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Abstract

We recall the systematic formulation of Eulerian mechanics in terms of Lie derivatives along the vector field of the material points. Using the abstract properties of Lie derivatives we show that the transport via Lie derivatives generates in a natural way a Poisson structure on the chosen phase space.

The evolution equations for thermo-viscoelastic-viscoplastic materials in the Eulerian setting is formulated in the abstract framework of GENERIC (General Equations for Non-Equilibrium Reversible Irreversible Coupling). The equations may not be new, but the systematic splitting between reversible Hamiltonian and dissipative effects allows us to see the equations in a new light that is especially useful for future generalizing of the system, e.g. for adding new effects like reactive species.

In memory of Wolfgang Dreyer

1 Introduction

Motivated by the related work [MiR25] we reconsider the theory of Lie derivatives for formulation dissipative continuum-mechanical systems in the Eulerian setting. For this we provide in Section 2 an introduction to the theory of Lie derivatives, first in the differential geometrical setting involving multilinear forms and then for vectors, co-vectors, and operators as they are used in continuum mechanics. The main point here is to gain an understanding that the usage of such objects may have different interpretations in differential geometry and hence need a suitable Lie derivative. It is well-known that there are different Lie derivatives for stress tensors (stress rates) but the issue is already relevant the distinction of extensive and intensive field variables and for vectors and co-vectors such that as the momentum.

In Section 3 we first recall the GENERIC framework, where the acronym GENERIC stands for *General Equation for Non-Equilibrium Reversible-Irreversible Coupling* and was introduced in [GrÖ97]. However, this class of models originates in the metriplectic theory developed in [Mor84, Mor86], cf. the survey [Mor09]. Over the last decade, GENERIC has proved to be a versatile modeling tool for various complex coupled models for fluids and solids, see e.g. [LJCV08, Mie11, HüS12, DPZ13, PKG18, PPK20, BeS19, Las21, PT*22, ZPT23] and the references therein. We also refer to [Mie15, KM*19] for applications in semiconductor and quantum devices. In [MPZ24] it is shown that a dissipative GENERIC system can be rigorously derived from an infinite-dimensional (non-dissipative) Hamiltonian systems.

In Section 3.2 we start to develop continuum mechanics system at finite strain in a systematic way that is compatible with Lie derivatives and GENERIC. In particular, we consider elastoplastic materials

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based on the multiplicative split $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}}\boldsymbol{F}_{\mathrm{p}}$ where $\boldsymbol{F}$ remains related with the velocity field $\boldsymbol{v}$ through the kinematic relation

$$\partial_t \boldsymbol{F} = -\mathfrak{L}^{\mathrm{ve}}_{\boldsymbol{v}}\boldsymbol{F} := -\boldsymbol{v}\cdot\nabla\boldsymbol{F} + (\nabla\boldsymbol{v})\boldsymbol{F},$$

which is indeed given by a suitable Lie derivative, see Lemma 3.1.

In addition to the momentum $\boldsymbol{\pi} = \rho\boldsymbol{v}$ (with $\rho = \rho_{\mathrm{ref}}/\det\boldsymbol{F}$) and $\boldsymbol{F}$, we use the plastic distorting $\boldsymbol{F}_{\mathrm{p}} \in \mathrm{GL}(\mathbb{R}^d)$ and a scalar thermodynamical field variable τ for form the state variable $q = (\boldsymbol{\pi}, \boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)$. The variable τ can be chosen quite general, e.g. as density of internal energy, enthalpy, or entropy or as the temperature.

In Section 3.3 we discuss how the total energy $\mathcal{E}(q)$ and the total entropy $\mathcal{S}(q)$ as main objects in the GENERIC framework generate thermodynamical driving forces that generate the evolution equations

$$\partial_t q = \mathbb{J}(q)\mathrm{D}\mathcal{E}(q) + \partial_\zeta\mathcal{R}^*\big(q, \mathrm{D}\mathcal{S}(q)\big),$$

which include the balance laws of mechanics as well as the constitutive relations. Here $\mathbb{J}$ is the Poisson operator, and $\mathcal{R}^*$ is the dual dissipation potential, see Section 3.1. All the material properties for the dissipative mechanisms are contained in $\mathcal{R}^*$ and the mapping $\xi \mapsto \boldsymbol{j} := \partial_\xi\mathcal{R}^*(q,\xi)$ is called the abstract kinetic relation between the vector ξ of all thermodynamic driving forces and the vector $\boldsymbol{j}$ of all corresponding fluxes.

The main theoretical result of this paper is Theorem 3.3 that shows that there is a straightforward way for the construction of a Poisson operator $q \mapsto \mathbb{J}(q)$ for Eulerian mechanics based on the general properties for Lie derivatives. It is usually difficult to show that $\mathbb{J}$ satisfies the so-called Jacobi identity (see (3.1)), but exploiting the calculation rules for general Lie derivatives (in particular the commutator rule (2.3)) they can be established without too much efforts.

In Section 3 we then follow the full program of the GENERIC framework and derive our Eulerian model for thermo-visco-elastoplasticity using the multiplicative split $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}}\boldsymbol{F}_{\mathrm{p}}$. One of the advantages of this framework is that we are able to derive the reversible Hamiltonian part and the irreversible dissipative parts independently, namely

$$\partial_t q = \boldsymbol{V}_{\mathrm{Ham}}(q) + \boldsymbol{V}_{\mathrm{diss}}(q) \text{ with } \boldsymbol{V}_{\mathrm{Ham}}(q) := \mathbb{J}(q)\mathrm{D}\mathcal{E}(q) \text{ and } \boldsymbol{V}_{\mathrm{diss}}(q) := \partial_\zeta\mathcal{R}^*\big(q, \mathrm{D}\mathcal{S}(q)\big).$$

This allows us to study the different physical effects separately, thus providing a better overview on the physical principles.

Our final system can be written in terms of the variables $q = (\boldsymbol{\pi}, \mathsf{w})$ where $\boldsymbol{\pi} = \rho\boldsymbol{v}$ is the momentum and $\mathsf{w} = (\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)$, and it takes the form

$$\partial_t\boldsymbol{\pi} = -\mathfrak{L}^{\mathrm{mo}}_{\boldsymbol{v}}\boldsymbol{\pi} + \mathrm{div}\,\big(\boldsymbol{\Sigma}_{\mathrm{Cauchy}}(\mathsf{w}) + \mathbb{D}_{\mathrm{visc}}(\mathsf{w})\boldsymbol{D}(\boldsymbol{v})\big), \tag{1.1a}$$

$$\partial_t\boldsymbol{F} = -\mathfrak{L}^{\mathrm{ve}}_{\boldsymbol{v}}\boldsymbol{F}, \tag{1.1b}$$

$$\partial_t\boldsymbol{F}_{\mathrm{p}} = -\mathfrak{L}^{\mathrm{in}}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{p}} + \boldsymbol{F}_{\mathrm{p}}\boldsymbol{L}_{\mathrm{vi.pl}}(\mathsf{w}), \tag{1.1c}$$

$$\partial_t\tau = j^S_{\mathrm{Ham}}(\mathsf{w}) + j^E_{\mathrm{diss}}(\mathsf{w}) - \frac{1}{\partial_\tau E(\mathsf{w})}\,\mathrm{div}\Big(\mathbb{K}_{\mathrm{heat}}(\mathsf{q})\nabla\frac{1}{\Theta(\mathsf{w})}\Big). \tag{1.1d}$$

The first equation is the momentum balance including the Cauchy stress tensor and the visco-elastic term with strain-rate tensor $\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^*\big)$. The second equation states that the deformation gradient $\boldsymbol{F}$ is simply transported by a (suitable) Lie derivative, and the third equation contains the visco-plastic flow rule encoded by $\boldsymbol{L}_{\mathrm{vi.pl}}$. The last equation is a scalar thermodynamic equation which reduces (i) to the energy balance if τ is chosen as the density of the internal energy, (ii) to the

heat equation for the choice $\tau = \theta =$temperature, or (iii) to the entropy (im)balance if we choose τ as entropy density. We refer to Section 4.3 for the explanation of all symbols.

At this stage we want to highlight that the first three equations clearly show the relevance of the Lie derivative $\mathfrak{L}_{\boldsymbol{v}}$, while in the fourth equation it will only appear, if one chooses τ as an intensive or an extensive variable. Here we have added the superscripts "mo", "ve", "in" for "momentum", "vector", and "internal variable" to the Lie derivative $\mathfrak{L}_{\boldsymbol{v}}$ to indicate that the Lie derivatives depend on the tensorial nature of the variables $\boldsymbol{\pi}$, $\boldsymbol{F}$, and $\boldsymbol{F}_{\mathrm{p}}$, respectively.

Section 5 concludes the paper by a discussion of the developed theory, in particular on the relevance of the proper usage of Lie derivatives and the GENERIC framework.

2 Lie derivatives

The theory of Lie derivatives is a well-known tool in differential geometry as well as in Eulerian fluid and solid mechanics. It is relevant when tensors (functions, vectors, forces, densities, etc.) are defined over a manifold and there is a vector field $\boldsymbol{v}$ on this manifold, see e.g. [Yan55], [Cha83, Cha. 1], [MaH94, Sec. 1.6], and [MaR99, Sec. 4.3]. See also [MaH94, footnote 26] for historical remarks concerning the first applications of Lie derivatives in continuum mechanics.

2.1 Differential geometric approach

In our case the manifold is the Eulerian space $\mathbb{R}^d$ or a subset S thereof, called the body in Eulerian description. The tangent and cotangent spaces are $\mathrm{T}_{\mathsf{x}}\mathsf{S} = \mathbb{R}^d$ and $\mathrm{T}^*_{\mathsf{x}}\mathsf{S} = \mathbb{R}^d_*$, respectively. A vector field $\boldsymbol{v}$ on S means that $\boldsymbol{v}(\mathsf{x}) \in \mathrm{T}_{\mathsf{x}}\mathsf{S}$ for all $\mathsf{x} \in \mathsf{S}$, and a co-vector field $\boldsymbol{\alpha}$ satisfies $\boldsymbol{\alpha}(\mathsf{x}) \in \mathrm{T}^*_{\mathsf{x}}\mathsf{S}$. In the differential geometric setting, a tensor $\boldsymbol{A}$ of type $(i_\circ, j_\circ)$ on S, written as $\boldsymbol{A} \in \mathfrak{T}^{i_\circ}_{j_\circ}(\mathsf{S})$, means that $\boldsymbol{A}(\mathsf{x})$ is a multi-linear mapping from $(\mathrm{T}_{\mathsf{x}}\mathsf{S})^{i_\circ} \times (\mathrm{T}^*_{\mathsf{x}}\mathsf{S})^{j_\circ}$, i.e. linear in each of the $i_\circ + j_\circ$ arguments. By the dual pairing

$$(\boldsymbol{v}, \boldsymbol{\alpha}) \mapsto \langle \boldsymbol{\alpha}, \boldsymbol{v} \rangle_{\mathsf{S}} : x \mapsto \langle \boldsymbol{\alpha}(\mathsf{x}), \boldsymbol{v}(\mathsf{x}) \rangle_{\mathrm{T}_{\mathsf{x}}\mathsf{S}}$$

we can identify $\boldsymbol{v}$ with a tensor in $\mathfrak{T}^0_1(\mathsf{S})$ and $\boldsymbol{\alpha}$ with a tensor in $\mathfrak{T}^1_0(\mathsf{S})$. The anti-symmetric tensors in $\mathfrak{T}^0_{j_\circ}(\mathsf{S})$ are the differentials forms denoted by the set $\Lambda_{j_0}(\mathsf{S})$.

For a general vector field $\boldsymbol{v}$ and a tensor field $\boldsymbol{A}$, the Lie derivative is defined by taking the derivative of $\boldsymbol{A}$ along the flow of $\boldsymbol{v}$. Throughout this section we assume that all objects like $\boldsymbol{v}$ and $\boldsymbol{A}$ are independent of time. First observe, that $\boldsymbol{v}$ defines a flow map $s \mapsto \boldsymbol{\Psi}(s, \cdot) : \mathsf{S} \to \mathbb{R}^d$ such that the following ODE is satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\Psi}(s, \mathsf{x}) = \boldsymbol{v}(\boldsymbol{\Psi}(s, \mathsf{x})) \text{ for } s \in]-\delta, \delta[, \qquad \boldsymbol{\Psi}(0, \mathsf{x}) = \mathsf{x}.$$

The Lie derivative of $\boldsymbol{A}$ along $\boldsymbol{v}$ is now defined via

$$\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{A} = \big(\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\Psi}(s, \cdot)^*\boldsymbol{A}\big)\Big|_{s=0}, \tag{2.1}$$

where the pull-back $\boldsymbol{\Phi}^*\boldsymbol{A}$ of $\boldsymbol{A}$ by $\boldsymbol{\Phi}$ is defined via

$$\boldsymbol{\Phi}^*\boldsymbol{A}(x)\big[\boldsymbol{v}_1, .., \boldsymbol{v}_{i_\circ}, \boldsymbol{\alpha}_1, .., \boldsymbol{\alpha}_{j_\circ}\big] = \boldsymbol{A}(\boldsymbol{\Phi}(x))\big[\mathrm{D}\boldsymbol{\Phi}\boldsymbol{v}_1, .., \mathrm{D}\boldsymbol{\Phi}\boldsymbol{v}_{i_\circ}; \big(\mathrm{D}\boldsymbol{\Phi}\big)^{-*}\boldsymbol{\alpha}_1, .., \big(\mathrm{D}\boldsymbol{\Phi}\big)^{-*}\boldsymbol{\alpha}_{j_\circ}\big],$$

where $\mathrm{D}\boldsymbol{\Phi}$ is evaluated at $x \in \mathcal{B}$, see Appendix A. In particular, $\mathfrak{L}_{\boldsymbol{v}}$ is again a tensor of the same order as $\boldsymbol{A}$.

