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Regularity for non-autonomous parabolic equations with right-hand side singular measures involved

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Abstract

This article provides a theory for non-autonomous parabolic equations the right-hand side of which includes singular measures – depending on the time parameter – on the spatial domain. In two space dimensions all bounded Radon measures are admissible as such. In higher dimensions the focus is on measures whose support is concentrated on l-sets in the sense of Jonsson and Wallin. It is shown that they may be interpreted as elements from a Sobolev space $W^{-1,q}(\Omega)$. So the right-hand side is considered as an element from $L^q(J; W^{-1,q}(\Omega)$. Having this at hand, previous results on maximal (non-autonomuous) maximal parabolic regularity apply and show that the solution lies in the corresponding space of maximal parabolic regularity. In contrast to other work in this field we only require absolute minimal smoothness for the data of the problem: the domain, the coefficients – and mixed boundary conditions are allowed. Under minimally stronger assumptions we even show the Hölder property in space and time. Overall, this work contains an interplay of geometric measure theory with advanced parabolic theory which delivers as much parabolic regularity for the solution as one can expect.

1 Introduction

In the meanwhile the consideration of generally non-autonomous parabolic equations

$$\frac{\partial u}{\partial t} + \mathcal{A}(\cdot)u = \varrho, \quad u(0) = 0 \tag{1}$$

with right-hand sides ϱ including measures is quite usual, see the pioneering paper [3], see also [8]. Here we consider the case where the (scalar) measure ϱ on the space-time cylinder is of the kind

$$C_0(]0, T[\times\Omega) \ni w \mapsto \int_0^T \int_\Omega w(t, x)\, d\rho_t(x)\, dt, \tag{2}$$

$\{\rho_t\}_{t\in]0,T]}$ being a suitable family of Radon measures on Ω, which is – in its dependence of t – weak* measurable. The procedure how to treat such parabolic equations is widely common: embed $\mathcal{M}(\Omega)$, the space of bounded Radon measures on Ω, into a space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ and identify the r.h.s. in this manner with a function $f \in L^s(]0, T[, W^{-1,q}_{\mathfrak{D}}(\Omega))$ ($W^{1,q}_{\mathfrak{D}}(\Omega)$ denoting the usual Sobolev space which includes a trace-zero condition on $\mathfrak{D} \subset \partial\Omega$ and $W^{-1,q}_{\mathfrak{D}}(\Omega)$ being the space of continuous antilinear forms on $W^{1,q'}_{\mathfrak{D}}(\Omega)$). For such right-hand sides one may – under certain conditions – apply maximal parabolic regularity of the operators involved to get a solution which belongs to the maximal parabolic regularity space (see (13) below). It is almost clear that this is widely optimal for the solution. Unfortunately, this has several drawbacks:

• In order to catch *all* bounded Radon measures, the q's to be chosen this way ly definitely below $\frac{d}{d-1}$, d being the space dimension. Therefore the domain of the elliptic second order divergence operator

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can be at best $W^{1,q}_{\mathfrak{D}}(\Omega)$ – with this limitation of q. This is any case more irregular than $W^{1,2}_{\mathfrak{D}}(\Omega)$. But even worse: in general it is extremely delicate – in view of pathologies which were discovered already by Serrin in [42] – to give the divergence operators on $W^{-1,q}_{\mathfrak{D}}(\Omega)$ a precise meaning at all, if $q < 2$ is far from 2.
▲ Only in special cases, concerning the domain Ω, the Dirichlet boundary part $\mathfrak{D}$ and the coefficient function μ it is true that the domain of $-\nabla \cdot \mu \nabla$ in fact is contained in the Sobolev space $W^{1,q}(\Omega)$. In general it is impossible to determine the domain explicitly.
■ Aside from more or less trivial cases – concerning 'smoothness' of the problem – one has no idea how to treat the *non-autonomous* case in generality. Sufficient concicitions are e.g. the continuity in time of the coefficient function with respect to the L^∞-norm (for an application of the Prüss/Schnaubelt theorem [40]) in combination with rather strong elliptic regularity results – fulfilled only in special cases.
In recent years also the numerics of such problems has been treated, see [31], [32]; see also [41], [35], [10]. In the first paper it is reflected that discontinuous diffusion coefficients allow the treatment of moving interfaces – what is often definitely required in applications. [32] and [10] discuss real world problems where the – time dependent – measures on the right-hand side of the parabolic equation are concentrated on hypersurfaces. Moreover, parabolic equations with measure valued right-hand sides have attracted attention also in optimal control, see e.g. [8], [9]. Here our ansatz in (2) is more or less identical with that in [8] concerning the measures involved. But, in contrast to [8], [9], our coefficient functions may depend *discontinuously* on space *and* time, so that in fact *non-autonomous* parabolic equations are the main subject of this paper.
Since the case where *any* bounded Radon measure, among them Dirac measures, can serve as a ρ_t in (2) is widely clear (inclusively the drawbacks in • ▲ ■), the focus is here on the following question:
Is there a good concept to confine the measures ρ_t to suitable subclasses $\widehat{\mathcal{M}}$ such that the embedding $\widehat{\mathcal{M}} \hookrightarrow W^{-1,q}_{\mathfrak{D}}(\Omega)$ is valid for considerably larger q than $\frac{d}{d-1}$ – at best close to 2?
'Good' means at least the following:
I) In view of the above mentioned applications it is necessary that measures, living on curves, surfaces..., should appear within such subclasses in a cognizable manner.
II) One can find clear and general analytical conditions on the measures ρ_t such that the resulting – via embedding – objects belong to $W^{-1,q}_{\mathfrak{D}}(\Omega)$ and, additionally, that the corresponding functions $J \ni t \mapsto \rho_t$ belong to a space $L^s(]0,T[; W^{-1,q}_{\mathfrak{D}}(\Omega))$. In particular, one has conditions on the sets M_t, carrying the corresponding measure ρ_t, such that the mapping $J \ni t \mapsto \rho_t \in W^{-1,q}_{\mathfrak{D}}$ is indeed measurable.
III) The spaces $W^{-1,q}_{\mathfrak{D}}(\Omega)$ – with q in the range of II) – have a good 'parabolic behavior' – i.e. the apparatus of *autonomous and non-autonomous* maximal parabolic regularity applies for the second order elliptic operators and one gets as much information as one can on the solution this way.

Astonishingly, one can find such concept when employing the old, but pioneering ideas of Jonsson/Wallin [29]. Namely one considers closed subsets M of Ω which are so-called **l-sets**. This means that they have to satisfy a condition

$$\mathfrak{c}_\bullet\, r^l \le \mathcal{H}_l(M \cap B(x,r)) \le \mathfrak{c}^\bullet\, r^l, \quad x \in M,\, r \in\,]0,1], \tag{3}$$

where $l \in \{1, \dots, d-1\}$ and $\mathcal{H}_l$ is the l-dimensional Hausdorff measure. So, M taken as an l-set, we consider measures $\sigma \mathcal{H}_l|_M$, σ being a function from $L^p(M; \mathcal{H}_l)$ with p suitably chosen. It is clear that this concept fulfills I): Lipschitzian curves, surfaces etc. are included as sets M (see [21, Ch. 3.3.4]), and on these density functions σ from adequate summability classes are allowed. But much more: one does not require any 'smoothness', e.g. any finite combination of surfaces or curves is admissible.

On the other hand, the pioneering results of Jonsson/Wallin admit even in this case the embeddings required for an adequate analysis. Here the most striking argument is the following: due to a classical result in [29] one gets the uniform boundedness for norms of the mappings

$$L^p(M;\mathcal{H}_l) \ni \sigma \mapsto \sigma\mathcal{H}_l|_M \in W^{-1,q}_{\mathfrak{D}}(\Omega)$$

if M runs through a class of subsets in Ω admitting a *uniform* upper l estimate, i.e. when the constant $\mathfrak{c}^\bullet$ in (3) may be chosen uniformly for all sets under consideration. Obviously, this comes into play when considering varying in time measures ρ_t on Ω and one needs that the corresponding right-hand sides indeed belong to $L^s(]0,T[;W^{-1,q}_{\mathfrak{D}}(\Omega))$ – in order to apply then maximal parabolic regularity for the second order operators on the space $W^{-1,q}_{\mathfrak{D}}(\Omega))$.
Now the right-hand sides being interpreted as functions in $L^s(]0,T[;W^{-1,q}_{\mathfrak{D}}(\Omega))$, let us discuss what one has to expect concerning the quality of the solution for the corresponding parabolic equation – in view of maximal parabolic regularity, autonomous and non-autonomous. Fortunately, in [24] and [15] we developped the sharp instruments for the treatment of even non-autonomous parabolic equations: on one hand is shown in [24] that the Lax-Milgram isomorphism

$$-\nabla\cdot\mu\nabla + 1 : W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{-1,q}_{\mathfrak{D}}(\Omega) \tag{4}$$

for $q = 2$ extrapolates to other integrability indices q close to 2 – including adequate estimates for the inverse operator and uniform in coefficient functions μ possessing the same L^∞ bound and the same ellipticity constant, see Prop. 2.6 below for details. Secondly, we exploit the central result of [15], namely that, for s,q close to 2

$$W^{1,s}_0(J;W^{-1,q}_{\mathfrak{D}}(\Omega)) \cap L^s(J;W^{1,q}_{\mathfrak{D}}(\Omega)) \ni w \mapsto \frac{\partial w}{\partial t} - div\hat{\mu}grad\, w \in L^s(J;W^{-1,q}_{\mathfrak{D}}(\Omega)) \tag{5}$$

is a topological isomorphism, extrapolating the classical Lions isomorphism for $r = q = 2$, see [11, Section XVIII.3 Remark 9].
Summing up, one can expect that III) fits well in general only if q is close to 2. But, good luck, exactly this happens in the following cases which we consider as the most relevant ones in applications:
♣ In $2d$ one has (see Lemma 3.1 below)

$$\mathcal{M}(\Omega) \hookrightarrow W^{-1,q}_{\mathfrak{D}}(\Omega), \tag{6}$$

for *every* $q \in]1,2[$. This makes the parabolic equation tractable in the sense of maximal parabolic regularity – then even knowing that the domains of the time dependent elliptic operators coincide with $W^{1,q}_{\mathfrak{D}}(\Omega)$ if q sufficiently close to 2.
♠ Consider again a two-dimensional domain, this time with a curve M within, this being an 1-set. Then $2 \sim q > 2$ is admissible, see Thm. 3.14.
★ In three dimensions the situation is as follows: if the supporting set M for the measure is a 'curve' – in the meaning of being a 1-set – then $q < 2$ may be taken arbitrarily close to 2.
If the supporting set M is a 'surface' – in the meaning of being a 2-set – then q may be taken even larger than 2 (even more is true, see Thm. 3.10). So in any of these two cases the available elliptic and parabolic regularity theorems apply.
Let us emphasize that this whole machinery works for *mixed* boundary conditions – where the cases of pure Dirichlet or pure Neumann conditions are, of course, included. Note that this requires for the coefficient function $t \mapsto \mu(t,\cdot)$ only boundeness, uniform (in t) ellipticity and an extremely weak measurability condition, see Assu. 2.13 below.

