First contact percolation

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Abstract

We study a version of first passage percolation on $\mathbb{Z}^d$ where the random passage times on the edges are replaced by contact times represented by random closed sets on $\mathbb{R}$. Similarly to the contact process without recovery, an infection can spread into the system along increasing sequences of contact times. In case of stationary contact times, we can identify associated first passage percolation models, which in turn establish shape theorems also for first contact percolation. In case of periodic contact times that reflect some reoccurring daily pattern, we also present shape theorems with limiting shapes that are universal with respect to the within-one-day contact distribution. In this case, we also prove a Poisson approximation for increasing numbers of within-one-day contacts. Finally, we present a comparison of the limiting speeds of three models – all calibrated to have one expected contact per day – that suggests that less randomness is beneficial for the speed of the infection. The proofs rest on coupling and subergodicity arguments.

1 Introduction

In *first passage percolation* (FPP) edges of a graph are assigned random *transition times* and one studies the minimal total time needed to connect two vertices, which can be seen as a *random metric* [2, 25, 26, 27, 33]. In particular, one might be interested in the set of vertices that are reached up to some fixed time t from a given vertex (say the origin o). This represents, for example, the set of wet sites at time t if o is interpreted as a water source and edges are pipes of random length, justifying the term *percolation*. Another common interpretation comes from the modeling of epidemiological processes, where an initially infected network component spreads a virus in a population with varying transition times between neighboring vertices.

However, one might argue that such a situation is represented more accurately by assigning *random contact times* between any pair of vertices connected by an edge. In this case, the time for an infection to reach another vertex is determined by the total time needed along an optimal path of edges with an increasing sequence of contact times. We call this model *first contact percolation* (FCP) and these notes are dedicated to the study of FCP with respect to the asymptotic speed of infection spread.

It is important to note that the connection times between pairs of vertices x, y no longer form a metric, since the time needed to spread from x to y may differ from the time needed to spread from y to x. This asymmetry arises as the traversal time of an edge depends on the time at which one arrives at its endpoint, which is unlike in FPP. The situation is perhaps best illustrated by a graphical representation, see Figure 1, where each edge is equipped with a point process of contact times and one follows increasing space-time paths.

An important special case arises when the point processes are i.i.d. (with respect to the edges) Poisson point processes. In this case, FCP becomes a Markov process in time, which simply
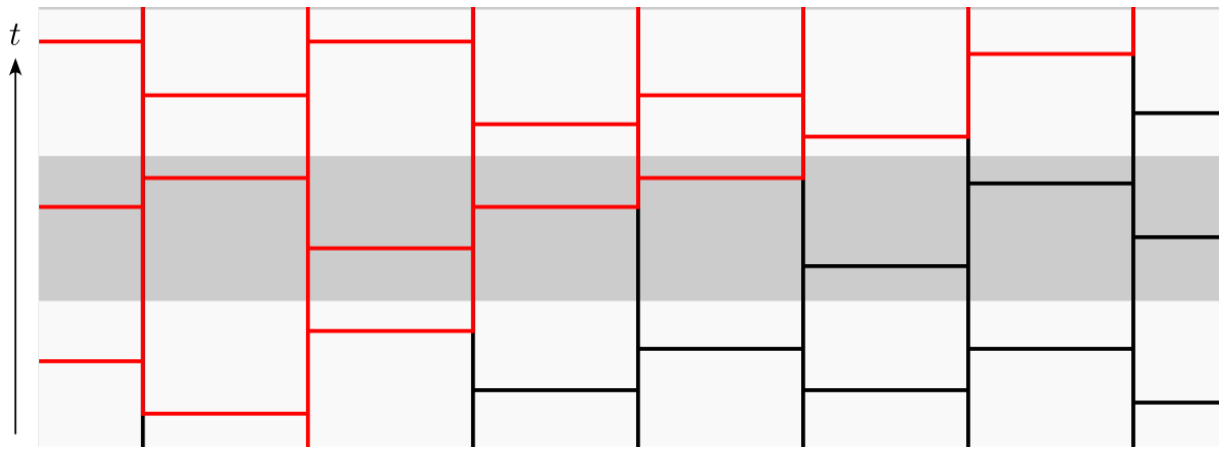
Figure 1: Graphical representation of an infection process. Vertical lines indicate vertices and their state. Horizontal lines indicate contact times between neighboring vertices. The infection (red) then spreads across the network along first contact times. In this realization, neighboring vertices have exactly one contact per integer interval (colored background).

corresponds to the classical *contact process* without recovery [13, 14, 15], i.e., its *pure-growth* version. In that sense, FCP can be seen as the pure-growth version of contact processes based on general point processes. In fact, one does not have to restrict oneself to point processes on the edges, instead more general *random closed sets* of times can be considered, which represent the time and duration of a meeting, see Figure 2 for an illustration. In this context, let us mention results on the contact process beyond exponential waiting times [6, 16, 17, 18, 19, 23, 30, 32], which primarily focus on either *renewal processes* or *time-dependent processes* as alternatives to the classical Poisson point processes.

The main focus of this study of FCP is to derive and compare the asymptotic shape of the infected vertices as time tends to infinity. More precisely, we consider FCP on the hypercubic lattice $\mathbb{Z}^d$ with $d \geq 1$, where each edge e is independently equipped with a random closed set of meeting times $\mathbb{X}_e \subset \mathbb{R}$ following the same distribution. Note that each edge must only be traversed once in order to accurately describe the distribution of the vertices reached up to a given time. This allows us to draw a connection to FPP and ultimately to recover an associated *shape theorem* when $\mathbb{X}_e$ is stationary. Such a direct connection is not possible for non-stationary $\mathbb{X}_e$. However, we establish a shape theorem also for specific models that obey stationarity with respect to shifts in $\mathbb{Z}$. Paradigmatically, we consider a situation in which the time axis is split into *days* $\{[x, x+1): x \in \mathbb{Z}\}$ and, $n \in \mathbb{N}$ i.i.d. meeting times are assigned per day and edge. We observe that the limiting shape of infected vertices in this model is independent of the underlying diffusive distribution of meeting times since only order statistics are relevant, which allows for the restriction to uniformly distributed meeting times. Furthermore, the properly rescaled limiting shape obeys a Poisson approximation, that is, it converges to the limiting shape of the classical *Richardson model* as n tends to infinity.

We further investigate how the asymptotic speed of the infection in FCP depends on the variability of the underlying contact-time distribution. For this, we compare the following three models. Two models with precisely one, uniformly distributed, contact time per day and edge, which is either fixed throughout the whole process or independently resampled for each day. The third model is the pure-growth model based on an intensity-one Poisson point process. Importantly, all models are calibrated to have one contact per day in expectation. It turns out that *rigidity accelerates infection spread*: The most rigid model with one fixed meeting time per
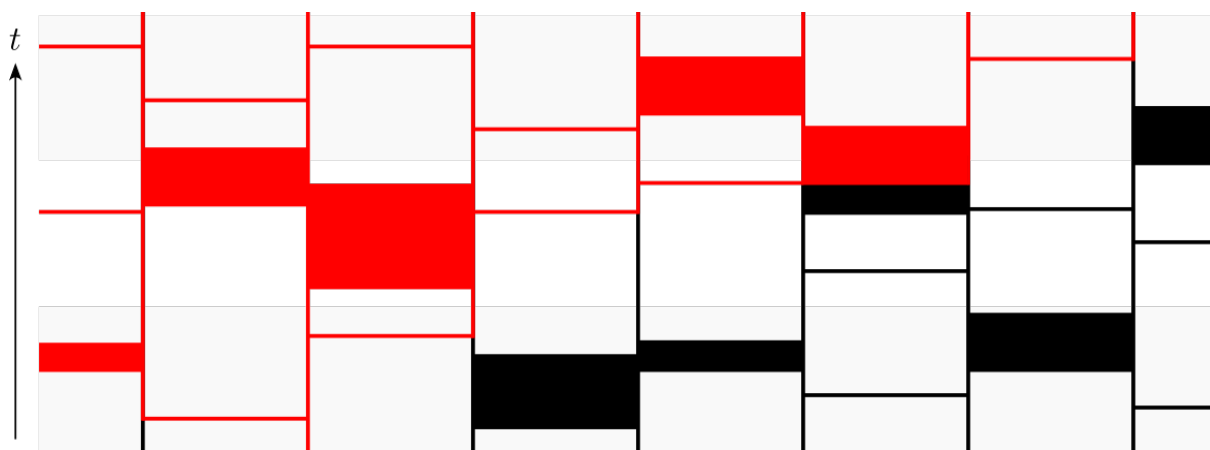
Figure 2: Graphical representation of an infection process where contact times can be arbitrary random closed sets, e.g., Boolean models with random radii based on simple point processes.

day and edge has the largest limiting set of infected vertices, while the Poisson process based model has the smallest such set, with the resampled meeting time model lying in between the two. In a nutshell, this is a consequence of the well-known waiting-time paradox. Hence, a population should organize its contact times as random as possible in order to slow down an infection. Let us mention that, for comparison, the speed of the infection in two-dimensional FPP is increasing *if the expected transition time is fixed* but the underlying i.i.d. passage-time distribution is made more variable, in the sense of convex ordering, see [4, 31]. In other words, roughly for FPP, *more randomness speeds up the infection*, since, in the presence of multiple possible paths, the larger variance provides also particularly fast paths.

The remainder of the paper is organized as follows. In Section 2 we present our setup and state our main results about shape theorems for stationary and periodic contact distributions as well as comparisons and connections between FPP and FCP. We conclude with a discussion on further research directions. Section 3 contains all proofs.