Because we have $\boldsymbol{\Psi}(s,\mathsf{x}) = \mathsf{x} + s\boldsymbol{v}(\mathsf{x}) + O(s^2)$ we find $\mathrm{D}\boldsymbol{\Psi}(s,\mathsf{x}) = \boldsymbol{I} + s\nabla\boldsymbol{v}(\mathsf{x}) + O(s^2)$ and conclude

$$\begin{aligned}\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{A}\big[\boldsymbol{v}_1,..,\boldsymbol{v}_{i_\circ};\boldsymbol{\alpha}_1,..,\boldsymbol{\alpha}_{j_\circ}\big] &= \big(\boldsymbol{v}\cdot\nabla\boldsymbol{A}\big)\big[\boldsymbol{v}_1,...,\boldsymbol{v}_{i_\circ};\boldsymbol{\alpha}_1,...,\boldsymbol{\alpha}_{j_\circ}\big]\\ &\quad+\sum_{i=1}^{i_\circ}\boldsymbol{A}\big[\boldsymbol{v}_1,...,(\nabla\boldsymbol{v})\boldsymbol{v}_i,...,\boldsymbol{v}_{i_\circ};\boldsymbol{\alpha}_1,...,\boldsymbol{\alpha}_{j_\circ}\big]\\ &\quad-\sum_{j=1}^{j_\circ}\boldsymbol{A}\big[\boldsymbol{v}_1,...,\boldsymbol{v}_{i_\circ};\boldsymbol{\alpha}_1,...,\big(\nabla\boldsymbol{v}\big)^*\boldsymbol{\alpha}_j,...,\boldsymbol{\alpha}_{j_\circ}\big].\end{aligned} \tag{2.2}$$

One of the fundamental properties of Lie derivatives is the commutator relation

$$\mathfrak{L}_{\boldsymbol{v}}\big(\mathfrak{L}_{\boldsymbol{w}}\boldsymbol{T}\big) - \mathfrak{L}_{\boldsymbol{w}}\big(\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{T}\big) = \mathfrak{L}_{[\![\boldsymbol{v},\boldsymbol{w}]\!]}\boldsymbol{T}, \tag{2.3}$$

where the commutator between vector fields is given by

$$[\![\boldsymbol{v},\boldsymbol{w}]\!] := (\nabla\boldsymbol{w})\boldsymbol{v} - (\nabla\boldsymbol{v})\boldsymbol{w} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} = \mathfrak{L}_{\boldsymbol{v}}\boldsymbol{w} = -\mathfrak{L}_{\boldsymbol{w}}\boldsymbol{v}.$$

It is easily checked that we have the Jacobi identity for vector fields, viz.

$$\big[\!\big[\boldsymbol{v}_1,[\![\boldsymbol{v}_2,\boldsymbol{v}_3]\!]\big]\!\big] + \big[\!\big[\boldsymbol{v}_2,[\![\boldsymbol{v}_3,\boldsymbol{v}_1]\!]\big]\!\big] + \big[\!\big[\boldsymbol{v}_3,[\![\boldsymbol{v}_1,\boldsymbol{v}_2]\!]\big]\!\big] = 0 \text{ for all } \boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3 \in \mathfrak{T}^1(\mathsf{S}). \tag{2.4}$$

The identities (2.3) and (2.4) will be extremely useful for showing that the the operator $\mathbb{J}$ to be constructed satisfies the Jacobi identity, and hence is a Poisson structure. It is not easy to find explicit statements of (2.3) in the literature, but it is an easy consequence of its validity for functions, vectors, and co-vectors (1-forms) and of the well-known derivation rule $\mathfrak{L}_{\boldsymbol{v}}(\boldsymbol{T}\otimes\boldsymbol{S}) = (\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{T})\otimes\boldsymbol{S}+\boldsymbol{T}\otimes(\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{S})$ for tensor products by doing induction over the rank of the tensors (see e.g. condition "(DO1)" in [AbM78, Ch. 2]).

Here the tensor product of $\boldsymbol{A}\in\mathfrak{T}^{i_\circ}_{j_\circ}(\mathsf{S})$ and $\boldsymbol{A}\in\mathfrak{T}^{n_\circ}_{m_\circ}(\mathsf{S})$ is the tensor $\boldsymbol{A}\otimes\boldsymbol{B}\in\mathfrak{T}^{i_\circ+n_\circ}_{j_\circ+m_\circ}(\mathsf{S})$ defined via

$$\begin{aligned}\boldsymbol{A}\otimes\boldsymbol{B}[\boldsymbol{v}_1,\dots,\boldsymbol{v}_{i_\circ+n_\circ};\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_{j_\circ+m_\circ}] &:= \boldsymbol{A}[\boldsymbol{v}_1,\dots,\boldsymbol{v}_{i_\circ};\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_{j_\circ}]\\ &\quad\cdot\boldsymbol{B}[\boldsymbol{v}_{i_\circ+1},\dots,\boldsymbol{v}_{i_\circ+n_\circ};\boldsymbol{\alpha}_{j_\circ+1},\dots,\boldsymbol{\alpha}_{j_\circ+m_\circ}].\end{aligned}$$

Inner products can be defined by tensor products and subsequent contraction of indices. For $\boldsymbol{A}\in\mathfrak{T}^{i_\circ}_{j_\circ}(\mathsf{S})$ and $n,m\in\mathbb{N}$ with $1\le n\le i_\circ$ and $1\le m\le j_\circ$ we can contract the nth vector slot with the m co-vector slot to obtain $\mathsf{C}^n_m\boldsymbol{A}\in\mathfrak{T}^{i_\circ-1}_{j_\circ-1}(\mathsf{S})$ defined via

$$\begin{aligned}&\mathsf{C}^n_m\boldsymbol{A}\big[\boldsymbol{v}_1,\dots,\boldsymbol{v}_{i_\circ-1};\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_{j_\circ-1}\big]\\ &\qquad:=\sum_{k=1}^{d}\boldsymbol{A}\big[\boldsymbol{v}_1,..,\boldsymbol{v}_{k-1},\boldsymbol{e}_k,\boldsymbol{v}_k,..,\boldsymbol{v}_{i_\circ-1};\boldsymbol{\alpha}_1,..,\boldsymbol{\alpha}_{m-1},\boldsymbol{\varepsilon}_k,\boldsymbol{\alpha}_m,\dots,\boldsymbol{\alpha}_{j_\circ-1}\big],\end{aligned}$$

where $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_d\}$ is an arbitrary basis in $\mathrm{T}_{\mathsf{x}}\mathsf{S}$ and $\boldsymbol{\varepsilon}_k\in\mathrm{T}^*_{\mathsf{x}}\mathsf{S}$ satisfy $\langle\boldsymbol{\varepsilon}_k,\boldsymbol{e}_l\rangle=\delta_{k,l}$ (the Kronecker symbol) for $k,l\in\{1,\dots,d\}$.

2.2 Differential forms

The special inner product with a vector field $\boldsymbol{w}$ is denoted by $\boldsymbol{i}_w : \mathfrak{T}^{i_\circ+1}_{j_\circ}(\mathsf{S}) \to \mathfrak{T}^{i_\circ}_{j_\circ}(\mathsf{S})$ and is given by

$$\boldsymbol{i}_{\boldsymbol{w}}\boldsymbol{A}\big[\boldsymbol{v}_1,..,\boldsymbol{v}_{i_\circ};\boldsymbol{\alpha}_1,..,\boldsymbol{\alpha}_{j_\circ}\big] = \boldsymbol{A}\big[\boldsymbol{w},\boldsymbol{v}_1,..,\boldsymbol{v}_{i_\circ};\boldsymbol{\alpha}_1,..,\boldsymbol{\alpha}_{j_\circ}\big],$$

i.e. by simple insertion into the first vector slot. Identifying $\boldsymbol{w}$ with the tensor $\boldsymbol{B}[\boldsymbol{\alpha}] = \langle \boldsymbol{\alpha}, \boldsymbol{w} \rangle$ we have the relation $\boldsymbol{i}_{\boldsymbol{w}}\boldsymbol{A} = \mathsf{C}_1^1\big(\boldsymbol{A}\otimes\boldsymbol{B}\big)$. It is not difficult to see that Lie derivatives commute with contraction, hence we also have the product rule for inner products:

$$\mathfrak{L}_{\boldsymbol{v}}\mathsf{C}_m^n\boldsymbol{A} = C_m^n\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{A} \quad \mathfrak{L}_{\boldsymbol{v}}\big(\boldsymbol{i}_{\boldsymbol{w}}\boldsymbol{A}\big) = \boldsymbol{i}_{\boldsymbol{w}}(\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{A}) + \boldsymbol{i}_{\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{w}}\boldsymbol{A}.$$

An important subclass of tensor is given by the differential forms that appear as k-forms for $k \in \{0, \dots, d\}$. They are given as tensors $\boldsymbol{\beta} \in \Lambda_m(\boldsymbol{S}) \subset \mathfrak{T}_k^0(\mathsf{S})$, where Λ_m denotes the subset of tensors that are anti-symmetric, i.e. interchanging any two arguments changes the result by a factor (-1). One can define the differential $\boldsymbol{d} : \Lambda_k(\boldsymbol{S}) \to \Lambda_{k+1}(\mathsf{S})$ and has $\boldsymbol{i}_{\boldsymbol{w}} : \Lambda_{k+1}(\boldsymbol{S}) \to \Lambda_k(\boldsymbol{S})$. For differential forms $\boldsymbol{\beta}$ one has the identities

$$\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{\beta} = \boldsymbol{i}_{\boldsymbol{v}}\boldsymbol{d}\boldsymbol{\beta} + \boldsymbol{d}(\boldsymbol{i}_{\boldsymbol{v}}\boldsymbol{\beta}) \quad \text{and} \quad \mathfrak{L}_{\boldsymbol{v}}(\boldsymbol{d}\boldsymbol{\beta}) = \boldsymbol{d}\big(\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{\beta}\big),$$

where the first one is called "Cartan's magic formula" (also known as Cartan's first structural equation) and the second follows easily from the first when using $\boldsymbol{d}\boldsymbol{d}\boldsymbol{\gamma} = 0$ for all differential forms $\boldsymbol{\gamma}$.

The wedge product $\wedge$ maps $\Lambda_k(\mathsf{S}) \times \Lambda_l(\mathsf{S})$ into $\Lambda_{k+l}(\mathsf{S})$ by $\boldsymbol{\beta} \wedge \boldsymbol{\gamma} = \mathrm{anti}(\boldsymbol{\beta}\otimes\boldsymbol{\gamma})$ and satisfies

$$\mathfrak{L}_{\boldsymbol{v}}(\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = \big(\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{\beta}\big) \wedge \boldsymbol{\gamma} + \boldsymbol{\beta} \wedge \big(\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{\gamma}\big).$$

2.3 Tensors with values in a linear space

The above theory can easily generalized to the case that the tensors $\boldsymbol{A}(x)$ do not map into $\mathbb{R}$ but into a general linear space $\boldsymbol{U}_{\mathsf{x}}$, i.e.

$$\boldsymbol{A}(x) : (\mathrm{T}_{\mathsf{x}}\mathsf{S})^{i_\circ} \times (\mathrm{T}_{\mathsf{x}}^*\mathsf{S})^{j_\circ} \to \boldsymbol{U}_{\mathsf{x}}$$

is still multi-linear in $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i_\circ}; \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{j_\circ})$. We will then speak of $\boldsymbol{U}$-valued tensors and remark that $\boldsymbol{U}_{\mathsf{x}}$ has to be independent of $\mathrm{T}_{\mathsf{x}}\mathsf{S}$ and $\mathrm{T}_{\mathsf{x}}^*\mathsf{S}$, which means that the vector-field $\boldsymbol{v}$ is not moving the points in $\boldsymbol{U}_{\mathsf{x}}$.

2.4 Lie derivatives of vectors, operators, and volume forms

A proper usage of differential geometric concepts in continuum mechanics is not standard, but we refer to [Seg23] for a careful and detailed approach in this direction. Traditionally, it is more common to use vector, co-vector and matrices, called tensors there. To distinguish these notions we use the name "operator" for matrices acting as linear mappings between linear spaces.

Even for scalar-valued functions the Lie derivatives is not simple, because functions in continuum mechanics occur in different forms. First one has *intensive* and *extensive* functions, which have the differential geometric interpretation as 0-forms (i.e. $f \in \Lambda_0(\mathsf{S})$) and volume or d-form (i.e. $\boldsymbol{\rho} \in \Lambda_d(\mathsf{S})$), respectively. In the latter case identify a function ρ with the volume form $\boldsymbol{\rho}[\boldsymbol{v}_1, \dots, \boldsymbol{v}_d] = \rho \det\big(\boldsymbol{v}_1|..|\boldsymbol{v}_d\big)$, and we obtain

$$\mathfrak{L}_{\boldsymbol{v}}f = \boldsymbol{v}\cdot\nabla f \quad \text{and} \quad \mathfrak{L}_{\boldsymbol{v}}\rho = \mathrm{div}(\rho\boldsymbol{v}). \tag{2.5}$$

In continuum mechanics the typical extensive variables are mass density ρ, internal-energy density e, entropy density s, or any one-homogeneous function of those. The typical intensive variables are concentrations, temperature $\theta = \mathrm{d}e/\mathrm{d}s$, velocity, pressure, or chemical potentials. For general functions,

not falling into the classes of intensive or extensive functions, the Lie derivative has to be obtained by representing it via intensive and extensive variables and then applying the chain rule and the appropriate Lie derivatives.

