(0,T;W)} + \|\mu_1 - \mu_2\|_{L^2(0,T;V)} \\ &\quad + \|w_1 - w_2\|_{H^1(0,T;H)} \le K_4 \, \|u_1 - u_2\|_{L^2(0,T;H)}, \end{aligned} \tag{2.24}$$

which involves larger function spaces for the differences in the φ and μ components.

3 The control-to-state operator

A consequence of the theorems of the previous section is that the control-to-state operator

$$\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) : u \mapsto \mathcal{S}(u) = (\mathcal{S}_1(u), \mathcal{S}_2(u), \mathcal{S}_3(u)) := (\varphi, \mu, w) \tag{3.1}$$

is Lipschitz continuous on the set $\mathcal{U}_R$ as a mapping between $L^2(0,T;H)$ and the Banach space

$$\mathcal{X} := \big(H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)\big) \times L^2(0,T;W) \times H^1(0,T;H).$$

We want to study the differentiability properties of the control-to-state operator $\mathcal{S}$. Recalling that $\mathcal{U} = L^\infty(Q)$, we also introduce the Banach spaces

$$\begin{aligned} \mathcal{Y} := {} & \big(W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)\big) \\ & \times \big(L^\infty(0,T;W) \cap L^2(0,T;H^3(\Omega))\big) \times H^1(0,T;H), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathcal{Z} := {} & \big\{(\varphi,\mu,w) \in \mathcal{Y} \cap \mathcal{U}^3 : \partial_t\varphi - \Delta\mu \in \mathcal{U}, \\ & \quad \tau\partial_t\varphi - \Delta\varphi - \mu - w \in \mathcal{U}, \ \ \gamma\partial_t w + w \in \mathcal{U}\big\}, \end{aligned} \tag{3.3}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are endowed with their standard norms and the norm in $\mathcal{Z}$ is given by

$$\begin{aligned} \|(\varphi,\mu,w)\|_{\mathcal{Z}} = {} & \|(\varphi,\mu,w)\|_{\mathcal{Y}} + \|(\varphi,\mu,w)\|_{\mathcal{U}^3} + \|\partial_t\varphi - \Delta\mu\|_{\mathcal{U}} \\ & + \|\tau\partial_t\varphi - \Delta\varphi - \mu - w\|_{\mathcal{U}} + \|\gamma\partial_t w + w\|_{\mathcal{U}}. \end{aligned} \tag{3.4}$$

In [31, Section 3] we show that under the assumptions **(A1)**–**(A3)** the operator $\mathcal{S}$ is twice continuously Fréchet differentiable on $\mathcal{U}$ as a mapping from $\mathcal{U}$ into $\mathcal{Z}$, where, for any control $u^* \in \mathcal{U}_R$, with associated state $(\varphi^*, \mu^*, w^*) =: \mathcal{S}(u^*)$, the first and second Fréchet derivatives $\mathcal{S}'(u^*) \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and $\mathcal{S}''(u^*) \in \mathcal{L}(\mathcal{U}, \mathcal{L}(\mathcal{U}, \mathcal{Z}))$ are given as follows:

(i) For any increment $h \in \mathcal{U}$, $(\xi, \eta, v) := \mathcal{S}'(u^*)[h] \in \mathcal{Z}$ is the unique solution to the linearized problem

$$\partial_t\xi - \Delta\eta = 0 \qquad \text{a.e. in } Q, \tag{3.5}$$

$$\tau\partial_t\xi - \Delta\xi - \eta - v = -f''(\varphi^*)\xi \qquad \text{a.e. in } Q, \tag{3.6}$$

$$\gamma\partial_t v + v = h \qquad \text{a.e. in } Q, \tag{3.7}$$

$$\partial_{\boldsymbol{n}}\eta = \partial_{\boldsymbol{n}}\xi = 0 \qquad \text{a.e. on } \Sigma, \tag{3.8}$$

$$\xi(0) = 0, \quad v(0) = 0 \qquad \text{a.e. in } \Omega. \tag{3.9}$$

(ii) For any pair of increments $h, k \in \mathcal{U}$, $(\psi, \nu, z) := \mathcal{S}''(u^*)[h,k] \in \mathcal{Z}$ is the unique solution to the bilinearized problem

$$\partial_t\psi - \Delta\nu = 0 \qquad \text{a.e. in } Q, \tag{3.10}$$

$$\tau\partial_t\psi - \Delta\psi - \nu - z = -f''(\varphi^*)\psi - f'''(\varphi^*)\xi^h\xi^k \qquad \text{a.e. in } Q, \tag{3.11}$$

$$\gamma\partial_t z + z = 0 \qquad \text{a.e. in } Q, \tag{3.12}$$

$$\partial_{\boldsymbol{n}}\nu = \partial_{\boldsymbol{n}}\psi = 0 \qquad \text{a.e. on } \Sigma, \tag{3.13}$$

$$\psi(0) = 0, \quad z(0) = 0 \qquad \text{a.e. in } \Omega, \tag{3.14}$$

where $(\xi^h, \eta^h, v^h) := \mathcal{S}'(u^*)[h]$ and $(\xi^k, \eta^k, v^k) := \mathcal{S}'(u^*)[k]$. We immediately note that the third component z of the solution (ψ, ν, z) to (3.10)–(3.14) fulfills $z = 0$ a.e. in Q due to (3.12) and (3.14).