Overall, combining ideas from geometric measure theory and advanced parabolic theory, we intend to give in this article a general and profound functional analytic investigation of – even nonautonomous

– parabolic equations with measure-valued right-hand sides which can serve as a basis for numerical treatment and even optimal control for such equations.

Throughout this paper we denote by d the dimension of the domain Ω and, for $l \in \{1, \dots, d-1\}$ the l-dimensional Hausdorff measure by $\mathcal{H}_l$. We recall that on smooth and Lipschitzian submanifolds of $\mathbb{R}^d$ the Hausdorff measure is identical with the measure defined by parametrizations on this manifold, see [21, Ch. 3.3/3.4]. Moreover, if $M \subset \Omega \subset \mathbb{R}^d$ then we abbreviate $L^p(M; \mathcal{H}_l|_M)$ by $L^p(M; \mathcal{H}_l)$ in all what follows, since misunderstandings seem to be excluded at this point. If $\Omega \subset \mathbb{R}^d$ is a bounded domain, then we denote by $\mathcal{M}(\Omega)$ the space of finite Radon measures on Ω.
Finally, for two Banach spaces X, Y, Y continuously embedded into X, we denote by $(X, Y)_{\theta,r}$ the usual real interpolation space and by $[X, Y]_\theta$ the corresponding complex interpolation space (see [44, Ch. I]).

Since in most applications the coefficient function is real-valued, we suppose this during this paper.

2 Elliptic and parabolic regularity in the $W^{-1,q}_{\mathfrak{D}}$ scale

In all what follows we suppose the following assumption to be in power.

Assumption 2.1. Let here and in the sequel $B(x, r)$ denote the ball in $\mathbb{R}^d$ with center x and radius r. Ω is a bounded domain in $\mathbb{R}^d$.

(a) $\mathfrak{D}$ is a closed subset of $\partial\Omega$ which satisfies the *Ahlfors–David condition*; that is, there are $c_0, c_1 > 0$ such that
$$c_0\, r^{d-1} \le \mathcal{H}_{d-1}(\mathfrak{D} \cap B(x, r)) \le c_1\, r^{d-1} \tag{7}$$
for all $x \in \mathfrak{D}$ and $r \in \,]0, 1]$.

(b) For every $x \in \overline{\partial\Omega \setminus \mathfrak{D}}$ there exists an open neighborhood U_x of x and a bi-Lipschitz map ϕ_x from U_x onto the cube $K := \,]-1, 1[^d$, such that the following three conditions are satisfied:
$$\phi_x(x) = 0,$$
$$\phi_x(U_x \cap \Omega) = \{x \in K : x_d < 0\},$$
$$\phi_x(U_x \cap \partial\Omega) = \{x \in K : x_d = 0\}.$$

(c) $\partial\mathfrak{D} := \mathfrak{D} \cap \overline{\partial\Omega \setminus \mathfrak{D})}$ is a $(d-2)$-set, i.e. there are constants $\underline{c}, \overline{c}$, such that
$$\underline{c}\, r^{d-2} \le \mathcal{H}_{d-2}\big(\partial\mathfrak{D} \cap B(x, r)\big) \le \overline{c}\, r^{d-2}.$$

2.1 Elliptic operators

Definition 2.2. We define $W^{1,q}_{\mathfrak{D}}(\Omega)$ as the closure of
$$C^\infty_{\mathfrak{D}}(\Omega) := \{\psi|_\Omega : \psi \in C^\infty_0(\mathbb{R}^d),\ supp\, \psi \cap \mathfrak{D} = \emptyset\}$$
in $W^{1,q}(\Omega)$.
By $W^{-1,q}_{\mathfrak{D}}(\Omega)$ we denote the (anti)dual of $W^{1,q'}_{\mathfrak{D}}(\Omega)$.

Remark 2.3. If only Assu. 2.1 (b) is fulfilled, then there exists a continuous extension operator $\mathfrak{E}$: $W^{1,q}_{\mathfrak{D}}(\Omega)$ into $W^{1,q}(\mathbb{R}^d)$. It can be chosen independently of $q \in [1,\infty[$.

Definition 2.4. Assume $q \in]1,\infty[$ and let μ be a measurable, bounded, $\mathbb{R}^{d\times d}$-valued function on Ω. Then we define the operator $-\nabla\cdot\mu\nabla : W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{-1,q}_{\mathfrak{D}}(\Omega)$ by

$$\langle -\nabla\cdot\mu\nabla\psi, \varphi\rangle = \int_\Omega \mu\nabla\psi\cdot\nabla\overline{\varphi}, \quad \psi\in W^{1,q}_{\mathfrak{D}}(\Omega),\ \varphi\in W^{1,q'}_{\mathfrak{D}}(\Omega). \tag{8}$$

Proposition 2.5. *i) $-\nabla\cdot\mu\nabla : W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{-1,q}_{\mathfrak{D}}(\Omega)$ is bounded with bound not larger than $\|\mu\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))}$ – entirely independent of q.*

ii) The operators are consistent with each other for different q's.

Proof. i) follows from Hölder's inequality and ii) is obvious. □

We use the same symbol $-\nabla\cdot\mu\nabla$ irrespective what q is. This is reasonable by ii) of the proposition.

Proposition 2.6. *(see [24])*

Let μ be a bounded, measurable, strongly elliptic coefficient function, i.e. one has

$$\operatorname*{ess\,inf}_{x\in\Omega} \Re\langle\mu(x)\xi,\xi\rangle \ge \mu_\bullet\|\xi\|^2, \quad \xi\in\mathbb{C}^d \tag{9}$$

for some positive constant $\mu_\bullet$.
Then, under Assumption 2.1, the set of q's for which

$$-\nabla\cdot\mu\nabla + I : W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{-1,q}_{\mathfrak{D}}(\Omega) \tag{10}$$

is a surjection (and, hence, topological isomorphism) is an open interval $\mathcal{I}_\mu =]2-\delta, 2+\varepsilon[\ni 2$.
For every set $\mathcal{C}$ of coefficient functions μ which admit a uniform L^∞ bound and also a uniform ellipticity constant $\mu_\bullet$, the numbers δ and ε may be taken uniformly with respect to all coefficient functions from $\mathcal{C}$.
In addition,

$$\sup_{\mu\in\mathcal{C}} \|(-\nabla\cdot\mu\nabla + I)^{-1}\|_{\mathcal{L}(W^{-1,q}_{\mathfrak{D}};W^{1,q}_{\mathfrak{D}})} < \infty$$

for all q from the corresponding uniform interval.

Let us state a permanence property for the operators $-\nabla\cdot\mu\nabla$.

Lemma 2.7. *Let μ^* be the adjoint coefficient function. Then*

$$-\nabla\cdot\mu^*\nabla + I : W^{1,q'}_{\mathfrak{D}}(\Omega) \to W^{-1,q'}_{\mathfrak{D}}(\Omega) \tag{11}$$

is the adjoint operator to (10).
(11) *is a topological isomorphism if and only if* (10) *is.*

Proof. This follows by a straight forward calculation. □

Striking examples (see [38] or [19, Ch. 4]) show that the result of Prop. 2.6 is optimal in general insofar as both, δ and ε, can become arbitrarily small in general. Nevertheless, one succeeds in establishing a large variety of geometries and coefficient functions on $\Omega \subset \mathbb{R}^3$, such that (10) is a topological isomorphism even for a $q > 3$, see [16], compare also [28], [18], [19], [22, Cor. 9.3], [45], [23]. The price one has to pay are severe restrictions on on the geometry of Ω, $\mathfrak{D}$ and on the coefficient function μ.

All of this makes it clear that it is more the exception than the rule that (10) is surjective. Even worse, this operator is for $q's$ outside the interval $]2-\delta, 2+\epsilon[$ not even a *closed* one and very strange phenomena can appear then, see [42]. In this paper we will avoid all these difficulties by restricting, for $q < 2$, the considerations to q's from $\mathcal{I}_\mu$.

In contrast to [2], $\mathfrak{D} \neq \emptyset \neq \partial\Omega \setminus \mathfrak{D}$ with $\mathfrak{D} \cap \overline{\partial\Omega \setminus \mathfrak{D}} \neq \emptyset$, i.e. mixed boundary conditions, are allowed – see in particular [25] for a model problem.

2.2 Non-autonomous maximal parabolic regularity: Definition and results

Throughout this paper let $T > 0$ and set $J =]0, T[$. Let us start by introducing the following (standard) definition.

Definition 2.8. If X is a Banach space and $s \in]1, \infty[$, then we denote by $L^s(J; X)$ the space of X-valued functions f on J which are Bochner-measurable and for which $\int_J \|f(t)\|^s \, dt$ is finite. We define $W^{1,s}(J; X) := \{u \in L^s(J; X) : \frac{\partial u}{\partial t} \in L^s(J; X)\}$, where $\frac{\partial u}{\partial t}$ is to be understood as the time derivative of u in the sense of X-valued distributions (cf. [1, Section III.1]). Moreover, we introduce the subspace

$$W^{1,s}_0(J; X) := \{u \in W^{1,s}(J; X) : u(0) = 0\}.$$

We equip this subspace always with the norm $u \mapsto \|\frac{\partial u}{\partial t}\|_{L^s(J;X)}$.