A vector field $\boldsymbol{w}$ and a co-vector field $\boldsymbol{\alpha}$ can be identified with tensors $\boldsymbol{A}_{\boldsymbol{w}} \in \mathfrak{T}^0_1(\mathsf{S})$ and $\boldsymbol{B}_{\boldsymbol{\beta}} \in \mathfrak{T}^1_0(\mathsf{S})$, respectively, by

$$\boldsymbol{A}_{\boldsymbol{w}}[\boldsymbol{\alpha}] = \langle \boldsymbol{\alpha}, \boldsymbol{w} \rangle \quad \text{and} \quad \boldsymbol{B}_{\boldsymbol{\beta}}[\boldsymbol{v}] = \langle \boldsymbol{\beta}, \boldsymbol{v} \rangle.$$

Using this, we can derive the form of the Lie derivatives from the tensor Lie derivatives, namely

$$\mathfrak{L}_{\boldsymbol{v}} \boldsymbol{w} = \boldsymbol{v} \cdot \nabla \boldsymbol{w} - (\nabla \boldsymbol{v}) \boldsymbol{w} = [\![\boldsymbol{v}, \boldsymbol{w}]\!] \quad \text{and} \quad \mathfrak{L}_{\boldsymbol{v}} \boldsymbol{\beta} = \boldsymbol{v} \cdot \nabla \boldsymbol{\beta} + (\nabla \boldsymbol{v})^* \boldsymbol{\beta}. \tag{2.6}$$

Moreover, in continuum mechanics the momentum vector $\boldsymbol{\pi} = \rho I_{\mathrm{R}} \boldsymbol{v}$ (where $I_{\mathrm{R}} : \mathrm{T}_{\mathsf{x}}\mathsf{S} \to \mathrm{T}^*_{\mathsf{x}}\mathsf{S}$) is the Riesz isomorphism) is a co-vector but its correct interpretation is as a $(d-1)$-form (by the Hodge star operator mapping $\Lambda_k(\mathsf{S})$ into $\Lambda_{d-k}(\mathsf{S})$), namely

$$\boldsymbol{\pi} = \rho I_{\mathrm{R}} \boldsymbol{v} = \boldsymbol{i}_{\boldsymbol{v}} \boldsymbol{\rho}. \tag{2.7a}$$

This leads to the Lie derivative

$$\mathfrak{L}_{\boldsymbol{v}} \boldsymbol{\pi} = \mathrm{div}(\boldsymbol{\pi} \otimes \boldsymbol{v}) + (\nabla \boldsymbol{v})^* \boldsymbol{\pi}, \tag{2.7b}$$

see [MiR25, Prop. 4.2]. A "vector" may also be considered as a tensor with values in the linear $\mathbb{R}^d$ which is not connected to $\mathrm{T}_{\mathsf{x}}\mathsf{S}$. Such a case is occurs in [BN*24, Eqn. (2.3)], where the magnetization vector $\boldsymbol{M}(t, \mathsf{x}) \in \mathbb{R}^3$ has the Lie derivative $\mathfrak{L}^{\mathrm{mag}}_{\boldsymbol{v}} \boldsymbol{M} = \boldsymbol{v} \cdot \nabla M + (\mathrm{div}\, \boldsymbol{v}) \boldsymbol{M} = \mathrm{div}\, \big(\boldsymbol{M} \otimes \boldsymbol{v} \big)$. The proper tensorial interpretation of "flux fields" is reported in [Seg23, p. 71]: "A flux field, such as the heat flux field, should be regarded as a 2-form in the three-dimensional body rather than as a vector field. As a 2-form, a heat flux field may be considered independently of the configuration of the body in space."

We have four types of operators, namely

$$\mathbb{B} : \mathrm{T}_{\mathsf{x}}\mathsf{S} \to \mathrm{T}_{\mathsf{x}}\mathsf{S}, \quad \mathbb{C} : \mathrm{T}_{\mathsf{x}}\mathsf{S} \to \mathrm{T}^*_{\mathsf{x}}\mathsf{S}, \quad \mathbb{D} : \mathrm{T}^*_{\mathsf{x}}\mathsf{S} \to \mathrm{T}^*_{\mathsf{x}}\mathsf{S}, \quad \mathbb{E} : \mathrm{T}^*_{\mathsf{x}}\mathsf{S} \to \mathrm{T}_{\mathsf{x}}\mathsf{S}.$$

They can be identified with tensors $\boldsymbol{B} \in \mathfrak{T}^1_1$, $\boldsymbol{C} \in \mathfrak{T}^2_0$, $\boldsymbol{D} \in \mathfrak{T}^1_1$, and $\boldsymbol{E} \in \mathfrak{T}^0_2$ via

$$\boldsymbol{B}[\boldsymbol{v}, \boldsymbol{\alpha}] = \langle \boldsymbol{\alpha}, \mathbb{B}\boldsymbol{v} \rangle, \quad \boldsymbol{C}[\boldsymbol{v}_1, \boldsymbol{v}_2] = \langle \mathbb{C}\boldsymbol{v}_1, \boldsymbol{v}_2 \rangle, \quad \boldsymbol{D}[\boldsymbol{v}, \boldsymbol{\alpha}] = \langle \mathbb{D}\boldsymbol{\alpha}, \boldsymbol{v} \rangle, \quad \boldsymbol{E}[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2] = \langle \boldsymbol{\alpha}_1, \mathbb{E}\boldsymbol{\alpha}_2 \rangle.$$

Thus, the corresponding Lie derivatives can be calculated by those for $\boldsymbol{B}$ to $\boldsymbol{E}$ and we obtain

$$\begin{aligned} &\mathfrak{L}_{\boldsymbol{v}} \mathbb{B} = \boldsymbol{v} \cdot \nabla \mathbb{B} + \mathbb{B}(\nabla \boldsymbol{v}) - (\nabla \boldsymbol{v}) \mathbb{B}, \qquad \mathfrak{L}_{\boldsymbol{v}} \mathbb{C} = \boldsymbol{v} \cdot \nabla \mathbb{C} + \mathbb{C}(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^* \mathbb{C}, \\ &\mathfrak{L}_{\boldsymbol{v}} \mathbb{D} = \boldsymbol{v} \cdot \nabla \mathbb{C} - \mathbb{D}(\nabla \boldsymbol{v})^* + (\nabla \boldsymbol{v})^* \mathbb{D}, \quad \mathfrak{L}_{\boldsymbol{v}} \mathbb{E} = \boldsymbol{v} \cdot \nabla \mathbb{E} - \mathbb{E}(\nabla \boldsymbol{v})^* - (\nabla \boldsymbol{v}) \mathbb{E}. \end{aligned} \tag{2.8}$$

Note that only $\mathbb{C}$ and $\mathbb{E}$ can be symmetric tensors and that in this case the two signs of the second and third terms are the same and one times we have $\nabla \boldsymbol{v}$ and the other time $(\nabla \boldsymbol{v})^*$. See also [MaH94, Ch. 1 Box 6.1].

As an example consider the Euclidean tensor $\boldsymbol{C}_{\mathrm{Euc}} \in \mathfrak{T}^2_0$ with $\boldsymbol{C}_{\mathrm{Euc}}[\boldsymbol{v}_1, \boldsymbol{v}_2] = (\boldsymbol{v}_1 | \boldsymbol{v}_2)_{\mathrm{Euc}}$ (the Euclidean scalar product). The associated operator $\mathbb{C}_{\mathrm{Euc}} = \mathbb{I}_{\mathrm{Riesz}} : \mathrm{T}_{\mathsf{x}}\mathbb{R}^3 \to \mathrm{T}^*_{\mathsf{x}}\mathbb{R}^3$ is the Riesz isomorphism. The Lie derivative provides twice the classical strain-rate tensor

$$\mathfrak{L}_{\boldsymbol{v}} \mathbb{C}_{\mathrm{Euc}} = 2\, \boldsymbol{D}(\boldsymbol{v}) \quad \text{with } \boldsymbol{D}(\boldsymbol{v}) := \frac{1}{2} \big(\mathbb{I}_{\mathrm{Riesz}} \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^* \mathbb{I}_{\mathrm{Riesz}} \big) \in \mathrm{Lin}_{\mathrm{sym}} \big(\mathbb{R}^3; (\mathbb{R}^3)^* \big). \tag{2.9}$$

There are various objective stress rates which arise from the fact that there are various stress tensor which have different tensorial properties. Similar to the construction of the momentum as a $(d-1)$

form (2.7) one also has to consider the Cauchy stress tensors $\boldsymbol{\Sigma}_{\mathrm{Cauchy}}$ as an extensive version, and its Lie derivative is the so-called Truesdell stress rate

$$\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{\Sigma}_{\mathrm{Cauchy}} = \boldsymbol{v}\cdot\nabla\boldsymbol{\Sigma}_{\mathrm{Cauchy}} + (\operatorname{div}\boldsymbol{v})\boldsymbol{\Sigma}_{\mathrm{Cauchy}} - \boldsymbol{\Sigma}_{\mathrm{Cauchy}}(\nabla\boldsymbol{v})^* - (\nabla\boldsymbol{v})\boldsymbol{\Sigma}_{\mathrm{Cauchy}}.$$

However, the Kirchhoff stress tensor $\boldsymbol{T}_{\mathrm{Kir}} = \det\boldsymbol{F}\,\boldsymbol{\Sigma}_{\mathrm{Cauchy}} = \frac{\rho_{\mathrm{ref}}}{\rho}\boldsymbol{\Sigma}_{\mathrm{Cauchy}}$ (see Section 3.2 for the last identity) has "intensive" properties, because it is the quotient of two extensive objects. Thus, $\boldsymbol{T}_{\mathrm{Kir}}$ has to be interpreted as element of $\mathfrak{T}^0_2$ like $\boldsymbol{E}$ in (2.8), and its Lie derivative is the upper convected or Oldroyd stress rate

$$\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{T}_{\mathrm{Kir}} = \boldsymbol{v}\cdot\nabla\boldsymbol{T}_{\mathrm{Kir}} - \boldsymbol{T}_{\mathrm{Kir}}(\nabla\boldsymbol{v})^* - (\nabla\boldsymbol{v})\boldsymbol{T}_{\mathrm{Kir}}.$$

Here, we do not dwell on this subject any further, but refer to [MaH94, Ch. 1 Box 6.1] for this and to [Fia08] for a way to write the Zaremba-Jaumann derivative as a Lie derivative.

3 GENERIC structures and Eulerian mechanics

The acronym GENERIC stands for

General Equation for Non-Equilibrium Reversible-Irreversible Coupling

and was introduced in [GrÖ97], but the class of models appears metriplectic systems already in [Mor84, Mor86], cf. the survey [Mor09]. Over the last decade, the GENERIC framework has proved to be a versatile modeling tool for various complex coupled models for fluids and solids, see e.g. [LJCV08, Mie11, HüS12, DPZ13, PKG18, BeS19, Las21, PT*22, ZPT23, MiR25] and the references therein.

3.1 Setup of GENERIC

We consider states q in a state space $\boldsymbol{Q}$ which is either a flat space or a smooth manifold. A GENERIC system is a quintuple $(\boldsymbol{Q}, \mathcal{E}, \mathcal{S}, \mathbb{J}, \mathbb{K})$, where the energy $\mathcal{E}$ and the entropy $\mathcal{S}$ are differentiable functions on $\boldsymbol{Q}$ with differentials $\mathrm{D}\mathcal{E}(q)$ and $\mathrm{D}\mathcal{S}(q)$ lie $\mathrm{T}_q\boldsymbol{Q}$. The geometric structures are the Poisson operator $\mathbb{J}$ for Hamiltonian dynamics and a (dual) dissipation potential $\mathcal{R}^*$, where $\mathbb{J}(q)$ maps $\mathrm{T}^*\boldsymbol{Q}$ to $\mathrm{T}\boldsymbol{Q}$ and $\mathcal{R}^*(q,\cdot) : \mathrm{T}^*_qQ \to [0,\infty]$ is a convex functional with $\mathcal{R}^*(q,0) = 0$. In many cases $\mathcal{R}^*(q,\cdot)$ is a quadratic functional given in terms of a symmetric, positive semi-definite operator $\mathbb{K}(q)$, namely $\mathcal{R}^*(q,\#xi) = \frac{1}{2}\langle\xi, \mathbb{K}(q)\xi\rangle$. Such operators $\mathbb{K}(q)$ are often called "Onsager operators" because of Onsager's fundamental reciprocal relations in [Ons31], earning him the Nobel Prize in Chemistry in 1968.

The evolution equation then takes the form

$$\frac{\partial q}{\partial t} = \mathbb{J}(q)\mathrm{D}\mathcal{E}(q) + \partial_\xi\mathcal{R}^*\big(q, \mathrm{D}\mathcal{S}(q)\big),$$

where $\partial_\xi\mathcal{R}^*(q,\cdot)$ denotes the convex subdifferential of $\mathcal{R}^*(q,\cdot)$, which is possibly set-valued (e.g. in plasticity).

The Poisson operator is defined by being skew-symmetric and satisfying the Jacobi identity, i.e.

$$\big\langle\zeta_1, \mathrm{D}\mathbb{J}(q)[\mathbb{J}(q)\zeta_2]\zeta_3\big\rangle + \text{cycl. perm.} \equiv 0 \quad \text{for all } \zeta_1,\zeta_2,\zeta_3 \in \mathrm{T}^*_q\boldsymbol{Q}. \tag{3.1}$$

The main condition for GENERIC are the so-called *non-interaction conditions*, namely

$$\text{(a) } \mathbb{J}(q)\mathrm{D}\mathcal{S}(q) \equiv 0 \qquad \text{and} \qquad \text{(b) } \mathcal{R}^*(q, \lambda\mathrm{D}\mathcal{E}(q)) \equiv 0 \text{ for all } \lambda \in \mathbb{R}. \tag{3.2}$$

By convexity, the latter condition implies $\mathcal{R}^*(q, \zeta+\lambda\mathrm{D}\mathcal{E}(q)) = \mathcal{R}^*(q, \zeta)$ for all $(q, \zeta) \in \mathrm{T}^*\boldsymbol{Q}$ and $\lambda \in \mathbb{R}$, and as a consequence we have $\big\langle \mathrm{D}\mathcal{E}(q)), \partial_\xi \mathcal{R}^*(q, \xi)\big\rangle = 0$ for all ξ.