2 Setting and main results

We consider the lattice $\mathbb{Z}^d$, with $d \geq 1$, and identify it with the *nearest-neighbor graph* in the usual way. That is, the vertices are given by the lattice sites and each lattice site is connected to its $2d$ nearest neighbors. Let $\{\mathbb{X}_e \subset \mathbb{R} \colon e = \{x, y\} \subset \mathbb{Z}^d \times \mathbb{Z}^d\}$ be a collection of independent, identically distributed *random closed sets* (under the Euclidean geometry) in $\mathbb{R}$ indexed by the lattice edges. The interpretation is as follows. Each site represents an agent of a network where two agents occasionally meet or contact each other if they share an edge. The times and durations of these contacts are modeled by the random closed sets assigned to the edges. Canonical examples are Poisson point processes or, more generally, renewal processes and Boolean models thereof. We are interested in the speed of an infection spreading through the network, where the infection is passed from an infected vertex to an uninfected one at the next contact time. Initially, only the agent at the origin o is assumed to be infected. The time it takes the infection to be passed from a vertex x, infected at time t_0, to some vertex y is

then given by

$$D_{t_0}(x,y) := \inf \Big\{ t \geq 0 \colon \exists \text{ path } \gamma = (x = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = y), \text{ such that}$$
$$t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = t_0 + t \colon t_i \in \mathbb{X}_{\{x_{i-1}, x_i\}} \text{ for all } 1 \leq i \leq n \Big\}. \tag{1}$$

Put differently, one may only traverse neighboring sites along increasing sequences of contact times. Correspondingly, we define the set of reachable vertices from some starting vertex x during the time interval $[t_0, t]$ as

$$I_{t_0}(x,t) := \{ y \in \mathbb{Z}^d \colon D_{t_0}(x,y) \leq t \}.$$

As we only assume the origin o to be infected at initial time 0, we abbreviate

$$D(x,y) := D_0(x,y) \quad \text{as well as} \quad I(t) := I_0(o,t) = \{ y \in \mathbb{Z}^d \colon D(o,y) \leq t \}.$$

We call this model *first contact percolation* (FCP) on $\mathbb{Z}^d$.

2.1 Stationary contact times

For comparison, we recall the basics of *first passage percolation* (FPP). Here, for μ being some probability measure on $[0, \infty]$, we assign to each edge $e = \{x, y\} \subset \mathbb{Z}^d$ an independent random variable X_e distributed according to μ. In this case, the transition time between two vertices is defined as the shortest path from x to y, i.e.,

$$D^{(\mu)}(x,y) := \min \Big\{ \sum_{e \in \gamma} X_e \colon \gamma \text{ is a path connecting } x \text{ and } y \Big\}.$$

The set of vertices reached up to time t from the origin is given by

$$J^{(\mu)}(t) = \{ x \in \mathbb{Z}^d \colon D^{(\mu)}(o,x) \leq t \}. \tag{2}$$

The most prominent example of an FPP model is the *Richardson model*, where the X_e are i.i.d. exponentially distributed random variables. For an overview and extensive discussion on FPP, we refer the reader to the book [2].

Let us note that while $D^{(\mu)}$ defines a metric, this is *not true* for D_{t_0} (as used in FCP). Particularly, the latter is not symmetric as the time it takes to pass the infection from x to y may differ considerably from the time it would take the infection to pass from y to x. Still, D_{t_0} fulfills a version of the triangle inequality as

$$D_{t_0}(x,z) \leq D_{t_0}(x,y) + D_{D_{t_0}(x,y)}(y,z). \tag{3}$$

Another crucial difference between FCP and FPP arises when processes are considered, in which edges are crossed multiple times, for instance a contact process on the graph that includes recoveries, see also the discussion in Section 2.4. In this work, we focus on the set of reachable vertices $I(t)$. In this regard, the link between FCP and FPP is more immanent: For stationary random closed sets, $I(t)$ corresponds to an FPP model as each edge is only traversed once.

Proposition 2.1 (FCP with stationary contacts is FPP)**.** *Consider the FCP based on some stationary random closed set $\mathbb{X}$. Then, the process $\{I(t) \colon t \geq 0\}$ of reachable vertices in the FCP model has the same distribution as $\{J^{(\mu)}(t) \colon t \geq 0\}$, where*

$$\mu([0,s]) = \mathbb{P}(\mathbb{X} \cap [0,s] \neq \emptyset) \qquad \text{for all } s \geq 0. \tag{4}$$

The proof (and all other proofs) is presented in Section 3. By now, it is a standard result that the continuum version $\widetilde{J}^{(\mu)}$ of $J^{(\mu)}$ satisfies a shape theorem (of the form (7) below) with limiting shape $\mathcal{B}^{(\mu)}$ under certain conditions on μ [2]. By Proposition 2.1, under those conditions, also I satisfies the same shape theorem. Before we discuss the relation between $\mathbb{X}$ and μ any further, let us give some examples.

Example 2.2. Let us mention some examples for which μ can be explicitly described.

(i) *Poisson point processes*: For $\mathbb{X}$ being the Poisson point process with intensity $\lambda > 0$, the corresponding FPP model is the Richardson model based on exponential transition times with parameter λ.

(ii) *Renewal processes:* If $\mathbb{X}$ is a stationary simple renewal process with inter-arrival distribution ν (with finite expectation), then μ is the distribution of the *forward recurrence time* of ν, i.e.,

$$\mu([0,s]) = \int_0^s \nu((x,\infty)) \mathrm{d}x \Big/ \int_0^\infty \nu((x,\infty)) \mathrm{d}x. \tag{5}$$

In particular, μ has the non-increasing density $f(s) = \nu((s,\infty))/\int_0^\infty \nu((x,\infty))\mathrm{d}x$.

(iii) *Shifted lattices*: Consider $\mathbb{X}$ being given by $L(\mathbb{Z} + U)$, $L > 0$, where U is a uniform random variable on $[0,1]$. Then, μ is the uniform distribution on $[0,L]$.

(iv) *Boolean models*: Let $\mathbb{X}'$ be a stationary random closed set with associated measure μ', given by Equation (4). Now, let $\mathbb{X}$ be the Boolean model based on $\mathbb{X}'$ with parameter $r > 0$, i.e., $\mathbb{X} := \bigcup_{x \in \mathbb{X}'} [x - r, x + r]$. Then, the corresponding FPP model has a transition-time distribution given by

$$\mu([0,s]) = \mu'([0, s + 2r]). \tag{6}$$

Indeed, this can be seen after assuming that the balls of radius r are attached "to the left" of $\mathbb{X}$, which does not change the distribution due to stationarity.

(v) *Cox processes*: Consider the random closed set $\mathbb{X}$ where, with probability $1 - \mathrm{e}^{-1}$, we have $\mathbb{X} = \mathbb{R}$ and otherwise $\mathbb{X}$ is sampled from a Poisson point process of unit intensity. The corresponding FPP has the transmission-time distribution

$$\mu([0,s]) = (1 - \mathrm{e}^{-1}) + \mathrm{e}^{-1}(1 - \mathrm{e}^{-s}) = 1 - \mathrm{e}^{-(s+1)}.$$

This also corresponds to the FCP given by the Boolean model of a Poisson point process of intensity one and radius $1/2$. In particular, this shows that different FCPs can correspond to the same FPP, with respect to the process of reachable vertices.

In view of the examples just described, our next result describes general properties of μ defined via (4) for given $\mathbb{X}$.

Lemma 2.3 (μ is concave and absolutely continuous)**.** *Let $\mathbb{X}$ be a stationary random closed set and μ its associated measure as given by Equation* (4)*. Then, $s \mapsto \mu((0,s])$ is concave. In particular, μ restricted to $(0,\infty)$ has a Lebesgue density f on $(0,\infty)$, which can be chosen to be non-increasing and right-continuous.*

What can be said about existence, uniqueness and other properties of stationary random closed sets giving rise to a prescribed transition-time measure μ? In view of Example 2.2 Part (v), we see that such $\mathbb{X}$ are not necessarily unique. Moreover, Lemma 2.3 requires concavity of μ. The following result establishes that concavity is also sufficient. Furthermore, we present

a construction of some $\mathbb{X}$ via randomly superposed translation-invariant lattices as well as renewal processes.

Proposition 2.4 (FCP from FPP). *Let μ be a distribution on $[0, \infty]$ whose restriction to $(0, \infty)$ has a non-increasing, right-continuous Lebesgue density f. Then, the following holds:*

(i) *There exists a stationary point process $\mathbb{X}$ such that the FCP under $\mathbb{X}$ corresponds to the FPP under μ. Furthermore, $\mathbb{X}$ can be chosen such that each realization is a shifted lattice of the form $L(\mathbb{Z} + U)$ for some $L \in [0, \infty]$ and $U \in [0, 1]$. Here, we make the convention that $0 \cdot \mathbb{Z} := \mathbb{R}$ and $\infty \cdot \mathbb{Z} := \emptyset$.*

(ii) *If μ has no atoms in $\{0, \infty\}$ and if f is bounded, then there exists a unique stationary and ergodic renewal process $\mathbb{X}$ such that Equation (4) holds.*

(iii) *If μ has no atom in ∞, then $\mathbb{X}$ can be chosen as a Boolean model of a stationary and ergodic renewal process.*

Let us note that the stationary sets $\mathbb{X}$ in Part (i) above are not necessarily ergodic. Also, we note that $\mathbb{X}$ in Part (iii) is not unique even within the set of Boolean models based on stationary renewal processes.