Definition 2.9. Let X, D be Banach spaces with D densely and continuously embedded in X. Let $J \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D; X)$ be a bounded and strongly measurable map and suppose that the operator $\mathcal{A}(t)$ is closed in X for all $t \in J$. Let $s \in]1, \infty[$. Then we say that the family $\{\mathcal{A}(t)\}_{t \in J}$ satisfies **(non-autonomous) maximal parabolic $L^s(J; X)$-regularity**, if for any $f \in L^s(J; X)$ there is a unique function $u \in L^s(J; D) \cap W^{1,s}_0(J; X)$ which satisfies

$$\frac{\partial u}{\partial t} + \mathcal{A}(t)u(t) = f(t) \tag{12}$$

for almost all $t \in J$. We write

$$\mathrm{MR}^s_0(J; D, X) := L^s(J; D) \cap W^{1,s}_0(J; X) \tag{13}$$

for the space of maximal parabolic regularity. The norm of $u \in \mathrm{MR}^s_0(J; D, X)$ is

$$\|u\|_{\mathrm{MR}^s_0(J;D,X)} = \|u\|_{L^s(J;D)} + \|\frac{\partial u}{\partial t}\|_{L^s(J;X)}.$$

Then $\mathrm{MR}^s_0(J; D, X)$ is a Banach space.

We emphasize that $\mathrm{Dom}(\mathcal{A}(t)) = D$ for all $t \in J$ in Definition 2.9. In particular, all operators $\mathcal{A}(t)$ have the same domain. If all operators $\mathcal{A}(t)$ are equal to one (fixed) operator A, and there exists *one*

$s \in]1, \infty[$ such that $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^s(J;X)$-regularity, then $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^s(J;X)$-regularity for *all* $s \in]1, \infty[$. In this case we say that A satisfies **maximal parabolic regularity on** X.

Let us recall some embedding property of the space of maximal par. regularity which will be needed later.

Proposition 2.10. *Let X, Y be Banach spaces and assume that Y is continuously embedded into X.*

i) If $s \in]1, \infty[$, then

$$W^{1,s}(J;X) \cap L^s(J;Y) \hookrightarrow C(\overline{J}; (X,Y)_{1-\frac{1}{s},s}),$$

(see [1, Ch. III Thm. 4.10].

ii) If $s \in]1, \infty[$ and $\theta \in]0, 1 - \frac{1}{s}[$, then

$$W^{1,s}(J;X) \cap L^s(J;Y) \hookrightarrow C^\beta(J; (X,Y)_{\theta,1}), \tag{14}$$

where $\beta = 1 - \frac{1}{s} - \theta$, (see [15, Lemma 2.11]).

In the sequel we primarily consider the case $D = Y = W^{1,q}_{\mathfrak{D}}(\Omega)$ and $X = W^{-1,q}_{\mathfrak{D}}(\Omega)$. For later purpose it is of essential importance to know compact embeddings of the maximal parabolic regularity space $\mathrm{MR}^s_0(J; W^{1,q}_{\mathfrak{D}}(\Omega), W^{-1,q}_{\mathfrak{D}}(\Omega))$ into suitable other functional spaces.

Lemma 2.11. *Assume $s > 2$.*

i) Suppose $q > 2$. Then $\mathrm{MR}^s_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ compactly embeds into $C(\overline{J}; L^q(\Omega))$.

ii) Suppose that $2 - q > 0$ is sufficiently small. Then $\mathrm{MR}^s_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ compactly embeds into $C(\overline{J}; L^2(\Omega))$.

Proof. We employ (see [5])

$$(W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{\frac{1}{2},1} \hookrightarrow [W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega)]_{\frac{1}{2}} = L^q(\Omega), \quad q \in]1, \infty[. \tag{15}$$

i) Chose $\theta \in]\frac{1}{2}, 1 - \frac{1}{s}[$ and use re-iteration (see [44, Ch. 1.10]) in combination with (15); this yields

$$\big(W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega)\big)_{\theta,1} = \big((W^{-1,q}_{\mathfrak{D}}(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{\frac{1}{2},1}, W^{1,q}_{\mathfrak{D}}(\Omega)\big)_{2\theta-1,1} \hookrightarrow$$

$$\hookrightarrow (L^q(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{2\theta-1,1}. \tag{16}$$

Since $W^{1,q}_{\mathfrak{D}}(\Omega)$ compactly embeds into $L^q(\Omega)$, also $(L^q(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega))_{2\theta-1,1}$ compactly embeds into $L^q(\Omega)$, see [44, Ch. 1.16.4]. This, in combination with (14) gives the assertion, due to the Arzela/Ascoli thm.

ii) Again chose $\theta \in]\frac{1}{2}, 1 - \frac{1}{s}[$. Take $q \in]2 - \delta, 2[$. The same argument as in i) gives the embedding (16) – but q now being *smaller* than 2. We define $\vartheta := d\big(\frac{1}{q} - \frac{1}{2}\big)$ and chose $q < 2$ so large that $\vartheta < 2\theta - 1$. Then we can again use re-iteration and continue (16), for some $\kappa \in]0, 1[$,

$$= \big([L^q(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega)]_\vartheta, W^{1,q}_{\mathfrak{D}}(\Omega)\big)_{\kappa,1} \hookrightarrow \big([L^q(\Omega), L^{q^*}(\Omega)]_\vartheta, W^{1,q}_{\mathfrak{D}}(\Omega)\big)_{\kappa,1}, \tag{17}$$

q^* being the Sobolev conjugated of q. But ϑ is chosen that $[L^q(\Omega), L^{q*}(\Omega)]_\vartheta$ equals $L^2(\Omega)$. So (17) equals $\big(L^2(\Omega), W^{1,q}_{\mathfrak{D}}(\Omega)\big)_{\kappa,1}$. This embeds compactly into $L^2(\Omega)$ by [44, Ch. 1.16.4]. So one can use (14), and the and the standard compactness argument via Arzela/Ascoli applies. □

Remark 2.12. It may seem somewhat surprising that one also gets in case ii) a compact embedding into $C(\overline{J}; L^2(\Omega))$ – although the supposed integrability index q is here *smaller* than 2. Of course, the reason is the higher integrability $s > 2$ with respect to time. The reader will see in a moment that this fits well to the subsequent extrapolation result Thm. 2.16.

Now we pass to the presentation of the (known) regularity results for the non-autonomous parabolic equations.

Assumption 2.13. Let $\hat{\mu}\colon J \times \Omega \to \mathbb{R}^{d\times d}$ be a bounded mapping, such that

$$J \in t \mapsto \hat{\mu}(t, \cdot) \in L^1(\Omega; \mathbb{R}^{d\times d}), \tag{18}$$

is measurable. The bound will be denoted by $\mu^\bullet$ in the sequel.

Note that the set of points in Ω where $\hat{\mu}(t, \cdot)$ is discontinuous may depend on t. In general the map $t \mapsto \mu(t, \cdot)$ from J into $L^\infty(\Omega; \mathbb{R}^{d\times d})$ is discontinuous at every time point t and therefore it cannot be measurable, see the example in [14, Ch. 7.1]. This is the reason for what we only demand L^1-measurability.
Let us make precise what we will understand under the non-autonomous parabolic operator.

Definition 2.14. Let $\hat{\mu}$ is as in Assu. 2.13. Then we define

$$\big(-div\hat{\mu}grad\, u\big)(t) := \big(-\nabla \cdot \hat{\mu}(t, \cdot)\nabla\big)\, u(t), \quad u \in L^s(J; W^{1,q}_{\mathfrak{D}}(\Omega)). \tag{19}$$

Lemma 2.15. *Adopt Assu. 2.13. Let $q, s \in (1, \infty)$. Then one has the following.*

i) The map $t \mapsto \nabla \cdot \mu(t, \cdot)\nabla u$ is (strongly) measurable from J into $W^{-1,q}_{\mathfrak{D}}(\Omega)$ for all $u \in W^{1,q}_{\mathfrak{D}}(\Omega)$.

ii) The map $\frac{\partial}{\partial t} - div\hat{\mu}grad$ is a bounded linear map from the space $\mathrm{MR}^s_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ into $L^s(J; W^{-1,q}_{\mathfrak{D}})$ with norm at most $1 + \mu^\bullet$.

Proof. see [14, Lemma 7.1] □

Proposition 2.16. *Adopt the Assumptions 2.1 (a)/(b) and 2.13. Moreover, suppose that the function* (18) *has a uniform in time ellipticity constant $\mu_\bullet$. Then there are two open intervals $\mathfrak{I}_1 \ni 2$ and $\mathfrak{I}_2 \ni 2$, such that*

$$\frac{\partial}{\partial t} - div\hat{\mu}grad : \mathrm{MR}^s_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}}) \to L^s(J; W^{-1,q}_{\mathfrak{D}}) \tag{20}$$

is a topological isomorphism for $s \in \mathfrak{I}_1 \ni 2$ and $q \in \mathfrak{I}_2 \ni 2$, i.e. $\big\{-div\hat{\mu}(t, \cdot)grad\big\}_{t\in J}$ satisfies non-autonomous maximal parabolic $L^s(J, W^{-1,q}_{\mathfrak{D}})$ regularity.

Proof. see [14, Thm. 7.3]. □

The reader should carefully notice that the extrapolation of non-autonomous maximal parabolic regularity is a very delicate matter. One cannot expect that this holds considerably beyond [14, Thm. 3.4] – neither what affects the Banach spaces nor the integrability exponent with respect to time, compare here [12] and [6].

3 Non-autonomous problems with 'wild' dependence of the coefficients on time and measure-valued functions as right-hand sides

3.1 Generalities

In this chapter, we investigate non-autonomous parabolic problems like

$$\frac{\partial u}{\partial t} - div\hat{\mu}grad\, u = \varrho, \quad u(0) = 0 \tag{21}$$

ρ being a function on J and taking in every time point t a bounded Radon measure ρ_t on Ω as its value.

It makes sense to consider mappings $J \ni t \mapsto \rho_t \in \mathcal{M}(\Omega)$, which are only weak* measurable, this means: mappings for which

$$J \ni t \mapsto \langle \rho_t, \psi \rangle, \quad \psi \in C(\overline{\Omega}) \tag{22}$$

are measurable (compare the discussion in [8, Ch. 2.1]). In the opposite case one excludes examples like this:

Let $J \ni t \mapsto x(t)$ be an injective curve in Ω. If one defines $\rho_t := \delta_{x(t)}$ – the Dirac measure in the point $x(t) \in \Omega$ – then the mapping $J \ni t \mapsto \rho_t$ is in *every* point discontinuous, if one equipps the space of (bounded) measures with the strong topology. Hence, it is not measurable if one defines the structure of measurability via this strong topology. On the contrary, if one considers the weak* topology and the induced concept of measurability, then the mapping $J \ni t \mapsto \delta_{x(t)}$ is at least measurable if the mapping $J \ni t \mapsto x(t)$ is measurable itself.