Using the chain rule, a simple consequence of Condition (3.2.b) (and $\mathbb{J} = -\mathbb{J}^*$) is the conservation of energy along solutions, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(q(t)) = 0$. Condition (3.2.a) (together with convexity of $\mathcal{R}^*(q, \cdot)$) implies entropy increase: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(q(t)) = \big\langle \mathrm{D}\mathcal{S}(q), \partial_\xi \mathcal{R}^*\big(q, \mathrm{D}\mathcal{S}(q)\big)\big\rangle \geq 0$, i.e. the second law of thermodynamics is automatically satisfied for GENERIC systems.

However, we emphasize that (3.2) is much stronger that energy conservation and entropy increase. We refer to [Ött05] and [Mie11, Sec. 2.2] for further properties of GENERIC systems, in particular concerning additional conservation laws and the maximum entropy principle providing thermal equilibrium states when maximizing $\mathcal{S}(q)$ subject to the conserved quantities. We also refer to [MG*00] for a discussion on the differences between GENERIC and other thermodynamical approaches.

An efficient way of constructing GENERIC systems is using the "special form" of GENERIC systems as described in [Ött05, Sec. 2.3.2] (following [Edw98]) and [Mie11, Sec. 2.4+4.3]. Having a suitable "simpler" Poisson structure $\mathbb{J}_{\mathrm{simple}}$ and a "simpler" dual dissipation potential $\mathcal{R}^*_{\mathrm{simple}}$, one can define more complex structures via

$$\mathbb{J}(q) = M_{\mathcal{S}}(q)\mathbb{J}_{\mathrm{simple}}(\Phi(q))M_{\mathcal{S}}(q)^* \quad \text{and} \quad \mathcal{R}^*(q, \xi) = \mathcal{R}^*_{\mathrm{simple}}\big(q, M_{\mathcal{E}}(q)^*\xi\big) \tag{3.3a}$$

where the following additional conditions have to be met:

$$M_{\mathcal{S}}(q)^*\mathrm{D}\mathcal{S}(q) = f_*, \quad \mathbb{J}_{\mathrm{simple}}(q)f_* = 0, \quad M_{\mathcal{S}}(q) = \big(\mathrm{D}\Phi(q)\big)^{-1}, \tag{3.3b}$$

$$M_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{E}(q) = g_*, \quad \mathcal{R}^*_{\mathrm{simple}}(q, \lambda g_*) = 0 \text{ for all } \lambda \in \mathbb{R}, \tag{3.3c}$$

where f_* and g_* are fixed vectors.

The typical application occurs in continuum systems where $q = (w, \tau)$ and τ is a scalar thermodynamical variable such as temperature θ, internal energy density u, or the entropy density s. Then, we set $\Phi(w, \tau) = (w, S(w, \tau))^\top$ and obtain

$$M_{\mathcal{S}}(w, \tau) = \begin{pmatrix} I & 0 \\ \partial_w S(w, \tau) & \partial_\tau S(w, \tau) \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -\frac{\langle \partial_w S(w,\tau), \square\rangle}{\partial_\tau S(w,\tau)} & \frac{1}{\partial_\tau S(w,\tau)} \end{pmatrix}.$$

Here and below the symbol "$\square$" indicates at which position the corresponding argument from matrix multiplication has to placed. In an example this means

$$\begin{pmatrix} a\,\nabla\square & \mathrm{e}^{\square\tau}V \\ \square\nabla\phi & M\,\nabla\square \end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix} a\nabla\xi + \mathrm{e}^{\eta\tau}V \\ \xi\nabla\phi + M\,\nabla\eta \end{pmatrix}.$$

With the similar construction for $\mathcal{E}$ we obtain

$$M_{\mathcal{S}}(w, \tau)^* = \begin{pmatrix} I & -\frac{\square}{\partial_\tau S(w,\tau)}\,\partial_w S(w, \tau) \\ 0 & \frac{1}{\partial_\tau S(w,\tau)} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\partial_w S \\ 1-\partial_\tau S \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{1}{\partial_\tau S} \end{pmatrix} \text{ and}$$

$$M_{\mathcal{E}}(w, \tau)^* = \begin{pmatrix} I & -\frac{\square}{\partial_\tau E(w,\tau)}\,\partial_w E(w, \tau) \\ 0 & \frac{1}{\partial_\tau E(w,\tau)} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\partial_w E \\ 1-\partial_\tau E \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{1}{\partial_\tau E} \end{pmatrix}.$$

Clearly, we have (3.3b) with $f_* = g_* = (0,1)^\top$. The remaining relation (3.3c) will follow from the classical balance equations via $\nabla 1 \equiv 0$.

Looking at $M_{\mathcal{S}}$ the best choice for τ is the entropy density $\tau = s$ such that $S(w,s) = s$, because this gives $M_{\mathcal{S}}(w,s)^* = \mathbb{I}$. However, this choice is not optimal for $M_{\mathcal{E}}$. In engineering often $\tau = \theta$ is chosen, which makes both matrices $M_{\mathcal{S}}$ and $M_{\mathcal{E}}$ nontrivial.

3.2 Eulerian thermo-elastoplasticity and its kinematics

In Lagrangian mechanics a body is described on a reference domain $\mathrm{M} \subset \mathbb{R}^d$ with material points $\mathrm{x} \in \mathrm{M}$. The time-dependent deformation is denoted by $\mathsf{x} = \boldsymbol{y}(t,\mathrm{x})$, where $\mathsf{x} \in \mathsf{S}$ is the spatial coordinate. Denoting the inverse mapping (also called return mapping) by $\mathrm{x} = \mathrm{Y}(t,\mathsf{x})$ the Eulerian velocity $\boldsymbol{v}$ and the Eulerian deformation gradient $\boldsymbol{F}$ are given by

$$\boldsymbol{v}(t,\mathsf{x}) = \frac{\partial}{\partial t}\boldsymbol{y}\big(t,\mathrm{Y}(t,\mathsf{x})\big) \quad \text{and} \quad \boldsymbol{F}(t,\mathsf{x}) = \nabla_{\mathrm{x}}\boldsymbol{y}\big(t,\mathrm{Y}(t,\mathsf{x})\big).$$

Note that $\boldsymbol{v}$ is a vector field on $\mathsf{S} \subset \mathbb{R}^d$, whereas $\boldsymbol{F}(t,x)$ is a two-point tensor mapping $\mathrm{T}_{\mathrm{Y}(t,\mathsf{x})}\mathrm{M}$ to $\mathrm{T}_{\mathsf{x}}\mathsf{S}$. Thus, we have to understand $\boldsymbol{F}$ as a $\boldsymbol{U}$-valued 1-form on $\mathcal{S}$, where $\boldsymbol{U}_{\mathsf{x}} = \mathrm{T}_{\mathrm{Y}(t,\mathsf{x})}\mathrm{M}$. The important kinematic relation is that $\boldsymbol{F}$ is directly transported by its Lie derivative. The same happens to the Eulerian density $\rho(t,\mathsf{x}) = \rho_{\mathrm{ref}}/\det\big(\boldsymbol{F}(t,\mathsf{x})\big)$, where the referential density ρ_{ref} is assumed to be constant.

These result are well-known, and we state them to highlight their Lie derivative aspect.

Lemma 3.1 (Kinematic relation) *Considering $\boldsymbol{v}$, $\boldsymbol{F}$, and ρ as above we have the relations (see* (2.5) *and* (2.6)*)*

$$\partial_t\rho = -\mathfrak{L}_{\boldsymbol{v}}\rho = -\operatorname{div}(\rho\boldsymbol{v}) \quad \textit{and} \quad \partial_t\boldsymbol{F} = -\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{F} = -\boldsymbol{v}\cdot\nabla\boldsymbol{F} + (\nabla\boldsymbol{v})\boldsymbol{F}.$$

In finite-strain elastoplasticity, the deformation tensor $\boldsymbol{F}$ is multiplicatively decomposed by the so-called Kröner-Lee-Liu form (cf. [Krö60, LeL67])

$$\boldsymbol{F}(t,\mathsf{x}) = \boldsymbol{F}_{\mathrm{e}}(t,\mathsf{x})\,\boldsymbol{F}_{\mathrm{p}}(t,\mathsf{x}), \tag{3.4}$$

where $\boldsymbol{F}_{\mathrm{p}}(t,\mathrm{x})$ maps $\mathrm{T}_{\mathrm{Y}(t,\mathsf{x})}\mathcal{B}$ into itself, so that the Lie derivatives are

$$\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{e}} = -\boldsymbol{v}\cdot\nabla\boldsymbol{F}_{\mathrm{e}} + (\nabla\boldsymbol{v})\boldsymbol{F}_{\mathrm{e}} \quad \text{and} \quad \mathfrak{L}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{p}} = -\boldsymbol{v}\cdot\nabla\boldsymbol{F}_{\mathrm{p}}.$$

Inserting these relations into the kinematic relation for $\partial_t\boldsymbol{F}$ we obtain an additive kinematic relation between the elastic and the plastic strain rates in the form

$$\boldsymbol{F}_{\mathrm{e}}^{-1}\big(\partial_t\boldsymbol{F}_{\mathrm{e}} + \mathfrak{L}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{e}}\big) + \big(\partial_t\boldsymbol{F}_{\mathrm{p}} + \mathfrak{L}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{p}}\big)\boldsymbol{F}_{\mathrm{p}}^{-1} = 0, \tag{3.5}$$

which is the counterpart to the often-used split in the Lagrangian setting, see e.g. [Lee69, BeV94, GuA05].

We will describe our full model for thermo-elastoplasticity by the state vector

$$q = \big(\boldsymbol{\pi}, \boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau\big)^\top,$$

where $\boldsymbol{\pi} = \rho \boldsymbol{v}$ is the momentum and τ is a scalar-valued thermodynamical field variable such as the internal energy, enthalpy, entropy, or temperature. At this stage the structure remains more transparent, when stay more general, see [Mie11, BeS19].

The class of models considered here can also be described by a smaller set of variables, namely $(\boldsymbol{\pi}, \boldsymbol{F}_{\mathrm{e}}, \tau)$, see [MiR25], where then (3.5) is rewritten as $\partial_t \boldsymbol{F}_{\mathrm{e}} = -\mathfrak{L}_{\boldsymbol{v}} \boldsymbol{F}_{\mathrm{e}} - \boldsymbol{F}_{\mathrm{e}} \boldsymbol{L}_{\mathrm{p}}$ and $\boldsymbol{L}_{\mathrm{p}}$ is modeled accordingly. We hope that the present formulation is clearer, as it treats $\boldsymbol{F}_{\mathrm{p}}$ as an integral part of the state of the system.

3.3 Energy, entropy and driving forces

The total energy is given as a the sum of the kinetic energy and the stored energy in the form

$$\mathcal{E}(\boldsymbol{\pi}, \boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) = \int_{\mathcal{S}} \Big(\frac{\det \boldsymbol{F}}{2\rho_{\mathrm{ref}}} |\boldsymbol{\pi}|^2 + E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) \Big) \, \mathrm{d}x. \tag{3.6}$$

Here we have written E as a general function of $\boldsymbol{F}$, $\boldsymbol{F}_{\mathrm{p}}$, and τ, however, often it is assumed that E depends on $\boldsymbol{F}$ only through $\boldsymbol{F}_{\mathrm{e}} = \boldsymbol{F}\boldsymbol{F}_{\mathrm{p}}^{-1}$ which is a consequence of the Kröner-Lee-Liu decomposition (3.4). A typical choice would be $E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) = W(\boldsymbol{F}\boldsymbol{F}_{\mathrm{p}}^{-1}, \tau) + H(\boldsymbol{F}_{\mathrm{p}}, \tau)$, where the latter term can be used to describe (kinematic) hardening effects. At this stage the structure remains clearer, if we stay with the general function $E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)$, but later on, we will specify to the case $E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) = \widetilde{E}(\boldsymbol{F}\boldsymbol{F}_{\mathrm{p}}^{-1}, \boldsymbol{F}_{\mathrm{p}}, \tau)$.

Similarly, the entropy is given in the form

$$\mathcal{S}(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) = \int_{\mathsf{S}} S(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) \, \mathrm{dx}, \tag{3.7}$$

where no dependence on the momentum $\boldsymbol{\pi}$ is present because of Galilean invariance. Here S is assume to be strictly increasing in τ, and the temperature is defined via

$$\theta = \Theta(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) := \frac{\partial_\tau E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)}{\partial_\tau S(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)} \quad \Longleftrightarrow \quad \tau = \widehat{\tau}(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \theta).$$

Note that E and S are an densities with respect to the spatial domain $\mathsf{S} \subset \mathbb{R}^3$ and that they must be frame indifferent, i.e. $E(\boldsymbol{Q}\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) = E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)$ and $S(\boldsymbol{Q}\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau) = S(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)$ for all $\boldsymbol{Q} \in \mathrm{SO}(\mathbb{R}^d)$, which implies that $\partial_{\boldsymbol{F}} E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)\boldsymbol{F}^*$ and $\partial_{\boldsymbol{F}} S(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)\boldsymbol{F}^*$ are symmetric. Here $\boldsymbol{F}^*$ is the adjoint operator to $\boldsymbol{F} : \mathrm{T}_{\mathrm{x}}\mathrm{M} \to \mathrm{T}_{\mathrm{x}}\mathsf{S}$ with $\mathrm{x} = \boldsymbol{y}(t, \mathrm{x})$, i.e. $\boldsymbol{F}^* : \mathrm{T}^*_{\mathrm{x}}\mathsf{S} \to \mathrm{T}^*_{\mathrm{x}}\mathrm{M}$, and we will also use $\boldsymbol{F}^{-*} = (\boldsymbol{F}^*)^{-1} = (\boldsymbol{F}^{-1})^* : \mathrm{T}^*_{\mathrm{x}}\mathrm{M} \to \mathrm{T}^*_{\mathrm{x}}\mathsf{S}$.