For this aim an approach based on the application of the implicit function theorem is used in [31, Section 3]. By this, the following result is proved.

Theorem 3.1. *Assume* **(A1)–(A3)**. *Then the control-to-state operator* $\mathcal{S}$ *is twice continuously Fréchet differentiable in* $\mathcal{U}_R$ *as a mapping from* $\mathcal{U}$ *into* $\mathcal{Z}$*. Moreover, for every* $u^* \in \mathcal{U}_R$ *and* $h, k \in \mathcal{U}$*, the functions* $(\xi, \eta, v) = \mathcal{S}'(u^*)[h] \in \mathcal{Z}$ *and* $(\psi, \nu, z) = \mathcal{S}''(u^*)[h, k] \in \mathcal{Z}$ *are the unique solutions to the linearized system* (3.5)–(3.9) *and the bilinearized system* (3.10)–(3.14)*, respectively.*

As for the argumentation used in the proof, the actual value of the constant $R > 0$ defining $\mathcal{U}_R$ does not matter. It turns out that $\mathcal{S}$ is twice continuously Fréchet differentiable as a mapping from $\mathcal{U}$ to $\mathcal{Z}$ on the entire space $\mathcal{U}$.

Important remarks follow.

- Due to the continuous embedding $\mathcal{Z} \subset \mathcal{Y}$, the control-to-state mapping $\mathcal{S}$ is also Fréchet differentiable from $\mathcal{U}$ to $\mathcal{Y}$ with the same expression for the Fréchet derivative, now regarded as an element of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$.
- As $\mathcal{U}$ is dense in $L^2(0, T; H)$, the operator $\mathcal{S}'(u^*) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ can be extended in the standard way to an operator belonging to $\mathcal{L}(L^2(0, T; H), \mathcal{Y})$. Denoting the extended operator by $\mathcal{S}'(u^*)$, we point out that $\mathcal{S}'(u^*)$ exists as a Fréchet derivative only on $\mathcal{U}$ and not in general on $L^2(0, T; H)$.
- By a density argument it is not difficult to check that $(\xi, \eta, v) = \mathcal{S}'(u^*)[h]$ coincides also for $h \in L^2(0, T; H)$ with the solution to (3.5)–(3.9).
- Analogously, the second Fréchet derivative $\mathcal{S}''(u^*)$ can be continuously extended, and this leads to an element of the space $\mathcal{L}(L^2(0, T; H), \mathcal{L}(L^2(0, T; H), \mathcal{Y}))$, still denoted by $\mathcal{S}''(u^*)$. We claim that $(\psi, \nu, z) = \mathcal{S}''(u^*)\,[h, k]$ solves (3.10)–(3.14) also for $h, k \in L^2(0, T; H)$.

For the extensions, the following result holds [31, Section 3].

Proposition 3.2. *Under the assumptions* **(A1)–(A3)***, let* $u^* \in \mathcal{U}_R$ *be fixed. Then, there is a constant* $K_5 > 0$*, depending only on* R *and the data, such that for every* $h, k \in L^2(0, T; H)$ *we have the estimates*

$$\|\mathcal{S}'(u^*)[h]\|_{\mathcal{Y}} \le K_5 \, \|h\|_{L^2(0,T;H)}, \quad \|\mathcal{S}''(u^*)[h, k]\|_{\mathcal{Y}} \le \, K_5 \, \|h\|_{L^2(0,T;H)} \, \|k\|_{L^2(0,T;H)}. \tag{3.15}$$

The following Lipschitz continuity properties of the extensions of the derivatives are crucial for the derivation of second-order sufficient optimality conditions.

Theorem 3.3. *The mappings*

$$\mathcal{U} \to \mathcal{L}(L^2(0, T; H), \mathcal{Y}), \; u \mapsto \mathcal{S}'(u)$$

and

$$\mathcal{U} \to \mathcal{L}(L^2(0, T; H), \mathcal{L}(L^2(0, T; H), \mathcal{Y})), \, u \mapsto \mathcal{S}''(u)$$

are Lipschitz continuous in the following sense: there exists a constant $K_6 > 0$, which depends only on R and the data, such that, for all controls $u_1, u_2 \in \mathcal{U}_R$ and all increments $h, k \in L^2(0,T;H)$, it holds that

$$\|(\mathcal{S}'(u_1) - \mathcal{S}'(u_2))[h]\|_{\mathcal{X}} \le K_6 \|u_1 - u_2\|_{L^2(0,T;H)} \|h\|_{L^2(0,T;H)}, \tag{3.16}$$

$$\| (\mathcal{S}''(u_1) - \mathcal{S}''(u_2)) [h,k]\|_{\mathcal{X}} \le K_6 \|u_1 - u_2\|_{L^2(0,T;H)} \|h\|_{L^2(0,T;H)} \|k\|_{L^2(0,T;H)}. \tag{3.17}$$

A detailed proof is given in [31, Section 3].