If $\mathcal{N}$ is a space of measures for which one knows an embedding $\mathcal{N} \hookrightarrow W^{-1,q}_{\mathfrak{D}}(\Omega)$, then the measurability of (22) is in particular true for functions $\psi \in C^\infty_{\mathfrak{D}}(\Omega)$ and, hence, carries over to all functions $\psi \in W^{1,q'}_{\mathfrak{D}}(\Omega)$ by density. But this means: the mapping $J \ni t \mapsto \rho_t \in W^{-1,q}_{\mathfrak{D}}(\Omega)$ is weakly measurable in this case. Then the separability of $W^{-1,q}_{\mathfrak{D}}(\Omega)$ implies, quite in contrast to the situation in $\mathcal{M}(\Omega)$, even the strong measurability. Thus one is, via embeddding, in a situation in which rather general mappings $J \ni t \mapsto \rho_t$ are admissible and, additionally, suit in the context of maximal parabolic regularity – even in the non-autonomous case.

However, the reader should carefully notice: weak* limits of measures, these being possibly concentrated on sets of lower Hausdorff dimension, can be of entirely different nature. E.g. *every* Radon measure on Ω is the weak* limit of linear combinations of Dirac measures on Ω. In other words: the affiliation of a measure to a class of measures, concentrated on lower dimensional objects, is by no means necessarily preserved for the weak* limit.

3.2 Interpretation of singular measures as elements from $W^{-1,q}_{\mathfrak{D}}(\Omega)$

As already explained, we intend to investigate the equation (21) by understanding the measures ρ_t as elements from $W^{-1,q}_{\mathfrak{D}}(\Omega)$ – a space on which the second order operators admit a well-behaved parabolic theory. Fortunately, the geometric measure theory, elaborated in particular by Jonsson and Wallin, provides the adequate instruments for this.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$, $\mathfrak{D}$ be a closed subset of $\partial\Omega$ which, together, fulfill Assu. 2.1 (b). Then the space of bounded Radon measures on Ω continuously embeds into any space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ if $q \in]1, d'[$.*

Proof. Thanks to Rem. 2.3 and the usual Sobolev embedding one has a continuous embedding $W^{1,q'}_{\mathfrak{D}}(\Omega) \hookrightarrow C(\overline{\Omega})$ for every $q > d$. So one gets for every bounded Radon measure $\mathfrak{m}$ on Ω

$$\|\mathfrak{m}\|_{W^{-1,q}_{\mathfrak{D}}} = \sup_{\|\psi\|_{W^{1,q'}_{\mathfrak{D}}}=1} \Big| \int_\Omega \psi \, d\mathfrak{m} \Big| \le \sup_{\|\psi\|_{W^{1,q'}_{\mathfrak{D}}}=1} \sup_{x\in\Omega} |\psi(x)| \, \|\mathfrak{m}\| \le c \, \|\mathfrak{m}\|.$$

□

Let us explicitly notice that an argument like this – to embed the space of bounded Radon measures in a space $W^{-1,q}$ – is not new at all, it appears at the latest in [7] – there even in the context of non-linear (elliptic) equations.
So far, this affects *general* bounded Radon measures on Ω irrespective of their singularity – even Dirac measures are admitted, compare e.g. [33], [41], [10].

In the sequel we restrict the class of measures which are admitted. The reason is the following: In full generality, Thm. 2.16 holds only for q close to 2. So, at least concerning non-autonomous parabolic equations, one is restricted to measures which can be considered as elements of $W^{-1,q}_{\mathfrak{D}}(\Omega)$ with $q \sim 2$ – if one is willing to exploit that theory. In two space dimensions it turns out that – besides the class of all bounded Radon measures – the measures situated on sets of Hausdorff dimension 1 deserve special attention; naively spoken: curves. In three space dimensions this affects the measures concentrated on 'surfaces' and 'curves' – in fact: 2-sets and 1-sets. In order to make this precise, we need some preparation. Recall first the definition of an l-set from the introduction.

Lemma 3.2. *If the closed set $M \subset \mathbb{R}^d$ is an l-set satisfying* (3)*, and one defines the measure ρ on $\mathbb{R}^d$ by $\rho(N) = \mathcal{H}_l(N \cap M)$ for every Borel set $N \subset \mathbb{R}^d$, then ρ satisfies $\rho(B(x,r)) \le 2^l \mathfrak{c}^\bullet \, r^l$ for $r \le 1$.*

Proof. For all $x \in \mathbb{R}^d$ with dist$(x, M) > 1/2$ one has $B(x,r) \cap M = \emptyset$ for $r \le 1/2$, so that $\rho(B(x,r)) = 0$ for these r. If dist$(x, M) = r \le 1/2$, then exists a $y \in M$ with $|x-y| = r \le 1/2$. But then $B(x,r) \subseteq B(y,2r)$ and the assertion follows. □

Proposition 3.3. *If $M \subset \Omega$ is a Borel set of finite Hausdorff measure $\mathcal{H}_l$, then the forming*

$$C_0(\Omega) \ni v \mapsto \int_M v \, d\mathcal{H}_l$$

is a bounded Radon measure on Ω.

Proof. Since $\mathcal{H}_l$ is a Borel measure on $\mathbb{R}^d$ (see [21, Ch. 2 Thm. 2.1]) and $\mathcal{H}_l(M)$ is finite, the restriction of $\mathcal{H}_l$ to M is even a (bounded) Radon measure on $\mathbb{R}^d$ (see [21, Ch. 2 Thm. 2.1]). It is clear that the restriction of this to Ω remains a (bounded) Radon measure. □

So, if in particular $M \subset \Omega$ is a Borelian l-set, then $\mathcal{H}_l|_M$ is a bounded Radon measure on Ω. Moreover, the total mass of $\overline{M} \supset M$ with respect to $\mathcal{H}_l$ can be estimated by $\mathfrak{c}^\bullet \times \tau$, where τ is the number of (shifted) unit balls required for a covering of $\overline{M}$.

Proposition 3.4. *Suppose $l \in \{1, \dots, d-1\}$. Let $M \subset \mathbb{R}^d$ be a closed set satisfying*

$$\mathcal{H}_l\big(M \cap B(x,r)\big) \le \mathfrak{c} \, r^l, \quad x \in M, \, r \in \,]0,1]. \tag{23}$$

Assume $\alpha \in \,]0,1]$, $0 < \alpha - \frac{d-l}{p}$ for some $p \in \,]1,\infty[$.

i) For $f \in L^p(\mathbb{R}^d)$ and $\varphi = G_\alpha \star f$ one has

$$\|\varphi\|_{L^p(M;\mathcal{H}_l)} \le c\,\|f\|_{L^p(\mathbb{R}^d)}, \tag{24}$$

G_α being the corresponding Bessel potential (see [43, Ch. V.3]). The constant c can be chosen independent of f.

ii) The constant c in (24) *may be taken even uniform for sets M obeying the estimate* (3) *with a uniform $\mathfrak{c}^\bullet$.*

Proof. i) is a special case of [29, Ch. VI. Lemma 6]. ii) follows by a careful inspection of that proof. For the convenience of the reader we give some comments how to read the proof of [29, Ch. VI. Lemma 6] in the special case under consideration here in the appendix. □

Corollary 3.5. *Let $M \subset \mathbb{R}^d$ be a Borel set with $l \in \{1,\dots,d-1\}$ which satisfies* (23)*. Let $\alpha \in]0,1]$, $0 < \alpha - \frac{d-l}{p}$ for some $p \in]1,\infty[$. Then one has a continuous embedding $H^\alpha_p(\mathbb{R}^d)$ into $L^p(M;\mathcal{H}_l)$, $H^\alpha_p(\mathbb{R}^d)$ being the well-known Bessel potential space (see [44, Ch. 2.3.3], compare also [43, Ch. V.3]). The embedding constants are uniformly bounded for different sets M obeying the estimate* (3) *with a uniform $\mathfrak{c}$.*

Proof. As is well-known, the space $H^\alpha_p(\mathbb{R}^d)$ can be defined as the set $\{G_\alpha \star f : f \in L^p(\mathbb{R}^d)\}$, this equipped with the corresponding graph norm (see [44, Ch. 2.3.4]). So (24) can be interpreted as

$$\|\varphi\|_{L^p(M;\mathcal{H}_l)} \le c\,\|\varphi\|_{H^\alpha_p(\mathbb{R}^d)}, \tag{25}$$

and the assertions follow. □

Theorem 3.6. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \ge 2$, and $\mathfrak{D}$ be a closed portion of $\partial\Omega$. Suppose that M is a closed subset of Ω, which satisfies* (23)*, $l \in \{1,\dots,d-1\}$. Assume $q \in]1,d[$. If $p \in]1, \frac{ql}{d-q}[$, then the usual trace mapping embeds $W^{1,q}_{\mathfrak{D}}(\Omega)$ continuously into $L^p(M;\mathcal{H}_l)$.*
The embedding constants may be taken uniform for different sets M as long as (23) *is satisfied for a uniform constant $\mathfrak{c}$.*

Proof. One first considers the mapping

$$W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{1,q}(\mathbb{R}^d) \to H^\alpha_p(\mathbb{R}^d), \tag{26}$$

the left being the extension operator $\mathfrak{E}$ (see Rem. 2.3) and the second Sobolev's embedding (see [44, Ch. 2.8.1 Rem. 2] or [29, Ch. 1.4]). Hence, q given, α and p are related via the well-known condition

$$\frac{1}{q} - \frac{1-\alpha}{d} = \frac{1}{p} \quad \text{or, equivalently,} \quad \alpha = \frac{d}{p} - \frac{d}{q} + 1. \tag{27}$$

Of course, our intention is to apply Cor. 3.5, what demands $\alpha > \frac{d-l}{p}$. A straight forward computation shows that for this latter, in combination with (27) the above condition for p is a necessary and sufficient one.