As GENERIC structures suggest, the main driving forces for the reversible (Hamiltonian) dynamics are $\mathrm{D}\mathcal{E}(q)$ and those for the dissipative dynamics are $\mathrm{D}\mathcal{S}(q)$:

$$\mathrm{D}\mathcal{E}(q) = \begin{pmatrix} \boldsymbol{v} \\ \partial_{\boldsymbol{F}} E(..) + \frac{\rho|\boldsymbol{v}|^2}{2} \boldsymbol{F}^{-*} \\ \partial_{\boldsymbol{F}_{\mathrm{p}}} E(..) \\ \partial_\tau E(..) \end{pmatrix} \quad \text{and} \quad \mathrm{D}\mathcal{S}(q) = \begin{pmatrix} 0 \\ \partial_{\boldsymbol{F}} S(..) \\ \partial_{\boldsymbol{F}_{\mathrm{p}}} S(..) \\ \partial_\tau S(..) \end{pmatrix}.$$

However, in light of the special GENERIC structure introduced at the end of Section 3.1, it will be more natural to look at the combined generalized driving forces $M_{\mathcal{S}}(q)\mathrm{D}\mathcal{E}(q)$ and $M_{\mathcal{E}}(q)\mathrm{D}\mathcal{S}(q)$.

Proposition 3.2 *With* $e_\tau = (0,0,0,1)^\top$ *we have the relations*

$$M_{\mathcal{S}}(q)^*\mathrm{D}\mathcal{E}(q) = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\Sigma}_{\mathrm{e}}^F + \frac{\rho|\boldsymbol{v}|^2}{2}\,\boldsymbol{F}^{-*} \\ \boldsymbol{\Sigma}_{\mathrm{p}}^F \\ \Theta \end{pmatrix} \text{ and } M_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{S}(q) = \frac{-1}{\Theta(q)} M_{\mathcal{E}}(q)\mathrm{D}\mathcal{S}(q) + \frac{1+\Theta}{\Theta} e_\tau, \tag{3.8}$$

where $\boldsymbol{\Sigma}_{\mathrm{e}}^F = \partial_{\boldsymbol{F}} E(..) - \Theta(..)\partial_{\boldsymbol{F}} S(..)$ *and* $\boldsymbol{\Sigma}_{\mathrm{p}}^F = \partial_{\boldsymbol{F}_{\mathrm{p}}} E(..) - \Theta(..)\partial_{\boldsymbol{F}_{\mathrm{p}}} S(..)$.

Here $\boldsymbol{\Sigma}_{\mathrm{e}}^F$ *and* $\boldsymbol{\Sigma}_{\mathrm{p}}^F$ *are the derivatives of the free energy* $F(q) = E(q) - \Theta(q)S(q)$ *at fixed temperature with respect to* $\boldsymbol{F}$ *and* $\boldsymbol{F}_{\mathrm{p}}$*, respectively. Inverting the relation* $\theta = \Theta(w,\tau)$ *into* $\tau = \widehat{\tau}(w,\theta)$ *and setting* $\Psi(w,\theta) := F\big(w, \widehat{\tau}(w,\theta)\big)$*, this means*

$$\boldsymbol{\Sigma}_{\mathrm{e}}^F = \partial_{\boldsymbol{F}} \Psi(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \theta)\big|_{\theta=\Theta(\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)} \quad \textit{and} \quad \boldsymbol{\Sigma}_{\mathrm{p}}^F = \partial_{\boldsymbol{F}_{\mathrm{p}}} \Psi(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \theta)\big|_{\theta=\Theta(\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)}.$$

Proof. The transformation matrices in Section 3.1 read

$$M_{\mathcal{S}}(q)^* = I - \tfrac{1}{\partial_\tau S}\big(\mathrm{D}\mathcal{S}(q) - e_\tau\big)\otimes e_\tau \text{ and } M_{\mathcal{E}}(q)^* = I - \tfrac{1}{\partial_\tau E}\big(\mathrm{D}\mathcal{E}(q) - e_\tau\big)\otimes e_\tau. \tag{3.9}$$

Thus, the generalized driving forces take the form

$$M_{\mathcal{S}}(q)^*\mathrm{D}\mathcal{E}(q) = \mathrm{D}\mathcal{E}(q) - \Theta\mathrm{D}\mathcal{S}(q) + \Theta e_\tau \quad \text{and} \quad M_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{S}(q) = \mathrm{D}\mathcal{S}(q) - \frac{1}{\Theta}\mathrm{D}\mathcal{E}(q) + \frac{1}{\Theta}\, e_\tau,$$

which means that the two driving forces, except for the last scalar component, are the same up to constant $-\Theta(..)$. Hence, the second statement in (3.8) is shown.

The first relation in (3.8) follows simply by using (3.9), the formulas for $\mathrm{D}\mathcal{E}$ and $\mathrm{D}\mathcal{S}$, and the definition of Θ.

The last statement about the free energy is established in [Mie11, Eqn. (2.13)]. ∎

In fact, for the dissipative driving forces, it will be more useful to generalize $M_{\mathcal{E}}(q)^*$ to the better adapted operator $N_{\mathcal{E}}$. We emphasize that the conditions (3.3b) for $M_{\mathcal{S}}$ are much more restrictive, because the third condition asks $M_{\mathcal{S}}$ to be the inverse of a derivative. In contrast, the condition (3.3c) for $M_{\mathcal{E}}$ are simpler, e.g. $M_{\mathcal{E}} \equiv 0$ would be allowed (but not really useful). In the following we will use the operator

$$N_{\mathcal{E}}(q)^* = \begin{pmatrix} \boldsymbol{D}(\square) & 0 & 0 & \frac{-\square}{\partial_\tau E}\,\boldsymbol{D}(\boldsymbol{v}) \\ 0 & 0 & \square & \frac{-\square}{\partial_\tau E}\,\partial_{\boldsymbol{F}_{\mathrm{p}}} E \\ 0 & 0 & 0 & \square/\partial_\tau E \end{pmatrix}, \text{ i.e. } N_{\mathcal{E}}(q)^* \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\xi} \\ \boldsymbol{\eta} \\ \kappa \end{pmatrix} = \begin{pmatrix} \boldsymbol{D}(\boldsymbol{w}) - \frac{\kappa}{\partial_\tau E}\,\boldsymbol{D}(\boldsymbol{v}) \\ \boldsymbol{\eta} - \frac{\kappa}{\partial_\tau E}\,\partial_{\boldsymbol{F}_{\mathrm{p}}} E \\ \frac{\kappa}{\partial_\tau E} \end{pmatrix}. \tag{3.10}$$

By construction, we have $N_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{E}(u) = (0,0,1)^\top$ and find the adapted driving forces

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_{\mathrm{m}} \\ \boldsymbol{\eta}_{\mathrm{p}} \\ \eta_{\mathrm{t}} \end{pmatrix} = N_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{S}(q) = \begin{pmatrix} -\frac{1}{\Theta}\,\boldsymbol{D}(\boldsymbol{v}) \\ \frac{1}{\Theta}\big(\partial_{\boldsymbol{F}_{\mathrm{p}}} S - \Theta\partial_{\boldsymbol{F}_{\mathrm{p}}} E\big) \\ 1/\Theta \end{pmatrix} = \frac{1}{\Theta}\begin{pmatrix} -\boldsymbol{D}(\boldsymbol{v}) \\ -\boldsymbol{\Sigma}_{\mathrm{p}}^F \\ 1 \end{pmatrix}. \tag{3.11}$$

The major reason for choosing $N_{\mathcal{E}}(q)^*$ is that the arising driving forces are Galilean invariant, in contrast to $M_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{S}(q)$ which involves $\boldsymbol{v}$ and $\rho|\boldsymbol{v}|^2$. Thus, we will have more flexibility in making physically reasonable choices for $\mathcal{R}^*_{\mathrm{simple}}$. Note also that $N_{\mathcal{E}}(q)^*$ is not quadratic and hence not invertible (in contrast to $M_{\mathcal{S}}(q)^*$). This reflects the fact that the equation of $\boldsymbol{F}$ is purely kinematic and thus will not include any dissipative effects.

3.4 The Poisson structure for the Hamiltonian part

The main result of paper states that skew-symmetric operators $\mathbb{J}$ build with Lie derivatives automatically satisfy the Jacobi identity and hence qualify as Poisson operators for Eulerian continuum mechanics.

Theorem 3.3 (Jacobi identity via Lie derivatives) *We consider the case that the state z is given by $z = (\boldsymbol{\pi}, \boldsymbol{X})$ where $\boldsymbol{\pi}$ is the momentum (considered as a $(d-1)$ form as in* (2.7)*) and a collection of variables comprised into a tuple $\boldsymbol{X}$. We define the state-dependent skew-symmetric operator $\mathbb{J}(\boldsymbol{\pi}, \boldsymbol{X})$ via*

$$\mathbb{J}(\boldsymbol{\pi}, \boldsymbol{X}) \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} -\mathfrak{L}_{\square}\boldsymbol{\pi} & \mathbb{B}(\boldsymbol{X})\square \\ -\mathfrak{L}_{\square}\boldsymbol{X} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} -\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{\pi} + \mathbb{B}(\boldsymbol{X})\boldsymbol{\xi} \\ -\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{X} \end{pmatrix},$$

where $\mathbb{B}(\boldsymbol{X})$ is defined by skew-symmetry, namely $\langle \boldsymbol{v}, \mathbb{B}(\boldsymbol{X})\boldsymbol{\xi} \rangle = \langle \boldsymbol{\xi}, \mathfrak{L}_{\boldsymbol{v}}\boldsymbol{X} \rangle$.

Then, $\mathbb{J}$ satisfies the Jacobi identity (3.1)*, and hence provides a Poisson structure.*

Proof. The proof relies in the rules for Lie derivatives, in particular the commutator rule (2.3), and the simple observation that the mapping $(\boldsymbol{\pi}, \boldsymbol{X}) \mapsto \mathbb{J}(\boldsymbol{\pi}, \boldsymbol{X})$ is linear. A special case of our result was already obtained in [MiR25, Prop. A.1], however the present proof is significantly shorter and still self-contained.

We consider on of the three terms in the Jacobi identity and simplify it using $\mathrm{D}\mathbb{J}(\boldsymbol{\pi}, \boldsymbol{X})[\boldsymbol{\pi}_2, \boldsymbol{X}_2] = \mathbb{J}(\boldsymbol{\pi}_2, \boldsymbol{X}_2)$ (because of linearity), where $(\boldsymbol{\pi}_2, \boldsymbol{X}_2)^\top = \mathbb{J}(\boldsymbol{\pi}, \boldsymbol{X})(\boldsymbol{v}_2, \boldsymbol{\xi}_2)^\top$. We obtain

$$\begin{aligned} \mathsf{T}_{1,2,3} &:= \big\langle \tbinom{\boldsymbol{v}_1}{\boldsymbol{\xi}_1}, \mathrm{D}\mathbb{J}(\boldsymbol{\pi}, \boldsymbol{X})[\boldsymbol{\pi}_2, \boldsymbol{X}_2] \tbinom{\boldsymbol{v}_3}{\boldsymbol{\xi}_3} \big\rangle = \langle \boldsymbol{v}_1, -\mathfrak{L}_{\boldsymbol{v}_3}\boldsymbol{\pi}_2 + \mathbb{B}(\boldsymbol{X}_2)\boldsymbol{\xi}_3 \rangle - \langle \boldsymbol{\xi}_1, \mathfrak{L}_{\boldsymbol{v}_3}\boldsymbol{X}_2 \rangle \\ &= \langle \mathfrak{L}_{\boldsymbol{v}_3}\boldsymbol{v}_1, \boldsymbol{\pi}_2 \rangle + \langle \boldsymbol{\xi}_3, \mathfrak{L}_{\boldsymbol{v}_1}\boldsymbol{X}_2 \rangle - \langle \boldsymbol{\xi}_1, \mathfrak{L}_{\boldsymbol{v}_3}\boldsymbol{X}_2 \rangle \end{aligned}$$

Inserting the definitions of $\boldsymbol{\pi}_2 = -\mathfrak{L}_{\boldsymbol{v}_2}\boldsymbol{\pi} + \mathbb{B}(\boldsymbol{X})\boldsymbol{\xi}_2$ and $\boldsymbol{X}_2 = -\mathfrak{L}_{\boldsymbol{v}_2}\boldsymbol{X}$ and using the definition of $\mathbb{B}(\boldsymbol{X})$ we obtain

$$\mathsf{T}_{1,2,3} = -\langle [\![\boldsymbol{v}_3, \boldsymbol{v}_1]\!], \mathfrak{L}_{\boldsymbol{v}_2}\boldsymbol{\pi} \rangle + \langle \boldsymbol{\xi}_2, \mathfrak{L}_{[\![\boldsymbol{v}_3, \boldsymbol{v}_1]\!]}\boldsymbol{X} \rangle - \langle \boldsymbol{\xi}_3, \mathfrak{L}_{\boldsymbol{v}_1}\mathfrak{L}_{\boldsymbol{v}_2}\boldsymbol{X} \rangle + \langle \boldsymbol{\xi}_1, \mathfrak{L}_{\boldsymbol{v}_3}\mathfrak{L}_{\boldsymbol{v}_2}\boldsymbol{X} \rangle.$$

The first term can be rewritten as $\langle \mathfrak{L}_{\boldsymbol{v}_2}[\![\boldsymbol{v}_3, \boldsymbol{v}_1]\!], \boldsymbol{\pi} \rangle = \big\langle \big[\!\big[\boldsymbol{v}_2, [\![\boldsymbol{v}_3, \boldsymbol{v}_1]\!]\big]\!\big], \boldsymbol{\pi} \big\rangle$. Hence adding the corresponding cyclic permutations, we can exploit the Jacobi identity for vector fields in (2.4) and find that the contribution of the terms linear in $\boldsymbol{\pi}$ in $\mathsf{C} := \mathsf{T}_{1,2,3} + \mathsf{T}_{2,3,1} + \mathsf{T}_{3,1,2}$ cancel each other.