Remark 3.4. The approach in [30] for the nonviscous case follows a direct method, working in $L^2(0,T;H)$ rather than in $\mathcal{U}$, and does not rely on the implicit function theorem. Instead, it employs ad hoc computations. The specific formulations of the linearized and bilinearized problems correspond to (3.5)–(3.9) and (3.10)–(3.14) in the present work but are adapted to the case where $\tau = 0$.

4 The optimal control problem

This section is concerned with the optimal control problem **(CP)** with the cost functional (1.7). In addition to the general conditions **(A1)**–**(A3)**, we also assume that

(A4) It holds $b_1 \ge 0$, $b_2 \ge 0$, $b_3 > 0$, and $\kappa > 0$.

(A5) The target functions satisfy $\varphi_Q \in L^2(Q)$ and $\varphi_\Omega \in V$.

Please note that $\kappa > 0$ in order to include the effects of sparsity: however, the theory of second-order conditions remains valid for $\kappa = 0$. The requirement $\varphi_\Omega \in V$ allows to have more regular solutions to the associated adjoint system (see below). In fact, it is not restrictive in view of the continuous embedding $(H^1(0,T;H) \cap L^2(0,T;W)) \subset C^0([0,T];V)$ which implies that $\varphi(T) \in V$.

The following existence result can be derived by standard arguments.

Theorem 4.1. *Suppose that* **(A1)**–**(A5)** *are fulfilled, and suppose that $G : L^2(Q) \to \mathbb{R}$ is nonnegative, convex and continuous. Then the optimal control problem* **(CP)** *admits a solution $u^* \in \mathcal{U}_{\mathrm{ad}}$.*

At this point, we introduce the adjoint system. In the sequel, $u^* \in \mathcal{U}_{\mathrm{ad}}$ often denotes a locally optimal control for **(CP)**, with the associated state $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$. We recall that a control $u^* \in \mathcal{U}_{\mathrm{ad}}$ is termed locally optimal in the sense of $L^p(Q)$ for some $p \in [1, +\infty]$ if and only if there is some $\varepsilon > 0$ such that $\mathcal{J}(u^*, \mathcal{S}_1(u^*)) \le \mathcal{J}(u, \mathcal{S}_1(u))$ for all $u \in \mathcal{U}_{\mathrm{ad}}$ with $\|u - u^*\|_{L^p(Q)} \le \varepsilon$. As can easily be seen, any locally optimal control in the sense of $L^p(Q)$ for some $1 \le p < +\infty$ is also locally optimal in the sense of $L^\infty(Q)$.

The corresponding adjoint system can be formally stated as

$$-\partial_t(p + \tau q) - \Delta q + f''(\varphi^*) q = b_1(\varphi^* - \varphi_Q) \quad \text{a.e. in } Q, \tag{4.1}$$

$$-\Delta p - q = 0 \quad \text{a.e. in } Q, \tag{4.2}$$

$$-\gamma \partial_t r + r - q = 0 \quad \text{a.e. in } Q, \tag{4.3}$$

$$\partial_{\boldsymbol{n}} p = \partial_{\boldsymbol{n}} q = 0 \quad \text{a.e. on } \Sigma, \tag{4.4}$$

$$(p + \tau q)(T) = b_2(\varphi^*(T) - \varphi_\Omega), \quad r(T) = 0 \quad \text{a.e. in } \Omega. \tag{4.5}$$

We immediately notice that the system is decoupled, meaning that r can be directly obtained from (4.3) using the terminal condition $r(T) = 0$, once q is determined. Additionally, the variational formulation of (4.1), (4.2), (4.4) is given by

$$-\int_\Omega \partial_t(p+\tau q)\rho + \int_\Omega \nabla q \cdot \nabla \rho + \int_\Omega f''(\varphi^*)q\rho = b_1 \int_\Omega (\varphi^* - \varphi_Q)\rho$$
$$\text{for a.e. } t \in (0,T) \text{ and every } \rho \in V, \tag{4.6}$$

$$\int_\Omega \nabla p \cdot \nabla \rho = \int_\Omega q\rho \quad \text{for a.e. } t \in (0,T) \text{ and every } \rho \in V. \tag{4.7}$$

The following result holds.

