Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

A stabilized finite element method for the Navier–Stokes/Darcy coupled problem

Rodolfo Araya[1], Cristian Cárcamo[2], Abner H. Poza[3]

submitted: March 6, 2025

[1] Departamento de Ingeniería Matemática & CI^{2}MA
Universidad de Concepción
Casilla 160-C Concepción, Chile
E-Mail: rodolfo.araya@udec.cl

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin, Germany
E-Mail: cristian.carcamosanchez@wias-berlin.de

[3] Departamento de Matemática y Física Aplicadas & $GIANuC^2$
Universidad Católica de la Santísima Concepción
Casilla 297, Concepción, Chile
E-Mail: apoza@ucsc.cl

No. 3183
Berlin 2025

2020 *Mathematics Subject Classification.* 35Q35, 76D07, 76S05, 65M60, 65M12.

Key words and phrases. Coupled Navier–Stokes/Darcy equation, stabilized finite element method, a priori error analysis.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

A stabilized finite element method for the Navier–Stokes/Darcy coupled problem

Rodolfo Araya, Cristian Cárcamo, Abner H. Poza

Abstract

In this work, we propose, analyze, and numerically verify a new stabilized finite element method for the Navier–Stokes/Darcy coupled problem, which models a fluid flowing through a free medium into a porous medium. At the interface between both domains, we impose mass conservation, the balance of normal forces, and the well-known Beavers–Joseph–Saffman conditions [9]. The stabilization terms are defined using Galerkin's least-squares stabilization for the Navier–Stokes equation and Masud–Hughes stabilization [30] for the Darcy equation. This new discrete scheme employs equal-order elements to approximate the velocity and pressure of the fluid and generalizes the scheme recently analyzed for the linear case in [4]. The well-posedness of the discrete problem is established via fixed-point theorems under small data conditions, and we prove optimal error estimates in natural norms. Finally, we present numerical examples to confirm the expected theoretical convergence orders in cases where a manufactured solution is available, as well as to demonstrate the effectiveness of our scheme in a physical model with varying permeability fields.

1 Introduction

In the past decade, the scientific community has shown increasing interest in studying the coupling phenomena between fluids flowing from a free medium into a porous medium due to its applications in various fields, such as medicine, geoscience, and industrial processes. For example, this phenomenon is relevant to blood filtration through arterial walls [7], the design and construction of different types of filters, and oil extraction processes in the industrial sector (for details, see [2, 22]). In geosciences, it plays a crucial role in studying groundwater contamination caused by saline water or industrial and domestic pollutants (see [31] and references therein), among other applications.

In this work, we consider a system based on the Navier–Stokes equations to describe the flow in one part of the domain, where the fluid moves freely, and the Darcy equations to model the flow in the porous medium. At the interface between these two media, we impose mass conservation, the balance of normal forces, and the well-known Beavers–Joseph–Saffman conditions [9, 28].

The purpose of this work is to propose and analyze a new stabilized finite element scheme in subspaces of equal-order approximation, based on the stabilization method proposed by Hughes and Masud in [30] for the Darcy equations and a Galerkin least-squares-type stabilization introduced by Franca and Hughes in [23] for the Navier–Stokes equations. This new finite element method extends the ideas recently proposed in [4] for the Stokes/Darcy coupled equations to the case where the fluid dynamics in the free part of the domain exhibit nonlinear behavior.

The pioneering works in which this nonlinear coupled problem was analyzed are [26] and [7]. In the first one, the coupled problem is reformulated, and using a Galerkin approximation, under small data conditions and fixed-point theorems, the existence and uniqueness of weak solutions were proven.

DOI 10.20347/WIAS.PREPRINT.3183

Furthermore, a discontinuous Galerkin method was proposed and analyzed. Almost simultaneously, in [7], where the problem is presented as a nonlinear interface equation, the existence and uniqueness of a solution were established in a closed convex set under a smallness condition on the normal velocity across the interface between both domains. Additionally, three different iterative conforming finite element schemes were proposed. Continuing with the first reference above, in [18], the authors proposed a finite element scheme using continuous elements in the free fluid medium and discontinuous elements in the porous medium while considering two different interface conditions for the equilibrium of normal forces. In [12], a decoupled and linearized two-grid scheme was introduced, with numerical analysis based on the existence of a nonsingular branch of solutions, similar to the case of the Navier–Stokes equation (for details, see [3, 6, 25]). Using a mixed formulation of the coupled problem, where the trace of the porous medium pressure is treated as a Lagrange multiplier, [20] proposed a conforming finite element method. The well-posedness and convergence of this method were established based on fixed-point theory, considering the Oseen/Darcy coupled problem as the linearized model under small data conditions. In [13], an augmented mixed finite element scheme was proposed and analyzed for the case where the viscosity of the free fluid is a nonlinear function. Using fixed-point theory and results on monotone operators, the authors proved the existence and uniqueness of continuous and discrete solutions under small data conditions, obtaining a priori estimates for the different unknowns of interest in specific choices of discrete subspaces. More recently, in [14], a hybridizable discontinuous Galerkin scheme for the coupled problem was presented and analyzed. This method is strongly conservative, and optimal convergence rates were proven, with a velocity error estimate independent of pressure and viscosity. Some extensions of this brief and incomplete list of finite element schemes for the Navier–Stokes/Darcy coupled problem to the time-dependent case can be found in [15, 16, 17].

In this work, we consider an augmented variational formulation of the continuous coupled problem and propose a new stabilized finite element scheme. This scheme allows us to employ equal-order approximation spaces, denoted by $\mathbb{P}_k^d \times \mathbb{P}_k$, for approximating velocities and pressures in the Navier–Stokes and Darcy domains. The stability analysis is inspired by the techniques proposed in [20], which involve rewriting the continuous and discrete problems as fixed-point equations, where the defined fixed-point operator arises from the Oseen/Darcy coupled problem. Convergence results are proven for any polynomial degree k using natural norms, and numerical examples confirm optimal convergence rates for different degrees of approximation in two and three dimensions. To conclude, the proposed scheme also demonstrates good performance and effectiveness in an example inspired by a physical model from the literature (for details, see [14] and [26]).

We conclude this introduction by outlining the structure of the remainder of this paper. In Section 2, we introduce the model problem, including various boundary conditions at the interface between the free and porous domains, along with its augmented weak formulation. Additionally, we recall several classic results from fixed-point theory and other preliminary results. The well-posedness analysis of the continuous problem is addressed in Sections 3 and 4, where we first define the linearized problem and establish its properties, followed by the formulation of the fixed-point problem and various technical results. In Section 5, we present the stabilized finite element method, and its well-posedness analysis is obtained by extending the ideas from the continuous case. Section 6 develops an a priori error analysis for this new scheme based on equal-order interpolation spaces, $\mathbb{P}_k^d \times \mathbb{P}_k$. Finally, in Section 7, numerical results confirm the theoretical convergence rate and demonstrate the good performance of our proposed scheme.

2 Model Problem and preliminary results

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. This domain is partitioned into two disjoint open subdomains Ω_{NS} and Ω_D, both with Lipschitz-continuous boundaries, such that $\overline{\Omega} = \overline{\Omega}_{NS} \cup \overline{\Omega}_D$. Here, Ω_{NS} and Ω_D represent the domains in the free and porous media, respectively. The interface between these two media is given by $\Gamma := \overline{\Omega}_{NS} \cap \overline{\Omega}_D$, as shown in Figure 1. The remaining parts of the boundaries are $\Gamma_{NS} := \partial\Omega_{NS} \backslash \Gamma$ and $\Gamma_D := \partial\Omega_D \backslash \Gamma$, where Γ_D is divided into $\Gamma_D^{\rm Dir}$ and $\Gamma_D^{\rm Neu}$, with $\Gamma_D^{\rm Dir} \cap \Gamma_D^{\rm Neu} = \emptyset$ and $\Gamma_D^{\rm Dir} \neq \emptyset$.

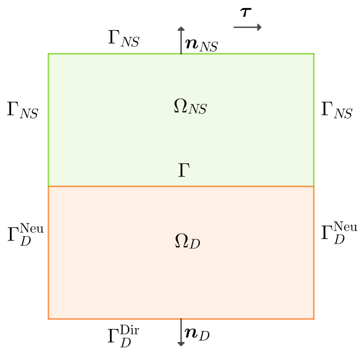

Figure 1: Representation of a possible computational domain Ω.

The Navier–Stokes/Darcy coupled problem consists of finding the velocities $\boldsymbol{u} := (\boldsymbol{u}_{NS}, \boldsymbol{u}_D)$ and the pressures $p := (p_{NS}, p_D)$ such that they satisfy the system of equations

$$(P) \quad \left\{ \begin{array}{rcll} -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_{NS}, p_{NS}) + (\nabla\boldsymbol{u}_{NS})\boldsymbol{u}_{NS} & = & \boldsymbol{f}_{NS} & \text{in } \Omega_{NS}, \\ \nabla\cdot\boldsymbol{u}_{NS} & = & 0 & \text{in } \Omega_{NS}, \\ \boldsymbol{u}_{NS} & = & \boldsymbol{0} & \text{on } \Gamma_{NS}, \\ \nu\,\boldsymbol{u}_D + \kappa\,\nabla p_D & = & \boldsymbol{0} & \text{in } \Omega_D, \\ \nabla\cdot\boldsymbol{u}_D & = & g_D & \text{in } \Omega_D, \\ p_D & = & 0 & \text{on } \Gamma_D^{\rm Dir}, \\ \boldsymbol{u}_D\cdot\boldsymbol{n}_D & = & \boldsymbol{0} & \text{on } \Gamma_D^{\rm Neu}, \end{array} \right. \tag{2.1}$$

where $\boldsymbol{\sigma}(\boldsymbol{u}_{NS}, p_{NS}) := 2\nu\,\boldsymbol{\varepsilon}(\boldsymbol{u}_{NS}) - p_{NS}\,\mathbf{I}$ is the stress rate tensor, $\mathbf{I}$ the identity matrix, $\boldsymbol{\varepsilon}(\boldsymbol{u}_{NS}) := \frac{1}{2}\left(\nabla\boldsymbol{u}_{NS} + \nabla\boldsymbol{u}_{NS}^t\right)$ the deformation rate tensor, $\nu > 0$ the viscosity of the fluid, $\kappa > 0$ the permeability of the porous medium, and $\boldsymbol{n}_D$ the outward unit normal vector on $\Gamma_D^{\rm Neu}$. Here, $\boldsymbol{f}_{NS} \in L^2(\Omega_{NS})^d$ and $g_D \in L^2(\Omega_D)$.

This problem is completed with mass conservation, equilibrium of normal forces, and the Beavers–Joseph–Saffman condition on Γ (for details, see [9]):

$$\begin{aligned} \boldsymbol{u}_D\cdot\boldsymbol{n}_D + \boldsymbol{u}_{NS}\cdot\boldsymbol{n}_{NS} &= 0, \\ -\boldsymbol{n}_{NS}\cdot\boldsymbol{\sigma}(\boldsymbol{u}_{NS}, p_{NS})\,\boldsymbol{n}_{NS} &= p_D, \\ -\boldsymbol{n}_{NS}\cdot\boldsymbol{\sigma}(\boldsymbol{u}_{NS}, p_{NS})\,\boldsymbol{\tau}_i &= \frac{\alpha_i}{\kappa^{1/2}}\,\boldsymbol{u}_{NS}\cdot\boldsymbol{\tau}_i, \qquad i = 1,\ldots,d-1, \end{aligned} \tag{2.2}$$

where α_i are non-dimensional positive constants, and $\boldsymbol{\tau}_i$ are the tangent vectors on Γ. In the sequel,

DOI 10.20347/WIAS.PREPRINT.3183

we will use the following Hilbert spaces:

$$\begin{aligned}
\boldsymbol{H}^{NS} &:= \left\{ \boldsymbol{v} \in H^1(\Omega_{NS})^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_{NS} \right\}, \quad \text{and} \quad \boldsymbol{H}^D := L^2(\Omega_D)^d, \\
Q^{NS} &:= L^2(\Omega_{NS}), \quad \text{and} \quad Q^D := \left\{ q \in H^1(\Omega_D) : q = 0 \text{ on } \Gamma_D^{\mathrm{Dir}} \right\}, \\
\boldsymbol{H} &:= \boldsymbol{H}^{NS} \times \boldsymbol{H}^D, \quad \text{and} \quad Q := Q^{NS} \times Q^D.
\end{aligned}$$

A mixed variational formulation of problem (2.1)–(2.2) can be written as: *Find* $(\boldsymbol{u}, p_D, p_{NS}) \in \boldsymbol{H} \times Q^D \times Q^{NS}$ *such that*

$$A_{\boldsymbol{u}_{NS}}(\boldsymbol{u}, p_D; \boldsymbol{v}, q_D) + B(\boldsymbol{v}, q_D; p_{NS}) = \tilde{F}(\boldsymbol{v}, q_D), \tag{2.3}$$

$$B(\boldsymbol{u}, p_D; q_{NS}) = 0, \tag{2.4}$$

for all $(\boldsymbol{v}, q_D, q_{NS}) \in \boldsymbol{H} \times Q^D \times Q^{NS}$, where given $\boldsymbol{w}_{NS} \in \boldsymbol{H}^{NS}$, $A_{\boldsymbol{w}_{NS}} : (\boldsymbol{H} \times Q^D) \times (\boldsymbol{H} \times Q^D) \longrightarrow \mathbb{R}$ is the form defined by

$$\begin{aligned}
A_{\boldsymbol{w}_{NS}}(\boldsymbol{u}, p_D; \boldsymbol{v}, q_D) &:= 2\nu\kappa\, (\varepsilon(\boldsymbol{u}_{NS}), \varepsilon(\boldsymbol{v}_{NS}))_{\Omega_{NS}} + \kappa\, ((\nabla \boldsymbol{u}_{NS})\boldsymbol{w}_{NS}, \boldsymbol{v}_{NS})_{\Omega_{NS}} + \kappa\, (p_D, \boldsymbol{v}_{NS} \cdot \boldsymbol{n}_{NS})_\Gamma \\
&+ \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i (\boldsymbol{u}_{NS} \cdot \boldsymbol{\tau}_i, \boldsymbol{v}_{NS} \cdot \boldsymbol{\tau}_i)_\Gamma + \nu\, (\boldsymbol{u}_D, \boldsymbol{v}_D)_{\Omega_D} + \kappa\, (\nabla p_D, \boldsymbol{v}_D)_{\Omega_D} - \kappa\, (\boldsymbol{u}_D, \nabla q_D)_{\Omega_D} \\
&- \kappa\, (\boldsymbol{u}_{NS} \cdot \boldsymbol{n}_{NS}, q_D)_\Gamma + \frac{1}{2\nu} (\nu\, \boldsymbol{u}_D + \kappa\, \nabla p_D, -\nu\, \boldsymbol{v}_D + \kappa\, \nabla q_D)_{\Omega_D},
\end{aligned}$$

for all $(\boldsymbol{u}, p_D), (\boldsymbol{v}, q_D) \in \boldsymbol{H} \times Q^D$, and $B : (\boldsymbol{H} \times Q^D) \times Q^{NS} \longrightarrow \mathbb{R}$ is the form defined by

$$B(\boldsymbol{v}, q_D; p_{NS}) := -\kappa\, (p_{NS}, \nabla \cdot \boldsymbol{v}_{NS})_{\Omega_{NS}},$$

for all $(\boldsymbol{v}, q_D) \in \boldsymbol{H} \times Q^D$ and $p_{NS} \in Q^{NS}$. Moreover, $\tilde{F} : \boldsymbol{H} \times Q^D \longrightarrow \mathbb{R}$ is the linear functional given by

$$\tilde{F}(\boldsymbol{v}, q_D) := \kappa(\boldsymbol{f}_{NS}, \boldsymbol{v}_{NS})_{\Omega_{NS}} + \kappa(g_D, q_D)_{\Omega_D},$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{H} \times Q$. The following results will be needed throughout the paper.

Lemma 1. *There exists a positive constant* C_{Korn}*, such that*

$$C_{\mathrm{Korn}}\, \|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}} \leq \|\varepsilon(\boldsymbol{v}_{NS})\|_{0,\Omega_{NS}} \leq \|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}},$$

for all $\boldsymbol{v}_{NS} \in \boldsymbol{H}^{NS}$.

Proof. See [19, Theorem 1.2-2]. □

We will also make use of the trace inequality (see [21, Theorem B.52]),

$$\|\boldsymbol{v}\|_{1/2,\partial\Omega_{NS}} \leq C_{\mathrm{tr}} \|\boldsymbol{v}\|_{1,\Omega_{NS}} \qquad \forall \boldsymbol{v} \in H^1(\Omega_{NS})^d, \tag{2.5}$$

where C_{tr} is a positive constant, and that the injections of $H^{1/2}(\partial\Omega_{NS})^d$ into $L^4(\partial\Omega_{NS})^d$ and $H^1(\Omega_{NS})^d$ into $L^4(\Omega_{NS})^d$, are continuous (see [20, 25]), i.e., there exist positive constants C_{iS} and C_q, such that

$$\|\boldsymbol{v}\|_{0,4,\partial\Omega_{NS}} \leq C_{\mathrm{iS}} \|\boldsymbol{v}\|_{1/2,\partial\Omega_{NS}} \qquad \forall \boldsymbol{v} \in H^{1/2}(\partial\Omega_{NS})^d, \tag{2.6}$$

$$\|\boldsymbol{v}\|_{0,4,\Omega_{NS}} \leq C_q \|\boldsymbol{v}\|_{1,\Omega_{NS}} \qquad \forall \boldsymbol{v} \in H^1(\Omega_{NS})^d. \tag{2.7}$$

Also, we need to recall the following identity:

$$((\nabla \boldsymbol{u})\boldsymbol{\alpha}, \boldsymbol{v})_{\Omega_{NS}} = -(\boldsymbol{u}, (\nabla \boldsymbol{v})\boldsymbol{\alpha})_{\Omega_{NS}} - ((\nabla \cdot \boldsymbol{\alpha})\boldsymbol{u}, \boldsymbol{v})_{\Omega_{NS}} + ((\boldsymbol{\alpha} \cdot \boldsymbol{n})\boldsymbol{u}, \boldsymbol{v})_{\partial\Omega_{NS}}, \tag{2.8}$$

which is valid for all $\boldsymbol{u}, \boldsymbol{\alpha}, \boldsymbol{v} \in H^1(\Omega_{NS})^d$. Additionally, let us recall some classical fixed point results.

Theorem 2 (Banach)**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a contractive operator, i.e. there exists $\lambda \in (0, 1)$ such that $d(f(x), f(y)) \leq \lambda\, d(x, y)$ for any $x, y \in X$. Then, f has a unique fixed point.*

Proof. See [8]. □

Theorem 3 (Schauder)**.** *Let W be a closed and convex subset of a Banach space X and let $f : W \to W$ be a continuous function such that $\overline{f(W)}$ is compact. Then f has at least one fixed point.*

Proof. See [34]. □

Theorem 4 (Brouwer)**.** *Let W be a compact and convex subset of a finite-dimensional Banach space V, and let $f : W \to W$ a continuous function. Then f has at least one fixed point.*

Proof. See [11]. □

Throughout this paper C and C_i, $i > 0$, will denote positive constants independent of the mesh size h, but who may depend on the physical parameters of the equation.

In Section 4, we will study the existence and uniqueness of the solution of variational problem (2.3)-(2.4), using Schauder's and Banach's fixed point theorems (see theorems 3 and 2) and the arguments introduced in [20]. To this end, we first study, in the next section, a linearized version of problem (P).