Thus, for the supposed p's one may apply Cor. 3.5, what proves the claim.
The reader should carefully observe here that M is a subset of the *open* set Ω so that the forming of the trace onto M when considered as a subset of Ω is compatible with the forming on whole $\mathbb{R}^d$. □

Our original intention is to identify the measures $\sigma\mathcal{H}_l$ with elements of $W^{-1,q}_{\mathfrak{D}}$. This follows from Thm. 3.6 by the following straight forward duality argument.

Theorem 3.7. *Let Ω be a bounded domain in $\mathbb{R}^d$ and $\mathfrak{D}$ be a closed portion of $\partial\Omega$. Let M be a closed subset of Ω of finite $\mathcal{H}_l$ measure, $l \in \{1, \ldots, d\}$. Suppose that $W^{1,q}_{\mathfrak{D}}(\Omega)$ continuously embeds into $L^p(M; \mathcal{H}_l)$ with embedding constant $\mathfrak{c}$. Then, for all $\sigma \in L^{p'}(M; \mathcal{H}_l)$, the measure $\sigma\mathcal{H}_l|_M$ belongs to $W^{-1,q'}_{\mathfrak{D}}(\Omega)$ and the mapping*

$$L^{p'}(M; \mathcal{H}_l) \ni \sigma \mapsto \sigma\mathcal{H}_l|_M =: \Psi \in W^{-1,q'}_{\mathfrak{D}}(\Omega) \tag{28}$$

is well-defined and has a norm not larger than $\mathfrak{c}$.

Proof. One has

$$|\langle \sigma\mathcal{H}_l|_M, \psi\rangle| \le \int_M |\psi|\,|\sigma|\,d\mathcal{H}_l \le \|\sigma\|_{L^{p'}(M;\mathcal{H}_l)}\,\|\psi\|_{L^p(M;\mathcal{H}_l)} \le$$

$$\mathfrak{c}\,\|\sigma\|_{L^{p'}(M;\mathcal{H}_l)}\,\|\psi\|_{W^{1,q}_{\mathfrak{D}}(\Omega)}, \quad \psi \in W^{1,q}_{\mathfrak{D}}(\Omega). \tag{29}$$

□

Let us make explicit, $l \in \{1, \ldots, d-1\}$ and $p \in \,]1, \infty[$ given, for which q's the embedding $L^p(M, \mathcal{H}_l) \hookrightarrow W^{-1,q}_{\mathfrak{D}}$ exists.

Theorem 3.8. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \ge 2$, and $\mathfrak{D}$ be a closed portion of $\partial\Omega$. Suppose that M is a closed subset of Ω, which satisfies* (23)*, $l \in \{1, \ldots, d-1\}$. If*

$$1 < q < \big(1 - \frac{1}{d}[l + 1 - \frac{l}{p}]\big)^{-1}, \tag{30}$$

then one has the embedding

$$L^p(M, \mathcal{H}_l) \hookrightarrow W^{-1,q}_{\mathfrak{D}}. \tag{31}$$

The embedding constants may be taken uniform for different sets M as long (23) *holds uniformly with the same constant $\mathfrak{c}$.*

Proof. From Thm. 3.6 and Thm. 3.7 it is clear that for the embedding (31) the condition $p' \le \frac{q'l}{d-q'}$ is a sufficient one, $q' < d$. A straight forward computation shows that this is true iff (30) holds. □

In the sequel we focus ourselves, for technical simplicity, to the most important cases

$$d = 2, l = 1 \tag{32}$$

and

$$d = 3, l \in \{1, 2\} \tag{33}$$

One immediately sees that the condition (30) becomes in case of $l = d-1$ the form $q < \frac{d}{d-1}p$ and in case of $l = d-2 > 0$ the form $q < d\big(1 + \frac{d-2}{p}\big)^{-1}$.

Lemma 3.9. *Let, for every $t \in J$, ρ_t be a bounded Borel measure on Ω, such that the mapping $J \ni t \mapsto \rho_t$ is weak* measurable. Suppose that, for every $t \in J$,*

$$\sup_{\psi \in C^\infty_{\mathfrak{D}}(\Omega), \|\psi\|_{W^{1,q'}_{\mathfrak{D}}} = 1} \Big| \int_\Omega \psi \, d\rho_t \Big| < \infty. \tag{34}$$

Then the forming $C^\infty_{\mathfrak{D}}(\Omega) \ni \psi \mapsto \int_\Omega \overline{\psi}\, d\rho_t$ extends by continuity to an element $\Psi_t \in W^{-1,q}_{\mathfrak{D}}(\Omega)$. Moreover, the mapping $J \ni t \mapsto \Psi_t \in W^{-1,q}_{\mathfrak{D}}(\Omega)$ is strongly *measurable.*

Proof. The first assertion is clear. Secondly, from the supposed weak* measurability it follows that $J \ni t \mapsto \int_\Omega \overline{\psi}\, d\rho_t = \langle \Psi_t, \psi \rangle$ is measurable as long as $\psi \in C^\infty_{\mathfrak{D}}(\Omega)$. But this latter set is dense in $W^{1,q'}_{\mathfrak{D}}(\Omega)$, so the measurability for general $\psi \in W^{1,q'}_{\mathfrak{D}}(\Omega)$ follows by taking the limit – what implies *weak* measurability of $J \ni t \mapsto \Psi_t \in W^{-1,q}_{\mathfrak{D}}(\Omega)$. Since $W^{1,q'}_{\mathfrak{D}}(\Omega)$ is separable and reflexive, its dual $W^{-1,q}_{\mathfrak{D}}(\Omega)$ also is, and the asserted strong measurability follows. □

Let us have a closer look on what kind of restriction the uniformity of the constant $\mathfrak{c}$ means in a simple example:
Consider a bounded domain $\Omega \subset \mathbb{R}^2$ which includes $0 \in R^2$ and a closed ball $\overline{B(0, r_0)}$ around. Take a sequence from the interval $[0, \pi]$, $\{\alpha_k\}_n$, which converges to zero. From this we form the set

$$\mathcal{N}_N := \cup_{k \le N} \{x \in \mathbb{R}^2 : x = re^{i\alpha_k}, r \in [0, r_0]\}.$$

Then the condition (41), there l taken as 1, obviously gives a bound for the admissible N. In any case, not the union over *all* k is admissible.
However: if one changes the above set to

$$\cup_k \{x \in \mathbb{R}^2 : x = re^{i\alpha_k}, r \in [0, r_k]\}$$

and chooses the r_k suitably, then (41) can indeed be satisfied in this case. Very roughly spoken, one can say: only finitely many 'curves' of a certain minimal length are admissible, but if the length may shrink to zero, then infinitely many may be admissible and still satisfy (3).
Let us now take a function $\eta \in C^\infty_0(\Omega)$ which is identical 1 on $\overline{B(0, r_0)}$. Then $\|\eta\|_{L^p(\mathcal{N}_N)} = N r_0^{1/p}$. This clearly shows that, in order to delimitate the embedding constant of $W^{1,q}_{\mathfrak{D}}(\Omega) \hookrightarrow L^p(\mathcal{N}_N)$ one must delimitate N. So an inequality like (3) seems not to be too far from a necessary one for the required embedding.
Clearly, one can construct analogous examples also in higher dimensions.

Lemma 3.10. *Let Ω be a bounded domain in $\mathbb{R}^3$ and $\mathfrak{D}$ be a closed portion of $\partial\Omega$.*

i) Let M be a closed subset of Ω fulfilling the condition

$$\mathcal{H}_1(\overline{M} \cap B(x, r)) \le \mathfrak{c}_1\, r, \quad x \in \overline{M}, \quad r \in]0, 1[. \tag{35}$$

a) $L^2(M; \mathcal{H}_1)$ continuously embeds into every space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ with q being any number from $]1, 2[$.

b) For every $p > 2$, $L^p(M; \mathcal{H}_1)$ continuously embeds into $W^{-1,q}_{\mathfrak{D}}(\Omega)$ as long as $q < \frac{3p}{p+1}$.

ii) Let M be a closed subset of Ω fulfilling the condition

$$\mathcal{H}_2(\overline{M} \cap B(x, r)) \le \mathfrak{c}_2\, r^2, \quad x \in \overline{M}, \quad r \in]0, 1[. \tag{36}$$

a) For every $\epsilon > 0$ there is a $\delta > 0$ such that $L^{2+\epsilon}(M; \mathcal{H}_2)$ continuously embeds into $W^{-1,3+\delta}_{\mathfrak{D}}(\Omega)$.
b) $L^2(M; \mathcal{H}_2)$ continuously embeds into every space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ with q being any number from $]1, 3[$.

iii) If different sets M do admit the same constant $\mathfrak{c}$ in (35)/(36)*, then the corresponding embedding constants in all cases may be taken uniformly.*

Proof. The proof follows from Thm. 3.8. □

Up to now we were primarily interested in individual measures $\sigma \mathcal{H}_l|_M$. Having parabolic equations with *varying in time* measures as right-hand sides in our general focus, we must find a concept which allows to identify the *time dependent*, measure-valued function as one with values in the Sobolev space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ – including suitable measurability and integrability properties. This is achieved in the next

Theorem 3.11. *Let Ω be a bounded domain in $\mathbb{R}^d$ and $\mathfrak{D}$ be a closed portion of $\partial\Omega$. Let, for every $t \in J$, M_t be a closed subset of Ω. Suppose that, for an $l \in \{1, \ldots, d-1\}$,*

$$\mathcal{H}_l(\overline{M}_t \cap B(x,r)) \leq \mathfrak{c}\, r^l, \quad x \in \overline{M}_t, \quad r \in]0,1] \tag{37}$$

holds with a uniform in t constant $\mathfrak{c}$.
Assume $p \in]1,\infty[$ and that q satisfies (30). *For every $t \in J$ let be given a function $\sigma_t \in L^p(M_t; \mathcal{H}_l)$ such that*
a) the mapping

$$J \ni t \mapsto \sigma_t \mathcal{H}_l|_{M_t} \in \mathcal{M}(\Omega) \tag{38}$$

is weak measurable*
and
b) the upper integral (see [13, Ch. 13.5]) $\int_J^* \|\sigma_t\|^s_{L^p(M_t,\mathcal{H}_l)}\, dt$ is finite.
Let $\Psi(t) \in W^{-1,q'}_{\mathfrak{D}}(\Omega)$ be the element which is associated to the measure $\sigma_t \mathcal{H}_l|_{M_t}$ by Thm. 3.8.