Theorem 4.2. *Assume* **(A1)–(A5)***, and let* $u^* \in \mathcal{U}_R$ *be a control with associated state* (φ^*, μ^*, w^*)*. Then the associated adjoint state system* (4.1)–(4.5) *has a unique strong solution* (p^*, q^*, r^*) *with the regularity*

$$p^* + \tau q^* \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{4.8}$$
$$p^* \in L^2(0,T;W \cap H^4(\Omega)), \tag{4.9}$$
$$q^* \in L^2(0,T;W), \tag{4.10}$$
$$r^* \in H^1(0,T;W). \tag{4.11}$$

Moreover, there is a constant $K_7 > 0$*, which depends only on* R *and the data, such that*

$$\|p^* + \tau q^*\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;H^2(\Omega))} + \|p^*\|_{L^2(0,T;H^4(\Omega))} + \|q^*\|_{L^2(0,T;H^2(\Omega))}$$
$$+ \|r^*\|_{H^1(0,T;H^2(\Omega))} \le K_7 \left(\|\varphi^* - \varphi_Q\|_{L^2(Q)} + \|\varphi^*(T) - \varphi_\Omega\|_V\right). \tag{4.12}$$

The proof can be found in [31, Section 4.1], and it uses a Faedo–Galerkin approximation along with well-known weak and weak-star compactness arguments to pass to the limit.

Here is a continuous dependence result that is needed for the proof of second-order sufficient optimality conditions.

Proposition 4.3. *Suppose that* **(A1)–(A5)** *are fulfilled, and let, for* $i = 1, 2$, $u_i \in \mathcal{U}_R$ *be given with the associated states* $(\varphi_i, \mu_i, w_i) = \mathcal{S}(u_i)$ *and adjoint states* (p_i, q_i, r_i)*. Then, there is constant* $K_8 > 0$*, depending only on* R *and the data, such that*

$$\|(p_1 + \tau q_1) - (p_2 + \tau q_2)\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;H^2(\Omega))} + \|p_1 - p_2\|_{L^2(0,T;H^4(\Omega))}$$
$$+ \|q_1 - q_2\|_{L^2(0,T;H^2(\Omega))} + \|r_1 - r_2\|_{H^1(0,T;H^2(\Omega))} \le K_8 \|u_1 - u_2\|_{L^2(0,T;H)}. \tag{4.13}$$

At least for one result, let us include a simple proof. Let $p = p_1 - p_2,\ q = q_1 - q_2,\ r = r_1 - r_2$. Then (p, q, r) is the unique strong solution to the system

$$-\partial_t(p+\tau q) - \Delta q + f''(\varphi_1)q = z_1 \qquad \text{a.e. in } Q, \tag{4.14}$$
$$-\Delta p - q = 0 \qquad \text{a.e. in } Q, \tag{4.15}$$
$$-\gamma\partial_t r + r - q = 0 \qquad \text{a.e. in } Q, \tag{4.16}$$
$$\partial_{\boldsymbol{n}} p = \partial_{\boldsymbol{n}} q = 0 \qquad \text{a.e. on } \Sigma, \tag{4.17}$$
$$(p+\tau q)(T) = z_2, \quad r(T) = 0 \qquad \text{a.e. in } \Omega, \tag{4.18}$$

where

$$z_1 = -(f''(\varphi_1) - f''(\varphi_2))q_2 + b_1(\varphi_1 - \varphi_2) \quad \text{and} \quad z_2 = b_2(\varphi_1(T) - \varphi_2(T)). \tag{4.19}$$

By a sequence of estimates performed in the proof of Theorem 4.2 for the derivation of (4.12), and now applied to the continuous system (4.14)–(4.18), it turns out that the assertion is proved as soon as we have that

$$\|z_1\|_{L^2(0,T;H)} + \|z_2\|_V \le C\,\|u_1 - u_2\|_{L^2(0,T;H)}\,.$$

However, this follows directly from the estimate (2.23) in Theorem 2.3. Indeed, by utilizing the continuity of the embedding $V \subset L^4(\Omega)$, we deduce that

$$\begin{aligned}
&\|z_1\|^2_{L^2(0,T;H)} + \|z_2\|^2_V \\
&\le C \int_0^T \|(\varphi_1 - \varphi_2)(s)\|^2_{L^4(\Omega)}\,\|q_2(s)\|^2_{L^4(\Omega)}\,ds \\
&\qquad + C\|\varphi_1 - \varphi_2\|^2_{L^2(0,T;H)} + C\|(\varphi_1 - \varphi_2)(T)\|^2_V \\
&\le C\,\|\varphi_1 - \varphi_2\|^2_{C^0([0,T];V)}\big(1 + \|q_2\|^2_{L^2(0,T;V)}\big) \le C\,\|u_1 - u_2\|^2_{L^2(0,T;H)}.
\end{aligned}$$

Then, the proof is complete.