3 The linearized coupled problem

To analyze the well-posedness of (2.3)–(2.4), we consider the mixed formulation of the Oseen-Darcy coupled problem: *Given $\boldsymbol{w}_{NS} \in \boldsymbol{H}^{NS}$, with $\nabla\cdot\boldsymbol{w}_{NS} = 0$ in Ω_{NS}, find $(\boldsymbol{u}, p_D, p_{NS}) \in \boldsymbol{H}\times Q^D\times Q^{NS}$ such that*

$$A_{\boldsymbol{w}_{NS}}(\boldsymbol{u}, p_D; \boldsymbol{v}, q_D) + B(\boldsymbol{v}, q_D; p_{NS}) = \tilde{F}(\boldsymbol{v}, q_D), \tag{3.9}$$

$$B(\boldsymbol{u}, p_D; q_{NS}) = 0, \tag{3.10}$$

for all $(\boldsymbol{v}, q_D, q_{NS}) \in \boldsymbol{H} \times Q^D \times Q^{NS}$.

3.1 Well-posedness of the linearized coupled problem

The following two results allow us to guarantee that the linearized problem is well-posed and that we have a continuous dependency result.

Lemma 5. *There exists a positive constant β_c, independent of κ and ν, such that*

$$\sup_{(\boldsymbol{v},q_D)\in\boldsymbol{H}\times Q^D} \frac{B(\boldsymbol{v}, q_D; q_{NS})}{\left\{\nu\kappa \,\|\boldsymbol{v}_{NS}\|^2_{1,\Omega_{NS}} + \nu\,\|\boldsymbol{v}_D\|^2_{0,\Omega_D} + \dfrac{\kappa^2}{\nu}\,|q_D|^2_{1,\Omega_D}\right\}^{1/2}} \geq \beta_c \sqrt{\frac{\kappa}{\nu}}\,\|q_{NS}\|_{0,\Omega_{NS}}$$

for all $q_{NS} \in Q^{NS}$.

Proof. The result is a simple consequence of [33, Proposition 5.3.2]. □

Lemma 6. *Let $\boldsymbol{w}_{NS} \in \boldsymbol{H}^{NS}$ with $\nabla \cdot \boldsymbol{w}_{NS} = 0$ in Ω_{NS} satisfying*

$$\|\boldsymbol{w}_{NS} \cdot \boldsymbol{n}_{NS}\|_{0,\Gamma} \leq \frac{2\,\nu\, C_{\mathrm{Korn}}^2}{C_{\mathrm{iS}}^2 C_{\mathrm{tr}}^2}.$$

Then, there exists a positive constant C_{coer}, independent of ν and κ, such that

$$A_{\boldsymbol{w}_{NS}}(\boldsymbol{v}, q_D; \boldsymbol{v}, q_D) \geq C_{\mathrm{coer}} \left\{ \nu\kappa \,\|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}}^2 + \nu \,\|\boldsymbol{v}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{\nu}\, |q_D|_{1,\Omega_D}^2 \right\}, \tag{3.11}$$

for all $(\boldsymbol{v}, q_D) \in \boldsymbol{H} \times Q^D$.

Proof. Let $(\boldsymbol{v}, q_D) \in \boldsymbol{H} \times Q^D$ and $\boldsymbol{w}_{NS} \in \boldsymbol{H}^{NS}$ that satisfies the previous hypothesis. Then, using the Cauchy-Schwarz and Korn inequalities, and the identity (2.8), we have

$$\begin{aligned} &A_{\boldsymbol{w}_{NS}}(\boldsymbol{v}, q_D; \boldsymbol{v}, q_D) \\ &\geq 2\nu\kappa \,\|\boldsymbol{\varepsilon}(\boldsymbol{v}_{NS})\|_{0,\Omega_{NS}}^2 + \frac{\nu}{2}\,\|\boldsymbol{v}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_D|_{1,\Omega_D}^2 + \frac{\kappa}{2}\int_\Gamma (\boldsymbol{w}_{NS}\cdot \boldsymbol{n}_{NS})|\boldsymbol{v}_{NS}|^2 \\ &\geq 2\nu\kappa\, C_{\mathrm{Korn}}^2\,\|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}}^2 + \frac{\nu}{2}\,\|\boldsymbol{v}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_D|_{1,\Omega_D}^2 - \frac{\kappa}{2}\left|\int_\Gamma (\boldsymbol{w}_{NS}\cdot \boldsymbol{n}_{NS})|\boldsymbol{v}_{NS}|^2\right|. \end{aligned} \tag{3.12}$$

Now, from Hölder's inequality, (2.6), and (2.5), it follows that

$$\begin{aligned} \frac{\kappa}{2}\left|\int_\Gamma (\boldsymbol{w}_{NS}\cdot \boldsymbol{n}_{NS})|\boldsymbol{v}_{NS}|^2\right| &\leq \frac{\kappa C_{\mathrm{iS}}^2}{2}\,\|\boldsymbol{w}_{NS}\cdot \boldsymbol{n}_{NS}\|_{0,\Gamma}\|\boldsymbol{v}_{NS}\|_{1/2,\partial\Omega_{NS}}^2 \\ &\leq \frac{\kappa C_{\mathrm{tr}}^2 C_{\mathrm{iS}}^2}{2}\,\|\boldsymbol{w}_{NS}\cdot \boldsymbol{n}_{NS}\|_{0,\Gamma}\|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}}^2. \end{aligned} \tag{3.13}$$

Thus, from (3.12) and (3.13), we have

$$\begin{aligned} A_{\boldsymbol{w}_{NS}}(\boldsymbol{v}, q_D; \boldsymbol{v}, q_D) \geq &\left(2\nu\kappa\, C_{\mathrm{Korn}}^2 - \frac{\kappa C_{tr}^2 C_{\mathrm{iS}}^2}{2}\,\|\boldsymbol{w}_{NS}\cdot \boldsymbol{n}_{NS}\|_{0,\Gamma}\right)\,\|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}}^2 \\ &+ \frac{\nu}{2}\,\|\boldsymbol{v}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_D|_{1,\Omega_D}^2 \\ \geq &C_{\mathrm{coer}}\left\{\nu\kappa\,\|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}}^2 + \nu\,\|\boldsymbol{v}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{\nu}\,|q_D|_{1,\Omega_D}^2\right\}, \end{aligned} \tag{3.14}$$

where $C_{\mathrm{coer}} := \min\left\{C_{\mathrm{Korn}}^2, \frac{1}{2}\right\}$, which completes the proof. □

In the sequel, we will use the following norm defined on the product space $\boldsymbol{H} \times Q$,

$$\|(\boldsymbol{w}, r)\| := \left\{\nu\kappa\,\|\boldsymbol{w}_{NS}\|_{1,\Omega_{NS}}^2 + \kappa\,\|r_{NS}\|_{0,\Omega_{NS}}^2 + \nu\,\|\boldsymbol{w}_D\|_{0,\Omega_D}^2 + \frac{\kappa^2}{\nu}\,|r_D|_{1,\Omega_D}^2\right\}^{1/2}, \tag{3.15}$$

for all $(\boldsymbol{w}, r) \in \boldsymbol{H} \times Q$.

Theorem 7. *Assume the hypothesis of Lemma 6. Then, problem* (3.9)–(3.10) *has a unique solution $(\boldsymbol{u}, p_D, p_{NS}) \in \boldsymbol{H} \times Q^D \times Q^{NS}$, and there exists a positive constant C_{dc} such that*

$$\|(\boldsymbol{u}, p)\| \leq C_{\mathrm{dc}}\left\{\|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D}\right\}. \tag{3.16}$$

Proof. The proof follows directly from Lemmas 5, 6, and the Babuška–Brezzi theory. □

4 The nonlinear continuous problem

As mentioned previously, the purpose of this section is to prove an existence and uniqueness result for the solution of problem (2.3)–(2.4). To achieve this, as is common, we will express $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ as the solution of a fixed-point problem (for similar approaches, see, for example, [5, 20]).

Specifically, we introduce the operator $T : \boldsymbol{H} \times Q \longrightarrow \boldsymbol{H} \times Q$, defined for all $(\boldsymbol{w}, r) \in \boldsymbol{H} \times Q$, with $\nabla \cdot \boldsymbol{w}_{NS} = 0$ in Ω_{NS}, as

$$T(\boldsymbol{w}, r) = (\bar{\boldsymbol{u}}, \bar{p}),$$

where $(\bar{\boldsymbol{u}}, \bar{p}) \in \boldsymbol{H} \times Q$ is the unique solution of the linear problem

$$A_{\boldsymbol{w}_{NS}}(\bar{\boldsymbol{u}}, \bar{p}_D; \boldsymbol{v}, q_D) + B(\boldsymbol{v}, q_D; \bar{p}_{NS}) = \tilde{F}(\boldsymbol{v}, q_D), \tag{4.17}$$

$$B(\bar{\boldsymbol{u}}, \bar{p}_D; q_{NS}) = 0, \tag{4.18}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{H} \times Q$. In this way, problem (2.3)–(2.4) can be rewritten as follows: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ *such that*

$$T(\boldsymbol{u}, p) = (\boldsymbol{u}, p). \tag{4.19}$$

In the sequel, we will establish several technical results necessary for applying fixed-point existence and uniqueness theorems. To this end, we assume that the data satisfy

$$C_{\mathrm{dc}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\} \leq \frac{2\nu^{3/2}\sqrt{\kappa}\, C_{\mathrm{Korn}}^2}{C_{\mathrm{tr}}^3 C_{\mathrm{iS}}^2}. \tag{4.20}$$

Lemma 8. *Let* $\boldsymbol{X}$ *be the closed and convex subset of* $\boldsymbol{H} \times Q$ *defined by*

$$\boldsymbol{X} := \left\{ (\boldsymbol{v}, q) \in \boldsymbol{H} \times Q : \ \nabla\cdot\boldsymbol{v}_{NS} = 0 \ \text{ in } \Omega_{NS} \ \text{ and } \|(\boldsymbol{v}, q)\| \leq C_{\mathrm{dc}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\} \right\}.$$

Then, $T(\boldsymbol{X}) \subseteq \boldsymbol{X}$.

Proof. Given $(\boldsymbol{w}, r) \in \boldsymbol{X}$, by (2.5) and (4.20), we obtain

$$\|\boldsymbol{w}_{NS} \cdot \boldsymbol{n}_{NS}\|_{0,\Gamma} \leq C_{\mathrm{tr}} \|\boldsymbol{w}_{NS}\|_{1,\Omega_{NS}} \leq \frac{C_{\mathrm{tr}}}{\sqrt{\nu\kappa}} C_{\mathrm{dc}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\} \leq \frac{2\,\nu\, C_{\mathrm{Korn}}^2}{C_{\mathrm{iS}}^2 C_{\mathrm{tr}}^2}. \tag{4.21}$$

Thus, the hypothesis of Lemma 6 holds. Next, by Theorem 7, there exists a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ satisfying

$$\|(\boldsymbol{u}, p)\| \leq C_{\mathrm{dc}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\}.$$

Therefore,

$$\|T(\boldsymbol{w}, r)\| = \|(\boldsymbol{u}, p)\| \leq C_{\mathrm{dc}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\}.$$

This completes the proof. □

Lemma 9. *There is a positive constant* C_1 *such that*

$$\|T(\boldsymbol{w}, q) - T(\tilde{\boldsymbol{w}}, \tilde{q})\| \leq C_1 \|T(\tilde{\boldsymbol{w}}, \tilde{q})\| \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{0,4,\Omega_{NS}} \qquad \forall (\boldsymbol{w}, q), (\tilde{\boldsymbol{w}}, \tilde{q}) \in \boldsymbol{X}.$$

Proof. Let $(\boldsymbol{w}, q), (\tilde{\boldsymbol{w}}, \tilde{q}) \in \boldsymbol{X}$, and define $(\boldsymbol{u}, p) := T(\boldsymbol{w}, q)$ and $(\tilde{\boldsymbol{u}}, \tilde{p}) := T(\tilde{\boldsymbol{w}}, \tilde{q})$. From the definition of the operator T, it is clear that

$$A_{\boldsymbol{w}_{NS}}(\boldsymbol{u}, p_D; \boldsymbol{v}, q_D) - A_{\tilde{\boldsymbol{w}}_{NS}}(\tilde{\boldsymbol{u}}, \tilde{p}_D; \boldsymbol{v}, q_D) + B(\boldsymbol{v}, q_D; p_{NS} - \tilde{p}_{NS}) = 0 \qquad \forall (\boldsymbol{v}, q_D) \in \boldsymbol{H} \times Q^D, \tag{4.22}$$

$$B(\boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D; q_{NS}) = 0 \qquad \forall q_{NS} \in Q^{NS}. \tag{4.23}$$

Setting $(\boldsymbol{v}, q_D) = (\boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D)$ in (4.22) and using (4.23), we conclude that

$$A_{\boldsymbol{w}_{NS}}(\boldsymbol{u}, p_D; \boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D) - A_{\tilde{\boldsymbol{w}}_{NS}}(\tilde{\boldsymbol{u}}, \tilde{p}_D; \boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D) = 0.$$

Rewriting the above, it follows that

$$\begin{aligned} &2\nu\kappa \, \|\boldsymbol{\varepsilon}(\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS})\|^2_{0,\Omega_{NS}} + \kappa \left((\nabla \boldsymbol{u}_{NS})\boldsymbol{w}_{NS} - (\nabla \tilde{\boldsymbol{u}}_{NS})\tilde{\boldsymbol{w}}_{NS}, \boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\right)_{\Omega_{NS}} \\ &+ \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i \|(\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}) \cdot \boldsymbol{\tau}_i\|^2_{0,\Gamma} + \frac{\nu}{2} \|\boldsymbol{u}_D - \tilde{\boldsymbol{u}}_D\|^2_{0,\Omega_D} + \frac{\kappa^2}{2\nu} |p_D - \tilde{p}_D|^2_{1,\Omega_D} = 0. \end{aligned}$$

Thus, we deduce that

$$\begin{aligned} &A_{\boldsymbol{w}_{NS}}(\boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D; \boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D) \\ &= 2\nu\kappa \, \|\boldsymbol{\varepsilon}(\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS})\|^2_{0,\Omega_{NS}} + \kappa \left(\nabla(\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS})\boldsymbol{w}_{NS}, \boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\right)_{\Omega_{NS}} \\ &\quad + \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i \|(\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}) \cdot \boldsymbol{\tau}_i\|^2_{0,\Gamma} + \frac{\nu}{2} \|\boldsymbol{u}_D - \tilde{\boldsymbol{u}}_D\|^2_{0,\Omega_D} + \frac{\kappa^2}{2\nu} |p_D - \tilde{p}_D|^2_{1,\Omega_D} \\ &= \kappa \left((\nabla \tilde{\boldsymbol{u}}_{NS})(\tilde{\boldsymbol{w}}_{NS} - \boldsymbol{w}_{NS}), \boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\right)_{\Omega_{NS}}. \end{aligned} \tag{4.24}$$

Since $\boldsymbol{w}_{NS}$ satisfies the hypothesis of Lemma 6, applying (2.7), (3.11), and (4.24), along with Hölder's and Young's inequalities (the latter with $\gamma > 0$), we obtain that there exists a positive constant C_2 such that

$$\nu\kappa \, \|\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}} + \nu \, \|\boldsymbol{u}_D - \tilde{\boldsymbol{u}}_D\|^2_{0,\Omega_D} + \frac{\kappa^2}{\nu} |p_D - \tilde{p}_D|^2_{1,\Omega_D} \leq C_2 \, \|\tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|^2_{0,4,\Omega_{NS}}. \tag{4.25}$$

In fact, using these arguments, we obtain

$$\begin{aligned} &\nu\kappa \, \|\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}} + \nu \, \|\boldsymbol{u}_D - \tilde{\boldsymbol{u}}_D\|^2_{0,\Omega_D} + \frac{\kappa^2}{\nu} \, |p_D - \tilde{p}_D|^2_{1,\Omega_D} \\ &\leq \frac{1}{C_{\text{coer}}} \, A_{\boldsymbol{w}_{NS}}(\boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D; \boldsymbol{u} - \tilde{\boldsymbol{u}}, p_D - \tilde{p}_D) \\ &= \frac{\kappa}{C_{\text{coer}}} \left((\nabla \tilde{\boldsymbol{u}}_{NS})(\tilde{\boldsymbol{w}}_{NS} - \boldsymbol{w}_{NS}), \boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\right)_{\Omega_{NS}} \\ &\leq \frac{\kappa}{C_{\text{coer}}} \, \|\tilde{\boldsymbol{u}}_{NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{0,4,\Omega_{NS}} \|\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\|_{0,4,\Omega_{NS}} \\ &\leq \frac{C_q \kappa}{C_{\text{coer}}} \, \|\tilde{\boldsymbol{u}}_{NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{0,4,\Omega_{NS}} \|\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\|_{1,\Omega_{NS}} \\ &\leq \frac{C_q^2 \kappa^2}{2 \, C_{\text{coer}}^2 \gamma} \, \|\tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|^2_{0,4,\Omega_{NS}} + \frac{\gamma}{2} \, \|\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}}, \end{aligned}$$

where we take $\gamma = \nu\kappa$ to obtain (4.25).
On the other hand, from (4.22), by applying the trace inequality (2.5) and Hölder's inequality, together with (2.7), (4.21), and (4.25), we obtain

$$
\begin{aligned}
& B(\boldsymbol{v}, q_D; p_{NS} - \tilde{p}_{NS}) \\
\leq & C \Big\{ \nu\kappa \, \|\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{NS} - \boldsymbol{u}_{NS})\|_{0,\Omega_{NS}} \|\boldsymbol{\varepsilon}(\boldsymbol{v}_{NS})\|_{0,\Omega_{NS}} + \kappa^{1/2} \, \|\tilde{\boldsymbol{u}}_{NS} - \boldsymbol{u}_{NS}\|_{1,\Omega_{NS}} \|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}} \\
& + \kappa \, |\tilde{p}_D - p_D|_{1,\Omega_D} \|\boldsymbol{v}_{NS}\|_{1,\Omega_{NS}} + \nu \, \|\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_D\|_{0,\Omega_D} \|\boldsymbol{v}_D\|_{0,\Omega_D} + \kappa \, |\tilde{p}_D - p_D|_{1,\Omega_D} \|\boldsymbol{v}_D\|_{0,\Omega_D} \\
& + \kappa \, \|\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_D\|_{0,\Omega_D} |q_D|_{1,\Omega_D} + \kappa \, \|\tilde{\boldsymbol{u}}_{NS} - \boldsymbol{u}_{NS}\|_{1,\Omega_{NS}} |q_D|_{1,\Omega_D} + \frac{\kappa^2}{\nu} \, |\tilde{p}_D - p_D|_{1,\Omega_D} |q_D|_{1,\Omega_D} \\
& + \kappa \, |\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}|_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS}\|_{0,4,\Omega_{NS}} \|\boldsymbol{v}_{NS}\|_{0,4,\Omega_{NS}} + \kappa \, \|\tilde{\boldsymbol{u}}_{NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{0,4,\Omega_{NS}} \|\boldsymbol{v}_{NS}\|_{0,4,\Omega_{NS}} \Big\} \\
\leq & C_3 \Big\{ \nu\kappa \, \|\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}} + \nu \, \|\boldsymbol{u}_D - \tilde{\boldsymbol{u}}_D\|^2_{0,\Omega_D} + \frac{\kappa^2}{\nu} \, |p_D - \tilde{p}_D|^2_{1,\Omega_D} \\
& + \nu\kappa \, \|\boldsymbol{u}_{NS} - \tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS}\|^2_{0,4,\Omega_{NS}} + \|\tilde{\boldsymbol{u}}_{NS}\|^2_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|^2_{0,4,\Omega_{NS}} \Big\}^{1/2} \\
& \times \Big\{ \nu\kappa \, \|\boldsymbol{v}_{NS}\|^2_{1,\Omega_{NS}} + \nu \, \|\boldsymbol{v}_D\|^2_{0,\Omega_D} + \frac{\kappa^2}{\nu} \, |q_D|^2_{1,\Omega_D} \Big\}^{1/2} \\
\leq & C_4 \, \|\tilde{\boldsymbol{u}}_{NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{0,4,\Omega_{NS}} \Big\{ \nu\kappa \, \|\boldsymbol{v}_{NS}\|^2_{1,\Omega_{NS}} + \nu \, \|\boldsymbol{v}_D\|^2_{0,\Omega_D} + \frac{\kappa^2}{\nu} \, |q_D|^2_{1,\Omega_D} \Big\}^{1/2} .
\end{aligned}
\tag{4.26}
$$

Finally, from the definition of the norm $\|(\cdot,\cdot)\|$ (cf. (3.15)), and using (4.25), (4.26), and Lemma 5, the proof follows. □

The following results allow us to ensure the existence and uniqueness of a fixed point $(\boldsymbol{u}, p) \in \boldsymbol{X}$ for the operator T.