Then the mapping $J \ni t \mapsto \Psi(t) \in W^{-1,q}_{\mathfrak{D}}(\Omega)$ is strongly measurable and one has

$$\int_J \|\Psi(t)\|^s_{W^{-1,q}_{\mathfrak{D}}(\Omega)}\, dt \leq \mathfrak{k} \int_J^* \|\sigma_t\|^s_{L^p(M_t,\mathcal{H}_l)} \tag{39}$$

for some constant $\mathfrak{k}$. Moreover, the constant $\mathfrak{k}$ is uniform with respect to all families $\{\sigma_t\}_{t \in J}$ for which only $\int_J^* \|\sigma_t\|^s_{L^p(M_t,\mathcal{H}_l)}\, dt < \infty$.

Proof. One may apply Lemma 3.9 – the suppositions of which are fulfilled according to Thm. 3.6 and Thm. 3.7. This first proves the asserted measurability. On the other hand, (37) implies, for every $t \in J$, $\|\Psi(t)\|_{W^{-1,q}_{\mathfrak{D}}(\Omega)} \leq \mathfrak{l}\|\sigma_t\|_{L^{p'}(M_t,\mathcal{H}_l)}$ with a *uniform* in t constant $\mathfrak{l}$, thanks to Thm. 3.6 and Thm. 3.7. Since we already know the measurability of the mapping in question this proves (39). □

Remark 3.12. The reader should carefully notice that – besides the weak* measurability of the function (38) *no* measurability condition is supposed for the function $t \mapsto \sigma_t$ and even not for $t \mapsto \|\sigma_t\|_{L^{p'}(M_t,\mathcal{H}_l)}$. To make such a measurability precise would be a challenging task – and not easy to control in examples. On the contrary, for the finiteness of the upper integral a uniform boundedness condition for the functions σ_t, for example, is a sufficient one since the function $J \ni t \mapsto \mathcal{H}_l(M_t)$ is bounded by the (supposed) uniform upper l-property of the sets M_t.

3.3 Regularity for non-autonomous parabolic equations with measure-valued right-hand sides

In this chapter, we prove parabolic regularity results for equations with measure valued right-hand sides. The crucial point is two-fold: on one hand, the results of the foregoing chapter allow to interpret suitable measures as elements of $W^{-1,q}_{\mathfrak{D}}(\Omega)$. Here in the two-dimensional case there are no restrictions concerning the measures under consideration: all bounded Radon measures are admissible. In the three dimensional case one is restricted in this concept to measures which live on sets with Hausdorff dimension one or two and are, additionally, absolutely continuous with respect to the corresponding Hausdorff measure there. Secondly, we are then in the position to apply the results of maximal *non-autonomous* parabolic regularity from Ch. 2.

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^2$ be a domain and $\mathfrak{D}$ be a closed portion of the boundary $\partial\Omega$. Assume $\varrho : J \to \mathcal{M}(\Omega)$ to be weakly* measurable with $\int_J^* \|\rho_t\|_{\mathcal{M}}^\tau dt < \infty$ for some $\tau > 2$. Then, for $s > 2$ and $q < 2$, both sufficiently close to 2, the solution of the problem* (21) *lies in the space $W_0^{1,s}(J; W_{\mathfrak{D}}^{-1,q}) \cap L^s(J; W_{\mathfrak{D}}^{1,q}) = \mathrm{MR}_0^s(J, W_{\mathfrak{D}}^{1,q}, W_{\mathfrak{D}}^{-1,q})$ – inclusively the appropriate estimate for the solution.*

Proof. In case of two space dimensions, the space of bounded Radon measures on Ω continuously embeds into *every* space $W_{\mathfrak{D}}^{-1,q}(\Omega)$ – as long $q \in]1, 2[$, see Lemma 3.1. So one associates to the function ϱ a function Ψ with values in $W_{\mathfrak{D}}^{-1,q}(\Omega)$ which is shown to be strongly measurable and admits the same integrability with respect to time, see Thm. 3.11. Investing this knowledge, one may apply Thm. 2.16. □

This result should not be far from optimal. Unfortunately, the range of admissible integrability exponents s with respect to time is restricted to numbers close to 2. The next result shows that the solution is more regular with respect to the spatial variable, if one restricts the admissible measures to those 'living on one-dimensional' subsets. In particular, for almost all $t \in J$, the function $u(t, \cdot)$ is Hölderian on Ω then.

Theorem 3.14. *Let $\Omega \subset \mathbb{R}^2$ be a domain and $\mathfrak{D}$ be a closed portion of the boundary $\partial\Omega$, together fulfilling Assu. 2.1. Moreover, we suppose the existence of a $p > 1$ with the following properties:*

(a) *For every $t \in J$ there is a closed 1-set M_t of Ω, and a function $\sigma_t \in L^p(M_t; \mathcal{H}_1)$ such that the mapping*

$$J \ni t \mapsto \sigma_t \mathcal{H}_1|_{M_t} =: \rho_t \in \mathcal{M}(\Omega) \tag{40}$$

is weak measurable.*

(b) *It exists a constant $\mathfrak{c}$, such that*

$$\mathcal{H}_1(\overline{M}_t \cap B(x, r)) \le \mathfrak{c}\, r, \quad x \in \overline{M}_t, \quad r \in]0, 1] \tag{41}$$

holds for all $t \in J$.

(c) *For a $\tau > 2$ one has $\int_J^* \|\sigma_t\|_{L^p(M_t;\mathcal{H}_1)}^\tau dt < \infty$.*

Then there exist $s, q > 2$, such that the solution of the problem (21) *lies in the space $W_0^{1,s}(J; W_{\mathfrak{D}}^{-1,q}) \cap L^s(J; W_{\mathfrak{D}}^{1,q}) = \mathrm{MR}_0^s(J, W_{\mathfrak{D}}^{1,q}, W_{\mathfrak{D}}^{-1,q})$ – inclusively the appropriate estimate for the solution.*

Proof. Thanks to Thm. 3.8/Thm. 3.11, the function (40) can be interpreted as a measurable one with values in $W_{\mathfrak{D}}^{-1,q}(\Omega)$ with the same integrability exponent in time, as long as $q \in [2, 2 + \epsilon[$. Possibly diminishing τ and q, one may now apply Thm. 2.16. □

We proceed with the corresponding results in three space dimensions.

Theorem 3.15. *Let $\Omega \subset \mathbb{R}^3$ and adopt the Assumptions 2.1 and 2.13.*

(a) *Let, for every $t \in J$, $M_t \subset \Omega$ be closed subsets of Ω. There is a constant $\mathfrak{c}_1$ such that*

$$\mathcal{H}_1(\overline{M}_t \cap B(x,r)) \le \mathfrak{c}_1 \, r, \quad x \in \overline{M}_t, \quad r \in]0,1] \tag{42}$$

holds for all $t \in J$.
Further, let, for some $p \ge 2$ and every $t \in J$, $\sigma_1(t) \in L^p(M_t, \mathcal{H}_1)$.

(b) *Let the mapping*

$$J \ni t \mapsto \sigma_1(t)\mathcal{H}_1|_{M_t} =: \rho_t \in \mathcal{M}(\Omega) \tag{43}$$

be weak measurable and suppose*

$$\int_J^* \|\sigma_1(t)\|^\tau_{L^p(M_t;\mathcal{H}_1)} dt < \infty \tag{44}$$

for some $\tau > 2$.

Then the following holds.

i) If $p = 2$, one can understand ρ_t in view of Thm. 3.8 as an element $\Psi(t) \in W^{-1,q}_{\mathfrak{D}}(\Omega)$ and the function ϱ in view of Thm. 3.11 as one from $L^\tau(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$, where q may be taken as any number from $]1,2[$.

ii) If $p > 2$, one can understand ρ_t in view of Thm. 3.8 as an element $\Psi(t) \in W^{-1,q}_{\mathfrak{D}}(\Omega)$ and the function ϱ in view of Thm. 3.11 as one from $L^\tau(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$, $q \in]2, \frac{3p}{p+1}[$.

iii) The equation

$$\frac{\partial u}{\partial t} - div\hat{\mu}gradu = \varrho, \quad u(0) = 0 \tag{45}$$

can be interpreted as

$$\frac{\partial u}{\partial t} - div\hat{\mu}gradu = \Psi, \quad u(0) = 0. \tag{46}$$

This equation has a unique solution which belongs to the space of maximal parabolic regularity $\mathrm{MR}^s_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ for some $s \in]2,\tau]$ and any *$q < 2$ in case of $p = 2$ and $q \in]2, \frac{3p}{p+1}[$ in case of ii)*

Proof. The proof follows from Thm. 3.8 and Thm. 3.11 in combination with Thm. 2.16. □

The reader should carefully observe that these considerations would not work for $p = 2$ if one would miss this certain flexibility concerning the integrability index q around 2 with regard of maximal parabolic regularity.

Theorem 3.16. *Let $\Omega \subset \mathbb{R}^3$ and adopt the Assumptions 2.1 and 2.13.*

(a) *Let, for every $t \in J$, $M_t \subset \Omega$ be closed subsets of Ω. There is a constant $\mathfrak{c}_2$ such that*

$$\mathcal{H}_1(\overline{M}_t \cap B(x,r)) \le \mathfrak{c}_2 \, r^2, \quad x \in \overline{M}_t, \quad r \in]0,1] \tag{47}$$

holds for all $t \in J$.
Further, let, for some $p > \frac{4}{3}$ and every $t \in J$, $\sigma_2(t) \in L^p(M_t, \mathcal{H}_2)$.

(b) *Assume the mapping*

$$J \ni t \mapsto \sigma_2(t)\mathcal{H}_1|_{M_t} =: \rho_t \in \mathcal{M}(\Omega) \tag{48}$$

to be weak measurable and suppose*

$$\int_J^* \|\sigma_2(t)\|^\tau_{L^p(M_t;\mathcal{H}_2)} dt < \infty \tag{49}$$

for some $\tau > 2$.

Then the following is true.

i) *One can understand* ρ_t *in view of Thm. 3.8 as an element* $\Psi(t) \in W^{-1,q}_{\mathfrak{D}}(\Omega)$ *and the function* ϱ *in view of Thm. 3.11 as one from* $L^\tau(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$, $q \in]1, \frac{3p}{2}[$.

ii) *The equation*

$$\frac{\partial u}{\partial t} - div \hat{\mu} grad\, u = \varrho, \quad u(0) = 0 \tag{50}$$

can be interpreted as

$$\frac{\partial u}{\partial t} - div \hat{\mu} grad\, u = \Psi, \quad u(0) = 0. \tag{51}$$

Its solution belongs belongs to the spaces of maximal parabolic regularity $\mathrm{MR}^s_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ *for some* $s \in]2, \tau]$ *and some* $q > 2$.