5 Optimality conditions and sparsity

Now, we aim at deriving first-order necessary optimality conditions for local minima of the optimal control problem **(CP)**. We assume that **(A1)–(A5)** hold and that $G : L^2(0,T;H) \to \mathbb{R}$ is a general nonnegative, convex and continuous functional. We define the reduced cost functionals associated with the functionals J and $\mathcal{J}$ introduced in (1.7) by

$$\widehat{J}(u) := J(\mathcal{S}_1(u), u), \quad \widehat{\mathcal{J}}(u) = \mathcal{J}(\mathcal{S}_1(u), u)\,. \tag{5.20}$$

Since $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is twice continuously Fréchet differentiable from $\mathcal{U}$ into the space $C^0([0,T];H)^3$ (which contains $\mathcal{Z}$), it follows from the chain rule that the smooth part $\widehat{J}$ of the reduced objective functional is a twice continuously Fréchet differentiable mapping from $\mathcal{U}$ into $\mathbb{R}$. Moreover, for every $u^* \in \mathcal{U}$ and every $h \in \mathcal{U}$, it holds that

$$\widehat{J}'(u^*)[h] = b_1 \iint_Q \xi(\varphi^* - \varphi_Q) + b_2 \int_\Omega \xi(T)(\varphi^*(T) - \varphi_\Omega) + b_3 \iint_Q u^* h\,, \tag{5.21}$$

where $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ is the state associated with u^* and $(\xi, \eta, v) = \mathcal{S}'(u^*)[h] \in \mathcal{Z}$ is the unique solution to the linearized system (3.5)–(3.9) associated with h.

Observe that the right-hand side of (5.21) is meaningful also for arguments $h \in L^2(0,T;H)$, where in this case (ξ, η, v) is still $\mathcal{S}'(u^*)[h]$, but with the operator $\mathcal{S}'(u^*)$ extended to $L^2(0,T;H)$. Then, in the light of the identity (5.21), we can also extend the operator $\widehat{J}'(u^*) \in \mathcal{U}^*$ to $L^2(0,T;H)$. This operator, still denoted by $\widehat{J}'(u^*)$, becomes an element of $L^2(0,T;H)^*$. In this way, expressions of the form $\widehat{J}'(u^*)[h]$ have a proper meaning also for $h \in L^2(0,T;H)$.

Now, let $u^* \in \mathcal{U}_{\rm ad}$ be a locally optimal control for **(CP)** in the sense of $\mathcal{U}$, that is, there is some $\varepsilon > 0$ such that

$$\widehat{\mathcal{J}}(u) \ge \widehat{\mathcal{J}}(u^*) \quad \text{for all } u \in \mathcal{U}_{\rm ad} \text{ satisfying } \|u - u^*\|_{\mathcal{U}} \le \varepsilon. \tag{5.22}$$

Notice that any locally optimal control in the sense of $L^p(Q)$ for some $1 \le p < \infty$ is also locally optimal in the sense of $\mathcal{U}$, since the topology of $\mathcal{U}$ is the finest among these spaces. Therefore, a result proved for locally optimal controls in the sense of $\mathcal{U}$ is also valid for locally optimal controls in the sense of $L^p(Q)$ for any $1 \le p < \infty$. The same statement holds for globally optimal controls as well.

A standard argument (detailed, e.g., in [52, 53]) of nondifferentiable convex optimization theory then shows that there is some $\lambda^* \in \partial G(u^*) \subset L^2(0,T;H)$ such that

$$\widehat{J}'(u^*)[u-u^*] + \kappa \iint_Q \lambda^*(u-u^*) \ \ge 0 \quad \forall\, u \in \mathcal{U}_{\rm ad}. \tag{5.23}$$

In a well-known way, the expression $\widehat{J}'(u^*)[u-u^*]$ in (5.23) can be simplified by the use of the adjoint state variables defined in (4.1)–(4.5), namely

$$\widehat{J}'(u^*)[u-u^*] = \iint_Q (r^* + b_3 u^*)\,(u-u^*)\,,$$

where r^* is part of the adjoint state.

In view of the linearized system (3.5)–(3.9), the following result can be shown.

Theorem 5.1. (Necessary optimality condition) *Assume* **(A1)–(A5)** *and let G be a nonnegative, convex and continuous function from $L^2(0,T;H)$ to $\mathbb{R}$. Moreover, let $u^* \in \mathcal{U}_{\rm ad}$ be a locally optimal control of* **(CP)** *in the sense of $\mathcal{U}$ with associated state $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ and adjoint state (p^*, q^*, r^*). Then there exists some $\lambda^* \in \partial G(u^*)$ such that, for all $u \in \mathcal{U}_{\rm ad}$,*

$$\iint_Q (r^* + \kappa\lambda^* + b_3 u^*)\,(u-u^*) \ \ge\ 0\,. \tag{5.24}$$

Once more, we emphasize that (5.24) is also necessary for all globally optimal controls and all controls that are locally optimal in the sense of $L^p(Q)$ with $p \ge 1$.

The convex function G in the objective functional plays a role in promoting the sparsity of optimal controls, meaning that any locally optimal control may be zero over certain regions of the space-time cylinder Q. The specific shape of this region depends on the particular choice of the functional G. The sparsity properties can be inferred from the variational inequality (5.24) and the explicit form of the subdifferential ∂G. In what follows, we focus on the case of *full sparsity* which is associated with the $L^1(Q)-$norm functional G introduced in (1.9). Its subdifferential is given by (see [49])

$$\partial G(u) = \left\{ \lambda \in L^2(Q) :\ \lambda(x,t) \in \begin{cases} \{1\} & \text{if } u(x,t) > 0 \\ [-1,1] & \text{if } u(x,t) = 0 \\ \{-1\} & \text{if } u(x,t) < 0 \end{cases} \text{for a.e. } (x,t) \in Q \right\}. \tag{5.25}$$

We then use the special form of this subdifferential in the variational inequality (5.24) to obtain the following result.