Lemma 10. *Assume that the data $\boldsymbol{f}_{NS} \in L^2(\Omega_{NS})^d$ and $g_D \in L^2(\Omega_D)$ satisfy the condition* (4.20). *Then, the mixed variational formulation* (2.3)–(2.4) *has at least one solution $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$.*

Proof. From Lemmas 8 and 9, and using the fact that the Sobolev embedding $H^1(\Omega_{NS}) \overset{c}{\hookrightarrow} L^4(\Omega_{NS})$ is compact, we deduce that $T : \boldsymbol{X} \longrightarrow \boldsymbol{X}$ is continuous. On the other hand, let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{X}$, which is clearly a bounded sequence in $\boldsymbol{H} \times Q$. Then, there exists a subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ of $\{\varphi_n\}_{n\in\mathbb{N}}$ and an element $\varphi \in \boldsymbol{H} \times Q$ such that $\varphi_{n_k} \overset{w}{\to} \varphi$. We proceed by defining $\varphi^{NS}_{n_k}$ and φ^{NS}_n as the first component of φ_{n_k} and φ_n, respectively. Due to the uniqueness of the weak limit, we obtain that $\varphi^{NS}_{n_k} \overset{w}{\to} \varphi^{NS}$. Now, using again the compact embedding $H^1(\Omega_{NS}) \overset{c}{\hookrightarrow} L^4(\Omega_{NS})$, we conclude that $\varphi^{NS}_{n_k} \longrightarrow \varphi^{NS}$ in $L^4(\Omega_{NS})$. Thus, by applying Lemma 9, we deduce that $T(\varphi_{n_k}) \longrightarrow T(\varphi)$, which proves that $\overline{T(\boldsymbol{X})}$ is compact. Applying Schauder's fixed point theorem (see Theorem 3), we obtain the existence of a solution $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ for (2.3)–(2.4). □

Theorem 11. *Assume that the data $\boldsymbol{f}_{NS} \in L^2(\Omega_{NS})^d$ and $g_D \in L^2(\Omega_D)$ satisfy the condition*

$$
\|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} < \frac{\sqrt{\nu\kappa}}{C_{\mathrm{dc}}} \min\left\{ \frac{2\,\nu\, C^2_{\mathrm{Korn}}}{C^2_{\mathrm{iS}} C^3_{\mathrm{tr}}}, \frac{1}{C_q C_1} \right\}. \tag{4.27}
$$

Then, the mixed variational formulation (2.3)–(2.4) *has a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ *and satisfies the estimate*

$$\|(\boldsymbol{u}, p)\| \leq C_{\mathrm{dc}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\}. \tag{4.28}$$

Proof. The main idea of the proof is to verify the hypotheses of Banach's fixed-point theorem. Since $\boldsymbol{X}$ is a Banach space and $T(\boldsymbol{X}) \subseteq \boldsymbol{X}$, it remains to prove that T is a contraction. In fact, as the inclusion of $H^1(\Omega_{NS})^d$ into $L^4(\Omega_{NS})^d$ is continuous, from Lemma 9 and condition (4.27), we obtain

$$\begin{aligned}
\|T(\boldsymbol{w}, q) - T(\tilde{\boldsymbol{w}}, \tilde{q})\| &\leq C_1 \|T(\tilde{\boldsymbol{w}}, \tilde{q})\| \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{0,4,\Omega_{NS}} \\
&\leq C_q C_1 \|T(\tilde{\boldsymbol{w}}, \tilde{q})\| \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} \\
&\leq C_{\mathrm{dc}} C_q C_1 \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\} \|\boldsymbol{w}_{NS} - \tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} \\
&\leq \frac{C_{\mathrm{dc}} C_q C_1}{\sqrt{\nu \kappa}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\} \|(\boldsymbol{w}, q) - (\tilde{\boldsymbol{w}}, \tilde{q})\|,
\end{aligned}$$

which completes the proof. □

5 A stabilized finite element method

From now on, we denote by $\{\mathcal{T}_h^{NS}\}_{h>0}$ and $\{\mathcal{T}_h^{D}\}_{h>0}$ two regular families of triangulations of $\bar{\Omega}_{NS}$ and $\bar{\Omega}_D$, respectively, composed of simplices that match at the interface Γ. For a triangulation $\mathcal{T}_h^{NS}$ or $\mathcal{T}_h^{D}$, we denote by K the elements of the triangulation and by $\mathcal{E}_h^{NS}$ the set of all edges (faces) of $\mathcal{T}_h^{NS}$, with the decomposition

$$\mathcal{E}_h^{NS} := \mathcal{E}_{\Omega_{NS}} \cup \mathcal{E}_{NS}^{\mathrm{Dir}} \cup \mathcal{E}_{\Gamma_h},$$

where $\mathcal{E}_{\Omega_{NS}}$ stands for the edges (faces) lying in the interior of Ω_{NS}, $\mathcal{E}_{NS}^{\mathrm{Dir}}$ stands for the edges (faces) on the boundary Γ_{NS}, and $\mathcal{E}_{\Gamma_h}$ stands for the edges (faces) on the boundary Γ.

Similarly, we denote by $\mathcal{E}_h^{D}$ the set of all edges (faces) of $\mathcal{T}_h^{D}$, with the decomposition

$$\mathcal{E}_h^{D} := \mathcal{E}_{\Omega_D} \cup \mathcal{E}_D^{\mathrm{Neu}} \cup \mathcal{E}_D^{\mathrm{Dir}} \cup \mathcal{E}_{\Gamma_h},$$

where $\mathcal{E}_{\Omega_D}$ stands for the edges (faces) lying in the interior of Ω_D, $\mathcal{E}_D^{\mathrm{Neu}}$ and $\mathcal{E}_D^{\mathrm{Dir}}$ stand for the edges (faces) on the boundaries Γ_D^{Neu} and Γ_D^{Dir}, respectively. As usual, we denote h_K as the diameter of K, $h := \max\limits_{K \in \mathcal{T}_h^{NS} \cup \mathcal{T}_h^{D}} h_K$, and $h_F := |F|$ for $F \in \mathcal{E}_h^{NS} \cup \mathcal{E}_h^{D}$. Finally, for each $K \in \mathcal{T}_h^{NS} \cup \mathcal{T}_h^{D}$ and $F \in \mathcal{E}_h^{NS} \cup \mathcal{E}_h^{D}$, we define $\mathcal{N}(K)$ as the set of nodes of K, $\mathcal{N}(F)$ as the set of nodes of F.

Additionally, we introduce the following neighborhoods:

$$\tilde{\omega}_K^{NS} := \bigcup_{\substack{\mathcal{N}(K) \cap \mathcal{N}(K') \neq \emptyset \\ K' \in \mathcal{T}_h^{NS}}} K', \qquad \tilde{\omega}_K^{D} := \bigcup_{\substack{\mathcal{N}(K) \cap \mathcal{N}(K') \neq \emptyset \\ K' \in \mathcal{T}_h^{D}}} K'.$$

We introduce the following finite element subspaces of $\boldsymbol{H}^{NS}$, $\boldsymbol{H}^{D}$, Q^{NS}, and Q^{D}, respectively:

$$\begin{aligned}
\boldsymbol{H}_h^{NS} &:= \left\{ \boldsymbol{v} \in C(\overline{\Omega}_{NS})^d \ : \ \boldsymbol{v}|_K \in \mathbb{P}_k(K)^d, \quad \forall K \in \mathcal{T}_h^{NS} \right\} \cap \boldsymbol{H}^{NS}, \\
\boldsymbol{H}_h^{D} &:= \left\{ \boldsymbol{v} \in C(\overline{\Omega}_{D})^d \ : \ \boldsymbol{v}|_K \in \mathbb{P}_k(K)^d, \quad \forall K \in \mathcal{T}_h^{D} \right\}, \\
Q_h^{NS} &:= \left\{ q \in C(\overline{\Omega}_{NS}) \ : \ q|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h^{NS} \right\}, \\
Q_h^{D} &:= \left\{ q \in C(\overline{\Omega}_{D}) \ : \ q|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h^{D} \right\} \cap Q^{D},
\end{aligned}$$

where $k \geq 1$, and $\mathbb{P}_k$ denotes the space of polynomials of total degree less than or equal to k.

We define the global spaces:

$$\boldsymbol{H}_h := \boldsymbol{H}_h^{NS} \times \boldsymbol{H}_h^{D}, \quad Q_h := Q_h^{NS} \times Q_h^{D}.$$

Let $\boldsymbol{u}_h := (\boldsymbol{u}_{h,NS}, \boldsymbol{u}_{h,D}) \in \boldsymbol{H}_h$ and $p_h := (p_{h,NS}, p_{h,D}) \in Q_h$. We consider the following discrete stabilized scheme: *Find* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ *such that*

$$\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = \mathcal{F}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{v}_h, q_h), \tag{5.29}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$, where for a given $\boldsymbol{w}_{h,NS} \in \boldsymbol{H}_h^{NS}$, the bilinear form $\mathcal{A}_{\boldsymbol{w}_{h,NS}} : (\boldsymbol{H}_h \times Q_h) \times (\boldsymbol{H}_h \times Q_h) \longrightarrow \mathbb{R}$ is defined as

$$\begin{aligned}
\mathcal{A}_{\boldsymbol{w}_{h,NS}}&(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) := 2\nu\kappa\, (\varepsilon(\boldsymbol{u}_{h,NS}), \varepsilon(\boldsymbol{v}_{h,NS}))_{\Omega_{NS}} + \kappa\, ((\nabla \boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS}, \boldsymbol{v}_{h,NS})_{\Omega_{NS}} \\
&+ \frac{\kappa}{2}\, ((\nabla\cdot \boldsymbol{w}_{h,NS})\, \boldsymbol{u}_{h,NS}, \boldsymbol{v}_{h,NS})_{\Omega_{NS}} - \kappa\, (p_{h,NS}, \nabla\cdot \boldsymbol{v}_{h,NS})_{\Omega_{NS}} + \kappa\, (q_{h,NS}, \nabla\cdot \boldsymbol{u}_{h,NS})_{\Omega_{NS}} \\
&+ \kappa^{1/2} \sum_{i=1}^{d-1} \alpha_i (\boldsymbol{u}_{h,NS}\cdot \boldsymbol{\tau}_i, \boldsymbol{v}_{h,NS}\cdot \boldsymbol{\tau}_i)_\Gamma + \kappa\, (p_{h,D}, \boldsymbol{v}_{h,NS}\cdot \boldsymbol{n}_{NS})_\Gamma \\
&+ \nu\, (\boldsymbol{u}_{h,D}, \boldsymbol{v}_{h,D})_{\Omega_D} + \kappa\, (\nabla p_{h,D}, \boldsymbol{v}_{h,D})_{\Omega_D} - \kappa\, (\boldsymbol{u}_{h,D}, \nabla q_{h,D})_{\Omega_D} - \kappa\, (\boldsymbol{u}_{h,NS}\cdot \boldsymbol{n}_{NS}, q_{h,D})_\Gamma \\
&+ \frac{1}{2\nu}\, (\nu\, \boldsymbol{u}_{h,D} + \kappa\, \nabla p_{h,D}, -\nu\, \boldsymbol{v}_{h,D} + \kappa\, \nabla q_{h,D})_{\Omega_D} + \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Bigg(- 2\nu\, \nabla\cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_{h,NS}) \\
&+ (\nabla \boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS} + \nabla p_{h,NS}, 2\nu\, \nabla\cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{w}_{h,NS} + \nabla q_{h,NS} \Bigg)_K .
\end{aligned} \tag{5.30}$$

for all $(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$.

Similarly, the linear form $\mathcal{F}_{\boldsymbol{w}_{h,NS}} : \boldsymbol{H}_h \times Q_h \longrightarrow \mathbb{R}$ is given by

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{v}_h, q_h) &:= \kappa(\boldsymbol{f}_{NS}, \boldsymbol{v}_{h,NS})_{\Omega_{NS}} + \kappa(g_D, q_{h,D})_{\Omega_D} \\
&+ \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Bigg(\boldsymbol{f}_{NS}, 2\nu\, \nabla\cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{w}_{h,NS} + \nabla q_{h,NS} \Bigg)_K .
\end{aligned} \tag{5.31}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$, where β is a positive constant. Moreover, it is known that there exists a positive constant C_I, independent of h, such that

$$C_I \sum_{K\in\mathcal{T}_h^{NS}} h_K^2\, \|\nabla\cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,K}^2 \leq \|\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,\Omega_{NS}}^2, \qquad \forall \boldsymbol{v}_{h,NS} \in \boldsymbol{H}_h^{NS}. \tag{5.32}$$

In the rest of this work, we define the following mesh-dependent norm over $\boldsymbol{H}_h \times Q_h$:

$$\begin{aligned}
|||(\boldsymbol{v}_h, q_h)|||_h := \Bigg\{ &\nu\kappa\, \|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}^2 + \kappa\, \|q_{h,NS}\|_{0,\Omega_{NS}}^2 + \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\, |q_{h,NS}|_{1,K}^2 \\
&+ \nu\, \|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{\nu}\, |q_{h,D}|_{1,\Omega_D}^2 \Bigg\}^{1/2},
\end{aligned} \tag{5.33}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$.

Moreover, we will use the following standard results:

Lemma 12. *There exist positive constants $\tilde{C}$, $C_{\rm inf}$, and $C_{\rm inv}$, independent of h, such that*

$$\|\boldsymbol{v}_{h,NS}\|_{l,p,K} \le \tilde{C}\, h_K^{m-l+d(1/p-1/q)}\, \|\boldsymbol{v}_{h,NS}\|_{m,q,K}, \tag{5.34}$$

$$\|\boldsymbol{v}_{h,NS}\|_{\infty,K} \le C_{\rm inf}\, h_K^{-1/2} |\boldsymbol{v}_{h,NS}|_{1,\Omega_{NS}}, \tag{5.35}$$

$$h_K\, |\boldsymbol{v}_{h,NS}|_{1,K} \le C_{\rm inv}\, \|\boldsymbol{v}_{h,NS}\|_{0,K}, \tag{5.36}$$

for all $\boldsymbol{v}_{h,NS} \in \boldsymbol{H}_h^{NS}$, where $0 \le m \le l$ and $1 \le p, q \le \infty$.

Proof. See [21, Lemma 1.138]. □

5.1 Well-posedness of the discrete linearized scheme

The purpose of this section is to prove an existence and uniqueness result for a solution of problem (5.29). To this end, as we did in the continuous case, we will write $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ as a solution of a fixed-point problem.

In fact, we introduce the operator $T_h : \boldsymbol{H}_h \times Q_h \longrightarrow \boldsymbol{H}_h \times Q_h$, such that for all $(\boldsymbol{w}_h, r_h) \in \boldsymbol{H}_h \times Q_h$, we define

$$T_h(\boldsymbol{w}_h, r_h) = (\bar{u}_h, \bar{p}_h),$$

where $(\bar{u}_h, \bar{p}_h) \in \boldsymbol{H}_h \times Q_h$ is the unique solution of the linear problem (see Theorem 16 below)

$$\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\bar{u}_h, \bar{p}_h; \boldsymbol{v}_h, q_h) = \mathcal{F}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{v}_h, q_h) \qquad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h. \tag{5.37}$$

In this way, the discrete problem (5.29) can be written as follows: *Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ such that*

$$T_h(\boldsymbol{u}_h, p_h) = (\boldsymbol{u}_h, p_h). \tag{5.38}$$

As in the continuous case, the following preliminary results will allow us to prove that the discrete operator T_h has a unique fixed point.

Lemma 13. *Assume that $\boldsymbol{w}_{h,NS} \in \boldsymbol{H}_h^{NS}$ satisfies*

$$\|\boldsymbol{w}_{h,NS} \cdot \boldsymbol{n}_{NS}\|_{0,\Gamma} \le \frac{2\,\nu\, C_{\rm Korn}^2}{C_{\rm iS}^2 C_{\rm tr}^2}, \tag{5.39}$$

and that the parameter β satisfies

$$\beta < \frac{C_I}{2}\, \min\left\{\frac{C_{\rm Korn}^2}{2}, 1\right\}.$$

Then, for sufficiently small $h > 0$, there exist positive constants C_1 and C_2, independent of h, such that

$$\begin{aligned} \mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{v}_h, q_h; \boldsymbol{v}_h, q_h) \ge{}& \nu\kappa\, C_1\, \|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}^2 + \frac{\nu}{2}\, \|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\, |q_{h,D}|_{1,\Omega_D}^2 \\ &+ C_2\, \kappa\, \beta \sum_{K \in \mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\, |q_{h,NS}|_{1,K}^2, \end{aligned} \tag{5.40}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$.