So far, we have only discussed processes $\mathbb{X}$ that are stationary with respect to shifts in $\mathbb{R}$, however those processes are not well suited to model the rather periodic behavior of a population with periods of sleep.

2.2   Periodic contact times

In order to accommodate periodic behavior, we consider the process based on the following *perturbed lattice*

$$\mathbb{X}^{(n)} = \bigcup_{k \in \mathbb{Z}} \bigcup_{i=1}^{n} \{k + U^{(k,i)}\}, \tag{PL}$$

where the $U^{(k,i)}$ are i.i.d. random variables on $[0, 1)$ with some Lebesgue density g. In other words, $\mathbb{X}^{(n)}$ models a situation in which every interval $[k, k+1)$ contains precisely n i.i.d. contacts. We require a Lebesgue density simply to avoid multiple contact times to coincide. Now, every edge is equipped with an independent copy of $\mathbb{X}^{(n)}$ and we write $\{I_n(t): t \geq 0\}$ for its process of reachable vertices. Additionally, we write $\widetilde{I_n}(t)$ for its continuum version obtained from $I_n(t)$ by identifying each lattice site x with the unit box centered around x. The following result establishes a shape theorem for $\widetilde{I_n}(t)$ in which the limiting shape does not depend on g. We present some simulated realizations on $\mathbb{Z}^2$ in Figure 3.

Theorem 2.5 (Shape theorem). *Consider the model based on (PL). For all $n \in \mathbb{N}$, there exists a convex and compact set $\mathcal{B}_n \subset \mathbb{R}^d$ such that, for all Lebesgue-densities g determining the distribution of the contact times and $\varepsilon > 0$, we have*

$$\mathbb{P}\Big(\exists\, T < \infty: (1-\varepsilon)\mathcal{B}_n \subset t^{-1}\widetilde{I_n}(t) \subset (1+\varepsilon)\mathcal{B}_n \text{ for all } t \geq T\Big) = 1. \tag{7}$$

The next statement establishes a Poisson-approximation result for the limiting shape. For this, let $J_R(t)$ denote the set of infected vertices at time t in the Richardson model, i.e., the FPP based on μ being the exponential distribution with parameter one, and $\mathcal{B}_R$ the limiting shape in the associated shape theorem.
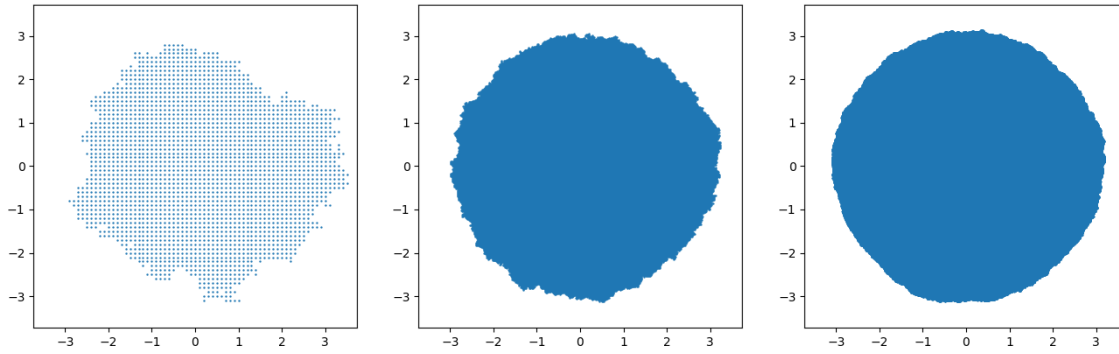
Figure 3: Realizations of $t^{-1}I(t)$ for $n = 1$ and $t = 10, 50, 190$. One can see the sets slowly converging to a deterministic limit set $\mathcal{B}_1$.

Theorem 2.6 (Poisson approximation). *Let $I_n(t)$ denote the set of infected vertices at time t and $\mathcal{B}_n$ the limit shape of the corresponding model. Then, there exists a coupling such that almost surely*

$$I_n(t/n) \supset J_R(t).$$

In particular $\mathcal{B}_n/n \supset \mathcal{B}_R$ for every $n \in \mathbb{N}$. Furthermore, with respect to the Hausdorff metric,

$$\mathcal{B}_n/n \to \mathcal{B}_R \qquad \text{as } n \to \infty.$$

Finally, we present a case study that shows that additional randomness in the process $\mathbb{X}$ decreases the set of infected vertices.

2.3 Comparison of infection speeds

Recall the model $\mathbb{X} = \mathbb{Z} + U$, with U being uniformly distributed in $[0, 1)$ from Example 2.2 Part (iii). In this situation, neighboring vertices decide only once independently for a contact time on the first day and then keep that time for all subsequent days. In order to compare this process with the process $\mathbb{X}^{(n)}$ defined in (PL), let us also consider the multi-contact version of $\mathbb{X} = \mathbb{Z} + U$, given by the *stationarized lattice*

$$\mathbb{X}'^{(n)} = \bigcup_{k \in \mathbb{Z}} \bigcup_{i=1}^{n} \{k + U^{(i)}\} = \bigcup_{i=1}^{n} \left(\mathbb{Z} + U^{(i)}\right), \tag{SL}$$

where the $U^{(i)}$ are i.i.d. uniforms in $[0, 1)$. We equip every edge independently with a copy of $\mathbb{X}'^{(n)}$ and write $I'_n(t)$ for the associated process of reachable vertices. Clearly, $\mathbb{X}'^{(n)}$ features less randomness than $\mathbb{X}^{(n)}$, which leads to more infections, as shown by the following result.

Proposition 2.7 (Randomness slows down). *Consider, for fixed $n \in \mathbb{N}$, the model based on (SL). Then, there exists a coupling such that almost surely $I'_n(t) \supset I_n(t)$.*

Combining Theorem 2.6 and Proposition 2.7, we have the following ordering of infected sets,

$$I'_n(t/n) \supset I_n(t/n) \supset J_R(t)$$

and thus also the inclusion

$$\mathcal{B}_R \subset \mathcal{B}_n/n \subset \mathcal{B}'_n/n,$$

where we used that $\mathbb{X}'^{(n)}$ is stationary and well behaved and thus obeys a shape theorem with limiting shape $\mathcal{B}'_n$. Meanwhile, setting $n = 1$, all three models have, in expectation, precisely one contact in the interval $[0, 1)$ and thus, we can conclude that in our situation, randomness slows down the infection process. As an illustration, let us note that one can explicitly calculate the asymptotic speeds in dimension one and for $n = 1$. In this case, one obtains the values 2 for the FCP model associated to $\mathbb{X}'^{(1)}$, $e - 1$ for the FCP model associated to $\mathbb{X}^{(1)}$, and 1 for the Richardson model, underlining the observed ordering.

We conclude this section by mentioning that, as the shape theorem for FPP holds for $I'(t)$, Proposition 2.7 also provides an alternative proof for the finiteness of the speed in the model based on (PL), see Lemma 3.4 below.

2.4   Further discussion and outlook

Let us elaborate on some open questions and connections of FCP to other processes.

2.4.1   Correspondences and differences between FCP and FPP

We have seen in Proposition 2.4 that we can find one-to-one correspondences between transition-time distributions μ in FPP and certain classes of stationary random closed sets $\mathbb{X}$. In that respect, we have considered super-positioned lattices $L(\mathbb{Z}+U)$, renewal processes and Boolean models thereof. However, it should be interesting to see what other properties on μ are necessary and sufficient to establish links to other classes of stationary random closed sets like, for example, Gibbs point processes or perturbed lattices, in particular ergodic ones. Let us mention that our model (PL) is indeed a perturbed lattice albeit not stationarized. On the other hand, the model (SL) is a shifted lattice and fits well into the framework of stationarized renewal processes.

Moreover, let us recall that many different stationary FCP models can lead to the same FPP exploration process. The main difference between FCP and FPP becomes apparent when we consider any characteristic that involves a revisiting of vertices. This begs the question: What other quantities and processes can we look at to easily distinguish the several models? Introducing recovery to consider contact processes is certainly one candidate.

2.5   Speed and rigidity

Theorem 2.6 and Proposition 2.7 have given us a nice ordering of the Richardson model and the FCP models given by $\mathbb{X}$ as in (PL) and $\mathbb{X}'$ in (SL) in terms of their time constants. We have observed that this also matches their "randomness". In what generality does this hold and can we make this statement rigorous?

As a small sanity check, let us briefly return to the one-dimensional case $d = 1$. Consider the stationarized version of $\mathbb{X}$, that is, $\mathbb{X} + U$ for a uniform random variable on $(0, 1)$. In a sense, this model is more random than $\mathbb{X}$, so we would expect its speed to be lower. Indeed, using (4), we see that its FPP transition-time distribution μ is given by

$$\mu([s, \infty)) = (1 - s) + s^3/6 \qquad \text{and} \qquad \mu([1 + s, \infty)) = (1 - s)^3/6$$

for $s \in [0, 1]$. Therefore, its inverse speed is given by $\int_0^\infty \mu([s, \infty))\mathrm{d}s = 7/12$. This underlines our assumption on the speeds since $12/7 < e - 1$.