Similarly, we can look at all terms in C involving $\boldsymbol{\xi}_1$ and find

$$\langle \boldsymbol{\xi}_1, \mathfrak{L}_{\boldsymbol{v}_3}\mathfrak{L}_{\boldsymbol{v}_2}\boldsymbol{X} \rangle - \langle \boldsymbol{\xi}_1, \mathfrak{L}_{\boldsymbol{v}_2}\mathfrak{L}_{\boldsymbol{v}_3}\boldsymbol{X} \rangle + \langle \boldsymbol{\xi}_1, \mathfrak{L}_{[\![\boldsymbol{v}_2, \boldsymbol{v}_3]\!]}\boldsymbol{X} \rangle = 0$$

by using the commutator rule (2.3). The same holds for the terms involving $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$, and hence the Jacobi identity for $\mathbb{J}$ is established. ∎

In the sense of the special form of GENERIC we define $\mathbb{J}_{\mathrm{simple}}$ by using the special choice $\tau = s$, namely the entropy density. The reason is that we know that the Hamiltonian dynamics does not change the entropy. Hence, we know that the evolution should be the simple transport along the Eulerian vector field $\boldsymbol{v}$ as a extensive variable. Note also that $\boldsymbol{F}_{\mathrm{p}}$ is an intensive variable that is $\boldsymbol{U}$-valued with $\boldsymbol{F}_{\mathrm{p}}(t, x) \in U_x = \mathrm{GL}(\mathrm{T}_x\mathrm{M}))$. Denoting the Lie derivatives for $\boldsymbol{\pi}$, $\boldsymbol{F}$, $\boldsymbol{F}_{\mathrm{p}}$, and s by $\mathfrak{L}^{\mathrm{mo}}_{\boldsymbol{v}}$,

$\mathfrak{L}^{\mathrm{ve}}_{\boldsymbol{v}}$, $\mathfrak{L}^{\mathrm{in}}_{\boldsymbol{v}}$, and $\mathfrak{L}^{\mathrm{ex}}_{\boldsymbol{v}}$, respectively, we set

$$\mathbb{J}_{\mathrm{simple}}(\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},s)=\begin{pmatrix}-\mathfrak{L}^{\mathrm{mo}}_{\square}\boldsymbol{\pi} & \mathbb{B}^{\mathrm{ve}}(\boldsymbol{F})\square & \mathbb{B}^{\mathrm{in}}(\boldsymbol{F}_{\mathrm{p}})\square & \mathbb{B}^{\mathrm{ex}}(s)\square\\ -\mathfrak{L}^{\mathrm{ve}}_{\square}\boldsymbol{F} & 0 & 0 & 0\\ -\mathfrak{L}^{\mathrm{in}}_{\square}\boldsymbol{F}_{\mathrm{p}} & 0 & 0 & 0\\ -\mathfrak{L}^{\mathrm{ex}}_{\square}s & 0 & 0 & 0\end{pmatrix}$$

where the operator in the first row are obtained by skew symmetry from the first column:

$$\mathbb{B}^{\mathrm{ve}}(\boldsymbol{F})\boldsymbol{\Xi}_{\mathrm{e}}=\nabla\boldsymbol{F}\mathbf{:}\boldsymbol{\Xi}_{\mathrm{e}}+\operatorname{div}\big(\boldsymbol{\Xi}_{\mathrm{e}}\boldsymbol{F}^{*}\big),\quad \mathbb{B}^{\mathrm{in}}(\boldsymbol{F}_{\mathrm{p}})\boldsymbol{\Xi}_{\mathrm{p}}=\nabla\boldsymbol{F}_{\mathrm{p}}\mathbf{:}\boldsymbol{\Xi}_{\mathrm{p}},\quad \mathbb{B}^{\mathrm{ex}}(s)\xi=-s\nabla\xi.$$

The full Poisson structure $\mathbb{J}(\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)$ is now obtain by using the transformation $s=S(\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)$ inducing the transformation operator $M_{\mathcal{S}}$ resulting in

$$\mathbb{J}(\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)=M_{\mathcal{S}}(\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)\mathbb{J}_{\mathrm{simple}}\big(\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},S(\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)\big)M_{\mathcal{S}}(\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)^{*}.$$

From Theorem 3.3 we know that $\mathbb{J}_{\mathrm{simple}}$ is a Poisson structure, hence the transformation shows that $\mathbb{J}$ is again a Poisson structure.

4 The Eulerian form of thermo-visco-elastoplasticity

We now collect the evolutionary equations for thermo-visco-elastoplasticity at finite strain from the GENERIC form. Abbreviating $q=(\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)$ we have

$$\begin{aligned}&\partial_t q=\boldsymbol{V}_{\mathrm{Ham}}(q)+\boldsymbol{V}_{\mathrm{diss}}(q)\\ &\text{with}\quad \boldsymbol{V}_{\mathrm{Ham}}(q)=\mathbb{J}(q)\mathrm{D}\mathcal{E}(q)\ \text{ and }\ \boldsymbol{V}_{\mathrm{diss}}(q)=N_{\mathcal{E}}(q)\partial_{\xi}\mathcal{R}^{*}_{\mathrm{simple}}\big(q,N_{\mathcal{E}}(q)^{*}\mathrm{D}\mathcal{S}(q)\big).\end{aligned}$$

One of the advantages of the GENERIC framework is that we can derive the Hamiltonian part and the dissipative part completely independently, thus separating the two quite different physical phenomena.

4.1 The Hamiltonian part of the model

We now discuss the terms arising from of the Hamiltonian part of the dynamics, namely $\frac{\partial q}{\partial t}=\boldsymbol{V}_{\mathsf{Ham}}(q)=\mathbb{J}(q)\mathrm{D}\mathcal{E}(q)$. Using the form of $\mathcal{E}$ and the definition of $\mathbb{J}$ we obtain

$$\boldsymbol{V}_{\mathsf{Ham}}(q)=\mathbb{J}(q)\mathrm{D}\mathcal{E}(q)=\begin{pmatrix}-\operatorname{div}\big(\rho\boldsymbol{v}\otimes\boldsymbol{v}\big)+\operatorname{div}\boldsymbol{\Sigma}_{\mathsf{Cauchy}}\\ -\boldsymbol{v}\cdot\nabla\boldsymbol{F}+(\nabla\boldsymbol{v})\,\boldsymbol{F}\\ -\boldsymbol{v}\cdot\nabla\boldsymbol{F}_{\mathrm{p}}\\ -\boldsymbol{v}\cdot\nabla\tau-\dfrac{S(..)\operatorname{div}\boldsymbol{v}+\partial_{\boldsymbol{F}}S(..)\boldsymbol{F}^{*}\mathbf{:}\boldsymbol{D}(\boldsymbol{v})}{\partial_{\tau}S(..)}\end{pmatrix},\tag{4.1}$$

where the Cauchy stress tensor $\boldsymbol{\Sigma}_{\mathsf{Cauchy}}$ is defined via the free-energy density $\Psi=E-\Theta S$:

$$\boldsymbol{\Sigma}_{\mathsf{Cauchy}}=\boldsymbol{\Sigma}^{F}_{\mathrm{e}}\boldsymbol{F}^{*}+\Psi\,\mathbb{I}\quad\text{with}\ \ \boldsymbol{\Sigma}^{F}_{\mathrm{e}}=\partial_{\boldsymbol{F}}E(\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)-\Theta(\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau)\,\partial_{\boldsymbol{F}}S(\boldsymbol{F},\boldsymbol{F}_{\mathrm{p}},\tau).$$

A simplified version of (4.1) was derived in in [HüT08, Sec. 3], where $\boldsymbol{F}_{\mathrm{p}}$ is neglected and $tau=\theta$ is chosen.

The rest of this subsection will explain the form of the four components of $\boldsymbol{V}_{\mathsf{Ham}}(q)$ in detail. For this it is important to keep the adapted reversible driving forces $M_{\mathcal{S}}(q)^*\mathrm{D}\mathcal{E}(q)$ from (3.8) in mind, where the first component remains $\boldsymbol{v}$. This vector will be applied to $M_{\mathcal{S}}(q)\mathbb{J}_{\mathrm{simple}}(\Phi(q))$ to yield $\boldsymbol{V}_{\mathsf{Ham}}(q) =$

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ -\frac{\partial_{\boldsymbol{v}} S}{\partial_\tau S} & -\frac{\partial_{\boldsymbol{F}} S}{\partial_\tau S} & -\frac{\partial_{\boldsymbol{F}_{\mathrm{p}}} S}{\partial_\tau S} & \frac{1}{\partial_\tau S} \end{pmatrix} \begin{pmatrix} -\mathfrak{L}^{\mathrm{mo}}_{\square}\boldsymbol{\pi} & \mathbb{B}^{\mathrm{ve}}(\boldsymbol{F}) & \mathbb{B}^{\mathrm{in}}(\boldsymbol{F}_{\mathrm{p}}) & \mathbb{B}^{\mathrm{ex}}(S) \\ -\mathfrak{L}^{\mathrm{ve}}_{\square}\boldsymbol{F} & 0 & 0 & 0 \\ -\mathfrak{L}^{\mathrm{in}}_{\square}\boldsymbol{F}_{\mathrm{p}} & 0 & 0 & 0 \\ -\mathfrak{L}^{\mathrm{ex}}_{\square} S & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\Sigma}^F_{\mathrm{e}} + \frac{\rho|\boldsymbol{v}|^2}{2}\boldsymbol{F}^{-*} \\ \boldsymbol{\Sigma}^F_{\mathrm{p}} \\ \Theta \end{pmatrix}$$

Note that the first matrix $M_{\mathcal{S}}(q)$ does not change the first three components; hence, the second and third components of $\boldsymbol{V}_{\mathsf{Ham}}(q)$ are simply given by $-\mathfrak{L}^{\mathrm{vec}}_{\boldsymbol{v}}\boldsymbol{F}$ and $-\mathfrak{L}^{\mathrm{int}}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{p}}$.

For the fourth component we can take advantage from $pl\boldsymbol{v}S \equiv 0$; hence it is a linear combination of the three Lie derivatives, namely

$$\frac{1}{\partial_\tau S}\Big(\partial_{\boldsymbol{F}} S{:}\mathfrak{L}^{\mathrm{ve}}_{\boldsymbol{v}}\boldsymbol{F} + \partial_{\boldsymbol{F}_{\mathrm{p}}} S{:}\mathfrak{L}^{\mathrm{in}}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{p}} - \mathfrak{L}^{\mathrm{ex}}_{\boldsymbol{v}} S\Big) = -\boldsymbol{v}{\cdot}\nabla\tau - \frac{S(..)\operatorname{div}\boldsymbol{v} + \partial_{\boldsymbol{F}} S(..)\boldsymbol{F}^*{:}\boldsymbol{D}(\boldsymbol{v})}{\partial_\tau S(..)}.$$

For the last identity, we use that the terms involving $\boldsymbol{v}{\cdot}\nabla\boldsymbol{F}$ and $\boldsymbol{v}{\cdot}\nabla\boldsymbol{F}_{\mathrm{p}}$ arising from $\mathfrak{L}^{\mathrm{ve}}_{\boldsymbol{v}}\boldsymbol{F}$ and $\mathfrak{L}^{\mathrm{in}}_{\boldsymbol{v}}\boldsymbol{F}_{\mathrm{p}}$, respectively, cancel with those arising from $\mathfrak{L}^{\mathrm{ex}}_{\boldsymbol{v}} S = \operatorname{div}(S(..)\boldsymbol{v})$. The last term in the above relation equals $\partial_{\boldsymbol{F}} S(..){:}\big((\nabla\boldsymbol{v})\boldsymbol{F}^*\big)$ because of the symmetry of $\partial_{\boldsymbol{F}} S(..)\boldsymbol{F}^*$.

Thus, it remains to establish the simple representation of the first component of $\boldsymbol{V}_{\mathsf{Ham}}(q)$ in (4.1). For this, we first calculate the terms involving $\boldsymbol{v}$ (see also [ZPT23, Eqn. (4.6)]), namely

$$\begin{aligned} -\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{\pi} + \mathbb{B}^{\mathrm{ve}}(\boldsymbol{F})\tfrac{\rho|\boldsymbol{v}|^2}{2}\boldsymbol{F}^{-*} &= -\operatorname{div}(\boldsymbol{\pi}\otimes\boldsymbol{v}) - (\nabla\boldsymbol{v})^*\boldsymbol{\pi} + \frac{\rho|\boldsymbol{v}|^2}{2}\boldsymbol{F}^{-*}{:}\nabla\boldsymbol{F} + \operatorname{div}\Big(\frac{\rho|\boldsymbol{v}|^2}{2}\mathbb{I}\Big) \\ &= -\operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \rho\,\nabla\big(\tfrac{|\boldsymbol{v}|^2}{2}\big) + \tfrac{\rho|\boldsymbol{v}|^2}{2\det\boldsymbol{F}}\nabla\det\boldsymbol{F} + \nabla\big(\rho\tfrac{|\boldsymbol{v}|^2}{2}\big) = -\operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}), \end{aligned}$$

because $\rho = \rho_{\mathrm{ref}}/\det\boldsymbol{F}$ implies $\nabla\rho = -(\rho/\det\boldsymbol{F})\nabla\det\boldsymbol{F}$, where ρ_{ref} is a constant.