Proof. The proof follows from Thm. 3.10 and Thm. 3.11 in combination with Thm. 2.16. □

The case of $p > 2$ will be of particular interest later. Here one gets $\Psi \in L^s(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$ with $q > 3$. This will be of substantial use for the derivation of Hölder continuity in space and time.

4 Hölder continuity in space and time

In this section we show, under very mild conditions on the data of the problem, the Hölder continuity of the solution for the parabolic equation simultaneously in space and time. The crucial point is that the domains of the elliptic operators, when being considered on $W^{-1,q}_{\mathfrak{D}}(\Omega)$, need not be known explicitly. By far they need not coincide with $W^{-1,q}_{\mathfrak{D}}(\Omega)$ – as we have used for $q \sim 2$ in the foregoing chapters. Fortunately, we have an elliptic Hölder regularity result at hand which implies the desired parabolic one.

We restrict the considerations here to the *autonomous* case. If one strengthens the geometric suppositions to those in [36, Assu. 2.2/Assu. 2.4], then gets Hölder regularity in space and time even in the *non-autonomous* case incusively corresponding estimates, see [36] for details. Since this brings presently nothing really new for measure valued right-hand sides, we will not go into details here.

In the subsequent context we consider constellations in which the operator (10) is not surjective. So we introduce the following

Definition 4.1. For $q > 2$ we define B_q as the part of $-\nabla \cdot \mu \nabla$ in $W^{-1,q}_{\mathfrak{D}}(\Omega)$, i.e.

$$\mathrm{Dom}(B_q) = \{\psi \in W^{-1,q}_{\mathfrak{D}}(\Omega) \cap W^{1,2}_{\mathfrak{D}}(\Omega) : -\nabla \cdot \mu \nabla \psi \in W^{-1,q}_{\mathfrak{D}}(\Omega)\} \tag{52}$$

and, for $\psi \in \mathrm{Dom}(B_q)$, one sets $B_q \psi = -\nabla \cdot \mu \nabla \psi$.

Proposition 4.2. *Adopt Assumption 2.1 and suppose that the coefficient function μ is measurable, bounded and strongly elliptic. Let $s \in]1, \infty[$ be arbitrary.*

Then, for all $q \in [2, \infty[$, the operator B_q satisfies maximal parabolic regularity on $W^{-1,q}_{\mathfrak{D}}(\Omega)$, i.e. the equation

$$\frac{\partial u}{\partial t} + B_q u = f \in L^s(J; W^{-1,q}_{\mathfrak{D}}(\Omega)), \quad u(0) = 0 \tag{53}$$

admits exactly one solution. This belongs to the max-par-reg space $\mathrm{MR}^s_0(J, \mathrm{Dom}(B_q); W^{-1,q}_{\mathfrak{D}})$, and the mapping $f \mapsto u$ between the corresponding spaces is continuous by the open mapping theorem.

Proof. see [4, Ch. 11]. □

The reader should carefully notice that it is – by far – *not* understood in the theorem that $\mathrm{Dom}(B_q)$ equals $W^{1,q}_{\mathfrak{D}}(\Omega)$.

Proposition 4.3. *Let Assumption 2.1 be satisfied and let $q > d$.*

i) *Suppose that $\mathrm{Dom}(B_q) \hookrightarrow C^\alpha(\Omega)$ for some $\alpha > 0$. Let $\kappa \in]0, \alpha[$ and $\iota \in \big(\frac{1}{2} + \frac{d}{2q} + \frac{\kappa}{\alpha}(\frac{1}{2} - \frac{d}{2q}), 1\big)$. Then one has*
$$\big(W^{-1,q}_{\mathfrak{D}}(\Omega), \mathrm{Dom}(B_q)\big)_{\iota,1} \hookrightarrow C^\kappa(\Omega).$$

ii) *If s is large enough, then there is a $\vartheta > 0$ such that the solution u of* (53) *even belongs to the space $C^\vartheta(J \times \Omega)$ and the mapping $L^s(J; W^{-1,q}_{\mathfrak{D}}(\Omega)) \ni f \mapsto u \in C^\vartheta(J \times \Omega)$ is continuous.*

Proof. i) see [26, Thm. 3.1]. ii) Thanks to Prop. 4.2 we know that B_q satisfies maximal parabolic regularity on $W^{-1,q}_{\mathfrak{D}}(\Omega)$, i.e. (53) has exactly one solution u, and the dependence of u, considered in the max.-par.-reg. space, on the right-hand side $f \in L^s(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$ is continuous. Moreover, according to Prop. 2.10, one has for $s \in]1, \infty[$ and $\theta \in]0, 1 - \frac{1}{s}[$,

$$W^{1,s}(J; W^{-1,q}_{\mathfrak{D}}(\Omega)) \cap L^s(J; \mathrm{Dom}(B_q)) \hookrightarrow C^\beta(J; (W^{-1,q}_{\mathfrak{D}}(\Omega), \mathrm{Dom}(B_q))_{\theta,1}), \tag{54}$$

where $\beta = 1 - \frac{1}{s} - \theta$. Combining this with i), one gets the result. □

This already enables us to formulate the final Hölder result in the autonomous case for two-dimensional domains Ω.

Theorem 4.4. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\mathfrak{D} \subset \partial\Omega$ a closed portion of the boundary which, together, fulfill Assu.2.1. Moreover, we suppose the existence of a $p > 1$ such that the suppositions of Thm. 3.14 are satisfied for sufficiently large s.*
Then there is a $q > 2$ such that the following holds:

i) *The function*
$$J \ni t \mapsto \sigma_t \mathcal{H}_1|_{M_t} =: \rho_t \in \mathcal{M}(\Omega) \tag{55}$$
can be interpreted as a measurable function Ψ with values in $W^{-1,q}_{\mathfrak{D}}(\Omega)$ with the same integrability exponent τ in time.

ii) The solution u of

$$\frac{\partial u}{\partial t} + B_q u = \Psi \in L^\tau(J; W_{\mathfrak{D}}^{-1,q}(\Omega)), \quad u(0) = 0 \tag{56}$$

exists, is unique and belongs to the maximal regularity space

$$\mathrm{MR}_0^\tau(J, \mathrm{Dom}(B_q); W_{\mathfrak{D}}^{-1,q}) = L^\tau(J; W_{\mathfrak{D}}^{1,q}(\Omega)) \cap W_0^{1,\tau}(J; W_{\mathfrak{D}}^{-1,q}(\Omega)).$$

One can diminish q, such that $\mathrm{Dom}(B_q) = W_{\mathfrak{D}}^{1,q}(\Omega)$, and, consequently,

$$\mathrm{MR}_0^\tau(J, \mathrm{Dom}(B_q); W_{\mathfrak{D}}^{-1,q}) = \mathrm{MR}_0^\tau(J, W_{\mathfrak{D}}^{1,q}; W_{\mathfrak{D}}^{-1,q}).$$

iii) Finally, if τ is large enough, then it exists a $\vartheta > 0$ such that this solution u even belongs to the space $C^\vartheta(J \times \Omega)$ and the C^ϑ-norm of u may be estimated as follows

$$\|u\|_{C^\vartheta(J\times\Omega)} \le c\Big(\int_J^* \|\sigma_t\|_{L^p(M_t;\mathcal{H}_1)}^\tau dt\Big)^{1/\tau}, \tag{57}$$

c being independent of $\{\sigma_t\}_{t\in J}$.

Proof. i) The first assertion follows from Thm. 3.11. ii) is implied by i) and Thm. 4.2. iii) One has an embedding $\mathrm{Dom}(B_q) \hookrightarrow C^\alpha(\Omega)$; for $q > 2 \sim q$ this follows from Prop. 2.6: $\mathrm{Dom}(B_q) = W_{\mathfrak{D}}^{1,q}(\Omega) \hookrightarrow C^\alpha(\Omega)$ and for larger q's from the smaller ones. So the result is obtained by means of Prop. 4.3. □

Unfortunately, in spaces with dimension $d > 2$ it is not that easy – simply by embedding $W_{\mathfrak{D}}^{1,q}(\Omega) \hookrightarrow C^\alpha(\Omega)$ – to achieve the Hölder property for elements of $\mathrm{Dom}(B_q)$, even if $q > d$. The fundamental problem lies in the fact that $\mathrm{Dom}(B_q)$ fails to coincide with $W_{\mathfrak{D}}^{1,q}(\Omega)$ in general – as already discussed above. But there is a way out of this dilemma: when investing, besides our general Assu. 2.1 two minimal additional ones, we indeed get $\mathrm{Dom}(B_q) \hookrightarrow C^\alpha(\Omega)$. These things were first elaborated in [20], but it exists a simplified version [26] for the dimensions up to 4. The latter avoids the highly non-trivial mechanisms of DeGiorgi estimates and Campanato spaces. The first additional assumption relies on the rather classical notion with a twist of saying that an open subset Λ of $\mathbb{R}^d$ is *of class* (A_ς) (at $\Upsilon \subseteq \partial\Lambda$) with a constant $\varsigma \in \,]0,1[$, if

$$|B(x;r) \setminus \Lambda| \ge \varsigma |B(x;r)| \qquad \text{for all } x \in \Upsilon,\ r \in (0,1].$$

This condition prevents inwards cusps of Λ at Υ. If $\Upsilon = \partial\Lambda$, we just refer to Λ being of class (A_ς). The second condition, rather intriguing, concerns the interface between the Dirichlet boundary part $\mathfrak{D}$ and the Neumann boundary part $N = \partial\Omega \setminus \mathfrak{D}$ in the boundary of Ω, here λ_{d-1} denoting the $(d-1)$-dimensional Lebesgue measure on the hyperplane $[z_d = 0]$.