2.5.1 Beyond stationary and periodic processes $\mathbb{X}$

Our main results only deal with contact-time processes $\mathbb{X}$ that are either stationary with respect to $\mathbb{R}$ or with respect to some lattice $L\mathbb{Z}$, $L > 0$. This has the advantage that the waiting time until the next crossing opportunity of an edge does not depend on the time at which we arrive at an endpoint of that edge. This is no longer the case if we abandon stationarity. Particularly, a shape theorem may no longer hold true, if $\mathbb{X}$, for example, is a renewal process with heavy-tailed inter-arrival times; a case mentioned in [23, Proposition 1.3], in the context of contact processes based on renewal processes replacing the usual Poisson processes in the graphical representation. Let $\mathbb{X}$ be a renewal processes started at time 0 such that, for some $0 < \varepsilon < 1$,

$$\mathbb{P}(\mathbb{X} \cap [t, t + t^\varepsilon] \neq \emptyset) \leq t^{-\varepsilon}, \qquad \text{for all } t \geq t_0,$$

whose existence is guaranteed by [17, Proposition 7]. In this case $S_t := \sup\{|y| : y \in I(t)\}$ behaves sublinearly in the sense that, almost surely for all $0 < c < \varepsilon$,

$$\limsup_{t \uparrow \infty} t^{c-1} S_t = 0.$$

However, in this example, no stationary version of $\mathbb{X}$ exists. If the inter-arrival times have a finite expectation, then the renewal process will eventually equilibrate and we expect a shape theorem to still hold.

2.5.2 Further directions of research

Natural further directions of research include the modeling and analysis of other interacting particle systems via graphical representations not based on Poisson processes, see e.g. [8, 28] in the context of the exclusion process, [6, 16, 17, 18, 19, 23, 30, 32] in the context of the contact process, or [20] for hydrodynamic limits of more general processes. In view of the related FPP, several properties can be investigated with respect to FCP, such as

- conditions under which we have continuity or monotonicity of the limiting shape with respect to varying random closed set distributions, see [9, 31],

- large-deviation type results for the probability to reach far-away points atypically fast or slow, see [3, 7],

- further properties of the geodesics in FCP such as their fluctuation around the straight line, see [1, 12] or the number of geodesics, see [22, 24],

- properties of the limiting shape for FPP models derived from FCP models, or of periodic FCP models, see [10, 21],

and many more.

3 Proofs

This section is devoted to the proofs of our main results regarding the model based on (PL) but also the correspondences between FCP and FPP. We start by proving the Propositions 2.1 and 2.4 in Section 3.1 as well as the auxiliary Lemma 2.3. Together, they provide necessary and sufficient requirements to compare FPP and certain FCP models. We prove the shape theorem (Theorem 2.5) in Section 3.2. This follows mostly from established arguments. In

Section 3.3, we prove the monotone coupling between monotonicity the models based on (PL) and (SL) (Proposition 2.7). Additionally, we show convergence of the model based on (PL) to the Richardson model (Theorem 2.6) as n tends to infinity.

3.1 Proofs of FCP-FPP correspondences

Proof of Proposition 2.1. Observe that $\{I(t)\colon t \geq 0\}$ can be seen as an exploration process of $\mathbb{Z}^d$ starting from the origin. Whenever the exploration discovers a new vertex that has not yet been seen, its edges incident to unexplored vertices and their respective next contact time are revealed. By assumption, the contact times on the edges are given by independent copies of the stationary random closed set $\mathbb{X}$. By the independence and stationarity, the next contact time on said edge has the distribution function μ independently of all other contact times. This concludes the proof.   □

Proof of Lemma 2.3. The proof is identical to the one in [5, Lemma 3.1.1] for stationary point processes, which we recapitulate here for convenience. Let $h > 0$ and $0 < a < b < \infty$. We first observe that

$$\begin{aligned}
\mu((a, a+h]) &= \mathbb{P}(\mathbb{X} \cap [0, a] = \emptyset \text{ and } \mathbb{X} \cap (a, a+h] \neq \emptyset) \\
&= \mathbb{P}(\mathbb{X} \cap [b-a, b] = \emptyset \text{ and } \mathbb{X} \cap (b, b+h] \neq \emptyset) \\
&\geq \mathbb{P}(\mathbb{X} \cap [0, b] = \emptyset \text{ and } \mathbb{X} \cap (b, b+h] \neq \emptyset) = \mu((b, b+h]),
\end{aligned}$$

where the second equality follows from stationarity of $\mathbb{X}$. Given arbitrary $0 < s < t < \infty$ and choosing $a = s$, $b = (s+t)/2$ and $h = (t-s)/2$ yields

$$\mu\big((s, (s+t)/2]\big) \geq \mu\big(((s+t)/2, t]\big) \iff \mu\big((0, (s+t)/2]\big) \geq 2^{-1}\big(\mu((0, s]) + \mu((0, t])\big),$$

which shows concavity of the function $F\colon (0, \infty) \to [0, 1]$, $s \mapsto \mu((0, s])$. Hence, a right derivative $f(s)$ exists for all $s > 0$. Additionally, f is decreasing and $F(c) - F(b) \leq f(a)|c - b|$ for all $0 < a < b < c$. Thus, F is absolutely continuous on any interval $[\delta, \infty)$, $\delta > 0$ and hence

$$F(b) = F(a) + \int_a^b f(s)\mathrm{d}s,$$

concluding the proof by taking the limit $a \downarrow 0$.   □

The following observation will be used in the proof of Proposition 2.4.

Lemma 3.1 (Limit behavior of f)**.** *Let $f\colon (0, \infty) \to (0, \infty)$ be monotone with $\int_0^\infty f(x)\mathrm{d}x < \infty$. Then,*

$$\lim_{x \downarrow 0} x f(x) = \lim_{x \uparrow \infty} x f(x) = 0.$$

Proof. It follows immediately that f must be non-increasing. Next, assume $\limsup_{x \uparrow \infty} x f(x) =: a > 0$. Then, there exists an increasing sequence $(x_k)_{k \in \mathbb{N}}$, $x_k \uparrow \infty$ such that $x_k f(x_k) \geq a$, i.e., $f(x_k) \geq a/x_k$. By choosing a subsequence, we may assume, without loss of generality, that $x_{k-1}/x_k < 1/2$. Then, due to monotonicity of f, we have that

$$\int_0^\infty f(x)\mathrm{d}x \geq \sum_{k=2}^\infty (x_k - x_{k-1})f(x_k) \geq \sum_{k=2}^\infty (1 - x_{k-1}/x_k)a \geq \sum_{k=2}^\infty a/2 = \infty,$$

which shows that $\limsup_{x \uparrow \infty} x f(x) = 0$ whenever $\int_0^\infty f(x)\mathrm{d}x < \infty$. Showing $\limsup_{x \downarrow 0} x f(x) = 0$ works analogously, which finishes the proof.   □

Proof of Proposition 2.4. We prove the statements individually.

Part (i)**:** Let us first additionally assume that $\mu(\{0, \infty\}) = 0$. Take L to be distributed according to a probability distribution ν on $(0, \infty)$ to be specified later and take U to be a uniform random variable on $(0, 1)$ independent of L. Our random closed set is going to be $\mathbb{X} := L(\mathbb{Z} + U)$, which is clearly stationary but not necessarily ergodic. Let us first define ν. Since f is non-increasing on $(0, \infty)$, we may define a measure Q on $(0, \infty)$ via

$$Q((a, b]) := f(a) - f(b), \qquad \text{for all } 0 < a < b < \infty.$$

Then, we define ν to be the measure that is absolutely continuous with respect to Q with density $x \mapsto x$, i.e.,

$$\nu(\mathrm{d}x) = xQ(\mathrm{d}x).$$

Let us first check that ν is indeed a probability measure. For this, we first see that Lemma 3.1 below applies to f, i.e., $\lim_{x \downarrow 0} xf(x) = \lim_{x \uparrow \infty} xf(x) = 0$. Further, we obtain by the integration-by-parts formula for Stieltjes integrals for $0 < a < b < \infty$,

$$\int_a^b x\, Q(\mathrm{d}x) = -\int_a^b x\, \mathrm{d}f(x) = af(a) - bf(b) + \int_a^b f(x)\, \mathrm{d}x = af(a) - bf(b) + \mu([a, b]).$$

Hence,

$$\nu((0, \infty)) = \int_0^\infty x\, Q(\mathrm{d}x) = \lim_{x \downarrow 0} xf(x) - \lim_{x \uparrow \infty} xf(x) + \mu((0, \infty)) = 1.$$

Let us verify Equation (4) next. The procedure is as follows: Given $s > 0$, the set $\mathbb{X} = L(\mathbb{Z}+U)$ will always have a point in $[0, s)$ if $s \geq L$. Otherwise, we need $LU \leq s$. Using integration by parts again, we infer

$$\mathbb{P}(\mathbb{X} \cap [0, s) \neq \emptyset) = \mathbb{P}(L(\mathbb{Z} + U) \cap [0, s) \neq \emptyset \text{ and } L > s) + \mathbb{P}(L(\mathbb{Z} + U) \cap [0, s) \neq \emptyset \text{ and } L \leq s)$$

$$= \int_s^\infty \mathbb{P}(xU \leq s)\, \nu(\mathrm{d}x) + \int_0^s \nu(\mathrm{d}x) = \int_s^\infty x\frac{s}{x}\, Q(\mathrm{d}x) + \int_0^s x\, Q(\mathrm{d}x)$$

$$= sQ((s, \infty)) - \int_0^s x\, \mathrm{d}f(x) = sf(s) - sf(s) + \lim_{x \downarrow 0} xf(x) + \int_0^s f(x)\, \mathrm{d}x = \mu([0, s]).$$

Now, let us consider the general case of $\mu(\{0, \infty\}) \geq 0$. In this case, we take

$$\nu := \mu(\{0\})\delta_0 + \mu(\{\infty\})\delta_\infty + (1 - \mu(\{0, \infty\}))\nu'$$

and interpret $0 \cdot \mathbb{Z} = \mathbb{R}$ and $\infty \cdot \mathbb{Z} := \emptyset$. In the case that $\mu(\{0, \infty\}) < 1$, we choose ν' as in the case above for the measure $\mu'([a, b]) := (1 - \mu(\{0, \infty\}))^{-1}\mu([a, b] \cap (0, \infty))$, $0 \leq a < b \leq \infty$, where the latter is indeed a probability measure satisfying the conditions required in the above first part. Thus, $\mathbb{X} = L(\mathbb{Z} + U)$ satisfies Equation (4) since $\mu(\{0\}) = \mathbb{P}(0 \in \mathbb{X}) = \mathbb{P}(L = 0)$ and analogously for $\mu(\{\infty\})$.