To simplify the the remaining terms in the first component we first use $\Psi = E - \Theta S$ to obtain

$$\begin{aligned} \nabla\Psi &= \nabla E - \Theta\nabla S - S\nabla\Theta \\ &= \nabla\boldsymbol{F}{:}(\partial_{\boldsymbol{F}} E - \Theta\partial_{\boldsymbol{F}} S) + \nabla\boldsymbol{F}_{\mathrm{p}}{:}(\partial_{\boldsymbol{F}_{\mathrm{p}}} E - \Theta\partial_{\boldsymbol{F}_{\mathrm{p}}} S) + \underbrace{(\partial_\tau E - \Theta\partial_\tau S)}_{=0}\nabla\tau - S\nabla\Theta \\ &= \nabla\boldsymbol{F}{:}\boldsymbol{\Sigma}^F_{\mathrm{e}} + \nabla\boldsymbol{F}_{\mathrm{p}}{:}\boldsymbol{\Sigma}^F_{\mathrm{p}} - S\nabla\Theta, \end{aligned}$$

where we recall $\boldsymbol{\Sigma}^F_{\mathrm{e,p}}$ from Proposition 3.2. Hence, the remain terms are

$$\begin{aligned} &\mathbb{B}^{\mathrm{ve}}(\boldsymbol{F})\boldsymbol{\Sigma}^F_{\mathrm{e}} + \mathbb{B}^{\mathrm{in}}(\boldsymbol{F}_{\mathrm{p}})\boldsymbol{\Sigma}^F_{\mathrm{p}} + \mathbb{B}^{\mathrm{ex}}(S)\Theta \\ &= \operatorname{div}(\boldsymbol{\Sigma}^F_{\mathrm{e}}\boldsymbol{F}^*) + \nabla\boldsymbol{F}{:}\boldsymbol{\Sigma}^F_{\mathrm{e}} + \nabla\boldsymbol{F}_{\mathrm{p}}{:}\boldsymbol{\Sigma}^F_{\mathrm{p}} - S\nabla\Theta = \operatorname{div}\big(\boldsymbol{\Sigma}^F_{\mathrm{e}}\boldsymbol{F}^* + \Psi I\big). \end{aligned}$$

Hence, also the desired form of the first component in (4.1) is established.

4.2 The dissipative part of Eulerian thermo-elastoplasticity

The modeling of the dissipative effects is considerably simpler than that of the reversible Hamiltonian part, because the their is much more freedom in choosing dissipation potentials than in choosing

Poisson structures. One of the main advantages of GENERIC is indeed the very structured modeling of dissipative processes allowing for general couplings. Using the subdifferential of the dual dissipation potential will automatically satisfy the so-called Onsager symmetries. In the quadratic case $\mathcal{R}^*(q,\xi) = \frac{1}{2}\langle \xi, \mathbb{A}(q)\xi\rangle$ this is seen by the fact that $\partial_\xi \mathcal{R}^*(q,\xi) = \mathbb{K}(q)\xi$ with $\mathbb{K}(q) = \frac{1}{2}\big(\mathbb{A}(q)+\mathbb{A}(q)\big) = \mathbb{K}(q)^*$, which is Onsager's reciprocal relation. Note that $\mathbb{K}(q) = \mathrm{D}^2\mathcal{R}^*(q,0)$ is the symmetric Hessian of $\mathcal{R}^*(q,\cdot)$.

Using the special form of GENERIC as described in (3.3) we can construct suitable nonlinear dissipation potentials $\mathcal{R}^*$ (or linear Onsager operators $\mathbb{K}$) by collecting the building blocks of the dissipative effects and combining them with a nontrivial operator $N_\mathcal{E}$ in the form

$$\mathcal{R}^*(q,\boldsymbol{\zeta}) = \mathcal{R}^*_{\mathrm{simple}}(q, N_\mathcal{E}(q)^*\boldsymbol{\zeta}) \quad \text{or} \quad \mathbb{K}(q) = N_\mathcal{E}(q)\mathbb{K}_{\mathrm{simple}}(q)N_\mathcal{E}(q)^*.$$

This strategy is propagated in [Ött05, Sec. 2.3.2] and follows [Edw98].

In our model we can have three dissipative processes:

(A) viscoelastic dissipation induced by $\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^\top\big)$,

(B) plastic dissipation induced by $\partial_t \boldsymbol{F}_{\mathrm{p}}$,

(C) heat flow induced by $\nabla(1/\theta)$.

Thus, we can construct a suitable dual dissipation potential in the additive form

$$\mathcal{R}^*_{\mathrm{simple}} = \mathcal{R}^*_{\mathrm{A}} + \mathcal{R}^*_{\mathrm{B}} + \mathcal{R}^*_{\mathrm{C}}.$$

However, we emphasize that this simplistic assumption is by far not necessary. Of course, it is possible to construct much more general thermodynamically consistent models where there is a strong interaction of the different dissipation mechanics, but to keep simplicity and clarity we will restrict our approach to the case of a simple block structure.

The advantage of using an operator $N_\mathcal{E}$ is three-fold. First, it is used to guarantee the second noninteraction condition by asking

$$N_\mathcal{E}(q)^*\mathrm{D}\mathcal{E}(q) = e_\tau = (0,...,0,1)^\top \quad \text{and} \quad \mathcal{R}^*_{\mathrm{simple}}\big(q, \lambda\, e_\tau\big)^\top\big) = 0 \ \text{ for all } \ \lambda \in \mathbb{R}. \tag{4.2}$$

Here $\lambda \in \mathbb{R}$ stand for the constant (reciprocal of the) temperature $1/\theta$, which does not generate any dissipation.

The second advantage is the fundamental observation in [Mie11, Sec. 4.3] that the dissipative (a.k.a. irreversible) driving forces are now given by the generalized driving forces

$$\boldsymbol{\eta} = N_\mathcal{E}(q)^*\mathrm{D}\mathcal{S}(q)\,,$$

which contains important thermodynamical information by identifying the correct combinations of $\mathrm{D}\mathcal{S}$ and $\mathrm{D}\mathcal{E}$ governing dissipative processes and satisfying Galilean invariance.

Finally, the operator $N_\mathcal{E}$ acting from the left on $\partial\mathcal{R}^*_{\mathrm{simple}}$ adjusts the dissipative terms in such a way that energy conservation holds for

$$\partial_\xi \mathcal{R}^*(q, \mathrm{D}\mathcal{S}(q)) = N_\mathcal{E}(q)\,\partial\mathcal{R}^*_{\mathrm{simple}}\big(q, N_\mathcal{E}(q)^*\mathrm{D}\mathcal{S}(q)\big).$$

Moreover, this form involving $N_\mathcal{E}$ and $N_\mathcal{E}^*$ guarantees the Onsager symmetries: if $\mathcal{R}^*_{\mathrm{simple}}(q,\boldsymbol{\eta}) = \frac{1}{2}\langle \boldsymbol{\eta}, \mathbb{K}_{\mathrm{simple}}(q)\boldsymbol{\eta}\rangle$ is quadratic, then $\mathcal{R}^*$ is still quadratic with $\mathcal{R}^*(q,\boldsymbol{\xi}) = \frac{1}{2}\langle \boldsymbol{\xi}, \mathbb{K}(q)\boldsymbol{\xi}\rangle$, where $\mathbb{K}(q) = N_\mathcal{E}(q)\mathbb{K}_{\mathrm{simple}}(q)N_\mathcal{E}(q)^* = \mathbb{K}(q)^*$ is again symmetric.

We recall our special choice for $N_{\mathcal{E}}(q)^*$ from (3.10) and find the adjoint

$$N_{\mathcal{E}}(q) = \begin{pmatrix} -\operatorname{div}(\square) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ \frac{-1}{\partial_\tau E}\boldsymbol{D}(\boldsymbol{v}){:}\square & \frac{-1}{\partial_\tau E}\partial_{\boldsymbol{F}_{\mathrm{p}}} E{:}\square & \frac{1}{\partial_\tau E} \end{pmatrix}.$$

The three 0's in the second row indicate that dissipative forces cannot change the kinematic relation of $\boldsymbol{F}$.

The full dual dissipation potential takes the form

$$\mathcal{R}^*(q,\boldsymbol{\zeta}) = \mathcal{R}^*_{\mathrm{simple}}(q, N_{\mathcal{E}}(q)^*\boldsymbol{\zeta}),$$

and now assume that $\mathcal{R}^*_{\mathrm{simple}}$ has a block structure

$$\mathcal{R}^*_{\mathrm{simple}}(q,\boldsymbol{\eta}) = \mathcal{R}^*_{\mathrm{vi.el}}(q,\boldsymbol{\eta}_{\mathrm{m}}) + \mathcal{R}^*_{\mathrm{vi.pl}}(q,\boldsymbol{\eta}_{\mathrm{p}}) + \mathcal{R}^*_{\mathrm{heat}}(q,\eta_{\mathrm{t}}).$$

Yet, we hasten to say that this is a simplification that is not necessary at all; in fact, it is one of the big advantages of the GENERIC framework that it can easily handle coupling phenomena between different effects.

Even for the these five blocks we only write the simplest forms and leave the study of more general dissipation potentials to future work.

$$\mathcal{R}^*_{\mathrm{vi.el}}(q,\boldsymbol{\eta}_{\mathrm{m}}) = \int_{\Omega} \frac{\Theta}{2}\boldsymbol{\eta}_{\mathrm{m}}{:}\mathbb{D}_{\mathrm{vi.el}}(q)\boldsymbol{\eta}_{\mathrm{m}}\,\mathrm{d}x, \quad \mathcal{R}^*_{\mathrm{vi.pl}}(q,\boldsymbol{\eta}_{\mathrm{p}}) = \int_{\Omega} R^*_{\mathrm{vi.pl}}\big(q,\Theta\boldsymbol{F}^*_{\mathrm{p}}\boldsymbol{\eta}_{\mathrm{p}}\big)\,\mathrm{d}x, \tag{4.3a}$$

$$\mathcal{R}^*_{\mathrm{heat}}(q,\eta_{\mathrm{t}}) = \int_{\Omega} \frac{1}{2}\nabla\eta_{\mathrm{t}}{\cdot}\mathbb{K}_{\mathrm{heat}}(q)\nabla\eta_{\mathrm{t}}\,\mathrm{d}x. \tag{4.3b}$$

Here the pre-multiplication of $\boldsymbol{\eta}_{\mathrm{p}}$ by $\boldsymbol{F}^*_{\mathrm{p}}$ will generate the so-called plastic indifference, see [Mie03, Sec. 3.1] and below.

With these choices the dissipative (irreversible) part $\boldsymbol{V}_{\mathrm{diss}}(q) = \partial_{\boldsymbol{\zeta}}\mathcal{R}^*\big(q,\mathrm{D}\mathcal{S}(q)\big)$ of the evolution takes the form

$$\boldsymbol{V}_{\mathrm{diss}}(q) = N_{\mathcal{E}}(q)\partial_{\boldsymbol{\eta}}\mathcal{R}^*_{\mathrm{simple}}\big(q,N_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{S}(q)\big) = \begin{pmatrix} -\operatorname{div}\big(\partial_{\boldsymbol{\eta}_{\mathrm{m}}}\mathcal{R}^*_{\mathrm{vi.el}}(q,\boldsymbol{\eta}_{\mathrm{m}})\big) \\ 0 \\ \partial_{\boldsymbol{\eta}_{\mathrm{p}}}\mathcal{R}^*_{\mathrm{vi.pl}}(q,\boldsymbol{\eta}_{\mathrm{p}}) \\ \frac{1}{\partial_\tau E}\,j_{\mathrm{ener}} \end{pmatrix}$$

$$\text{with } j_{\mathrm{ener}} = -\partial_{\boldsymbol{\eta}_{\mathrm{m}}}\mathcal{R}^*_{\mathrm{vi.el}}(..){:}\boldsymbol{D}(\boldsymbol{v}) - \partial_{\boldsymbol{\eta}_{\mathrm{p}}}\mathcal{R}^*_{\mathrm{vi.pl}}(..){:}\partial_{\boldsymbol{F}_{\mathrm{p}}}E(q) + \partial_{\eta_{\mathrm{t}}}R^*_{\mathsf{heat}}(q,\eta_{\mathrm{t}}),$$

Inserting $\boldsymbol{\eta} = N_{\mathcal{E}}(q)^*\mathrm{D}\mathcal{S}(q)$ from (3.11) and the choices (4.3) we arrive at

$$\boldsymbol{V}_{\mathrm{diss}}(q) = \begin{pmatrix} \operatorname{div}\big(\mathbb{D}_{\mathrm{visc}}(q)\boldsymbol{D}\big) \\ 0 \\ \boldsymbol{F}_{\mathrm{p}}\boldsymbol{L}_{\mathrm{vi.pl}} \\ \frac{1}{\partial_\tau E}\Big(\boldsymbol{D}{:}\mathbb{D}_{\mathrm{visc}}\boldsymbol{D} + (\boldsymbol{F}_{\mathrm{p}}\boldsymbol{L}_{\mathrm{vi.pl}}){:}\partial_{\boldsymbol{F}_{\mathrm{p}}}E(q) - \operatorname{div}\big(\mathbb{K}_{\mathrm{heat}}\nabla\frac{1}{\Theta}\big)\Big) \end{pmatrix},$$

where $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\big(\mathbb{I}_{\mathrm{Riesz}}\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^*\mathbb{I}_{\mathrm{Riesz}}\big)$ (cf. (2.9)) and $\boldsymbol{L}_{\mathrm{vi.pl}} = \Theta(q)\partial_{\boldsymbol{\xi}_{\mathrm{p}}}R^*_{\mathrm{vi.pl}}\big(q,-\boldsymbol{F}^*_{\mathrm{p}}\boldsymbol{\Sigma}^F_{\mathrm{p}}\big)$ with $\boldsymbol{\Sigma}^F_{\mathrm{p}} = \partial_{\boldsymbol{F}_{\mathrm{p}}}E(q) - \Theta(q)\partial_{\boldsymbol{F}_{\mathrm{p}}}S(q)$.