Theorem 5.2. (Full sparsity) *Suppose that the assumptions* **(A1)–(A5)** *are fulfilled and assume that $\underline{u}$ and $\overline{u}$ are constants such that $\underline{u} < 0 < \overline{u}$. Let $u^* \in \mathcal{U}_{\rm ad}$ be a locally optimal control in the sense of $\mathcal{U}$ for the problem* **(CP)** *with the functional G defined in* (1.9)*, and with associated state*

$(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ *solving* (1.1)–(1.5) *and adjoint state* (p^*, q^*, r^*) *solving* (4.1)–(4.5)*. Then there exists a function* $\lambda^* \in \partial G(u^*)$ *satisfying* (5.24)*, and for a.e.* $(x,t) \in Q$ *it holds that*

$$u^*(x,t) = 0 \qquad \Longleftrightarrow \qquad |r^*(x,t)| \le \kappa\,. \tag{5.26}$$

Moreover, if r^* *and* λ^* *are given, then* u^* *is obtained from the projection formula*

$$u^*(x,t) \quad = \quad \max\left\{\underline{u}, \min\left\{\overline{u}, -b_3^{-1}\left(r^* + \kappa\,\lambda^*\right)(x,t)\right\}\right\} \;\text{ for a.e. } (x,t) \in Q.$$

Regarding the proof, we note that the projection formula follows directly from the variational inequality (5.24). The remaining task is to establish the validity of (5.26). To do so, we apply the projection formula along with the fact that $\underline{u} < 0 < \overline{u}$. For almost every $(x,t) \in Q$, we consider the following cases:

- if $u^*(x,t) = 0$, then it follows that $-b_3^{-1}(r^*(x,t) + \kappa\lambda^*(x,t)) = 0$, where $\lambda^*(x,t) \in [-1,1]$. Consequently, we obtain

  $$|r^*(x,t)| = \kappa|\lambda^*(x,t)| \le \kappa.$$

- Now, assume that $|r^*(x,t)| \le \kappa$. If $u^*(x,t) > 0$, then $\lambda^*(x,t) = 1$ and, by the projection formula, we get

  $$-b_3^{-1}(r^*(x,t) + \kappa) \ge u^*(x,t) > 0.$$

  This implies that $r^*(x,t) + \kappa < 0$ leading to

  $$|r^*(x,t)| = -r^*(x,t) > \kappa,$$

  which contradicts our assumption.

- Similarly, assuming $u^*(x,t) < 0$ leads to a contradiction using the same reasoning.

Therefore, we must conclude that $u^*(x,t) = 0$, completing the proof.

We now demonstrate that all locally optimal controls in the sense of $\mathcal{U}$ must be identically zero when the sparsity parameter is sufficiently large. Indeed, the global estimate (2.15) for solutions to the state system holds for all controls $u \in \mathcal{U}_{\rm ad}$, and the same applies to the global estimate (4.12). Consequently, there exists a constant $C^* > 0$ such that

$$\|r^*\|_{H^1(0,T;H^2(\Omega))} \;\le\; C^* \quad \forall\, u^* \in \mathcal{U}_{\rm ad}.$$

Furthermore, due to the continuity of the embedding $H^1(0,T;H^2(\Omega)) \subset C^0(\overline{Q})$, we also obtain

$$\|r^*\|_{C^0(\overline{Q})} \;\le\; \kappa^* \quad \forall\, u^* \in \mathcal{U}_{\rm ad},$$

for some sufficiently large $\kappa^* > 0$, which proves our claim.

We conclude this note by deriving second-order sufficient optimality conditions. Specifically, we establish conditions that guarantee the local optimality of functions u^* that satisfy the first-order necessary optimality conditions outlined in Theorem 5.1. These second-order sufficient optimality conditions rely on a coercivity condition imposed on the smooth part J of $\mathcal{J}$ within a certain critical cone. The non-smooth component G contributes to sufficiency through its convexity. Throughout our analysis, we generally assume that conditions **(A1)**–**(A5)** are fulfilled. Our approach closely follows the methodology in [7], where a second-order analysis was conducted for sparse control of the FitzHugh–Nagumo system. In particular, it is possible to adapt the proof of [7, Thm. 3.4] to our setting of a viscous Cahn–Hilliard system.