Part (ii)**:** Assume $\mu(\{0, \infty\}) = 0$ and $f(0) < \infty$ for the right-continuous, monotone Lebesgue density of μ. We aim to finding a stationarized (simple) renewal process $\mathbb{X}$ with inter-arrival distribution ν generating μ. Inverting Equation (5) yields

$$\nu((s, \infty)) := f(0)^{-1}f(s).$$

The validity of Equation (4) is easy to check.

Part (iii)**:** We allow for $\mu(\{0\}) =: c > 0$ and want to use a Boolean model of a stationary and simple renewal process instead. If $c = 1$, simply take $\nu := \delta_1$ and radius $r = \infty$. Otherwise, μ has a Lebesgue density f on $(0, \infty)$. Consider the non-increasing, right-continuous function $f^* \colon [0, \infty) \to [0, \infty)$ as

$$f^*(x) := f\Big(\max(0, x - c/f(0)) \Big).$$

We see that f^* is a probability density as

$$\int_0^\infty f^*(x)\mathrm{d}x = \int_0^\infty f(x)\mathrm{d}x + \frac{c}{f(0)}f(0) = \mu((0,\infty)) + \mu(\{0\}) = 1.$$

Choosing ν as in Part (ii) but according to f^* and taking $r = c/(2f(0))$ proves the claim by Equation (6). Note that many other candidates for f^* exist that yield a smaller radius r, hence $\mathbb{X}$ is not unique. $\qquad \square$

3.2   Proof of shape theorem

During the whole section, we fix some $n \in \mathbb{N}$ for the number of contact times per day and may drop it from the notation whenever convenient. Furthermore, we assume that the contact times per day are independently and identically distributed according to some Lebesgue-density g. However, we shall see that the particular choice of g has no influence on the result.

The proof of the shape theorem then follows established arguments, in which the first step is to establish linear speed in a fixed direction. That is, for each fixed direction $x \in \mathbb{R}^d$, there exists some $\varphi(x)$ such that the time it takes the infection to reach a site close to tx is roughly $t\varphi(x)$. The quantity $\varphi(x)$ is hence also called the *inverse speed* in direction x. For $x \in \mathbb{R}^d$, let us write $[x]$ for its closest neighbor in $\mathbb{Z}^d$. In case of ambiguity, any tie-breaker will do.

Lemma 3.2 (Existence of inverse speed for lattice points)**.** *For all* $x \in \mathbb{Z}^d$, *there exists* $\varphi(x) := \varphi^{(n)}(x) \in [0, \infty)$ *such that almost surely*

$$\lim_{t \uparrow \infty} t^{-1} D(o, [tx]) = \varphi(x).$$

Furthermore,

$$\varphi(x) = \lim_{t \uparrow \infty} t^{-1}\mathbb{E}D(o, [tx]) = \inf_{t \geq 1} t^{-1}\mathbb{E}D(o, [tx]). \tag{8}$$

Proof. In order to make use of the independence between days, it will be useful to consider an integer-valued version of the travel duration D, defined as

$$\widetilde{D}_{t_0}(y, z) := \lceil D_{t_0}(y, z) \rceil = \inf\{t \in \mathbb{N} \colon z \in I_{t_0}(y, t)\},$$

for a time $t_0 \geq 0$. That is, the infection starting in x at time t_0 reaches vertex y in $\widetilde{D}_{t_0}(y, z)$ days or during the $\lfloor t_0 \rfloor + \widetilde{D}_{t_0}(y, z)$-th day of the process respectively. Note that the limits $D(0, [tx])/t$ and $\widetilde{D}_0(0, [tx])/t$ (as $t \uparrow \infty$) coincide if they exist. Given $0 \leq s \leq t \in \mathbb{N}$, we define the random variables

$$X_{s,t} := \widetilde{D}_{\widetilde{D}_0(o, [sx])}([sx], [tx]),$$

representing the number of days required for the infection to spread from $[sx]$ to $[tx]$ when started the first day after the infection has reached $[sx]$ from the origin. In the remainder of the proof, we shall identify sx and tx with $[sx]$ and $[tx]$ respectively to ease notation. The proof is now concluded by applying the subadditive ergodic theorem [29, Theorem 2.6] in the version of [11, Theorem 4.2]. For this, we have to verify the following conditions:

(i) $X_{0,t} \leq X_{0,s} + X_{s,t}$,

(ii) $0 \leq \mathbb{E}X_{0,t} < \infty$ for every t,

(iii) the distribution of $\{X_{s,s+k} : k \in \mathbb{N}\}$ does not depend on s, and

(iv) $\limsup_{t \uparrow \infty} X_{0,tk}/t \leq \mathbb{E}X_{0,k}$ for each $k \in \mathbb{N}$.

Under the assumption that the Conditions (i)–(iv) are satisfied, we infer from [11, Theorem 4.2]

$$\varphi(x) := \lim_{t \to \infty} t^{-1}\mathbb{E}X_{0,t} = \inf_{t \geq 1} t^{-1}\mathbb{E}X_{0,t},$$

implying (8), and $\varphi(x) = \lim_{t \to \infty} t^{-1}X_{0,t}$ almost surely. It hence remains to verify these conditions.

On (i) This is a direct consequence of the triangle inequality (3).

On (ii) This is immediate from the trivial bounds $0 \leq X_{0,t} \leq t|x|_1$.

On (iii) Note that we have for all measurable sets $A \subset \mathbb{N}^{\mathbb{N}}$

$$
\begin{aligned}
\mathbb{P}\big(X_{s,s+1}, X_{s,s+2}, \cdots \in A\big) &= \sum_{j \geq 1} \mathbb{P}\big(\widetilde{D}_k(sx, (s+1)x), \widetilde{D}_k(sx, (s+2)x), \cdots \in A, \widetilde{D}_0(0, sx) = j\big) \\
&= \sum_{j \geq 1} \mathbb{P}\big(\widetilde{D}_k(sx, (s+1)x), \widetilde{D}_k(sx, (s+2)x), \cdots \in A\big) \mathbb{P}(\widetilde{D}_0(0, sx) = j) \\
&= \sum_{j \geq 1} \mathbb{P}\big(\widetilde{D}_0(0, x), \widetilde{D}_0(0, 2x), \cdots \in A\big) \mathbb{P}(\widetilde{D}_0(0, sx) = j) \\
&= \mathbb{P}\big(X_{0,1}, X_{0,2}, \ldots \in A\big),
\end{aligned}
$$

where we used the independence of the contact times with respect to different days as well as the lattice shift invariance of the system on every individual day in the penultimate step.

On (iv) Define recursively

$$Y_1^k := \widetilde{D}_0(0, kx) = X_{0,k} \qquad \text{and} \qquad Y_i^k := \widetilde{D}_{Y_1^k + \cdots + Y_{i-1}^k}((i-1)kx, ikx)$$

and note that for every realization

$$X_{0,nk} \leq \sum_{i=1}^{n} Y_i^k.$$

As the Y_i^k all depend on mutually disjoint days, they form an i.i.d. sequence. Thus, the claim follows by the strong law of large numbers.

This concludes the proof. $\square$

Next, we extend the established inverse speed in $\mathbb{Z}^d$ directions to *all* directions.

Lemma 3.3 (Existence of inverse speed for general directions). *Under the assumptions and definitions of Lemma 3.2, almost surely, for all $x \in \mathbb{R}^d$, the limit $\varphi(x) = \varphi^{(n)}(x)$ exists. Furthermore, $\varphi \colon \mathbb{R}^d \to [0, \infty)$ is almost surely continuous.*

Proof. Since we travel at least one edge per day, the reverse triangle inequality implies $|D_0(o, y) - D_0(o, z)| \leq |z - y|_1$ for every $y, z \in \mathbb{Z}^d$. In particular, this yields for arbitrary reals $y, z \in \mathbb{R}^d$,

$$|D_0(o, [y]) - D_0(o, [z])| \leq |z - y|_1 + 2d. \tag{9}$$

Let us first consider the case $x \in \mathbb{Q}^d$. Then, there exists some $s \in \mathbb{N}$ such that $sx \in \mathbb{Z}^d$. We will show that $\varphi(x)$ exists and

$$\varphi(x) = \lim_{t \uparrow \infty} t^{-1} D_0(o, [tx]) = \lim_{t \uparrow \infty} (st)^{-1} D_0(o, stx) = s^{-1} \varphi(sx).$$