Proof. Let $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$ and $\boldsymbol{w}_{h,NS} \in \boldsymbol{H}_h^{NS}$ that satisfies (5.39). Then, from the definition of the bilinear form $\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\cdot,\cdot)$, using Cauchy-Schwarz inequality and the equality (2.8), we have

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{v}_h, q_h; \boldsymbol{v}_h, q_h)\\
&\geq 2\nu\kappa \,\|\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,\Omega_{NS}}^2 + \frac{\nu}{2}\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_{h,D}|_{1,\Omega_D}^2 + \frac{\kappa}{2}\int_\Gamma (\boldsymbol{w}_{h,NS}\cdot\boldsymbol{n}_{NS})|\boldsymbol{v}_{h,NS}|^2\\
&\quad + \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\Bigg\{ -4\nu^2\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,K}^2 + \|(\nabla\boldsymbol{v}_{h,NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 + |q_{h,NS}|_{1,K}^2\\
&\quad + 2\left((\nabla\boldsymbol{v}_{h,NS})\boldsymbol{w}_{h,NS}, \nabla q_{h,NS}\right)_K\Bigg\}.
\end{aligned} \tag{5.41}$$

Hence, using inverse inequality (5.32), Lemma 1 and Young's inequality, we get

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{v}_h, q_h; \boldsymbol{v}_h, q_h)\\
&\geq 2\nu\kappa \,\|\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,\Omega_{NS}}^2 + \frac{\nu}{2}\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_{h,D}|_{1,\Omega_D}^2 - \frac{\kappa}{2}\left|\int_\Gamma (\boldsymbol{w}_{h,NS}\cdot\boldsymbol{n}_{NS})|\boldsymbol{v}_{NS}|^2\right|\\
&\quad + \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\left\{ -4\nu^2\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,K}^2 - \|(\nabla\boldsymbol{v}_{h,NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 + \frac{1}{2}|q_{h,NS}|_{1,K}^2\right\}\\
&\geq 2\nu\kappa \,\|\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,\Omega_{NS}}^2 + \frac{\nu}{2}\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_{h,D}|_{1,\Omega_D}^2 - \frac{\kappa}{2}\left|\int_\Gamma (\boldsymbol{w}_{h,NS}\cdot\boldsymbol{n}_{NS})|\boldsymbol{v}_{h,NS}|^2\right|\\
&\quad - \frac{4\nu\kappa\beta}{C_I}\,\|\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})\|_{0,\Omega_{NS}}^2 - \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,|\boldsymbol{v}_{h,NS}|_{1,K}^2\|\boldsymbol{w}_{h,NS}\|_{\infty,K}^2 + \frac{1}{2}\,\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}|q_{h,NS}|_{1,K}^2\\
&\geq 2C_{\text{Korn}}^2\nu\kappa\left(1-\frac{2\beta}{C_I}\right)\|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}^2 + \frac{\nu}{2}\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_{h,D}|_{1,\Omega_D}^2 - \frac{\kappa}{2}\left|\int_\Gamma (\boldsymbol{w}_{h,NS}\cdot\boldsymbol{n}_{NS})|\boldsymbol{v}_{h,NS}|^2\right|\\
&\quad - \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K}{\nu} C_{\text{inf}}^2\,|\boldsymbol{v}_{h,NS}|_{1,K}^2|\boldsymbol{w}_{h,NS}|_{1,\Omega_{NS}}^2 + \frac{1}{2}\,\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}|q_{h,NS}|_{1,K}^2.
\end{aligned}$$

Finally, under the assumption (5.39) and the estimate (3.13), we obtain

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{v}_h, q_h; \boldsymbol{v}_h, q_h)\\
&\geq \left(2\nu\kappa\, C_{\text{Korn}}^2 - \frac{4\nu\kappa\beta}{C_I} - \frac{\kappa C_{tr}^2 C_{\text{iS}}^2}{2}\,\|\boldsymbol{w}_{h,NS}\cdot\boldsymbol{n}_{NS}\|_{0,\Gamma} - \kappa\,\beta\,\frac{h}{\nu}\,C_{\text{inf}}^2\,|\boldsymbol{w}_{h,NS}|_{1,\Omega_{NS}}^2\right)\|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}^2\\
&\quad + \frac{\nu}{2}\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_{h,D}|_{1,\Omega_D}^2 + \frac{1}{2}\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,|q_{h,NS}|_{1,K}^2\\
&\geq \nu\kappa\left(C_{\text{Korn}}^2 - \frac{4\beta}{C_I} - \beta\,\frac{h}{\nu^2}\,C_{\text{inf}}^2\,|\boldsymbol{w}_{h,NS}|_{1,\Omega_{NS}}^2\right)\|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}^2\\
&\quad + \frac{\nu}{2}\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_{h,D}|_{1,\Omega_D}^2 + \frac{1}{2}\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,|q_{h,NS}|_{1,K}^2\\
&\geq \nu\kappa\, C_1\,\|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}^2 + \frac{\nu}{2}\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2 + \frac{\kappa^2}{2\nu}\,|q_{h,D}|_{1,\Omega_D}^2 + C_2\,\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,|q_{h,NS}|_{1,K}^2,
\end{aligned}$$

assuming h that is sufficiently small. □

As is well known, the subspaces $\boldsymbol{H}_h^{NS}$ and Q_h^{NS} do not satisfy a discrete inf–sup condition; however, they satisfy the following weak inf–sup condition.

Lemma 14. *There exist positive constants C_3 and C_4, independent of ν and h, such that*

$$\sup_{\boldsymbol{v}_{h,NS}\in \boldsymbol{H}_h^{NS}\cap H_0^1(\Omega_{NS})^d} \frac{(q_{h,NS}, \nabla\cdot \boldsymbol{v}_{h,NS})_{\Omega_{NS}}}{\|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}} \geq C_3\,\|q_{h,NS}\|_{0,\Omega_{NS}} - C_4 \left\{ \beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2\, |q_{h,NS}|_{1,K}^2 \right\}^{1/2}, \tag{5.42}$$

for all $q_{h,NS}\in Q_h^{NS}$.

Proof. See [24, Lemma 3.3] or [4, Lemma 2]. □

The following result will be necessary to prove the well-posedness of the discrete stabilized scheme. The proof is based on similar arguments to those used in [4, Lemma 3].

Lemma 15. *Assume the hypothesis of Lemma 13. Then, for $h>0$ sufficiently small, there exists a positive constant $C_{\mathcal{A}_h}$, independent of h, such that*

$$\sup_{(\boldsymbol{v}_h,q_h)\in \boldsymbol{H}_h\times Q_h} \frac{\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h,p_h;\boldsymbol{v}_h,q_h)}{|||(\boldsymbol{v}_h,q_h)|||_h} \geq C_{\mathcal{A}_h}\, |||(\boldsymbol{u}_h,p_h)|||_h,$$

for all $(\boldsymbol{u}_h,p_h)\in \boldsymbol{H}_h\times Q_h$.

Proof. Let $\tilde{\boldsymbol{w}}_{NS}\in \boldsymbol{H}_h^{NS}\cap H_0^1(\Omega_{NS})^d$ such that satisfies Lemma 14 and suppose that $\|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} = \|p_{h,NS}\|_{0,\Omega_{NS}}$. Then

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h,p_h;-\tilde{\boldsymbol{w}}_{NS},\mathbf{0},0,0)\\
=&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h,0;-\tilde{\boldsymbol{w}}_{NS},\mathbf{0},0,0) + \mathcal{A}_{\boldsymbol{w}_{h,NS}}(\mathbf{0},p_h;-\tilde{\boldsymbol{w}}_{NS},\mathbf{0},0,0)\\
=&-2\nu\kappa\,(\varepsilon(\boldsymbol{u}_{h,NS}),\varepsilon(\tilde{\boldsymbol{w}}_{NS}))_{\Omega_{NS}} - \kappa\,((\nabla\boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS},\tilde{\boldsymbol{w}}_{NS})_{\Omega_{NS}} - \frac{\kappa}{2}\,((\nabla\cdot\boldsymbol{w}_{h,NS})\,\boldsymbol{u}_{h,NS},\tilde{\boldsymbol{w}}_{NS})_{\Omega_{NS}} +\\
&\kappa\,(p_{h,NS},\nabla\cdot\tilde{\boldsymbol{w}}_{NS})_{\Omega_{NS}} + 4\nu\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \Bigg(\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,NS}),\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{w}}_{NS})\Bigg)_K\\
&+2\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \Bigg(\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,NS}),(\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS}\Bigg)_K\\
&-\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Bigg((\nabla\boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS} + \nabla p_{h,NS}, 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{w}}_{NS}) + (\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS}\Bigg)_K\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{5.43}$$

Using Cauchy–Schwarz inequality, the inverse inequality (5.32) and the compact inclusion (2.7), we deduce that the first five terms of (5.43) can be estimated from below as follows

$$\begin{aligned}
I_1 + I_2 \geq& -2\nu\kappa\left[1 + \frac{C_q^2}{\nu}\,\|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}} + \frac{2\beta}{C_I}\right] \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} + \kappa\,(p_{h,NS},\nabla\cdot\tilde{\boldsymbol{w}}_{NS})_{\Omega_{NS}}\\
\geq& -\nu\kappa M_1\,\|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}}\|p_{h,NS}\|_{0,\Omega_{NS}} + \kappa\,(p_{h,NS},\nabla\cdot\tilde{\boldsymbol{w}}_{NS})_{\Omega_{NS}},
\end{aligned} \tag{5.44}$$

where M_1 is a positive constant that depend on $\|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}}$ which is bounded.
On the other hand, it is straightforward to verify that, by applying Hölder's inequality and (5.35), the following holds

$$\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|(\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 \leq \beta\, C_{\inf} h\, \|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}}^2, \tag{5.45}$$

$$\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|(\nabla\boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 \leq \beta\, C_{\inf} h\, \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}}^2. \tag{5.46}$$

Since that $\beta \leq C_I/2$, we conclude by the inverse estimate (5.32) that

$$\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{w}}_{NS})\|_{0,K}^2 \leq \frac{1}{2}\,\|\boldsymbol{\varepsilon}(\tilde{\boldsymbol{w}}_{NS})\|_{0,\Omega_{NS}}^2. \tag{5.47}$$

Then applying Cauchy–Schwarz inequality and (5.45) and (5.47), we can estimate from below I_3 as follows

$$\begin{aligned}
I_3 &= 2\kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \left(\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,NS}), (\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS}\right)_K \\
&\geq -\,2\kappa \left\{\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{h,NS})\|_{0,K}^2\right\}^{1/2} \left\{\beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|(\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2\right\}^{1/2} \\
&\geq -\,2\kappa\beta^{1/2}\, C_{\inf} h^{1/2}\, \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}} \|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} \\
&\geq -\,\nu\kappa M_2 h^{1/2}\, \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|p_{h,NS}\|_{0,\Omega_{NS}},
\end{aligned} \tag{5.48}$$

where M_2 is a positive constant that depend on $\|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}}$.
Employing similar arguments as above, but this time utilizing (5.45)-(5.47), we can estimate I_4 as

follows

$$
\begin{aligned}
I_4 = &- \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Bigg((\nabla \boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS} + \nabla p_{h,NS}, 2\nu\, \nabla\cdot \boldsymbol{\varepsilon}(\tilde{\boldsymbol{w}}_{NS}) + (\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS} \Bigg)_K \\
\geq &- 2\kappa \left\{ \beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|(\nabla \boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 \right\}^{1/2} \left\{ \beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{w}}_{NS})\|_{0,K}^2 \right\}^{1/2} - \\
&\frac{\kappa}{\nu} \left\{ \beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|(\nabla \boldsymbol{u}_{h,NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 \right\}^{1/2} \left\{ \beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|(\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 \right\}^{1/2} - \\
&2\kappa^{1/2}\,\nu^{1/2} \left\{ \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \,|p_{h,NS}|_{1,K}^2 \right\}^{1/2} \left\{ \beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{w}}_{NS})\|_{0,K}^2 \right\}^{1/2} - \\
&\frac{\kappa^{1/2}}{\nu^{1/2}} \left\{ \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \,|p_{h,NS}|_{1,K}^2 \right\}^{1/2} \left\{ \beta \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \,\|(\nabla\tilde{\boldsymbol{w}}_{NS})\boldsymbol{w}_{h,NS}\|_{0,K}^2 \right\}^{1/2} \\
\geq &- \kappa\beta^{1/2}\, C_{\inf} h^{1/2}\, \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}} \left[2\,\|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} + \frac{\beta^{1/2}}{\nu}\, C_{\inf} h^{1/2}\, \|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}} \|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} \right] - \\
&\kappa^{1/2} \left\{ \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \,|p_{h,NS}|_{1,K}^2 \right\}^{1/2} \left[2\,\|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} + \frac{\beta^{1/2}}{\nu^{1/2}}\, C_{\inf} h^{1/2}\, \|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}} \|\tilde{\boldsymbol{w}}_{NS}\|_{1,\Omega_{NS}} \right] \\
\geq &- \nu\kappa M_3 h^{1/2}\, \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|p_{h,NS}\|_{0,\Omega_{NS}} - M_4 \left\{ \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \,|p_{h,NS}|_{1,K}^2 \right\}^{1/2} \|p_{h,NS}\|_{0,\Omega_{NS}},
\end{aligned}
\tag{5.49}
$$

where M_3 and M_4 are positive constants that depend on $\|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}}$. Now, by connecting (5.44), (5.48), and (5.49) with (5.43) and applying Lemma 14, we obtain

$$
\begin{aligned}
&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h, p_h; -\tilde{\boldsymbol{w}}_{NS}, \mathbf{0}, 0, 0) \\
&\geq - \nu\kappa \left[M_1 + h^{1/2}(M_2 + M_3) \right] \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|p_{h,NS}\|_{0,\Omega_{NS}} + C_3\,\kappa\, \|p_{h,NS}\|_{0,\Omega_{NS}}^2 - \\
&\quad (M_4 + C_4 \kappa^{1/2}\nu^{1/2}) \left\{ \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \,|p_{h,NS}|_{1,K}^2 \right\}^{1/2} \|p_{h,NS}\|_{0,\Omega_{NS}} \\
&\geq - \nu\kappa\, M_5\, \|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \|p_{h,NS}\|_{0,\Omega_{NS}} + C_3\,\kappa\, \|p_{h,NS}\|_{0,\Omega_{NS}}^2 \\
&\quad - M_6 \left\{ \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \,|p_{h,NS}|_{1,K}^2 \right\}^{1/2} \|p_{h,NS}\|_{0,\Omega_{NS}},
\end{aligned}
\tag{5.50}
$$

where M_5 and M_6 are positives constants that depend of $\|\boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}}$, which is bounded. Now,

DOI 10.20347/WIAS.PREPRINT.3183

using Young inequality, we have

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h, p_h; -\tilde{\boldsymbol{w}}_{NS}, \mathbf{0}, 0, 0)\\
&\geq \kappa \left[C_3 - \frac{\nu M_5 \gamma_1}{2} - M_6 \frac{\gamma_2}{2\kappa}\right] \|p_{h,NS}\|^2_{0,\Omega_{NS}} - \nu\kappa \frac{M_5}{2\gamma_1} \|\boldsymbol{u}_{h,NS}\|^2_{1,\Omega_{NS}} - \frac{M_6}{2\gamma_2} \left\{\kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} |p_{h,NS}|^2_{1,K}\right\}\\
&\geq \kappa\, C_5 \|p_{h,NS}\|^2_{0,\Omega_{NS}} - C_6\, \nu\kappa \|\boldsymbol{u}_{h,NS}\|^2_{1,\Omega_{NS}} - C_7 \left\{\kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} |p_{h,NS}|^2_{1,K}\right\}, \qquad (5.51)
\end{aligned}$$

with γ_1 and γ_2 chosen small enough.

Defining $(\boldsymbol{v}_h, q_h) := (\boldsymbol{u}_h, p_h) + \delta\,((-\tilde{\boldsymbol{w}}_{NS}, \mathbf{0}), (0, 0))$, and combining, (5.40) and (5.51), we get

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = \mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h, p_h; \boldsymbol{u}_h, q_h) + \delta\, \mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h, p_h; -\tilde{\boldsymbol{w}}_{NS}, \mathbf{0}, 0, 0)\\
&\geq \nu\kappa\,(C_1 - \delta C_6)\, \|\boldsymbol{u}_{h,NS}\|^2_{1,\Omega_{NS}} + C_5 \delta\, \kappa\, \|p_{h,NS}\|^2_{0,\Omega_{NS}} + (C_2 - \delta\, C_7)\, \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} |p_{h,NS}|^2_{1,K}\\
&\quad + \frac{\nu}{2} \|\boldsymbol{u}_{h,D}\|^2_{0,\Omega_D} + \frac{\kappa^2}{2\nu} |p_{h,D}|^2_{1,\Omega_D}\\
&\geq C\, |||(\boldsymbol{u}_h, p_h)|||^2_h, \qquad (5.52)
\end{aligned}$$

choosing $0 < \delta < \min\left\{\frac{C_1}{C_6}, \frac{C_2}{C_7}, \frac{1}{\nu^{1/2}}\right\}$. Finally, it is clear that $|||(\boldsymbol{v}_h, q_h)|||_h \leq C\, |||(\boldsymbol{u}_h, p_h)|||_h$, which combined with (5.52) proves the stability estimate. □

Theorem 16. *Assume the hypothesis of Lemma 15. Then, for $h > 0$ sufficiently small, the problem* (5.37) *has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$. Moreover, there exists a positive constant $C_{\rm dcd}$, which depends on the physical parameters but is independent of h, such that*

$$|||(\boldsymbol{u}_h, p_h)|||_h \leq C_{\rm dcd} \left\{\|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D}\right\}.$$

Proof. The result follows directly from Lemma 15. □

5.2 Existence and uniqueness of the discrete solution

As in the continuous case, we assume that

$$C_{\rm dcd}\, \{\|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D}\}\, \frac{2\,\nu^{3/2} \kappa^{1/2}\, C^2_{\rm Korn}}{C^2_{\rm iS} C^3_{\rm tr}}. \qquad (5.53)$$

Lemma 17. *For $h > 0$ sufficiently small, let $\boldsymbol{X}_h$ the closed and convex subset of $\boldsymbol{H} \times Q$ defined by*

$$\boldsymbol{X}_h := \left\{(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h :\ |||(\boldsymbol{v}_h, q_h)|||_h \leq C_{\rm dcd}\, \{\|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D}\}\right\}.$$

Then $T_h(\boldsymbol{X}_h) \subseteq \boldsymbol{X}_h$.

Proof. The argument follows analogous to the continuous case (c.f. Lemma 8). □

Lemma 18. *Assume that $h > 0$ is sufficiently small. Then, the following inequality holds:*

$$
\begin{aligned}
&\|T_h(\boldsymbol{w}_h, q_h) - T_h(\tilde{\boldsymbol{w}}_h, \tilde{q}_h)\|_h \\
&\le \frac{1}{C_{\mathcal{A}_h}} \left\{ C_{\inf}\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^{3/2}}{\nu^2}\, \|\boldsymbol{f}_{NS}\|_{0,K} + \frac{3C_q^2}{\nu^{3/2}\kappa^{1/2}}\, \|T_h(\tilde{\boldsymbol{w}}_h, \tilde{q}_h)\|_h \right\} \|(\boldsymbol{w}_h - \tilde{\boldsymbol{w}}_h, q_h - \tilde{q}_h)\|_h,
\end{aligned}
$$

for all $(\boldsymbol{w}_h, q_h), (\tilde{\boldsymbol{w}}_h, \tilde{q}_h) \in \boldsymbol{X}_h$.

Proof. Let $(\boldsymbol{w}_h, q_h), (\tilde{\boldsymbol{w}}_h, \tilde{q}_h) \in \boldsymbol{X}_h$ be arbitrary, and define $(\boldsymbol{u}_h, p_h) := T_h(\boldsymbol{w}_h, q_h)$ and $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) := T_h(\tilde{\boldsymbol{w}}_h, \tilde{q}_h)$, which satisfy the following conditions:

$$
\begin{aligned}
\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, r_h) &= F_{\boldsymbol{w}_{h,NS}}(\boldsymbol{v}_h, r_h), \\
\mathcal{A}_{\tilde{\boldsymbol{w}}_{h,NS}}(\tilde{\boldsymbol{u}}_h, \tilde{p}_h; \boldsymbol{v}_h, r_h) &= F_{\tilde{\boldsymbol{w}}_{h,NS}}(\boldsymbol{v}_h, r_h),
\end{aligned}
$$

for all $(\boldsymbol{v}_h, r_h) \in \boldsymbol{H}_h \times Q_h$.