4.3 The Eulerian model for thermo-visco-elastoplasticity

We can now assemble the whole continuum model for non-isothermal finite-strain visco-elastoplasticity as obtained from the GENERIC framework:

$$\dot{q} = \boldsymbol{V}_{\mathrm{Ham}}(q) + \boldsymbol{V}_{\mathrm{diss}}(q) = \mathbb{J}(q)\mathrm{D}\mathcal{E}(q) + \partial_\zeta \mathcal{R}^*\big(q, \mathrm{D}\mathcal{S}(q)\big).$$

Combining the results from Sections 4.1 and 4.2, this leads to the following system for $(\boldsymbol{v}, \mathsf{q})$ with $\mathsf{q} = (\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, \tau)$:

$$\partial_t(\rho\boldsymbol{v}) = -\operatorname{div}\big(\rho\boldsymbol{v}\otimes\boldsymbol{v}\big) + \operatorname{div}\big(\boldsymbol{\Sigma}_{\mathrm{Cauchy}}(\mathsf{q}) + \mathbb{D}_{\mathrm{visc}}(\mathsf{q})\boldsymbol{D}\big), \tag{4.4a}$$

$$\partial_t \boldsymbol{F} = -\mathfrak{L}_{\boldsymbol{v}}\boldsymbol{F} = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{F} + (\nabla\boldsymbol{v})\boldsymbol{F}, \tag{4.4b}$$

$$\partial_t \boldsymbol{F}_{\mathrm{p}} = -\boldsymbol{v}\cdot\nabla\boldsymbol{F}_{\mathrm{p}} + \boldsymbol{F}_{\mathrm{p}}\boldsymbol{L}_{\mathrm{vi.pl}}(\mathsf{q}), \tag{4.4c}$$

$$\partial_t \tau = j^S_{\mathrm{Ham}}(\mathsf{q}) + j^E_{\mathrm{diss}}(\mathsf{q}) - \frac{1}{\partial_\tau E(\mathsf{q})}\operatorname{div}\Big(\mathbb{K}_{\mathrm{heat}}(\mathsf{q})\nabla\frac{1}{\Theta(\mathsf{q})}\Big), \tag{4.4d}$$

where $\rho = \rho_{\mathrm{ref}}/\det\boldsymbol{F}$ and where we have

$$\boldsymbol{\Sigma}_{\mathrm{Cauchy}}(\mathsf{q}) = \big(\partial_{\boldsymbol{F}}E(\mathsf{q}) - \Theta(\mathsf{q})\partial_{\boldsymbol{F}}S(\mathsf{q})\big)\boldsymbol{F}^* + \big(E(\mathsf{q}) - \Theta(\mathsf{q})S(\mathsf{q})\big)\mathbb{I}, \tag{4.5a}$$

$$\boldsymbol{L}_{\mathrm{vi.pl}}(\mathsf{q}) = \Theta(\mathsf{q})\partial_{\boldsymbol{\xi}_{\mathrm{p}}}R^*_{\mathrm{vi.pl}}\big(\mathsf{q}, -\boldsymbol{F}^*_{\mathrm{p}}\boldsymbol{\Sigma}^F_{\mathrm{p}}\big) \text{ with } \boldsymbol{\Sigma}^F_{\mathrm{p}} = \partial_{\boldsymbol{F}_{\mathrm{p}}}E(\mathsf{q}) - \Theta(\mathsf{q})\partial_{\boldsymbol{F}_{\mathrm{p}}}S(\mathsf{q}), \tag{4.5b}$$

$$j^S_{\mathrm{Ham}}(\mathsf{q}) = -\boldsymbol{v}\cdot\nabla\tau - \frac{1}{\partial_\tau S(\mathsf{q})}\Big(S(\mathsf{q})\operatorname{div}\boldsymbol{v} + \partial_{\boldsymbol{F}}S(\mathsf{q})\boldsymbol{F}^*\!:\!\boldsymbol{D}(\boldsymbol{v})\Big), \tag{4.5c}$$

$$j^E_{\mathrm{diss}}(\mathsf{q}) = \frac{1}{\partial_\tau E(\mathsf{q})}\Big(\boldsymbol{D}(\boldsymbol{v})\!:\!\mathbb{D}_{\mathrm{visc}}(\mathsf{q})\boldsymbol{D}(\boldsymbol{v}) + \big(\boldsymbol{F}_{\mathrm{p}}\boldsymbol{L}_{\mathrm{vi.pl}}(\mathsf{q})\big)\!:\!\partial_{\boldsymbol{F}_{\mathrm{p}}}E(\mathsf{q})\Big). \tag{4.5d}$$

Specifying the choice of τ either to $\tau = e$, the density of the internal energy giving $E(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, e) = e$, or to $\tau = s$, the density of the entropy giving $S(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}, s) = s$, reveals more structure to the terms $j^S_{\mathrm{Ham}}(\mathsf{q}) + j^E_{\mathrm{diss}}(\mathsf{q})$. In the first case we can read of the heating contribution through mechanical processes, and in the second case we see the entropy production through plasticity. We refer to [MiR25, Sec. 4.6] for a more elaborate discussion.

5 Discussion and outlook

In this paper we have brought together the theory of Lie derivatives and the GENERIC framework to provide the right modeling tools for Eulerian mechanics of solid materials.

The GENERIC framework takes care that the model is thermodynamically consistent in the sense that energy is conserved (first law of thermodynamics) and that the entropy is non-decreasing (second law of thermodynamics). However, the GENERIC framework enforces an additional structural condition on the reversible/conservative part and on the irreversible/dissipative part of the system. The reversible part has to be strictly Hamiltonian which is more restrictive than being energy and entropy preserving. For this it would be sufficient to have the skew-symmetry $\mathbb{J}(q)^* = -\mathbb{J}(q)$ and the non-interaction condition $\mathbb{J}(q)\mathrm{D}\mathcal{S}(q) \equiv 0$, but GENERIC imposes the Jacobi identity (3.1). The irreversible part is restricted by enforcing that the kinetic relations for the dissipative processes derive from the (sub)-differential of a dissipation potential, which automatically enforces Onsager's symmetry relations.

Thus, GENERIC systems form a subclass of all thermodynamically consistent systems. The additional structural conditions may be restrictive but appear to be valid for many continuum models. In particular,

in situations where complex coupled models are to be developed, the framework provides easy ways for the derivation of good coupling terms between different variables; see e.g. [KM*19, Sec. 3.4+4.6], where a reaction-diffusion system for charge carriers is coupled to a quantum mechanical system. We expect that the GENERIC framework will also be very useful in solid mechanics when in addition to deformation and temperature also internal variables such as a plastic distortion, phase indicators, dislocation densities, or magnetization have to be modeled, see e.g. [GK*16, BN*24].

This approach was used already very successfully in [Mie11, HüS12, GK*16] for models based on the Lagrangian description, where a referential body M is used for describing all the fields. While the Eulerian perspective working in the spatial domain $\mathsf{S} \subset \mathbb{R}^d$ is commonly used in fluid mechanics, see e.g. [GrÖ97, Ött05, ZPT23], the combination of GENERIC, Eulerian description, and solid mechanics was brought together only recently in [MiR25]. One of the reasons for using the Eulerian description for solids is the growing mathematical interest in geophysical flows in the upper mantle of our planet Earth, see e.g. [RoT21, Rou23, MRS23]. Depending on the time scale under consideration the solid rock behaves visco-elastically or visco-plastically, but there are many other variables to be taken into account, like concentrations of different chemicals, water contents, temperature, or aging variables, see e.g. [GeY07, HGV18, SB*21].

The Eulerian description of solids is intrinsically linked to the notion of Lie derivatives, because the variables are transported with the velocity $\boldsymbol{v}$ of the points in the moving and deforming body. This observation from [MiR25] served as a motivation to provide a short and self-contained introduction to Lie derivatives with a focus on the applications in continuum thermodynamics in the present paper. As an major outcome we found Theorem 3.3, which states that in the Eulerian setting the Jacobi identity 3.1 for Poisson operators $\mathbb{J}$ is intimately related to the commutator relation (2.3) for Lie derivatives. The result is now more general and the proof is shorter and more direct as for [MiR25, Thm. 4.1].

A Pullbacks and push-forwards in differential geometry

A diffeomorphism $\boldsymbol{\Phi} : M \to N$ defines pull-backs $\boldsymbol{\Phi}^*$ and push-forwards $\boldsymbol{\Phi}_*$ as follows:

0. Functions:

$$\big(\boldsymbol{\Phi}^* g\big)(m) = g(\boldsymbol{\Phi}(m)) \quad \text{and} \quad \big(\boldsymbol{\Phi}_* f\big)(n) = f(\boldsymbol{\Phi}^{-1}(n)).$$

1. Vector fields:

$$\begin{aligned}\big(\boldsymbol{\Phi}^* \boldsymbol{w}\big)(m) &= \big(\mathrm{D}\boldsymbol{\Phi}(\boldsymbol{\Phi}(m))\big)^{-1} \boldsymbol{w}(\boldsymbol{\Phi}(m)) = \big(\mathrm{D}\boldsymbol{\Phi}(m)\big)^{-1} \boldsymbol{w}(\boldsymbol{\Phi}(m)) \in \mathrm{T}_m M \\ \big(\boldsymbol{\Phi}_* \boldsymbol{v}\big)(n) &= \mathrm{D}\boldsymbol{\Phi}(\boldsymbol{\Phi}^{-1}(n)) \boldsymbol{v}(\boldsymbol{\Phi}^{-1}(n)) = \mathrm{D}(\boldsymbol{\Phi}^{-1})(n) \boldsymbol{v}(\boldsymbol{\Phi}^{-1}(n)) \in \mathrm{T}_n N\end{aligned}$$

1*. one-forms:

$$\begin{aligned}\big(\boldsymbol{\Phi}^* \boldsymbol{\beta}\big)(m) &= \mathrm{D}\boldsymbol{\Phi}(m)^* \boldsymbol{\beta}(\boldsymbol{\Phi}(m)) \in \mathrm{T}_m^* M \\ \big(\boldsymbol{\Phi}_* \boldsymbol{\alpha}\big)(n) &= \big(\mathrm{D}\boldsymbol{\Phi}(\boldsymbol{\Phi}^{-1}(m))\big)^{-*} \boldsymbol{\alpha}(\boldsymbol{\Phi}^{-1}(n)) \in \mathrm{T}_n N\end{aligned}$$

Here for a matrix H we denote by H^{-*} the transposed of the inverse, namely $H^{-*} = \big(H^{-1}\big)^* = (H^*)^{-1}$. Pull-backs and push-forwards are consistent with dual pairings, namely

$$\begin{aligned}&\boldsymbol{\Phi}^*\big(\langle \boldsymbol{\beta}, \boldsymbol{w} \rangle_N\big)(m) = \big\langle \boldsymbol{\Phi}^* \boldsymbol{\beta}(m), \boldsymbol{\Phi}^* \boldsymbol{w}(m) \big\rangle_{\mathrm{T}_m M} \\ &\boldsymbol{\Phi}_*\big(\langle \boldsymbol{\alpha}, \boldsymbol{v} \rangle_M\big)(n) = \big\langle \boldsymbol{\Phi}_* \boldsymbol{\alpha}(\phi^{-1}(n)), \boldsymbol{\Phi}_* \boldsymbol{v}(\phi^{-1}(n)) \big\rangle_{\mathrm{T}_n N}, \\ &\langle \boldsymbol{\Phi}^* \boldsymbol{\beta}, \boldsymbol{v} \rangle_M = \langle \boldsymbol{\beta}, \boldsymbol{\Phi}_* \boldsymbol{v} \rangle_N \quad \text{and} \quad \langle \boldsymbol{\alpha}, \boldsymbol{\Phi}^* \boldsymbol{w} \rangle_M = \langle \boldsymbol{\Phi}_* \boldsymbol{\alpha}, \boldsymbol{w} \rangle_N.\end{aligned}$$

General tensors $\boldsymbol{A} \in \mathfrak{T}^{i_\circ}_{j_\circ}(M)$ with $\boldsymbol{A}(m) \in \mathrm{ML}\big((\mathrm{T}_m M)^{i_\circ} \times (\mathrm{T}_m M)^{j_\circ}\big)$
and $\boldsymbol{B} \in \mathfrak{T}^{i_\circ}_{j_\circ}(N)$ with $\boldsymbol{B}(n) \in \mathrm{ML}\big((\mathrm{T}_n N)^{i_\circ} \times (\mathrm{T}_n N)^{j_\circ}\big)$:

$$\boldsymbol{\Phi}^* \boldsymbol{B}(m)[\boldsymbol{v}_1, .., \boldsymbol{v}_{i_\circ}, \boldsymbol{\alpha}_1, .., \boldsymbol{\alpha}_{j_0}] = \boldsymbol{B}(n)\big[\boldsymbol{\Phi}_* \boldsymbol{v}_1, .., \boldsymbol{\Phi}_* \boldsymbol{v}_{i_\circ}, \boldsymbol{\Phi}_* \boldsymbol{\alpha}_1, .., \boldsymbol{\Phi}_* \boldsymbol{\alpha}_{j_\circ}\big]\Big|_{n=\boldsymbol{\Phi}(m)},$$
$$\boldsymbol{\Phi}_* \boldsymbol{A}(n)[\boldsymbol{w}_1, .., \boldsymbol{w}_{i_\circ}, \boldsymbol{\beta}_1, .., \boldsymbol{\beta}_{j_0}] = \boldsymbol{A}(m)\big[\boldsymbol{\Phi}^* \boldsymbol{w}_1, .., \boldsymbol{\Phi}^* \boldsymbol{w}_{i_\circ}, \boldsymbol{\Phi}^* \boldsymbol{\beta}_1, .., \boldsymbol{\Phi}^* \boldsymbol{\beta}_{j_\circ}\big]\Big|_{m=\boldsymbol{\Phi}^{-1}(n)}.$$

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