By Lemma 3.2, $\varphi(x)$ exists almost surely. Furthermore, for each $t > 0$, we pick $s = s(t)$ such that $t \in [(k-1)s, ks)$. Then, using the triangle inequality (3) and (9), we infer

$$\lim_{t \uparrow \infty} |s^{-1}\varphi(sx) - t^{-1} D_0(o, [tx])|$$

$$\leq \lim_{t \uparrow \infty} \left| s^{-1}\varphi(sx) - (ks)^{-1} D_0(o, ksx) \right| + \lim_{t \uparrow \infty} \left| (ks)^{-1} D_0(o, ksx) - t^{-1} D_0(o, [tx]) \right|$$

$$\leq 0 + \lim_{t \uparrow \infty} (ks)^{-1} \left| D_0(o, ksx) - D_0(o, [tx]) \right| + \lim_{t \uparrow \infty} \left| (ks)^{-1} - t^{-1} \right| D_0(o, [tx])$$

$$\leq \lim_{t \uparrow \infty} (ks)^{-1} \left[(ks - t)|x|_1 + 2d \right] + \lim_{t \to \infty} \left| (ks)^{-1} - t^{-1} \right| \left(|tx|_1 + d \right)$$

$$\leq \lim_{t \uparrow \infty} (ks)^{-1} \left[s|x|_1 + 2d \right] + \lim_{t \uparrow \infty} \left(\tfrac{1}{(k-1)s} - \tfrac{1}{ks} \right) \left(ks|x|_1 + d \right)$$

$$= 0,$$

using $k \to \infty$ with $n \to \infty$. This shows the claim for each rational direction $x \in \mathbb{Q}^d$. The proof extends to real directions $x \in \mathbb{R}^d$ as

$$\lim_{t \uparrow \infty} t^{-1} D_0(o, [tx])$$

is continuous in x since (9) yields

$$t^{-1} |D_0(o, [tx]) - D_0(o, [ty])| \leq |y - x|_1 + 2d/t,$$

which concludes the proof. $\square$

Having established the inverse speed in all directions, we proceed by showing its positivity, i.e., the infection has finite speed. Let us mention that, unlike before, the dependence on n becomes relevant now, which we shall emphasize in the notation. In order to prove finite speed, we require the notion of a path's *traveling time*: Given a path $\gamma = (x_0, \ldots, x_k)$, we define its traveling time from $t_0 \geq 0$ by

$$T_{t_0}^{(n)}(\gamma) := \inf \left\{ t \geq 0 : \exists t_0 \leq t_1 \leq \cdots \leq t_k = t_0 + t \text{ such that } t_i \in \mathbb{X}_{(x_{i-1}, x_i)}^{(n)} \text{ for all } 0 < i \leq k \right\} \tag{10}$$

Clearly, the minimum in (10) is always realized.

Lemma 3.4 (Finite speed). *For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ with $|x|_\infty \geq 1$, we have $\varphi^{(n)}(x) \geq (2dne)^{-1}$. In particular, the speed is finite.*

Proof. We aim to show that for all $a < (2dne)^{-1}$, we have $\lim_{t \to \infty} \mathbb{P}(D^{(n)}(o, [tx]) \geq at) = 1$. First, a union bound over self-avoiding paths γ starting in the origin o yields

$$\mathbb{P}(D^{(n)}(o, [tx]) < at) \leq \sum_{k \geq t} \sum_{\gamma : |\gamma| = k} \mathbb{P}(T^{(n)}(\gamma) < at). \tag{11}$$

In order to bound the probability on the right-hand side, let us introduce some fixed, infinite, self-avoiding path $\gamma_\infty = (x_0, x_1, \dots)$, for example $\gamma_\infty = (o, e_1, e_2, \dots)$. Observe that the time it takes to travel a given path only depends on the order statistics of the meeting times along the edges but not on the path itself. Hence, for any $|\gamma| = k$, we have

$$\mathbb{P}(T^{(n)}(\gamma) < at) \leq \mathbb{P}(\text{At least the first } k \text{ edges of } \gamma_\infty \text{ are traversed in } \lceil at \rceil \text{ days}).$$

Let $\tau_i^{(n)}$ be the (random) number of edges that γ_∞ can travel during the i-th day. The distribution of $\tau_i^{(n)}$ depends on neither i nor on the choice of γ_∞. As a result, $\{\tau_i^{(n)} : i \in \mathbb{N}\}$ forms an i.i.d. sequence. Note that

$$\mathbb{P}(\tau_1^{(n)} \geq j) \leq n^j/j! \quad \text{for all } j \in \mathbb{N},$$

as there are n contact times on each of the j edges, yielding n^j possibilities to choose one contact time from each edge with each such choice forming an increasing sequence with probability $1/j!$. Note that $\tau_1^{(n)}$ has all exponential moments and in particular

$$\mathbb{E}\tau_1^{(n)} \leq \mathrm{e}^n \quad \text{and} \quad \mathbb{E}\mathrm{e}^{s\tau_1^{(n)}} \leq \exp(n\mathrm{e}^s) < \infty \text{ for any } s \in \mathbb{R}.$$

Hence, by an exponential moment bound

$$\mathbb{P}(T^{(n)}(\gamma) \leq at) \leq \mathbb{P}\Big(\sum_{i=1}^{\lceil at \rceil} \tau_i^{(n)} \geq k\Big) \leq \prod_{i=1}^{\lceil at \rceil} \mathbb{E}\exp\Big(s\tau_i^{(n)}\Big) \cdot \exp(-ks) \leq \exp\big(\lceil at \rceil n\mathrm{e}^s - ks\big).$$

Note that there are no more than $(2d)^k$ paths of length k. Choosing $s = \log(2d) + 1$ yields

$$
\begin{aligned}
\mathbb{P}(D(0, [tx]) < at) &\leq \sum_{k \geq t} \sum_{\gamma : |\gamma| = k} \exp\big(\lceil at \rceil n\mathrm{e}^s - ks\big) \\
&\leq \sum_{k \geq t} (2d)^k \exp(-k[\log(2d) + 1] + \lceil at \rceil n\mathrm{e}^{\log(2d)+1}) \\
&\leq \sum_{k \geq t} \mathrm{e}^{-k} \exp(2de(at+1)n) \leq \mathrm{e}^{-t} \frac{\mathrm{e}^{1+2den}}{\mathrm{e}-1} \exp(2deatn),
\end{aligned}
$$

which tends to zero as $t \to \infty$ for every $a < 1/(2dne)$. This concludes the proof. $\qquad\square$

Corollary 3.5 (Exponential decay). *For every $a > 0$ and $n, t \in \mathbb{N}$, we have*

$$\mathbb{P}\Big(I(t) \not\subset t[-1/a, 1/a]^d\Big) \leq \mathrm{e}^{-t} \frac{\mathrm{e}^{1+2den}}{\mathrm{e}-1} \exp(2deatn).$$

Proof. This follows from the exact same calculations as in Lemma 3.4. $\qquad\square$

Lemma 3.6 (Properties of the inverse speed). *Let $x, y \in \mathbb{R}^d$ and $c \in \mathbb{R}$. Then,*

   (i) φ is invariant under the symmetries of $\mathbb{Z}^d$ that fix the origin,

   (ii) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ and particularly $|\varphi(x) - \varphi(y)| \leq \varphi(x - y)$,

   (iii) φ is continuous,

   (iv) $\varphi(cx) = |c|\varphi(x)$,

   (v) $(2dne)^{-1}|x|_\infty \leq \varphi(x) \leq |x|_1 \leq d|x|_\infty$,

(vi) φ *is Lipschitz continuous,*

(vii) φ *does not depend on the density g of the contact-time distribution.*

Proof. Part (i) follows from the fact that the function φ inherits the symmetry of the underlying model on $\mathbb{Z}^d$.

In order to prove subadditivity in Part (ii), recall

$$\varphi(x) = \lim_{t\uparrow\infty} t^{-1} D(0, [tx]) = \lim_{t\uparrow\infty} t^{-1} \widetilde{D}_0(0, [tx]).$$

We further obtain

$$t^{-1}\widetilde{D}_0(0, [t(x+y)]) \le t^{-1}\widetilde{D}_0(0, [tx]) + t^{-1}\widetilde{D}_{\widetilde{D}_0(0,[tx])}([tx], [t(x+y)]).$$

Clearly, the left-hand side converges almost surely to $\varphi(x + y)$ and the first term on the right-hand side converges almost surely to $\varphi(x)$. Therefore, it remains to show

$$\liminf_{t\uparrow\infty} t^{-1}\widetilde{D}_{\widetilde{D}_0(0,[tx])}([tx], [t(x+y)]) \le \varphi(y).$$

For the second term on the right-hand side, we use shift-invariance on the lattice and shift-invariance over days to see that the term is equal in law to

$$t^{-1}\widetilde{D}_0(0, [t(x+y)] - [tx]).$$

We aim to compare this term to $\widetilde{D}_0(0, [ty])/t$, which converges to $\varphi(y)$ almost surely. To this end, observe that

$$t^{-1}\left|\widetilde{D}_0(0, [t(x+y)] - [tx]) - \widetilde{D}_0(0, [ty])\right| \to 0,$$

in distribution, as $t \to \infty$, since $|([t(x + y)] - [tx]) - [ty]| \le 3$. In particular, this implies

$$\widetilde{D}_{\widetilde{D}_0(0,[tx])}([tx], [t(x+y)]) \to \varphi(y),$$

in distribution. Since $\varphi(y)$ is a deterministic limit, the established convergence also holds in probability and we can therefore find a subsequence along which the convergence holds almost surely. This finishes the proof of subadditivity. The second statement in Part (ii) simply follows from symmetry and the reverse triangle inequality.

The continuity of φ is a result of Lemma 3.3, proving Part (iii).

In order to prove Part (iv), observe that we have, for all $s \in \mathbb{N}$,

$$\varphi(sx) = \lim_{t\uparrow\infty} t^{-1} D(0, tsx) = \lim_{t\uparrow\infty} s t^{-1} D(0, tx) = s\varphi(x)$$

and by distributional symmetry clearly $\varphi(-x) = \varphi(x)$. Further, for each rational $c = p/m$

$$m\varphi(xp/m) = \varphi(px) = p\varphi(x),$$

implying $\varphi(cx) = c\varphi(x)$. The claim for arbitrary $c \in \mathbb{R}$ then follows from the continuity in Part (iii).