Since the objective is to apply the discrete inf-sup condition of Lemma 15, it is easy to see that

$$
\mathcal{A}_{\boldsymbol{w}_{h,NS}}(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h, p_h - \tilde{p}_h; \boldsymbol{v}_h, r_h) = (F_{\boldsymbol{w}_{h,NS}} - F_{\tilde{\boldsymbol{w}}_{h,NS}})(\boldsymbol{v}_h, r_h) + (\mathcal{A}_{\tilde{\boldsymbol{w}}_{h,NS}} - \mathcal{A}_{\boldsymbol{w}_{h,NS}})(\tilde{\boldsymbol{u}}_h, \tilde{p}_h; \boldsymbol{v}_h, r_h). \tag{5.54}
$$

Thus, both terms on the right-hand side of (5.54) will be bounded below. In fact, from the definition of $F_{\boldsymbol{w}_{h,NS}}$ and $F_{\tilde{\boldsymbol{w}}_{h,NS}}$ (cf. (5.31)), Hölder's inequality and the inverse inequality (5.35), we get that

$$
\begin{aligned}
&(F_{\boldsymbol{w}_{h,NS}} - F_{\tilde{\boldsymbol{w}}_{h,NS}})(\boldsymbol{v}_h, r_h) \\
&= \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \left(\boldsymbol{f}_{NS}, 2\nu\, \nabla\cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{w}_{h,NS} + \nabla r_{h,NS} \right)_K \\
&\quad - \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \left(\boldsymbol{f}_{NS}, 2\nu\, \nabla\cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\tilde{\boldsymbol{w}}_{h,NS} + \nabla r_{h,NS} \right)_K \\
&= \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \left(\boldsymbol{f}_{NS}, (\nabla \boldsymbol{v}_{h,NS})(\boldsymbol{w}_{h,NS} - \tilde{\boldsymbol{w}}_{h,NS}) \right)_K \\
&\le \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\, \|\boldsymbol{f}_{NS}\|_{0,K} |\boldsymbol{v}_{h,NS}|_{1,K} \|\boldsymbol{w}_{h,NS} - \tilde{\boldsymbol{w}}_{h,NS}\|_{\infty,K} \\
&\le C_{\inf}\, \kappa\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^{3/2}}{\nu}\, \|\boldsymbol{f}_{NS}\|_{0,K} |\boldsymbol{v}_{h,NS}|_{1,K} |\boldsymbol{w}_{h,NS} - \tilde{\boldsymbol{w}}_{h,NS}|_{1,\Omega_{NS}} \\
&\le C_{\inf}\, \kappa^{1/2}\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^{3/2}}{\nu^{3/2}}\, \|\boldsymbol{f}_{NS}\|_{0,K} \|(\boldsymbol{v}_h, r_h)\|_h |\boldsymbol{w}_{h,NS} - \tilde{\boldsymbol{w}}_{h,NS}|_{1,\Omega_{NS}} \\
&\le C_{\inf}\, \beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^{3/2}}{\nu^2}\, \|\boldsymbol{f}_{NS}\|_{0,K} \|(\boldsymbol{v}_h, r_h)\|_h \|(\boldsymbol{w}_h - \tilde{\boldsymbol{w}}_h, q_h - \tilde{q}_h)\|_h.
\end{aligned} \tag{5.55}
$$

On the other hand, to bound the second term in (5.54), we use the definitions of $\mathcal{A}_{\tilde{\boldsymbol{w}}_{h,NS}}$ and $\mathcal{A}_{\boldsymbol{w}_{h,NS}}$ (cf. (5.30)) to decompose the expression as follows

$$\begin{aligned}
&(\mathcal{A}_{\tilde{\boldsymbol{w}}_{h,NS}} - \mathcal{A}_{\boldsymbol{w}_{h,NS}})(\tilde{\boldsymbol{u}}_h, \tilde{p}_h; \boldsymbol{v}_h, r_h) \\
=&\kappa\, \big((\nabla\tilde{\boldsymbol{u}}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS}), \boldsymbol{v}_{h,NS}\big)_{\Omega_{NS}} + \frac{\kappa}{2}\, \big(\nabla\cdot(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS})\,\tilde{\boldsymbol{u}}_{h,NS}, \boldsymbol{v}_{h,NS}\big)_{\Omega_{NS}} \\
&+ \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Big(-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS}) + \nabla\tilde{p}_{h,NS}, (\nabla\boldsymbol{v}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS})\Big)_K \\
&+ \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Big((\nabla\tilde{\boldsymbol{u}}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS}), 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + \nabla q_{h,NS}\Big)_K \\
&+ \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Big((\nabla\tilde{\boldsymbol{u}}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS}), (\nabla\boldsymbol{v}_{h,NS})\tilde{\boldsymbol{w}}_{h,NS}\Big)_K \\
&+ \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Big((\nabla\tilde{\boldsymbol{u}}_{h,NS})\boldsymbol{w}_{h,NS}, (\nabla\boldsymbol{v}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS})\Big)_K \\
=&I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{5.56}$$

To estimate the first term, we use Hölder's inequality and (2.7), to get

$$\begin{aligned}
I_1 &= \kappa\, \big((\nabla\tilde{\boldsymbol{u}}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS}), \boldsymbol{v}_{h,NS}\big)_{\Omega_{NS}} + \frac{\kappa}{2}\, \big(\nabla\cdot(\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS})\,\tilde{\boldsymbol{u}}_{h,NS}, \boldsymbol{v}_{h,NS}\big)_{\Omega_{NS}} \\
&\le \kappa\,|\tilde{\boldsymbol{u}}_{h,NS}|_{1,\Omega_{NS}}\|\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS}\|_{0,4,\Omega_{NS}}\|\boldsymbol{v}_{h,NS}\|_{0,4,\Omega_{NS}} \\
&\quad + \frac{\kappa\sqrt{d}}{2}\,|\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS}|_{1,\Omega_{NS}}\|\tilde{\boldsymbol{u}}_{h,NS}\|_{0,4,\Omega_{NS}}\|\boldsymbol{v}_{h,NS}\|_{0,4,\Omega_{NS}} \qquad (5.57)\\
&\le \kappa\, C_q^2\left(1 + \frac{\sqrt{d}}{2}\right)\, \|\tilde{\boldsymbol{u}}_{h,NS}\|_{1,\Omega_{NS}}\|\tilde{\boldsymbol{w}}_{h,NS} - \boldsymbol{w}_{h,NS}\|_{1,\Omega_{NS}}\|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}} \\
&\le \frac{2C_q^2}{\nu^{3/2}\kappa^{1/2}}\, |\!|\!|(\tilde{\boldsymbol{u}}_h, \tilde{p}_h)|\!|\!|_h |\!|\!|(\boldsymbol{w}_h - \tilde{\boldsymbol{w}}_h, q_h - \tilde{q}_h)|\!|\!|_h |\!|\!|(\boldsymbol{v}_h, r_h)|\!|\!|_h. \qquad (5.58)
\end{aligned}$$

Using Cauchy-Schwarz and Hölder's inequalities together with (5.35), we obtain

$$
\begin{aligned}
I_2 &= \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\Bigg(-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS}) + \nabla\tilde{p}_{h,NS}, (\nabla\boldsymbol{v}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS})\Bigg)_K \\
&\le \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS}) + \nabla\tilde{p}_{h,NS}\|_{0,K}|\boldsymbol{v}_{h,NS}|_{1,K}\|\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}\|_{\infty,K} \\
&\le \frac{\kappa\beta}{\nu^{1/2}}\left\{\sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS}) + \nabla\tilde{p}_{h,NS}\|_{0,K}^2\right\}^{1/2} \\
&\quad\times\left\{\sum_{K\in\mathcal{T}_h^{NS}} h_K^2|\boldsymbol{v}_{h,NS}|_{1,K}^2\|\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}\|_{\infty,K}^2\right\}^{1/2} \qquad (5.59)\\
&\le \frac{C_{\inf}\kappa\beta}{\nu^{1/2}}\left\{\sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS}) + \nabla\tilde{p}_{h,NS}\|_{0,K}^2\right\}^{1/2} \\
&\quad\times\left\{|\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}|_{1,\Omega_{NS}}^2\sum_{K\in\mathcal{T}_h^{NS}} h_K|\boldsymbol{v}_{h,NS}|_{1,K}^2\right\}^{1/2} \qquad (5.60)\\
&\le \frac{C_{\inf}\kappa^{1/2}\beta^{1/2}h^{1/2}}{\nu^{1/2}}\left\{\kappa\beta\sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS}) + \nabla\tilde{p}_{h,NS}\|_{0,K}^2\right\}^{1/2} \qquad (5.61)\\
&\quad\times|\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}|_{1,\Omega_{NS}}|\boldsymbol{v}_{h,NS}|_{1,\Omega_{NS}}. \qquad (5.62)
\end{aligned}
$$

Now, from triangle inequality, the inverse inequality (5.32) and the definition of the norm $|\!|\!|(\cdot,\cdot)|\!|\!|_h$, it is clear that

$$
\begin{aligned}
&\kappa\,\beta\sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS}) + \nabla\tilde{p}_{h,NS}\|_{0,K}^2 \\
&\le 8\kappa\nu\,\beta\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS})\|_{0,K}^2 + 2\kappa\,\beta\sum_{K\in\mathcal{T}_h^{NS}}\frac{h_K^2}{\nu}\,|\tilde{p}_{h,NS}|_{1,K}^2 \\
&\le 4\kappa\nu\,\|\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_{h,NS})\|_{0,\Omega_{NS}}^2 + 2\kappa\,\beta\sum_{K\in\mathcal{T}_h^{NS}}\frac{h_K^2}{\nu}\,|\tilde{p}_{h,NS}|_{1,K}^2 \\
&\le 4\,|\!|\!|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)|\!|\!|_h^2. \qquad (5.63)
\end{aligned}
$$

Next, from (5.62) and (5.63), we conclude that

$$
I_2 \le \frac{2C_{\inf}\beta^{1/2}h^{1/2}}{\nu^{3/2}\kappa^{1/2}}|\!|\!|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)|\!|\!|_h|\!|\!|(\boldsymbol{v}_h,r_h)|\!|\!|_h|\!|\!|(\boldsymbol{w}_h-\tilde{\boldsymbol{w}}_h,q_h-\tilde{q}_h)|\!|\!|_h. \qquad (5.64)
$$

To estimate I_3, we use similar arguments thus we have

$$
\begin{aligned}
I_3 &= \kappa\,\beta\sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\Bigg((\nabla\tilde{\boldsymbol{u}}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}), 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + \nabla q_{h,NS}\Bigg)_K \\
&\le \frac{2C_{\inf}\beta^{1/2}h^{1/2}}{\nu^{3/2}\kappa^{1/2}}|\!|\!|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)|\!|\!|_h|\!|\!|(\boldsymbol{v}_h,r_h)|\!|\!|_h|\!|\!|(\boldsymbol{w}_h-\tilde{\boldsymbol{w}}_h,q_h-\tilde{q}_h)|\!|\!|_h. \qquad (5.65)
\end{aligned}
$$

To estimate I_4, we again employ similar arguments, leading to:

$$\begin{aligned}
I_4 &= \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \left((\nabla\tilde{\boldsymbol{u}}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}), (\nabla\boldsymbol{v}_{h,NS})\tilde{\boldsymbol{w}}_{h,NS} \right)_K \\
&\leq \kappa\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} |\tilde{\boldsymbol{u}}_{h,NS}|_{1,K} \|\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}\|_{\infty,K} |\boldsymbol{v}_{h,NS}|_{1,K} \|\tilde{\boldsymbol{w}}_{h,NS}\|_{\infty,K} \\
&\leq \frac{\kappa\beta}{\nu} \left\{ \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \, |\tilde{\boldsymbol{u}}_{h,NS}|_{1,K}^2 \|\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}\|_{\infty,K}^2 \right\}^{1/2} \left\{ \sum_{K\in\mathcal{T}_h^{NS}} h_K^2 \, |\boldsymbol{v}_{h,NS}|_{1,K}^2 \|\tilde{\boldsymbol{w}}_{h,NS}\|_{\infty,K}^2 \right\}^{1/2} \\
&\leq \frac{C_{\inf}^2\kappa\beta}{\nu} \left\{ |\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}|_{1,\Omega_{NS}}^2 \sum_{K\in\mathcal{T}_h^{NS}} h_K \, |\tilde{\boldsymbol{u}}_{h,NS}|_{1,K}^2 \right\}^{1/2} \left\{ |\tilde{\boldsymbol{w}}_{h,NS}|_{1,\Omega_{NS}}^2 \sum_{K\in\mathcal{T}_h^{NS}} h_K \, |\boldsymbol{v}_{h,NS}|_{1,K}^2 \right\}^{1/2} \\
&\leq \frac{C_{\inf}^2\beta h}{\nu^3\kappa} \|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\|_h \|(\tilde{\boldsymbol{w}}_h,\tilde{q}_h)\|_h \|(\boldsymbol{v}_h,r_h)\|_h \|(\boldsymbol{w}_h-\tilde{\boldsymbol{w}}_h,q_h-\tilde{q}_h)\|_h. \qquad (5.66)
\end{aligned}$$

The same reasoning can be applied to the last term, resulting in

$$\begin{aligned}
I_5 &= \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \left((\nabla\tilde{\boldsymbol{u}}_{h,NS})\boldsymbol{w}_{h,NS}, (\nabla\boldsymbol{v}_{h,NS})(\tilde{\boldsymbol{w}}_{h,NS}-\boldsymbol{w}_{h,NS}) \right)_K \\
&\leq \frac{C_{\inf}^2\beta h}{\nu^3\kappa} \|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\|_h \|(\boldsymbol{w}_h,q_h)\|_h \|(\boldsymbol{v}_h,r_h)\|_h \|(\boldsymbol{w}_h-\tilde{\boldsymbol{w}}_h,q_h-\tilde{q}_h)\|_h. \qquad (5.67)
\end{aligned}$$

Finally, since $(\boldsymbol{w}_h, q_h) \in \boldsymbol{X}_h$ and assuming that the data condition (5.53) holds, the hypotheses of Lemma 15 are satisfied. Combining these with (5.54)–(5.58) and (5.64)–(5.67), we conclude that

$$\begin{aligned}
& C_{\mathcal{A}_h} \|(\boldsymbol{u}_h-\tilde{\boldsymbol{u}}_h, p_h-\tilde{p}_h)\|_h \\
&\leq \Bigg\{ C_{\inf}\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^{3/2}}{\nu^2} \|\boldsymbol{f}_{NS}\|_{0,K} + \frac{2C_q^2}{\nu^{3/2}\kappa^{1/2}} \|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\|_h + \frac{4C_{\inf}\beta^{1/2}h^{1/2}}{\nu^{3/2}\kappa^{1/2}} \|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\|_h \\
&\quad + \frac{C_{\inf}^2\beta h}{\nu^3\kappa} \|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\|_h \Big[\|(\tilde{\boldsymbol{w}}_h,\tilde{q}_h)\|_h + \|(\boldsymbol{w}_h,q_h)\|_h \Big] \Bigg\} \|(\boldsymbol{w}_h-\tilde{\boldsymbol{w}}_h,q_h-\tilde{q}_h)\|_h \\
&\leq \Bigg\{ C_{\inf}\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^{3/2}}{\nu^2} \|\boldsymbol{f}_{NS}\|_{0,K} + \frac{3C_q^2}{\nu^{3/2}\kappa^{1/2}} \|(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\|_h \Bigg\} \|(\boldsymbol{w}_h-\tilde{\boldsymbol{w}}_h,q_h-\tilde{q}_h)\|_h, \qquad (5.68)
\end{aligned}$$

under the assumption that h is sufficiently small. □

Lemma 19. *Let $\boldsymbol{X}_h$ be as in Lemma 17. Under the assumption* (5.53) *and assuming that the stabilization parameter β satisfies the condition of Lemma 13, the stabilized finite element scheme* (5.29) *has at least one solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h$.*

Proof. From Lemmas 17 and 18, the operator $T_h : \boldsymbol{X}_h \longrightarrow \boldsymbol{X}_h$ is continuous. Then, by Brouwer's fixed-point theorem (cf. Theorem 4), it has at least one fixed point. □

Theorem 20. *Let $\boldsymbol{X}_h$ be as in Lemma 17. Under the assumption* (5.53) *and assuming that the stabilization parameter β satisfies the condition of Lemma 13, suppose also that the data $\boldsymbol{f}_{NS} \in L^2(\Omega_{NS})^d$*

and $g_D \in L^2(\Omega_D)$ satisfy the following condition:

$$\frac{1}{C_{\mathcal{A}_h}} \left\{ C_{\inf} \beta \sum_{K \in \mathcal{T}_h^{NS}} \frac{h_K^{3/2}}{\nu^2} \|\boldsymbol{f}_{NS}\|_{0,K} + \frac{3C_q^2}{\nu^{3/2}\kappa^{1/2}} C_{\mathrm{dcd}} \left(\|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right) \right\} < 1. \quad (5.69)$$

Then, for $h > 0$ sufficiently small, the stabilized finite element scheme (5.29) *has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h$.*

Proof. From Lemmas 17 and 18, and under the assumption (5.69), the mapping $T_h : \boldsymbol{X}_h \to \boldsymbol{X}_h$ is a contraction. Therefore, applying Banach's fixed-point theorem (cf. Theorem 2), the result follows. □

6 Convergence analysis of the stabilized scheme

In this section, we present an a priori error analysis for the stabilized finite element scheme (5.29). We consider $(\boldsymbol{u}, p) \in (\boldsymbol{H} \times Q) \cap \boldsymbol{X}$ as the unique solution of problem (2.3)–(2.4), or equivalently:

$$\mathcal{B}_{\boldsymbol{u}_{NS}}(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{G}(\boldsymbol{v}, q) \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{H} \times Q, \quad (6.70)$$

where, given $\boldsymbol{w}_{NS} \in \boldsymbol{H}^{NS}$ with $\nabla \cdot \boldsymbol{w}_{NS} = 0$ in Ω_{NS}, the bilinear form $\mathcal{B}_{\boldsymbol{w}_{NS}} : (\boldsymbol{H} \times Q) \times (\boldsymbol{H} \times Q) \longrightarrow \mathbb{R}$ and the functional $\mathcal{G} : \boldsymbol{H} \times Q \longrightarrow \mathbb{R}$ are defined as

$$\mathcal{B}_{\boldsymbol{w}_{NS}}(\boldsymbol{u}, p; \boldsymbol{v}, q) := A_{\boldsymbol{w}_{NS}}(\boldsymbol{u}, p_D; \boldsymbol{v}, q_D) + B(\boldsymbol{v}, q_D; p_{NS}) - B(\boldsymbol{u}, \bar{p}_D; q_{NS}), \quad (6.71)$$

for all $(\boldsymbol{u}, p), (\boldsymbol{v}, q) \in \boldsymbol{H} \times Q$, and

$$\mathcal{G}(\boldsymbol{v}, q) := \tilde{F}(\boldsymbol{v}, q_D), \quad (6.72)$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{H} \times Q$. Moreover, we assume that problem (6.70) has additional regularity, meaning that there exists a positive constant $\tilde{C}_{\mathrm{dc}}$ such that

$$\|\boldsymbol{u}_{NS}\|_{2,\Omega_{NS}} + \|p_{NS}\|_{0,\Omega_{NS}} + \|\boldsymbol{u}_D\|_{0,\Omega_D} + |p_D|_{1,\Omega_D} \le \tilde{C}_{\mathrm{dc}} \left\{ \|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D} \right\}. \quad (6.73)$$

Additionally, we recall a local trace theorem, as well as the definition and properties of some interpolation operators that will be used in the sequel.

Lemma 21. *There exists a positive constant C, independent of h_K, such that*

$$\|\psi\|_{0,\partial K}^2 \le C \left\{ h_K^{-1} \|\psi\|_{0,K}^2 + h_K |\psi|_{1,K}^2 \right\},$$

for all $K \in \mathcal{T}_h^{NS} \cup \mathcal{T}_h^D$ and all $\psi \in H^1(K)$.