To proceed, we fix a control u^* that satisfies the first-order necessary optimality conditions and set $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$. Next, we introduce the *cone of feasible directions*, denoted as

$$C(u^*) = \{v \in L^2(0,T;H) \,:\, v \text{ satisfies the sign conditions (5.27) a.e. in } Q\}.$$

These sign conditions are given by

$$v(x,t) \begin{cases} \geq 0 & \text{if } u^*(x,t) = \underline{u} \\ \leq 0 & \text{if } u^*(x,t) = \overline{u} \end{cases} . \tag{5.27}$$

This cone forms a convex and closed subset of $L^2(0,T;H)$. Additionally, we need the *directional derivative* of G at $u \in L^2(0,T;H)$ in the direction $v \in L^2(0,T;H)$, which is given by

$$G'(u,v) = \lim_{t \searrow 0} \frac{1}{t}(G(u+tv) - G(u)) . \tag{5.28}$$

According to the definition of the critical cone in [7, Sect. 3.1], we define

$$C_{u^*} = \{v \in C(u^*) : \widehat{J}'(u^*)[v] + \kappa G'(u^*, v) = 0\} , \tag{5.29}$$

and this is also a closed and convex subset of $L^2(0,T;H)$. In the light of [7, Sect. 3.1], the set C_{u^*} consists of all $v \in C(u^*)$ satisfying

$$v(x,t) \begin{cases} = 0 & \text{if } |r^*(x,t) + b_3 u^*(x,t)| \neq \kappa \\ \geq 0 & \text{if } u^*(x,t) = \underline{u} \text{ or } (r^*(x,t) = -\kappa \text{ and } u^*(x,t) = 0) \\ \leq 0 & \text{if } u^*(x,t) = \overline{u} \text{ or } (r^*(x,t) = \kappa \text{ and } u^*(x,t) = 0) \end{cases} . \tag{5.30}$$

At this stage, we derive an explicit expression for $\widehat{J}''(u)[h,k]$ for arbitrary $u, h, k \in \mathcal{U}$. Our approach follows an argument similar to that presented in [57, Sect. 5.7]. First, we observe that for every $((\varphi, \mu, w), u) \in (C^0([0,T];H))^3 \times \mathcal{U}$ and for any $\boldsymbol{y} = (y_1, y_2, y_3)$, $\boldsymbol{z} = (z_1, z_2, z_3)$ such that

$$(\boldsymbol{y}, u_1), (\boldsymbol{z}, u_2) \in (C^0(0,T;H))^3 \times \mathcal{U},$$

the quadratic functional J satisfies the relation

$$\begin{aligned} & J''((\varphi,\mu,w),u)[(\boldsymbol{y},u_1),(\boldsymbol{z},u_2)] \\ & \quad = b_1 \iint_Q y_1 z_1 \,+\, b_2 \int_\Omega y_1(T) z_1(T) \,+\, b_3 \iint_Q u_1\, u_2. \end{aligned} \tag{5.31}$$

For the second-order derivative of the reduced cost functional $\widehat{J}$ at a given control u^*, we set $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ and obtain

$$\begin{aligned} \widehat{J}''(u^*)[h,k] &= D_{(\varphi,\mu,w)} J((\varphi^*,\mu^*,w^*),u^*)[(\psi,\nu,z)] \\ & \quad + J''((\varphi^*,\mu^*,w^*),u^*)[((\xi^h,\eta^h,v^h),h),((\xi^k,\eta^k,v^k),k)], \end{aligned} \tag{5.32}$$

Here, (ξ^h, η^h, v^h), (ξ^k, η^k, v^k), and (ψ, ν, z) represent the unique solutions to the linearized system (3.5)–(3.9) corresponding to h and k, and to the bilinearized system (3.10)–(3.14), respectively. From the definition of the cost functional (1.7) we readily deduce that

$$\begin{aligned} &D_{(\varphi,\mu,w)} J((\varphi^*, \mu^*, w^*), u^*)[(\psi, \nu, z)] \\ &\quad = b_1 \iint_Q (\varphi^* - \varphi_Q)\psi \,+\, b_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\psi(T). \end{aligned} \tag{5.33}$$

We now assert that, with the associated adjoint state (p^*, q^*, r^*), the following relation holds:

$$b_1 \iint_Q (\varphi^* - \varphi_Q)\psi \,+ b_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\psi(T) \,=\, - \iint_Q f^{(3)}(\varphi^*)\xi^h \xi^k q^* \,. \tag{5.34}$$

To establish this claim, we proceed by multiplying equation (3.10) by p^*, (3.11) by q^*, (3.12) by r^*. Summing the resulting equalities and integrating over Q, while performing integration by parts where necessary, leads to