The upper bound in Part (v) is trivial since at least one edge per day is traversed. Furthermore, the shortest path from the origin o to any $[kx]$ has length at most $|[kx]|_1 \le d(k|x|_\infty + 1)$ and the lower bound is thus a consequence of Lemma 3.4 and Part (iii).

For the Lipschitz-continuity, observe that we have for any $x \neq y$ by (ii) and (iv),

$$\left|\varphi(x) - \varphi(y)\right| \leq \varphi(x - y) \leq |x - y|_1 \varphi\left(\frac{x - y}{|x - y|_1}\right) \leq |x - y|_1 \cdot \varphi(e_1),$$

using $\sup_{|v|_1 = 1} \varphi(v) = \varphi(e_1)$ and thus proving Part (vi).

Finally, the independence of φ of f follows from the observation that φ only depends on the order statistics of the contact times, which is the same for all absolutely continuous distributions w.r.t. the Lebesgue measure. This concludes the proof. $\qquad\square$

We have now collected all the results required to prove Theorem 2.5. Let us first define the limiting shape $\mathcal{B} = \mathcal{B}_n$ via

$$\mathcal{B} := \{x \in \mathbb{R}^d \colon \varphi(x) \leq 1\}, \tag{12}$$

and note that, due to Lemma 3.6, $\mathcal{B}$ is indeed closed and convex, $\mathcal{B} \subset [-b, b]^d$ for $b = 2dne$, and $\mathcal{B}$ does not depend on the distribution of the contact times.

Proof of Theorem 2.5. We argue by contradiction. Assume that the theorem does not hold. That is, there exists some $\varepsilon > 0$ and an event Ω_ε of positive probability, on which for infinitely many $t \in \mathbb{N}$, there exists a $z_t \in \mathbb{Z}^d$ such that either

$$z_t \in I(t) \text{ and } z_t \notin (1 + \varepsilon)t\mathcal{B} \tag{13}$$

or

$$z_t \notin I(t) \text{ and } z_t \in (1 - \varepsilon)t\mathcal{B}. \tag{14}$$

Let us focus on the case that (14) happens for infinitely many t. The case (13) is handled similarly as explained below.

As $\mathcal{B}$ is bounded, we find some $x \in \mathbb{R}^d$ and a subsequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that

$$z_{t_k} \in (1 - \varepsilon)t\mathcal{B} \qquad \text{and} \qquad \lim_{k \uparrow \infty} z_{t_k}/t_k = x.$$

Furthermore, since $\mathcal{B}$ is closed, we have $x \in (1 - \varepsilon)\mathcal{B}$. In particular, by Lemma 3.3 and Definition (12), we have

$$\lim_{t \uparrow \infty} t^{-1} D_0(0, [tx]) = \varphi(x) < 1 - \varepsilon.$$

However, $z_{t_k} \notin \tilde{I}(t_k)$ is equivalent to $D_0(0, z_{t_k}) > t_k$, i.e.,

$$D_0(0, z_{t_k})/t_k > 1,$$

which yields, by use of the reverse triangle inequality (9) for z_{t_k} and $t_k x$,

$$1 < D_0(0, z_{t_k})/t_k \leq |z_{t_k}/t_k - x|_1 + 2d/t_k + D_0(0, [t_k x])/t_k \xrightarrow{k \to \infty} \varphi(x) \leq 1 - \varepsilon,$$

providing a contradiction. Let us briefly discuss the remaining case, and assume that (13) happens infinitely often. For some fixed $a < 2dne$, Corollary 3.5 together with the Borel–Cantelli Lemma implies $\tilde{I}(t)/t \subset [-1/a, 1/a]^d$ almost surely for large t. We infer the existence of a convergent subsequence $z_{t_k}/t_k \to x$, as $k \to \infty$. Analogous calculations yield the contradiction

$$1 \geq D_0(0, z_{t_k})/t_k \geq D_0(0, [t_k x])/t_k - \left(|z_{t_k}/t_k - x|_1 + 2d/t_k\right) \xrightarrow{k \to \infty} \varphi(x) \geq 1 + \varepsilon.$$

This concludes the proof. $\qquad\square$

3.3   Proof of couplings

We provide various couplings of the model based on (PL) in this section. This will ultimately result in showing Proposition 2.7 and Theorem 2.6, i.e., verifying that the rescaled shape $\mathcal{B}_n/n$ converges w.r.t. the Hausdorff metric towards the limiting shape $\mathcal{B}_R$ of the Richardson model.

We start with the proof of Proposition 2.7, which enables a simpler proof of Theorem 2.6 later. Loosely speaking, the rigid model (SL) is faster since not being able to traverse an edge in a given day will give a bias to traverse said edge earlier the next day.

Proof of Proposition 2.7. Let $(U_e^{(1)})_{e \in E}$ and $(U_e^{(2)})_{e \in E}$ be two independent sequences of in-dependent uniformly on $(0, 1)$ distributed random variables. We start by describing the explo-ration processes corresponding to the models (18) and (SL) respectively, thus coupling the two explorations. To this end, denote by

$$F_{n,p}(\cdot) = \sum_{i=0}^{\lfloor \cdot \rfloor} \binom{n}{i} p^i (1-p)^{n-i}$$

the distribution function of a binomially distributed random variable. For each edge e, we define the function $K_e : [0, 1] \to \{0, \ldots, n\}$ via

$$K_e(t) = \max \left\{ k \in \{0, \ldots, n\} \colon U_e^{(1)} \le F_{n,(1-t)}(k) \right\},$$

and extend K_e to the whole real line by way of $K_e(t) := K_e(t - \lfloor t \rfloor)$. The quantity $K_e(t)$ thus models the number of contacts left on the edge e during $[t, \lceil t \rceil)$, where $\lceil t \rceil = \lfloor t \rfloor + 1$. Put differently, if the infection arrives at an edge for the first time on the $\lfloor t \rfloor$-th day at time $t - \lfloor t \rfloor$, then there are $K_e(t)$ many contacts still happening until the end of that day. By definition, it is easy to see that for $0 < s < t < 1$, we have $K_e(s) \ge K_e(t)$ since $1 - t < 1 - s$.

For $\tau > 0$ and $K \in \mathbb{N}$, we additionally define the random variable

$$T_e(\tau, K) = \tau \left(1 - \sqrt[K]{U_e^{(2)}} \right),$$

noting that it equals, in distribution, the minimum of K independent random variables dis-tributed uniformly on $[0, \tau]$.

We proceed by describing the exploration process I associated to (PL) in terms of K_e and T_e. Recall that, whenever the exploration discovers a new vertex that has not yet be seen at some time s, its edges incident to yet unexplored vertices are revealed. Consider such a fixed edge e. If there is at least one more contact happening during $[s, \lceil s \rceil]$, then the other end-vertex becomes revealed at the minimum of the remaining contact times. If there is no more contact happening, the other end-vertex becomes revealed at the minimum of the n newly sampled contact times within $[\lceil s \rceil, \lceil s \rceil + 1]$. However, by the above, the probability of seeing at least one more contact during the day of discovery coincides with the probability of $\{K_e(s) \ge 1\}$. Moreover, conditionally on $\{K_e(s) \ge 1\}$, the next contact has the same distribution as

$$s + T_e\big(\lceil s \rceil - s, K_e(s)\big) = \lceil s \rceil - \big(\lceil s \rceil - s\big)^{K_e(s)} \sqrt{U_e^{(2)}}. \tag{15}$$

Similarly, if there is no more contact happening on the same day, meaning $K_e(s) = 0$, the first contact on the $\lceil s \rceil$-th day has the same distribution as

$$\lceil s \rceil + T_e(1, n) = \lceil s \rceil + 1 - \sqrt[n]{U_e^{(2)}}. \tag{16}$$

Analogously, we describe I', the exploration process associated to (SL). First of all, if an edge e is considered for the first time at time s and there is still a contact left during $[\lceil s \rceil, \lceil s \rceil + 1]$, the situation is precisely as in (15). However, conditionally on $K_e(s) = 0$, the information that no contact falls into $[s, \lceil s \rceil]$ yields that the first contact time on the next day equals in distribution

$$\lceil s \rceil + T_e\big(s - \lfloor s \rfloor, n\big) = s + 1 - \big(s - \lfloor s \rfloor\big) \sqrt[n]{U_e^{(2)}}. \qquad (17)$$

Having established the coupling, it remains to prove its monotonicity in the sense that $I(t) \subset I'(t)$ for all $t \geq 0$. We work on the probability space where I and I' are jointly defined by the previous coupling. Let $(u_e^{(1)})_e$ and $(u_e^{(2)})$ be realizations of the used sequences of independent random variables. Let $(s_v)_{v \in \mathbb{Z}^d}$ and $(s'_v)_{v \in \mathbb{Z}^d}$ be the collection of first times each vertex is explored by the processes I and I' respectively. Let $0 = \sigma'_0 < \sigma'_1, \dots$ be the elements of $(s'_v)_v$ sorted from smallest to largest value. We now show inductively that for all times σ'_n, we have $I(\sigma'_n) \subset I'(\sigma'_n)$, and all edges that have been revealed but not yet been traversed in I' are traversed in I' no later than in I. This particularly implies $I(t) \subset I'(t)$ for all $t \geq 0$. We start with the *base case* $\sigma'_0 = 0$. At time 0, we have $I(0) = I'(0) = \{o\}$, i.e., only the origin has been discovered yet. Moreover, all incident edges and their next contact time become revealed. By the coupling, the next contact time on each of these edges coincide.