Proof. See [1, Theorem 3.10] or [10, (10.3.8)]. □

We consider the Lagrange interpolation operator $\boldsymbol{\Pi}_h^{NS} : H^{k+1}(\Omega_{NS})^d \cap \boldsymbol{H}^{NS} \longrightarrow \boldsymbol{H}_h^{NS}$, and the Clément interpolation operator $\mathcal{C}_h^{NS} : H^k(\Omega) \longrightarrow Q_h^{NS}$, such that (see [21] for details), for all $K \in \mathcal{T}_h^{NS}$, we have

$$|\boldsymbol{u}_{NS} - \boldsymbol{\Pi}_h^{NS} \boldsymbol{u}_{NS}|_{l,K} \le C h_K^{s-l} |\boldsymbol{u}_{NS}|_{s,K}, \quad (6.74)$$

$$\|p_{NS} - \mathcal{C}_h^{NS} p_{NS}\|_{0,K} \le C h_K^{s} \|p_{NS}\|_{s,\tilde{\omega}_K^{NS}}, \quad (6.75)$$

for all $\boldsymbol{u}_{NS} \in H^s(K)^d$ and all $p_{NS} \in H^s(\tilde{\omega}_K^{NS})$, with $0 \leq l \leq 1$ and $1 \leq s \leq k+1$. Here, C is a positive constant independent of h. Similarly, we consider the Clément interpolation operator $\boldsymbol{\mathcal{C}}_h^D : H^k(\Omega_D)^d \longrightarrow \boldsymbol{H}_h^D$, and the Lagrange interpolation operator $\Pi_h^D : H^{k+1}(\Omega_D) \cap Q^D \longrightarrow Q_h^D$, such that for all $K \in \mathcal{T}_h^D$, we have

$$\|\boldsymbol{u}_D - \boldsymbol{\mathcal{C}}_h^D \boldsymbol{u}_D\|_{0,K} \leq C h_K^s \, \|\boldsymbol{u}_D\|_{s,\tilde{\omega}_K^D}, \tag{6.76}$$

$$|p_D - \Pi_h^D p_D|_{l,K} \leq C h_K^{s-l} \, |p_D|_{s,K}, \tag{6.77}$$

for all $\boldsymbol{u}_D \in H^s(\tilde{\omega}_K^D)^d$ and all $p_D \in H^s(K)$, with $0 \leq l \leq 1$ and $1 \leq s \leq k+1$. Again, C is a positive constant independent of h.

Lemma 22. *Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of* (6.70) *and* (5.29)*, respectively. Assume the regularity hypothesis* (6.73) *and that $(\boldsymbol{u}, p) \in \left[\left(H^{k+1}(\Omega_{NS})^d \cap \boldsymbol{H}^{NS}\right) \times H^k(\Omega_D)^d\right] \times \left[H^k(\Omega_{NS}) \times \left(H^{k+1}(\Omega_D) \cap Q^D\right)\right]$, with $(\boldsymbol{u}, p) \in \boldsymbol{X}$ and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h$. If we additionally assume that*

$$\frac{3 C_q^2 \tilde{C}_{\mathrm{dc}}}{\nu \kappa^2} \, \{\|\boldsymbol{f}_{NS}\|_{0,\Omega_{NS}} + \|g_D\|_{0,\Omega_D}\} < \frac{C_{\mathcal{A}_h}}{2}, \tag{6.78}$$

then

$$\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v}_h, q_h) \leq \frac{C_{\mathcal{A}_h}}{2} \, \nu^{1/2} \, \kappa^{1/2} \, \|\boldsymbol{u}_{NS} - \boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}} \, |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_h, \tag{6.79}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$.

Proof. Given $\boldsymbol{u}_{h,NS} \in \boldsymbol{H}_h^{NS}$, by definition of the bilinear form $\mathcal{A}_{\boldsymbol{u}_{h,NS}}$ (cf. (5.30)), and as $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ satisfies (6.70) and (5.29), respectively, we get

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v}_h, q_h) \\
&= \mathcal{B}_{\boldsymbol{u}_{NS}}(\boldsymbol{u}, p; \boldsymbol{v}_h, q_h) + \kappa\,((\nabla \boldsymbol{u}_{NS})(\boldsymbol{u}_{h,NS} - \boldsymbol{u}_{NS}), \boldsymbol{v}_{h,NS})_{\Omega_{NS}} + \frac{\kappa}{2}\,(\nabla \cdot (\boldsymbol{u}_{h,NS} - \boldsymbol{u}_{NS})\boldsymbol{u}_{NS}, \boldsymbol{v}_{h,NS})_{\Omega_{NS}} \\
&\quad - \mathcal{A}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + \kappa\,\beta \sum_{K \in \mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Bigg(- 2\nu\, \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_{NS}) \\
&\quad + (\nabla \boldsymbol{u}_{NS})\boldsymbol{u}_{h,NS} + \nabla p_{NS}, 2\nu\, \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS} + \nabla q_{h,NS} \Bigg)_K \\
&= \tilde{F}(\boldsymbol{v}_h, q_{h,D}) + \kappa\,((\nabla \boldsymbol{u}_{NS})(\boldsymbol{u}_{h,NS} - \boldsymbol{u}_{NS}), \boldsymbol{v}_{h,NS})_{\Omega_{NS}} \\
&\quad + \frac{\kappa}{2}\,(\nabla \cdot (\boldsymbol{u}_{h,NS} - \boldsymbol{u}_{NS})\boldsymbol{u}_{NS}, \boldsymbol{v}_{h,NS})_{\Omega_{NS}} - \mathcal{A}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + \qquad (6.80) \\
&\quad \kappa\,\beta \sum_{K \in \mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Bigg(\boldsymbol{f} - (\nabla \boldsymbol{u}_{NS})\boldsymbol{u}_{NS} + (\nabla \boldsymbol{u}_{NS})\boldsymbol{u}_{h,NS}, 2\nu\, \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS} + \nabla q_{h,NS} \Bigg)_K \\
&= \kappa\,((\nabla \boldsymbol{u}_{NS})(\boldsymbol{u}_{h,NS} - \boldsymbol{u}_{NS}), \boldsymbol{v}_{h,NS})_{\Omega_{NS}} + \frac{\kappa}{2}\,(\nabla \cdot (\boldsymbol{u}_{h,NS} - \boldsymbol{u}_{NS})\boldsymbol{u}_{NS}, \boldsymbol{v}_{h,NS})_{\Omega_{NS}} \\
&\quad + \kappa\,\beta \sum_{K \in \mathcal{T}_h^{NS}} \frac{h_K^2}{\nu} \Bigg((\nabla \boldsymbol{u}_{NS})(\boldsymbol{u}_{h,NS} - \boldsymbol{u}_{NS}), 2\nu\, \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS} + \nabla q_{h,NS} \Bigg)_K \\
&= I_1 + I_2. \qquad (6.81)
\end{aligned}$$

By applying similar arguments as in (5.58), it follows that

$$
\begin{aligned}
I_1 =& \kappa\left((\nabla \boldsymbol{u}_{NS})(\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}), \boldsymbol{v}_{h,NS}\right)_{\Omega_{NS}} + \frac{\kappa}{2}\left(\nabla\cdot(\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS})\boldsymbol{u}_{NS}, \boldsymbol{v}_{h,NS}\right)_{\Omega_{NS}} \\
\le& \frac{2C_q^2}{\nu^{1/2}\kappa^{1/2}}\|\boldsymbol{u}_{NS}\|_{1,\Omega_{NS}}\|\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}\|_{1,\Omega_{NS}}|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h.
\end{aligned}
\tag{6.82}
$$

Now using Hölder inequality, (2.7), (5.63), (5.35), we get

$$
\begin{aligned}
I_2 =& \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\left((\nabla \boldsymbol{u}_{NS})(\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}), 2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS} + \nabla q_{h,NS}\right)_K \\
\le& \kappa\,\beta \sum_{K\in\mathcal{T}_h^{NS}} \frac{h_K^2}{\nu}\|\nabla\boldsymbol{u}_{NS}\|_{0,4,K}\|\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}\|_{0,4,K}\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS} + \nabla q_{h,NS}\|_{0,K} \\
\le& \frac{\kappa^{1/2}\beta^{1/2}}{\nu^{1/2}}\left\{\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|\nabla\boldsymbol{u}_{NS}\|_{0,4,K}^2\|\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}\|_{0,4,K}^2\right\}^{1/2} \times \\
& \left\{\kappa\,\beta\sum_{K\in\mathcal{T}_h^{NS}}\frac{h_K^2}{\nu}\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla \boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS} + \nabla q_{h,NS}\|_{0,K}^2\right\}^{1/2} \\
\le& \frac{\kappa^{1/2}\beta^{1/2}}{\nu^{1/2}}\left\{\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|\nabla\boldsymbol{u}_{NS}\|_{0,4,K}^4\right\}^{1/4}\left\{\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}\|_{0,4,K}^4\right\}^{1/4} \times \\
& \left\{2\kappa\,\beta\sum_{K\in\mathcal{T}_h^{NS}}\frac{h_K^2}{\nu}\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + \nabla q_{h,NS}\|_{0,K}^2 + 2\kappa\,\beta\sum_{K\in\mathcal{T}_h^{NS}}\frac{h_K^2}{\nu}\|(\nabla \boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS}\|_{0,K}^2\right\}^{1/2} \\
\le& \frac{\kappa^{1/2}\beta^{1/2}hC_q^2}{\nu^{1/2}}\left\{8+\frac{2\kappa^{1/2}\beta C_{\inf}^2 h}{\nu^{3/2}}\right\}^{1/2}\|\boldsymbol{u}_{NS}\|_{2,\Omega_{NS}}\|\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}\|_{1,\Omega_{NS}}|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h.
\end{aligned}
\tag{6.83}
$$

Next, from (6.81)-(6.83), we obtain

$$
\begin{aligned}
&\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h; \boldsymbol{v}_h, q_h) \\
\le& C_q^2\,\kappa^{1/2}\left\{\frac{2}{\nu^{1/2}\kappa}\|\boldsymbol{u}_{NS}\|_{1,\Omega_{NS}} + \frac{\beta^{1/2}h}{\nu^{1/2}}\left\{8+\frac{2\kappa^{1/2}\beta C_{\inf}^2 h}{\nu^{3/2}}\right\}^{1/2}\right. \\
&\left.\times\ \|\boldsymbol{u}_{NS}\|_{2,\Omega_{NS}}\right\}\|\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}\|_{1,\Omega_{NS}}\,|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h, \qquad (6.84)\\
\le& \frac{3C_q^2}{\nu^{1/2}\kappa^{3/2}}\|\boldsymbol{u}_{NS}\|_{1,\Omega_{NS}}\,\|\boldsymbol{u}_{h,NS}-\boldsymbol{u}_{NS}\|_{1,\Omega_{NS}}\,|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h,
\end{aligned}
$$

under the assumption that h is sufficiently small, and applying (6.73) and (6.78), the result is obtained. □

Theorem 23. *Let* $(\boldsymbol{u},p)$ *and* $(\boldsymbol{u}_h,p_h)$ *be the solutions of* (6.70) *and* (5.29)*, respectively. Assume the regularity hypothesis* (6.73) *and that* $(\boldsymbol{u},p)\in\left[\left(H^{k+1}(\Omega_{NS})^d\cap\boldsymbol{H}^{NS}\right)\times\ H^k(\Omega_D)^d\right]\times\left[H^k(\Omega_{NS})\times\right.$, $\left.\left(H^{k+1}(\Omega_D)\cap Q^D\right)\right]$ *with* $(\boldsymbol{u},p)\in\boldsymbol{X}$ *and* $(\boldsymbol{u}_h,p_h)\in\boldsymbol{X}_h$*. Additionally, assume the data condition* (6.78)*.*

Then, there exist positive constants h_0 *and* C*, independent of* h*, such that*

$$
\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\| \le C\,h^k\left(\|\boldsymbol{u}_{NS}\|_{k+1,\Omega_{NS}} + \|p_{NS}\|_{k,\Omega_{NS}} + \|\boldsymbol{u}_D\|_{k,\Omega_D} + \|p_D\|_{k+1,\Omega_D}\right) \quad \forall h\le h_0.
$$

DOI 10.20347/WIAS.PREPRINT.3183

Proof. We consider the following notations:

$$\begin{aligned}
\eta^{\boldsymbol{u}_{NS}} &:= \boldsymbol{u}_{NS} - \boldsymbol{\Pi}_h^{NS}\boldsymbol{u}_{NS}, \quad \eta^{p_{NS}} := p_{NS} - \mathcal{C}_h^{NS} p_{NS},\\
\eta^{\boldsymbol{u}_D} &:= \boldsymbol{u}_D - \boldsymbol{\mathcal{C}}_h^{D}\boldsymbol{u}_D, \quad \eta^{p_D} := p_D - \Pi_h^D p_D,\\
\eta^{\boldsymbol{u}} &:= (\eta^{\boldsymbol{u}_{NS}}, \eta^{\boldsymbol{u}_D}), \quad \eta^{p} := (\eta^{p_{NS}}, \eta^{p_D}),\\
e_h^{\boldsymbol{u}} &:= (\boldsymbol{u}_{h,NS} - \boldsymbol{\Pi}_h^{NS}\boldsymbol{u}_{NS}, \boldsymbol{u}_{h,D} - \boldsymbol{\mathcal{C}}_h^{D}\boldsymbol{u}_D), \quad e_h^{p} := (p_{h,NS} - \mathcal{C}_h^{NS} p_{NS}, p_{h,D} - \Pi_h^D p_D).
\end{aligned}$$

Using the definition of $\mathcal{A}_{\boldsymbol{u}_{h,NS}}$ given in (5.30), and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\eta^{\boldsymbol{u}}, \eta^{p}; \boldsymbol{v}_h, q_h)\\
&\leq C\Bigg\{\|\eta^{\boldsymbol{u}_{NS}}\|^2_{1,\Omega_{NS}} + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{\boldsymbol{u}_{NS}}\|^2_{0,F} + \sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{p_D}\|^2_{0,F} + \|\eta^{\boldsymbol{u}_D}\|^2_{0,\Omega_D} + |\eta^{p_D}|^2_{1,\Omega_D}\\
&\quad + \|\eta^{p_{NS}}\|^2_{0,\Omega_{NS}} + \|\nu\,\eta^{\boldsymbol{u}_D} + \kappa\,\nabla\eta^{p_D}\|^2_{0,\Omega_D} + \sum_{K\in\mathcal{T}_h^{NS}} h_K^2|\eta^{\boldsymbol{u}_{NS}}|^2_{1,K} + \sum_{K\in\mathcal{T}_h^{D}} h_K^2|\eta^{\boldsymbol{u}_D}|^2_{1,K}\\
&\quad + \sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_{NS}}) + (\nabla\eta^{\boldsymbol{u}_{NS}})\boldsymbol{u}_{h,NS} + \nabla\eta^{p_{NS}}\|^2_{0,K}\Bigg\}^{1/2}\\
&\Bigg\{\nu\kappa\,\|\boldsymbol{v}_{h,NS}\|^2_{1,\Omega_{NS}} + \nu\kappa\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|\boldsymbol{v}_{h,NS}\|^2_{0,F} + \nu\,\|\boldsymbol{v}_{h,D}\|^2_{0,\Omega_D} + \frac{\kappa^2}{\nu}\,|q_{h,D}|^2_{1,\Omega_D}\\
&\quad + \frac{\kappa^2}{\nu}\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|q_{h,D}\|^2_{0,F} + \kappa\,\|q_{h,NS}\|^2_{0,\Omega_{NS}} + \|-\nu\,\boldsymbol{v}_{h,D} + \kappa\,\nabla q_{h,D}\|^2_{0,\Omega_D}\\
&\quad + \sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS}) + (\nabla\boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS} + \nabla q_{h,NS}\|^2_{0,K}\Bigg\}^{1/2}.
\end{aligned}$$

Now, using mesh regularity, (6.74), and Lemma 21, we get

$$\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{\boldsymbol{u}_{NS}}\|^2_{0,F} \leq C\sum_{K\in\mathcal{T}_h^{NS}} h_K^{-1}\left\{h_K^{-1}\,\|\eta^{\boldsymbol{u}_{NS}}\|^2_{0,K} + h_K\,|\eta^{\boldsymbol{u}_{NS}}|^2_{1,K}\right\} \leq C\,h^{2k}\|\boldsymbol{u}_{NS}\|^2_{k+1,\Omega_{NS}}. \tag{6.85}$$

Similar arguments allow us to obtain the following estimate:

$$\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{p_D}\|^2_{0,F} \leq C\,h^{2k}\|p_D\|^2_{k+1,\Omega_D}. \tag{6.86}$$

Additionally, applying the triangle and Hölder inequalities along with (5.35) and (6.74)–(6.75), we obtain

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_{NS}})+(\nabla\eta^{\boldsymbol{u}_{NS}})\boldsymbol{u}_{h,NS}+\nabla\eta^{p_{NS}}\|_{0,K}^2\\
&\le C\left\{\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_{NS}})\|_{0,K}^2+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|(\nabla\eta^{\boldsymbol{u}_{NS}})\boldsymbol{u}_{h,NS}\|_{0,K}^2+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,|\eta^{p_{NS}}|_{1,K}^2\right\}\\
&\le C\left\{\|\eta^{\boldsymbol{u}_{NS}}\|_{1,\Omega_{NS}}^2+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,|\eta^{\boldsymbol{u}_{NS}}|_{1,K}^2\|\boldsymbol{u}_{h,NS}\|_{\infty,K}^2+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,|\eta^{p_{NS}}|_{1,K}^2\right\}\\
&\le C\left\{\|\eta^{\boldsymbol{u}_{NS}}\|_{1,\Omega_{NS}}^2+\|\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}}^2\sum_{K\in\mathcal{T}_h^{NS}} h_K\,|\eta^{\boldsymbol{u}_{NS}}|_{1,K}^2+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,|\eta^{p_{NS}}|_{1,K}^2\right\}\\
&\le C\,h^{2k}\left\{\|\boldsymbol{u}_{NS}\|_{k+1,\Omega_{NS}}^2+\|p_{NS}\|_{k,\Omega_{NS}}^2\right\}. \qquad (6.87)
\end{aligned}$$

Thus, from (6.85)-(6.87) it follows that

$$\begin{aligned}
&\|\eta^{\boldsymbol{u}_{NS}}\|_{1,\Omega_{NS}}^2+\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{\boldsymbol{u}_{NS}}\|_{0,F}^2+\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F^{-1}\|\eta^{p_D}\|_{0,F}^2+\|\eta^{\boldsymbol{u}_D}\|_{0,\Omega_D}^2+|\eta^{p_D}|_{1,\Omega_D}^2\\
&+\|\eta^{p_{NS}}\|_{0,\Omega_{NS}}^2+\|\nu\,\eta^{\boldsymbol{u}_D}+\kappa\,\nabla\eta^{p_D}\|_{0,\Omega_D}^2+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2|\eta^{\boldsymbol{u}_{NS}}|_{1,K}^2+\sum_{K\in\mathcal{T}_h^{D}} h_K^2|\eta^{\boldsymbol{u}_D}|_{1,K}^2\\
&+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_{NS}})+(\nabla\eta^{\boldsymbol{u}_{NS}})\boldsymbol{u}_{h,NS}+\nabla\eta^{p_{NS}}\|_{0,K}^2\\
&\le C\,h^{2k}\left\{\|\boldsymbol{u}_{NS}\|_{k+1,\Omega_{NS}}^2+\|p_{NS}\|_{k,\Omega_{NS}}^2+\|\boldsymbol{u}_D\|_{k,\Omega_D}^2+\|p_D\|_{k+1,\Omega_D}^2\right\}. \qquad (6.88)
\end{aligned}$$

On the other hand, using the inverse and Poincaré inequalities, mesh regularity, Lemma 21, and proceeding as in (6.83), we obtain

$$\begin{aligned}
&\nu\kappa\,\|\boldsymbol{v}_{h,NS}\|_{1,\Omega_{NS}}^2+\nu\kappa\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|\boldsymbol{v}_{h,NS}\|_{0,F}^2+\nu\,\|\boldsymbol{v}_{h,D}\|_{0,\Omega_D}^2+\frac{\kappa^2}{\nu}\,|q_{h,D}|_{1,\Omega_D}^2\\
&+\frac{\kappa^2}{\nu}\sum_{F\in\mathcal{E}_{\Gamma_h}} h_F\|q_{h,D}\|_{0,F}^2+\kappa\,\|q_{h,NS}\|_{0,\Omega_{NS}}^2+\|-\nu\,\boldsymbol{v}_{h,D}+\kappa\,\nabla q_{h,D}\|_{0,\Omega_D}^2+\\
&\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})+(\nabla\boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS}+\nabla q_{h,NS}\|_{0,K}^2\\
&\le C\left\{|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h^2+\sum_{K\in\mathcal{T}_h^{NS}} h_K^2\,\|2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_{h,NS})+(\nabla\boldsymbol{v}_{h,NS})\boldsymbol{u}_{h,NS}+\nabla q_{h,NS}\|_{0,K}^2\right\}\\
&\le C\left\{|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h^2+\frac{C_{\inf}^2 h}{\nu\kappa}\,|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h^2\right\}\\
&\le C\,|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h^2, \qquad (6.89)
\end{aligned}$$

under the assumption that h is sufficiently small.