$$\begin{aligned} 0 \,=\, &\int_\Omega p^*(T)\psi(T) - \iint_Q \partial_t p^* \psi \,-\, \iint_Q \nu\, \Delta p^* \,+ \int_\Omega \tau q^*(T)\psi(T) \,-\, \iint_Q \tau \partial_t q^* \psi \\ &- \iint_Q \psi \Delta q^* \,-\, \iint q^*(\nu + z) \,+\, \iint f''(\varphi^*)\psi q^* \,+\, \iint_Q f^{(3)}(\varphi^*)\xi^h \xi^k q^* \\ &+ \int_\Omega \gamma r^*(T) z(T) \,-\, \iint_Q \gamma \partial_t r^* z \,+\, \iint_Q r^* z \\ = \,&\int_\Omega b_2(\varphi^*(T) - \varphi_\Omega)\psi(T) \,+\, \iint_Q \psi \Big[- \partial_t(p^* + \tau q^*) - \Delta q^* + f''(\varphi^*) q^* \Big] \\ &+ \iint_Q \nu \Big[- \Delta p^* - q^* \Big] \,+\, \iint_Q z \Big[- \gamma \partial_t r^* + r^* - q^* \Big] \\ = \,&b_1 \iint_Q (\varphi^* - \varphi_Q)\psi \,+\, b_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\psi(T) \,+\, \iint_Q f^{(3)}(\varphi^*)\xi^h \xi^k q^* \,. \end{aligned}$$

Thus, the claim follows, since (p^*, q^*, r^*) satisfies the adjoint system (4.1)–(4.5). From this characterization, together with (5.32) and (5.33), we conclude that

$$\widehat{J}''(u^*)[h, k] \,=\, \iint_Q \big(b_1 - f^{(3)}(\varphi^*) q^*\big) \xi^h \, \xi^k \,+\, b_2 \int_\Omega \xi^h(T) \xi^k(T) \,+\, b_3 \iint_Q h\, k \,. \tag{5.35}$$

Observe that the expression on the right-hand side of (5.35) remains well-defined even for increments $h, k \in L^2(Q)$. Indeed, in this case, the terms

$$(\xi^h, \eta^h, v^h) = \mathcal{S}'(u^*)[h], \; (\xi^k, \eta^k, v^k) = \mathcal{S}'(u^*)[k], \text{ and } (\psi, \nu, z) = \mathcal{S}''(u^*)[h, k]$$

have an interpretation in the sense of the extended operators $\mathcal{S}'(u^*)$ and $\mathcal{S}''(u^*)$, as introduced after Theorem 3.1. Consequently, the operator $\widehat{J}''(u^*)$ can be extended to the space $L^2(Q) \times L^2(Q)$ via the identity (5.35) . This extension, which will still be denoted by $\widehat{J}''(u^*)$, is frequently used in the following. Next, we show that this operator is continuous. Specifically, we claim that for all $h, k \in L^2(Q)$, the inequality

$$\Big| \widehat{J}''(u^*)[h, k] \Big| \,\le\, \widehat{C}\, \|h\|_{L^2(Q)} \, \|k\|_{L^2(Q)} \tag{5.36}$$

holds, where the constant $\widehat{C} > 0$ is independent of the choice of $u^* \in \mathcal{U}_R$. To establish this, we note that only the first integral on the right-hand side of (5.35) requires further analysis. Applying Hölder's inequality, the continuity of the embedding $V \subset L^4(\Omega)$, and the global estimates (2.18), (3.15), and (4.12), we obtain

$$\Big| \iint_Q f^{(3)}(\varphi^*)\xi^h\xi^k q^* \Big| \leq C \int_0^T \|\xi^h(t)\|_{L^4(\Omega)} \|\xi^k(t)\|_{L^4(\Omega)} \|q^*(t)\|_{L^2(\Omega)} \, dt$$

$$\leq C \|\xi^h\|_{C^0([0,T];V)} \|\xi^k\|_{C^0([0,T];V)} \|q^*\|_{L^2(0,T;H)} \leq C \|h\|_{L^2(Q)} \|k\|_{L^2(Q)} ,$$

as claimed.

In the following, we will use the coercivity condition:

$$\widehat{J}''(u^*)[v,v] > 0 \quad \forall \, v \in C_{u^*} \setminus \{0\} . \tag{5.37}$$

Condition (5.37) is a direct extension of similar conditions that are commonly encountered in finite-dimensional nonlinear optimization. It was first introduced in the context of optimal control for partial differential equations in [8].

We now present the following result.

Theorem 5.3. (Second-order sufficient condition) *Assume that* **(A1)–(A5)** *are satisfied and let* $u^* \in \mathcal{U}_{\rm ad}$*, with the corresponding state* $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ *and adjoint state* (p^*, q^*, r^*)*, fulfill the first-order necessary optimality conditions of Theorem 5.1. If* u^* *additionally satisfies the coercivity condition* (5.37)*, then there exist constants* $\varepsilon > 0$ *and* $\zeta > 0$ *such that the quadratic growth condition*

$$\widehat{\mathcal{J}}(u) \geq \widehat{\mathcal{J}}(u^*) + \zeta \, \|u - u^*\|^2_{L^2(Q)} \tag{5.38}$$

holds for all $u \in \mathcal{U}_{\rm ad}$ *with* $\|u - u^*\|_{L^2(Q)} < \varepsilon$*. As a result,* u^* *is a locally optimal control in the* $L^2(Q)$ *sense.*

We note that the proof of Theorem 5.3 is thoroughly explained in [31, Section 4.4] and is inspired by the proof of [7, Thm. 3.4].

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