Now assume for the *induction step* that our assumption holds for σ'_{n-1}. We note that $I(\sigma'_{n-1}) \subset I'(\sigma'_{n-1})$ implies that all edges revealed but not traversed in I are either already traversed or at least revealed in I' as well. The second assumption then guarantees that the edge traversed at time σ'_n by I' has not been traversed earlier by I and thus $I(\sigma'_n) \subset I'(\sigma'_n)$ still. Let v be the vertex explored at time σ'_n, i.e., $s'_v = \sigma'_n$ and therefore $s_v \geq s'_v$. Consider the edges incident to v that have not yet been revealed and we denote by t_e, t'_e the time of the first contact on e after being revealed by the respective explorations. To finish the proof of the induction step, we have to show that $t_e \geq t'_e$ for all newly revealed edges. To this end, we make use of (15), (16), and (17) and distinguish the following cases:

(i) If $\lceil s_v \rceil = \lceil s'_v \rceil$ and $K_e(s_v) \geq 1$, then

$$t'_e = \lceil s'_v \rceil - \big(\lceil s'_v \rceil - s'_v\big)^{K_e(s'_v)}\hspace{-1em}\sqrt{u_e^{(2)}} \leq \lceil s_v \rceil - \big(\lceil s_v \rceil - s_v\big)^{K_e(s_v)}\hspace{-1em}\sqrt{u_e^{(2)}} = t_e,$$

using $K_e(s'_v) \geq K_e(s_v)$ here, as outlined above.

(ii) If $\lceil s_v \rceil = \lceil s'_v \rceil$ and $K_e(s'_v) > K_e(s_v) = 0$, then $t'_e \leq \lceil s_v \rceil \leq t_e$.

(iii) If $\lceil s_v \rceil = \lceil s'_v \rceil$ and $K_e(s'_v) = K_e(s_v) = 0$, then

$$t'_e = \lceil s_v \rceil + T_e(s'_v - \lceil s'_v \rceil, n) \leq \lceil s_v \rceil + T_e(1, n) = t_e.$$

(iv) In all remaining cases, we have $\lceil s'_v \rceil < s_v$. If additionally either $s_v \geq \lceil s'_v \rceil + 1$ or $K_e(s_v) = 0$, then $t'_e \leq \lceil s'_v \rceil + 1 \leq t_e$. Similarly, if $K_e(s'_v) > 0$, then $t'_e \leq \lceil s'_v \rceil \leq s_v \leq t_e$. Finally, if $K_e(s'_v) = 0$, $K_e(s_v) > 0$, and $\lceil s'_v \rceil = \lfloor s_v \rfloor$, then $\lceil s'_v \rceil + 1 = \lceil s_v \rceil$ and thus

$$t'_e = s'_v + 1 - \big(s'_v - \lfloor s'_v \rfloor\big)\sqrt[n]{u_e^{(2)}} \leq \lceil s'_v \rceil + 1 - \big(s'_v - \lfloor s'_v \rfloor\big)^{K_e(s_v)}\hspace{-1em}\sqrt{u_e^{(2)}} - \big(\lceil s'_v \rceil - s'_v\big)^{K_e(s_v)}\hspace{-1em}\sqrt{u_e^{(2)}}$$

$$= \lceil s_v \rceil - \big(\lceil s'_v \rceil - \lfloor s'_v \rfloor\big)^{K_e(s_v)}\hspace{-1em}\sqrt{u_e^{(2)}} \leq \lceil s_v \rceil - \big(\lceil s_v \rceil - s_v\big)^{K_e(s_v)}\hspace{-1em}\sqrt{u_e^{(2)}} = t_e.$$

Summarizing, we have $t'_e \leq t_e$ for all newly revealed edges, which finishes the induction and thus the proof of Proposition 2.7. $\qquad\square$

Now, let us move towards the proof of Theorem 2.6. We begin with the following observations. First, we may restrict ourselves without loss of generality to the special case of the model based on (PL), in which the contact times are distributed uniformly on $[0, 1)$ as this does not influence the limiting shape $\mathcal{B}_n$. Put differently, the random variables $\{U_e^{(k,i)} \colon k \in \mathbb{N}_0, i = 1, \ldots, n\}$ on each edge form i.i.d. sequences of $\mathrm{Uniform}(0, 1)$ random variables. Secondly, we want to slow down time by a factor of n. Instead of assigning each edge n contact times per day (an interval of unit length), we assign the n contact times to a period of n days represented by an interval of the form $[kn, (k + 1)n)$. Formally, assign independently to each edge e the independent random variables $\{\overline{U}_e^{(k,i)} \colon k \in \mathbb{N}_0, i = 1, \ldots, n\}$ distributed uniformly on $[kn, (k+1)n)$. Then, the corresponding FCP is defined via the point processes

$$\overline{\mathbb{X}}_e^{(n)} := \bigcup_{k=0}^{\infty} \bigcup_{i=1}^{n} \left\{\overline{U}_e^{(k,i)}\right\}. \tag{18}$$

Let us denote the corresponding process of reachable vertices and the limiting shape by $\overline{I}(t)$, respectively $\overline{\mathcal{B}}_n$, which are equal to $I(t/n)$, respectively $\mathcal{B}_n/n$.

In order to prove Theorem 2.6, we rely on the following continuity result [9, Theorem 3].

Theorem 3.7 (Continuity of the time constant in FPP, Theorem 3 [9]). *Let μ_n and μ be distributions supported on $[0, \infty)$ such that $\mu_n \xrightarrow{w} \mu$. Then, for all $x \in \mathbb{Z}^d$*

$$\lim_{n \uparrow \infty} \psi^{(\mu_n)}(x) = \psi^{(\mu)}(x),$$

where $\psi^{(\mu)}(x)$ denotes the time constant in direction $x \in \mathbb{R}^d$, meaning

$$\psi^{(\mu)}(x) = \lim_{t \uparrow \infty} t^{-1} D^{(\mu)}(o, [kx]).$$

Remark 3.8. The analogous continuity of the inverse speed as established for FPP in Theorem 3.7 does not hold in general for FCP: Consider a version of the FCP model in which the contact times are distributed uniformly on $[0, 1/k]$. By Theorem 2.5 we observe the same inverse speed and the same limiting shape for each fixed k. However, as $k \to \infty$, the limiting distribution has an atom at zero, which results in instant transmission across all of $\mathbb{Z}^d$.

Proof of Theorem 2.6. Let us first construct the coupling between the time-rescaled model of the model based on (PL) (i.e., having n contacts in n days) and the Richardson model. For an edge e and time $t \geq 0$, let $\overline{X}_{e,t}^{(n)}$ be the time duration until the next contact from t on the edge e. That is,

$$\overline{X}_{e,t}^{(n)} = \min\left\{s \geq 0 \colon t + s \in \overline{\mathbb{X}}_e^{(n)}\right\}.$$

Given some $t \in [0, n)$, we have for any $s \in [0, n)$ with $s + t \leq n$,

$$\mathbb{P}(\overline{X}_{e,t}^{(n)} > s) = \mathbb{P}\left(\overline{\mathbb{X}}_e^{(n)} \cap [t, t+s] = \emptyset\right) = (1 - s/n)^n, \tag{19}$$

while we have for $s \in [0, 2n]$ with $2n \geq t + s \geq n$, by the independence of the contact times in the intervals $[0, n)$ and $[n, 2n]$,

$$\begin{aligned}
\mathbb{P}(\overline{X}_{e,t}^{(n)} > s) &= \mathbb{P}(\overline{\mathbb{X}}_e^{(n)} \cap [t, t+s] = \emptyset) \\
&= \mathbb{P}\left(\overline{\mathbb{X}}_e^{(n)} \cap [t, n) = \emptyset\right) \mathbb{P}\left(\mathbb{X}_e^{(n)} \cap [n, t+s) = \emptyset\right) \\
&= \left(\frac{t}{n}\right)^n \left(\frac{2n - (t+s)}{n}\right)^n.
\end{aligned} \tag{20}$$

Note that this is true for any $t \in [kn, (k+1)n)$ for some k by shift invariance. Furthermore, note that (19) is included in (20) by choosing $t = n$. We observe from Equation (20) that

$$\mathbb{P}\left(\overline{X}_{e,t}^{(n)} > s\right) = \left(\frac{t}{n}\right)^n \left(\frac{2n-(t+s)}{n}\right)^n = \left(1 + \frac{t-n}{n}\right)^n \left(1 + \frac{n-(t+s)}{n}\right)^n \leq \mathrm{e}^{t-n+n-(t+s)} = \mathrm{e}^{-s},$$

which shows the claim $\overline{I}(t) \supset J_R(t)$ via a standard coupling in the exploration process.

Next, we show the convergence of $\overline{\mathcal{B}}_n/n$. Using, Proposition 2.7, it suffices to show $\mathcal{B}'_n/n \to \mathcal{B}_R$. As the latter is a FCP model given by an underlying stationary random closed set, Proposition 2.1 allows us to equivalently consider an FPP model with waiting time distribution $\mu^{(n)}$ given by (4). We see that for arbitrary $s \geq 0$ (and $n \geq s$), we have

$$\mu^{(n)}([s,\infty)) = (1 - s/n)^n \xrightarrow{n\to\infty} \mathrm{e}^{-s}.$$

Since the (deterministic) limiting shapes are solely defined via the (deterministic) time constants, Theorem 3.7 finishes the proof. □

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