Next, connecting (6.88) and (6.89) with (6.85), we conclude that

$$\begin{aligned}&\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\eta^{\boldsymbol{u}},\eta^{p};\boldsymbol{v}_h,q_h)\\&\le C\,h^k\left\{\|\boldsymbol{u}_{NS}\|_{k+1,\Omega_{NS}}+\|p_{NS}\|_{k,\Omega_{NS}}+\|\boldsymbol{u}_D\|_{k,\Omega_D}+\|p_D\|_{k+1,\Omega_D}\right\}\,|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h.\end{aligned} \tag{6.90}$$

Now, since $\boldsymbol{u}_{h,NS}\in\boldsymbol{X}_h$, it follows from Lemma 17 that the hypotheses of Lemma 15 are satisfied. Therefore, by the definitions of the norms $\|(\cdot,\cdot)\|$ and $|\!|\!|(\cdot,\cdot)|\!|\!|_h$ (cf. (3.15) and (5.33), respectively), lemmas 15 and 22, and the estimate (6.90), we have

$$\begin{aligned}&\|(e_h^{\boldsymbol{u}},e_h^{p})\|\\&\le\frac{1}{C_{\mathcal{A}_h}}\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{H}_h\times Q_h}\frac{\mathcal{A}_{\boldsymbol{u}_{h,NS}}(e_h^{\boldsymbol{u}},e_h^{p};\boldsymbol{v}_h,q_h)}{|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h}\\&=\frac{1}{C_{\mathcal{A}_h}}\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{H}_h\times Q_h}\frac{\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\eta^{\boldsymbol{u}},\eta^{p};\boldsymbol{v}_h,q_h)-\mathcal{A}_{\boldsymbol{u}_{h,NS}}(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;\boldsymbol{v}_h,q_h)}{|\!|\!|(\boldsymbol{v}_h,q_h)|\!|\!|_h}\\&\le C\,h^k\left\{\|\boldsymbol{u}_{NS}\|_{k+1,\Omega_{NS}}+\|p_{NS}\|_{k,\Omega_{NS}}+\|\boldsymbol{u}_D\|_{k,\Omega_D}+\|p_D\|_{k+1,\Omega_D}\right\}+\frac{1}{2}\|(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h)\|.\end{aligned} \tag{6.91}$$

Finally, the result follows using triangle inequality, interpolation properties (6.74)–(6.77) and (6.91). □

7 Numerical experiments

In this section, we present three numerical experiments to evaluate the performance of our stabilized scheme. Since the free medium follows the Navier-Stokes equations, we implement our stabilized scheme using a Picard iteration method with a tolerance of $\mathrm{tol}=10^{-6}$.

For the implementation, we use the Python libraries `Multiphenics` and `FEniCS` [29]. The stabilization constant is defined as $\beta=\dfrac{1}{24}$, if $k=1$, and $\beta=\dfrac{1}{384}$, if $k=2$.

7.1 A smooth solution in two dimensions

For the first test case, we consider an analytical solution to the model problem. The parameters are set $\kappa=1$, $\nu=10^{-2}$ or 10^{-4}. The computational domain, as shown in Figure 2, is given in the free medium domain by $\Omega_{NS}:=(-1/2,1/2)\times(0,2)$, and in the porous medium domain by $\Omega_D:=(1/2,3/2)\times(0,2)$. The interface is given by $\Gamma:=\{(1/2,y)\in\mathbb{R}^2:0<y<2\}$ and the boundaries $\Gamma_{NS}:=\partial\Omega_{NS}\backslash\Gamma$ and Γ_D are divided as follows

$$\Gamma_D^{\mathrm{Dir}}:=\left\{(3/2,y)\in\mathbb{R}^2:0<y<2\right\}\cup\left\{(x,2)\in\mathbb{R}^2:1/2<x<3/2\right\},$$

$$\Gamma_D^{\mathrm{Neu}}:=\left\{(x,0)\in\mathbb{R}^2:1/2<x<3/2\right\}.$$

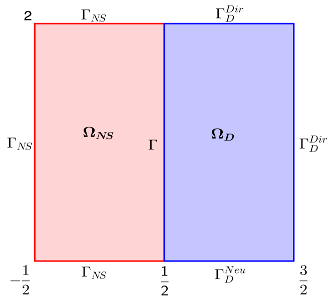

Figure 2: Configuration of the computational domain Ω.

On the Beavers–Joseph–Saffman interface condition we take $\alpha_1 = 1$ and in the domain Ω_{NS} we consider the well–known Kovasznay solution $(\boldsymbol{u}_{NS}, p_{NS})$ (see [27]), defined by

$$\boldsymbol{u}_{NS}(x,y) := \left(1 - e^{\lambda x}\cos(2\pi y), \frac{\lambda}{2\pi}e^{\lambda x}\sin(2\pi y)\right), \quad p_{NS}(x,y) := \frac{1}{2}e^{\lambda x} - \frac{1}{\lambda}(e^{3\lambda/2} - e^{-\lambda/2}),$$

where the parameter $\lambda := \frac{1}{2\nu} - \sqrt{\frac{1}{4\nu^2} + 4\pi^2}$, and in the porous medium Ω_D we consider a manufactured analytical solution $(\boldsymbol{u}_D, p_D)$, that satisfies the mass conservation on Γ, given by

$$\boldsymbol{u}_D(x,y) := \left(\frac{3}{2} - x + e^{\lambda/2}\sin(\pi x)\cos(2\pi y), xy^2\right), \quad p_D(x,y) := 2\lambda\nu e^{\lambda x}\cos(2\pi y) + \frac{1}{2}e^{\lambda x} - \frac{1}{\lambda}(e^{3\lambda/2} - e^{-\lambda/2}).$$

With these expressions, the functions $\boldsymbol{f}_{NS}$ and g_D are constructed so that $(\boldsymbol{u}, p)$ is the solution of problem (P). Furthermore, since under this configuration, the equilibrium of normal forces and the Beavers–Joseph–Saffman condition are not satisfied on Γ, appropriate source terms must be added to the proposed stabilized finite element scheme (5.29).

h	$\|\boldsymbol{u}_{NS} - \boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_{NS} - \boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}}$	order	$\|p_{NS} - p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,\Omega_D}$	order	$\|p_D - p_{h,D}\|_{1,\Omega_D}$	order
0.1250	2.4157e-01	–	2.7741e+00	–	6.4083e-02	–	5.9228e-01	–	1.3249e-02	–
0.0625	7.2909e-02	1.7283	1.3428e+00	1.0468	1.9017e-02	1.7526	2.0484e-01	1.5318	6.6656e-03	0.9911
0.0312	1.9679e-02	1.8895	6.4224e-01	1.0640	5.4180e-03	1.8115	6.6233e-02	1.6289	3.3349e-03	0.9991
0.0156	5.0842e-03	1.9525	3.1096e-01	1.0464	1.4996e-03	1.8532	2.1764e-02	1.6056	1.6666e-03	1.0008
0.0078	1.2880e-03	1.9809	1.5347e-01	1.0188	4.1372e-04	1.8578	7.3480e-03	1.5665	8.3287e-04	1.0007
0.0039	3.2356e-04	1.9931	7.6403e-02	1.0063	1.1743e-04	1.8168	2.5314e-03	1.5374	4.1631e-04	1.0004

Table 1: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 10^{-2}$ and $k = 1$.

h	$\|\boldsymbol{u}_{NS} - \boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_{NS} - \boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}}$	order	$\|p_{NS} - p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,\Omega_D}$	order	$\|p_D - p_{h,D}\|_{1,\Omega_D}$	order
0.1250	1.7699e-02	–	4.3038e-01	–	2.4339e-03	–	5.7976e-03	–	3.3023e-05	–
0.0625	1.8282e-03	3.2751	1.0256e-01	2.0691	4.1829e-04	2.5407	7.1133e-04	3.0269	3.0119e-06	3.4547
0.0312	2.1800e-04	3.0680	2.5209e-02	2.0245	1.0692e-04	1.9680	9.0043e-05	2.9818	3.8160e-07	2.9806
0.0156	2.6930e-05	3.0170	6.2461e-03	2.0129	2.7165e-05	1.9767	1.1309e-05	2.9931	4.8742e-08	2.9688
0.0078	3.3568e-06	3.0041	1.5568e-03	2.0044	6.7978e-06	1.9986	1.4160e-06	2.9976	6.1583e-09	2.9846
0.0039	4.1927e-07	3.0011	3.8887e-04	2.0012	1.6990e-06	2.0004	1.7737e-07	2.9970	7.7956e-10	2.9818

Table 2: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 10^{-2}$ and $k = 2$.

h	$\|\boldsymbol{u}_{NS} - \boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_{NS} - \boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}}$	order	$\|p_{NS} - p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,\Omega_D}$	order	$\|p_D - p_{h,D}\|_{1,\Omega_D}$	order
0.1250	2.1253e-01	–	2.7347e+00	–	4.0912e-03	–	9.6563e-02	–	12.5301e-03	–
0.0625	7.2909e-02	1.5435	1.3428e+00	1.0262	1.9017e-02	-2.2167	2.0484e-01	-1.0850	6.6656e-03	-9.9954
0.0312	1.9679e-02	1.8895	6.4224e-01	1.0640	5.4180e-03	1.8115	6.6233e-02	1.6289	3.3349e-03	0.9991
0.0156	5.0842e-03	1.9525	3.1096e-01	1.0464	1.4996e-03	1.8532	2.1764e-02	1.6056	1.6666e-03	1.0008
0.0078	1.2880e-03	1.9809	1.5347e-01	1.0188	4.1372e-04	1.8578	7.3480e-03	1.5665	8.3287e-04	1.0007
0.0039	3.2356e-04	1.9931	7.6403e-02	1.0063	1.1743e-04	1.8168	2.5314e-03	1.5374	4.1631e-04	1.0004

Table 3: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 10^{-4}$ and $k = 1$.

h	$\|\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\|_{1,\Omega_{NS}}$	order	$\|p_{NS}-p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_{D}-\boldsymbol{u}_{h,D}\|_{0,\Omega_{D}}$	order	$\|p_{D}-p_{h,D}\|_{1,\Omega_{D}}$	order
0.1250	2.1654e-02	–	5.5512e-01	–	3.3463e-04	–	7.8163e-03	–	3.6449e-07	–
0.0625	1.8282e-03	3.5661	1.0256e-01	2.4363	4.1829e-04	-0.3219	7.1133e-04	3.4579	3.0119e-06	-3.0468
0.0312	2.1800e-04	3.0680	2.5209e-02	2.0245	1.0692e-04	1.9680	9.0043e-05	2.9818	3.8160e-07	2.9806
0.0156	2.6930e-05	3.0170	6.2461e-03	2.0129	2.7165e-05	1.9767	1.1309e-05	2.9931	4.8742e-08	2.9688
0.0078	3.3568e-06	3.0041	1.5568e-03	2.0044	6.7978e-06	1.9986	1.4160e-06	2.9976	6.1583e-09	2.9846
0.0039	4.1927e-07	3.0011	3.8887e-04	2.0012	1.6990e-06	2.0004	1.7737e-07	2.9970	7.7956e-10	2.9818

Table 4: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 10^{-4}$ and $k = 2$.

We can observe in Tables 1- 4, the expected rates of convergence for our proposed scheme under quasi-uniform refinement. It is worth noting that the error norms $\|\boldsymbol{u}_D - \boldsymbol{u}_{h,D}\|_{0,\Omega_D}$ and $\|p_{NS} - p_{h,NS}\|_{0,\Omega_{NS}}$ exhibit better behavior than predicted by Theorem 23 when polynomial functions of order $k = 1$ are employed. We hypothesize that this is due to the smoothness of the solutions in this example. The isovalues of velocity and pressure in both media, which are not shown in this paper, are very close to the exact ones.

7.2 A coupled flow with different types of permeability

Inspired by the numerical results presented in [26], in this section we consider a series of experiments that simulate the interaction of fluid flowing from a free medium to a porous medium where the permeability of the latter varies from a constant value, under different boundary conditions, to a highly oscillatory case (for a similar numerical example, see also [14]).

The computational domain $\Omega := (0,10)^2$ is partitioned into $\Omega_{NS} := (0,10)\times(6,10)$, $\Omega_D := (0,10)\times(0,6)$, and the definitions of the different boundaries can be seen in Figure 3.

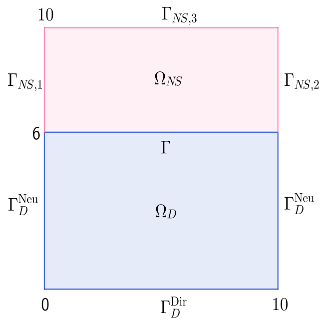

Figure 3: Configuration of the computational domain Ω.

For the first case=, we set $\kappa = 1$, and $\nu = 1$, imposing the following boundary conditions on Γ_{NS}:

$$\boldsymbol{u}_{NS}(x,y)\begin{cases}\left(10\sin\left(\frac{\pi}{8}(y-6)\right),\, 0\right), & \text{on } \Gamma_{NS,1},\\[2ex] \left(\sin\left(\frac{\pi}{8}(y-6)\right),\, 0\right), & \text{on } \Gamma_{NS,2},\\[2ex] (1,\, 0), & \text{on } \Gamma_{NS,3}.\end{cases}$$

Moreover, on the boundary $\Gamma_D^{\rm Neu}$ we impose the impermeability condition $\boldsymbol{u}_D \cdot \boldsymbol{n}_D = 0$. On $\Gamma_D^{\rm Dir}$ we prescribe the Dirichlet boundary condition $p_D(x,y) = 20 - x$. To complete the specifications of this problem, the source terms $\boldsymbol{f}_{NS}$, g_D are set to zero and the parameter α_1 in the Beavers–Joseph–Saffman condition is taken as 1.

In Figure 4 we present the streamlines and the isovalues of the computed velocity magnitude and pressure, using an interpolation order $k = 2$ on a mesh with 25,600 elements. We observe that the numerical solution recovers the expected behaviour, closely matching the results established in [26].

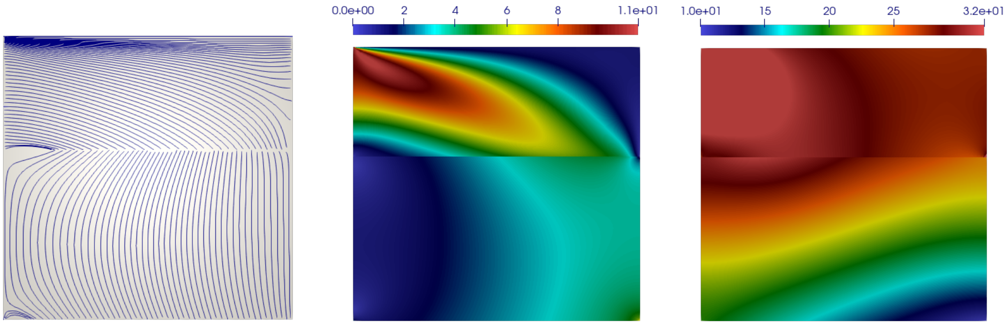

Figure 4: The coupled flow problem with permeability $\kappa = 1$. Streamline (left), isovalues of velocity magnitude (center) and pressure (right). Here $\nu = 1$ with interpolation order $k = 2$ on a mesh with 25,600 elements.

In Figure 5, we consider a permeability of $\kappa = 10^{-6}$ in the porous medium, and we replace the boundary condition on $\Gamma_{NS,2}$ with that defined on $\Gamma_{NS,1}$. With this configuration, the domain Ω_D provides significant resistance to the flow, which explains the behavior of the streamlines in the free medium.

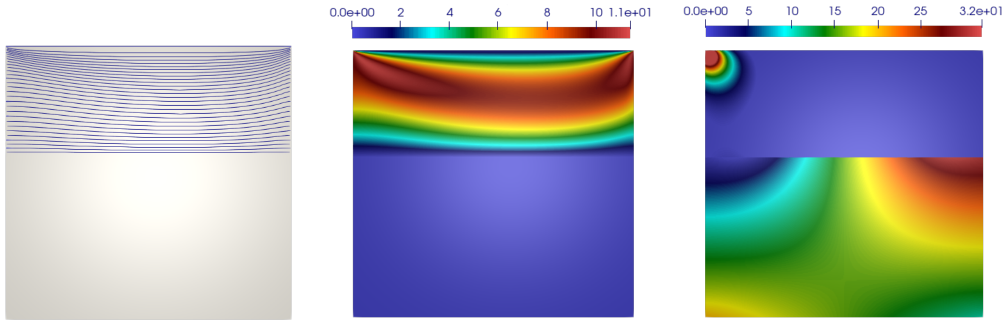

Figure 5: The coupled flow problem with permeability $\kappa = 10^{-6}$. Streamline (left), isovalues of velocity magnitude (center) and pressure (right). Here $\nu = 1$ with interpolation order $k = 2$ on a mesh with 25,600 elements.

Finally, we use the same configuration and boundary conditions as in the previous experiment, but now consider a coarse mesh with $6,400$ triangles for Ω_D where the permeability κ is randomly defined within the interval $[0.0005, 0.1]$ (see Figure 6 on the left). In this situation, the behavior of the fluid in the porous medium becomes more complex and less predictable. Figure 6 presents the numerical results obtained using our stabilized scheme on a refined mesh with $102,400$ triangles, generated from the initial coarse mesh where the permeability was defined. The results are displayed using different scales for each domain to enhance visualization. The streamlines tend to follow paths of least resistance, leading to irregular flow patterns. The velocity isovalues highlight regions of high permeability, where the medium allows rapid fluid flow, whereas in areas of low permeability, the velocity remains restricted. In contrast, in areas with low permeability, the velocity of the fluid is limited. Concerning the heterogeneous pressure distribution on the porous medium, in this same figure, the isovalues show high gradients near the interface between both mediums.

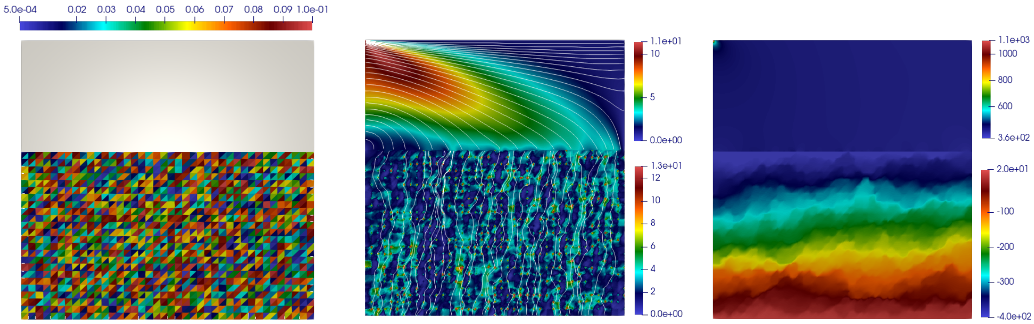

Figure 6: Random permeability field $\kappa \in [0.0005, 0.1]$ (left) on a coarse mesh of Ω_D with $6,400$ triangles. Isovalues of velocity magnitude with streamlines (center) and isovalues of pressure (right) computed on a mesh with 102,400 elements. Here, $\nu = 1$ with interpolation order $k = 2$.

7.3 An analytical solution in three dimensions

This final test illustrates the performance of our stabilized finite element scheme in the case where problem (P) is defined in a bounded solid region of $\mathbb{R}^3$. For this, we consider the computational domain $\Omega := (0,1) \times (-1/2, 1/2) \times (0,1)$, with the free-flow region given by $\Omega_{NS} := (0,1) \times (-1/2, 0) \times (0,1)$ and the porous medium region by $\Omega_D := (0,1) \times (0, 1/2) \times (0,1)$. Moreover, as shown in Figure 7, the interface between both media is defined as $\Gamma := \overline{\Omega}_{NS} \cap \overline{\Omega}_D$, the boundaries $\Gamma_{NS} := \partial\Omega_{NS} \backslash \Gamma$ and Γ_D are divided as follows

$$\Gamma_D^{\mathrm{Neu}} := \left\{ (x, y, 0) \in \mathbb{R}^3 : 0 < x < 1,\, 0 < y < 1/2 \right\}, \qquad \Gamma_D^{\mathrm{Dir}} := \Gamma_D \backslash \overline{\Gamma_D^{\mathrm{Neu}}}.$$

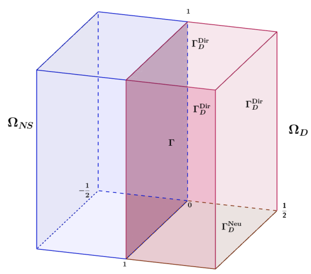

Figure 7: Configuration of the computational domain Ω.

In the Beavers–Joseph–Saffman condition, we take $\alpha_1 = \alpha_2 = 1$, the viscosity of the fluid is $\nu = 1$ or $\nu = 10^{-2}$, with a permeability of the porous medium $\kappa = 1$. The data $\boldsymbol{f}_{NS}$ and g_D are such that the exact solution is given by

$$\begin{aligned}
\boldsymbol{u}_{NS}(x,y) &:= (e^z \sin y, -e^z \sin y, e^z \cos y - e^z \cos x), \\
p_{NS}(x,y) &:= -\frac{1}{2} e^{2z} + \frac{1}{4}(e^2 - 1), \\
\boldsymbol{u}_D(x,y) &:= (\cos(\pi z)\sin(\pi x)\sin(\pi y), \sin(\pi z)\cos(\pi x)\sin(\pi y), -2\sin(\pi z)\sin(\pi x)\cos(\pi y)), \\
p_D(x,y) &:= \sin(\pi z)\sin(\pi x)\cos(\pi y).
\end{aligned}$$

As in the two-dimensional case, this solution satisfies mass conservation but does not satisfy the other interface conditions on Γ, so appropriate right-hand sides must be constructed such that $(\boldsymbol{u}, p)$ is the solution of the coupled problem. The calculations were performed using interpolation order $k = 1$ and $k = 2$. In Tables 5-8 we have the errors for the proposed scheme when considering a family of triangulations with different viscosity values ν. The convergence rates for each of the approximate unknowns behave as expected. Finally, in Figures 8 and 9, we can compare some isovalues of velocity and pressure when considering interpolation order $k = 1$ in a mesh with 196,608 tetrahedrons. These visualizations confirm that our proposed stabilized scheme is effective in approximating the exact solution.

h	$\|\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\lvert\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\rvert_{1,\Omega_{NS}}$	order	$\|p_{NS}-p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\Omega_D}$	order	$\lvert p_D-p_{h,D}\rvert_{1,\Omega_D}$	order
0.8660	5.3192e-02	–	6.2219e-01	–	6.5204e-01	–	8.0469e-01	–	1.0529e+00	–
0.4330	1.4257e-02	1.90	3.1540e-01	0.99	2.4651e-01	1.40	3.2748e-01	1.30	6.4546e-01	0.70
0.2165	3.9227e-03	1.86	1.5687e-01	1.00	9.0379e-02	1.45	9.5318e-02	1.78	3.4112e-01	0.93
0.1083	1.0520e-03	1.90	7.7838e-02	1.01	2.9480e-02	1.62	2.8676e-02	1.73	1.7227e-01	0.99
0.0541	2.7258e-04	1.95	3.8746e-02	1.00	9.1414e-03	1.69	9.1784e-03	1.64	8.6256e-02	1.00

Table 5: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 1$ and $k = 1$.

h	$\|\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\lvert\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\rvert_{1,\Omega_{NS}}$	order	$\|p_{NS}-p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\Omega_D}$	order	$\lvert p_D-p_{h,D}\rvert_{1,\Omega_D}$	order
0.8660	2.9085e-03	–	5.2558e-02	–	3.0839e-02	–	2.5161e-01	–	3.9347e-01	–
0.4330	3.8854e-04	2,90	1.3652e-02	1.95	5.7556e-03	2.43	6.3930e-02	1.98	1.1783e-01	1.72
0.2165	5.6971e-05	2.77	3.8494e-03	1.83	1.4205e-03	2.01	1.9272e-02	1.73	3.1735e-02	1.90
0.1083	8.3535e-06	2.77	1.3977e-03	1.47	3.3506e-04	2.08	5.6271e-03	1.78	8.1205e-03	1.97
0.0541	1.2285e-06	2.76	3.4943e-04	1.99	8.2694e-05	2.01	1.5163e-03	1.90	2.0388e-03	2.00

Table 6: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 1$ and $k = 2$.

h	$\|\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\lvert\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\rvert_{1,\Omega_{NS}}$	order	$\|p_{NS}-p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\Omega_D}$	order	$\lvert p_D-p_{h,D}\rvert_{1,\Omega_D}$	order
0.8660	5.6107e-02	–	6.2576e-01	–	4.4445e-01	–	7.9286e-01	–	1.0511e+00	–
0.4330	1.4257e-02	1.9765	3.1540e-01	0.9884	2.4651e-01	0.8504	3.2748e-01	1.2757	6.4546e-01	0.7035
0.2165	3.9227e-03	1.8618	1.5687e-01	1.0076	9.0379e-02	1.4476	9.5318e-02	1.7806	3.4112e-01	0.9200
0.1083	1.0793e-03	1.9765	7.7838e-02	1.0110	2.9480e-02	1.6163	2.8676e-02	1.7329	1.7227e-01	0.9856
0.0541	2.9696e-04	1.9765	3.8746e-02	1.0064	9.1414e-03	1.6892	9.1784e-03	1.6435	8.6256e-02	0.9980

Table 7: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 10^{-2}$ and $k = 1$.

h	$\|\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\|_{0,\Omega_{NS}}$	order	$\lvert\boldsymbol{u}_{NS}-\boldsymbol{u}_{h,NS}\rvert_{1,\Omega_{NS}}$	order	$\|p_{NS}-p_{h,NS}\|_{0,\Omega_{NS}}$	order	$\|\boldsymbol{u}_D-\boldsymbol{u}_{h,D}\|_{0,\Omega_D}$	order	$\lvert p_D-p_{h,D}\rvert_{1,\Omega_D}$	order
0.8660	1.8767e-02	–	2.1266e-01	–	1.2217e-02	–	2.5908e+01	–	4.0869e-01	–
0.4330	3.8860e-04	5.5938	1.3652e-02	3.9614	5.7540e-03	1.0863	6.4538e-02	8.649	1.1898e-01	1.7803
0.2165	5.6964e-05	2.7702	3.8490e-03	1.8266	1.4203e-03	2.0184	1.9315e-02	1.7404	3.1813e-02	1.903
0.1083	8.3502e-06	2.7702	1.0852e-03	1.8265	3.5058e-04	2.0184	5.6302e-03	1.7785	8.1255e-03	1.9691
0.0541	1.2240e-06	2.7702	3.1392e-04	1.7895	8.6598e-05	2.0173	1.5165e-03	1.8924	2.0391e-03	1.9945

Table 8: Mesh sizes, errors and rates of convergence of the solutions for $\nu = 10^{-2}$ and $k = 2$.

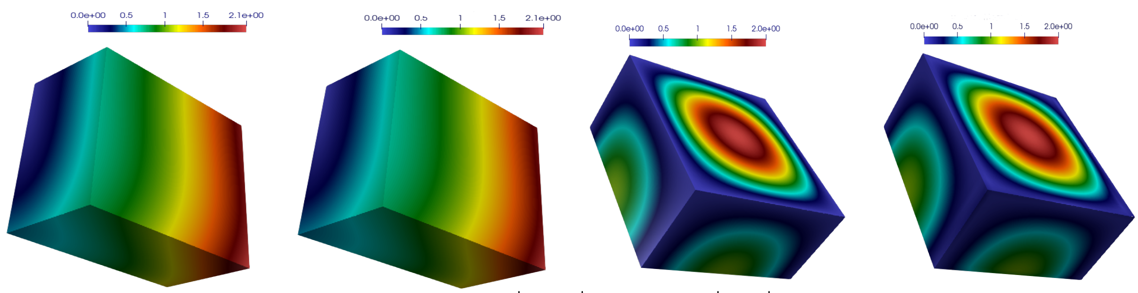

Figure 8: Isovalues of the velocity magnitudes $|\boldsymbol{u}_{NS}|$ (first) and $|\boldsymbol{u}_D|$ (third) compared with the approximated solutions $|\boldsymbol{u}_{h,NS}|$ (second) and $|\boldsymbol{u}_{h,D}|$ (fourth), respectively, on a mesh with 196,608 tetrahedrons. Here, $\nu = 1$ and the interpolation order $k = 1$.

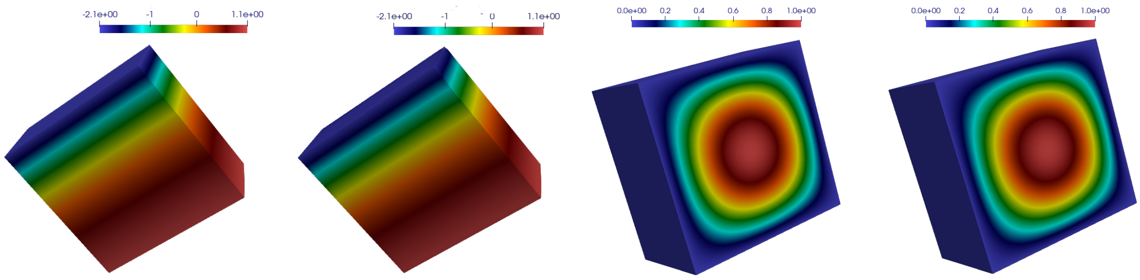

Figure 9: Isovalues of the pressures p_{NS} (first) and p_D (third) compared with the approximated solutions $p_{h,NS}$ (second) and $p_{h,D}$ (fourth), respectively, on a mesh with 196,608 tetrahedrons. Here, $\nu = 1$ and the interpolation order $k = 1$.

Conclusions

In this work, we have proposed a new stabilized finite element scheme for the Navier–Stokes/Darcy coupled problem in Lagrange spaces with equal-order approximation. To achieve this, we define an augmented variational formulation, inspired by the recently published work [4] on the Stokes/Darcy coupled problem. Using fixed-point theory, we establish the existence and uniqueness of the solution to the continuous variational problem within a bounded ball. This new formulation is stabilized, and by extending these techniques to the discrete case, we also prove the existence and uniqueness of the proposed scheme's solution under small data conditions. Furthermore, assuming standard regularity conditions for the exact solution, we demonstrate the convergence of the discrete scheme. Numerical validation confirms, in both two and three dimensions, the good approximation properties expected when the model problem has an analytical solution. Additionally, in the case where the problem is inspired by a physical setting, our stabilized finite element method yields good results.

For future work, it remains to explore the decoupling of the proposed scheme, as well as to study and improve the convergence order of the Picard method for the nonlinear scheme by incorporating and analyzing acceleration strategies [32].

Acknowledgments

The first author was partially funded by ANID-Chile through the projects Centro de Modelamiento Matemático (FB210005) of the PIA Program: Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal, and Fondecyt Regular No 1211649. The second author was partially supported by Dirección de Investigación of the Universidad Católica de la Santísima Concepción through project FGII 06/2024 and FONDECYT Regular No 1211649.

References

[1] S. Agmon. *Lectures on elliptic boundary value problems*. AMS Chelsea Publishing, Providence, RI, 2010.

[2] M. Amara, D. Capatina, and L. Lizaik. Coupling of Darcy-Forchheimer and compressible Navier-Stokes equations with heat transfer. *SIAM J. Sci. Comput.*, 31(2):1470–1499, 2009.

[3] R. Araya, G.R. Barrenechea, A. H. Poza, and F. Valentin. Convergence analysis of a residual local projection finite element method for the Navier-Stokes equations. *SIAM J. Numer. Anal.*, 50(2):669–699, 2012.

[4] R. Araya, C. Cárcamo, A. H. Poza, and E. Vino. An adaptive stabilized finite element method for the Stokes–Darcy coupled problem. *J. Comput. Appl. Math.*, 443:115753, 2024.

[5] R. Araya, C. Cárcamo, and A. H. Poza. A stabilized finite element method for the Stokes–Temperature coupled problem. *Appl. Numer. Math.*, 187:24–49, 2023.

[6] R. Araya, A. H. Poza, and F. Valentin. A low-order local projection method for the incompressible Navier-Stokes equations in two- and three-dimensions. *IMA J. Numer. Anal.*, 36(1):267–295, 2016.

[7] L. Badea, M. Discacciati, and A. Quarteroni. Numerical analysis of the Navier-Stokes/Darcy coupling. *Numer. Math.*, 115(2):195–227, 2010.

[8] S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.*, 3:133–181, 1922.

[9] Gordon S. Beavers, Daniel D. Joseph, G.S. Beavers, and D.D. Joseph. Boundary conditions at a naturally permeable wall. *J. Fluid Mech.*, 30(1):197–207, 1967.

[10] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[11] L. E. J. Brouwer. Über abbildung von mannigfaltigkeiten. *Math. Ann.*, 71:97–115, 1911.

[12] M. Cai, M. Mu, and J. Xu. Numerical solution to a mixed Navier–Stokes/Darcy model by the two-grid approach. *SIAM J. Numer. Anal.*, 47(5):3325–3338, 2009.

[13] S. Caucao, G. N. Gatica, R. Oyarzúa, and I. Šebestová. A fully-mixed finite element method for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity. *J. Numer. Math.*, 25(2):55–88, 2017.

[14] A. Cesmelioglu and S. Rhebergen. A hybridizable discontinuous Galerkin method for the coupled Navier–Stokes and Darcy problem. *J. Comput. Appl. Math.*, 422:114923, 2023.

[15] A. Cesmelioglu and B. Rivière. Analysis of time-dependent Navier–Stokes flow coupled with Darcy flow. *J. Numer. Math.*, 16(4):249–280, 2008.

[16] A. Cesmelioglu and B. Rivière. Primal discontinuous Galerkin methods for time-dependent coupled surface and subsurface flow. *J. Sci. Comput.*, 40(1):115–140, 2009.

[17] N. Chaabane, V. Girault, Ch. Puelz, and B. Rivière. Convergence of IPDG for coupled time-dependent Navier–Stokes and Darcy equations. *J. Comput. Appl. Math.*, 324:25–48, 2017.

[18] P. Chidyagwai and B. Rivière. On the solution of the coupled Navier–Stokes and Darcy equations. *Comput. Methods Appl. Mech. Engrg.*, 198(47):3806–3820, 2009.

[19] P. G. Ciarlet. *Mathematical elasticity. Vol. II. Theory of plates*, volume 27 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1997.

[20] M. Discacciati and R. Oyarzúa. A conforming mixed finite element method for the Navier–Stokes/Darcy coupled problem. *Numer. Math.*, 135(2):571–606, 2017.

[21] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[22] V. J. Ervin, E. W. Jenkins, and S. Sun. Coupled generalized nonlinear Stokes flow with flow through a porous medium. *SIAM J. Numer. Anal.*, 47(2):929–952, 2009.

[23] L. P. Franca and T. J. R. Hughes. Convergence analyses of Galerkin least-squares methods for symmetric advective-diffusive forms of the Stokes and incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 105(2):285–298, 1993.

[24] L.P. Franca and R. Stenberg. Error analysis of Galerkin least squares methods for the elasticity equations. *SIAM J. Numer. Anal.*, 28(6):1680–1697, 1991.

[25] V. Girault and P.A. Raviart. *Finite element methods for Navier–Stokes equations: Theory and algorithms*. Springer Series in Computational Mathematics. Springer-Verlag Berlin Heidelberg, 1986.

[26] Vivette Girault and Béatrice Rivière. DG approximation of coupled Navier–Stokes and Darcy equations by Beaver–Joseph–Saffman interface condition. *SIAM J. Numer. Anal.*, 47(3):2052–2089, 2009.

[27] L. I. G. Kovasznay. Laminar flow behind a two-dimensional grid. *Mathematical Proceedings of the Cambridge Philosophical Society*, 44(1):58–62, 1948.

[28] Thérèse Levy and Enrique Sánchez-Palencia. On boundary conditions for fluid flow in porous media. *Internat. J. Engrg. Sci.*, 13(11):923–940, 1975.

[29] A. Logg, G. Wells, and K.A. Mardal, editors. *Automated solution of differential equations by the finite element method. The FEniCS book*, volume 84 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag Berlin Heidelberg, 2012.

[30] A. Masud and T. J. R. Hughes. A stabilized mixed finite element method for Darcy flow. *Comput. Methods Appl. Mech. Engrg.*, 191(39–40):4341–4370, 2002.

[31] Manoranjan Mishra. Hydrodynamic instabilities in porous media flows. *Math. Student*, 88(1-2):49–66, 2019.

[32] S. Pollock, L. G. Rebholz, and M. Xiao. Anderson-accelerated convergence of Picard iterations for incompressible Navier–Stokes equations. *SIAM J. Numer. Anal.*, 57(2):615–637, 2019.

[33] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford University Press, 1999.

[34] J. Schauder. Der fixpunktsatz in funktionalraümen. *Stud. Math.*, 2(1):171–180, 1930.