Assumption 4.5. [(a)]

1 There is some $\varsigma \in (0,1)$ such that Ω is of class (A_ς) at $\mathfrak{D}$.

2 Using the notation of Assumption 2.1 (b), there are two constants $c_0 \in \,]0,1[$ and $c_1 > 0$ such that for any point $x \in \Pi := \mathfrak{D} \cap \overline{N}$, every $y \in \mathbb{R}^{d-1}$ such that $(y,0) \in \phi_x(\Pi \cap V_x)$ and every $s \in \,]0,1]$ it holds

$$\lambda_{d-1}\Big(\big\{z \in \mathbf{B}_r(y)\colon\ \mathrm{dist}\big(z, \phi_x(N \cap V_x)\big) > c_0 r\big\}\Big) \ge c_1 r^{d-1}. \tag{58}$$

Here $\mathbf{B}_r(y)$ denotes the open ball of radius r in $\mathbb{R}^{d-1}$ with its center at $y \in \mathbb{R}^{d-1}$, and in the distance function we tacitly consider $\phi_x(N \cap V_x) \subset [z_d = 0]$ as a subset of $\mathbb{R}^{d-1}$ in the obvious manner.

Very roughly spoken, condition (58) demands that $\mathfrak{D}$ is 'sufficiently rich' in a neighborhood of $\mathfrak{D} \cap \overline{(\partial\Omega \setminus \mathfrak{D})}$ – in a certain quantitative sense. It is clear that this condition is tailor suited for mixed boundary conditions: the pure Dirichlet case is known for decades (see [30]), and the pure Neumann case was treated in [39].

Theorem 4.6. *Assume* $d = 3$.

(a) *Adopt the Assumptions 2.1 and 2.13 and 4.5.*

(b) *Let, for every* $t \in J$, $M_t \subset \Omega$ *be a closed* 2*-set of* Ω *with the property*

$$\mathcal{H}_2(\overline{M}_t \cap B(x,r)) \le \mathfrak{c}_2\, r^2, \quad x \in \overline{M}_t,\ r \in]0,1] \tag{59}$$

for a constant $\mathfrak{c}_2$ *and all* $t \in J$.

(c) *There is some* $p > 2$ *and, for all* $t \in J$, *a function* $\sigma_t \in L^p(M_t, \mathcal{H}_2)$ *together satisfying*

$$\int_J^* \|\sigma_t\|^s_{L^p(M_t;\mathcal{H}_2)} dt, < \infty, \tag{60}$$

such that the mapping

$$J \ni t \mapsto \sigma_2 \mathcal{H}_2|_{M_t} =: \rho_t \in \mathcal{M}(\Omega) \tag{61}$$

is weakly continuous.*

i) Then the equation

$$\frac{\partial u}{\partial t} + B_q u = \rho \quad u(0) = 0 \tag{62}$$

admits exactly one solution u. *This solution belongs to the maximal parabolic reg. space* $\mathrm{MR}^s_0(J, \mathrm{Dom}(B_q); W^{-1,q}_{\mathfrak{D}})$.

ii) If s *is large enough, then there is a* $\vartheta > 0$ *such that this solution* u *even belongs to the space* $C^\vartheta(J \times \Omega)$. *Finally, one has the estimate*

$$\|u\|_{C^\vartheta(J\times\Omega)} \le c \left(\int_J^* \|\sigma_t\|^s_{L^p(M_t);\mathcal{H}_2)} dt \right)^{1/s} \tag{63}$$

for some constant c.

Proof. According to the Theorems 3.10/3.11 one can understand ρ_t as an element of $W^{-1,q}_{\mathfrak{D}}(\Omega)$ and the function ρ as one from $L^s(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$ with a $q > 3$. So the result is obtained by Thm. 4.5 in [15]. □

5 Measurability in time

Up to now we considered parabolic equation with prescribed right-hand side $\sigma_t \mathcal{H}_l|_{M_t}$ here only demanding the finiteness of the upper integral

$$\int_J^* \|\sigma\|^s_{L^p(M_t;\mathcal{H}_l)} dt < \infty \tag{64}$$

and, secondly, the measurability of the mappings

$$J \ni t \mapsto \langle \sigma_t \mathcal{H}_l, \psi \rangle_{W_{\mathfrak{D}}^{-1,q} \times W_{\mathfrak{D}}^{1,q'}} = \int_{M_t} \sigma_t \, \overline{\psi}|_{M_t} \, d\mathcal{H}_l, \ \psi \in W_{\mathfrak{D}}^{1,q'}(\Omega). \tag{65}$$

Unfortunately, the upper integral does single out a suitable space of functions nor a 'practical' norm on it. This can be seen as follows: if one has a family $\{\sigma_t\}_t$, chose a scalar **non-measurable** function f on J, taking only the values ± 1. Then, if (65) is measurable, then it is not if σ_t is replaced there by $f(t)\sigma_t$ there. This cries for a suitable restriction of the functions σ_t in the context of (65). In the sequel we offer one version of this. Let us first inspect the case where all sets M_t are identical, i.e. $M_t = M$ for a fixed M. Then it is clear that the desired w*-measurability is achieved if and only if the mapping $J \ni t \mapsto \sigma(t) \in L^p(M, \mathcal{H}_l)$ itself is measurable. This shows that the family $\{\sigma_t\}_t$ has, firstly, to satisfy some 'measurability in itself'. Of course, it would be not satisfying to restrict the considerations to this case. In practice one would like to investigate the case where, firstly, the M_t's 'move in time' and, secondly, are allowed to 'deform'. It is the intention of this section to offer a concept within that is allowed.

So, throughout this section we make the following general

Assumption 5.1. There is an l-set $M \subset \Omega$ such that, for all $t \in J$, there is a bi-Lipschitz diffeomorphism ϕ_t from M onto M_t. The Lipschitz constants l_t of the ϕ_t's and their inverses ϕ_t^{-1}, l_t^-, are uniformly (in t) bounded.

Lemma 5.2. *Consider the image, named ϖ_t, of the Hausdorff measure $\mathcal{H}_l$ on M under ϕ_t on M_t. Then ϖ_t is of the form $\varpi_t = \varsigma_t \mathcal{H}_l$, where ς_t is $\mathcal{H}_l$-measurable and is bounded from above and below by constants, uniform in t.*

Proof. One has, for any $\mathcal{H}_l$-measurable subset $\mathcal{A} \subset M$

$$\gamma_t \mathcal{H}_l(\phi_t(\mathcal{A})) \le \mathcal{H}_l(\mathcal{A})) \le \gamma_t^{-1} \mathcal{H}_l(\phi_t(\mathcal{A})), \quad t \in J \tag{66}$$

where γ_t is determined by the Lipschitz constants of the mappings ϕ_t, ϕ_t^{-1}, see [21, Ch. 2.4.1]. In particular, the sets $\{\gamma_t\}_t$ and $\{\gamma_t^{-1}\}_t$ are bounded. This shows, in particular, that ϖ_t is absolutely continuous with respect to $\mathcal{H}_l$ on M_t and, hence, admits a density ς_t by the Radon-Nikodym theorem. It is clear that (66) implies the (uniform in t) boundedness of the ς_t's from above and below by positive constants. □

Consider now the space $L^s(J; L^p(M; \mathcal{H}_l))$. Let us introduce, for every t, the mapping $V_t : L^p(M; \mathcal{H}_l) \to L^p(M_t; \mathcal{H}_l)$ defined by

$$\big(V_t(\varphi)\big)(x) = \varsigma_t(x) \, \varphi(\phi_t^{-1} x), \quad \varphi \in L^p(M; \mathcal{H}_l), \, x \in M_t. \tag{67}$$

Then the definition of the image of a measure together with Lemma 5.2 shows that, for every $t \in J$, V_t is a linear and *bounded* mapping from $L^p(M; \mathcal{H}_l)$ onto $L^p(M_t; \mathcal{H}_l)$. So it is straight forward to check that the set of mappings

$$J \ni t \mapsto V_t \sigma_t \in L^p(M_t; \mathcal{H}_l), \quad \sigma \in L^s(J; L^p(M; \mathcal{H}_l)),$$

topologized by the pre-images from $L^s(J; L^p(M; \mathcal{H}_l))$ gives rise to a suitable space – formally denoted as a direct integral $\int_J^{\oplus} L^p(M_t; \mathcal{H}_l) \, dt$.

The crucial point is now the measurability – or not – of the mappings

$$J \ni t \mapsto \langle \sigma_t \mathcal{H}_l, \psi \rangle_{W_{\mathfrak{D}}^{-1,q} \times W_{\mathfrak{D}}^{1,q'}} = \int_{M_t} \sigma_t \, \overline{\psi}|_{M_t} \, d\mathcal{H}_l, \quad \sigma \in \int_J^{\oplus} L^p(M_t; \mathcal{H}_l) \, dt, \tag{68}$$

for all $\psi \in W^{1,q'}_{\mathfrak{D}}$. Since $C^\infty_{\mathfrak{D}}(\Omega)$ is dense in $W^{1,q'}_{\mathfrak{D}}(\Omega)$ we may restrict ourselves to $\psi \in C^\infty_{\mathfrak{D}}(\Omega)$. For the inspection of (68) it is essential to observe that $\sigma \in \int_J^\oplus L^p(M_t;\mathcal{H}_l)\, dt$ is necessarily of the form $\sigma_t = \varsigma_t\, v_t(\phi_t^{-1}(\cdot))$ with $v \in L^s(J; L^p(M;\mathcal{H}_l))$. So one may calculate

$$\int_{M_t} \sigma_t\, \overline{\psi}|_{M_t}\, d\mathcal{H}_l = \int_{M_t} \varsigma_t\, v_t(\phi_t^{-1}(\cdot))\, \overline{\psi}|_{M_t}\, d\mathcal{H}_l = \int_{M_t} v_t(\phi_t^{-1}(\cdot))\, \overline{\psi}|_{M_t}\, d\varpi_t =$$

$$= \int_M v_t\, \overline{\psi}(\phi_t(\cdot))\, d\mathcal{H}_l, \quad \psi \in C^\infty_{\mathfrak{D}}(\Omega) \tag{69}$$

because ϖ_t was the image of the measure $\mathcal{H}_l|_M$ under ϕ_t. The reader should carefully notice that the function $M \ni x \mapsto \psi(\phi_t(x)) \to \mathbb{C}$ is bounded and continuous – hence measurable with respect to $\mathcal{H}_l$. Since $\mathcal{H}_l(M)$ is finite, the function, consequently, belongs to $L^2(M,\mathcal{H}_l)$, and the last term in (69) is well defined – irrespective of the Hausdorff dimension of M. Since the functions v run through the whole space $L^s(J; L^p(M;\mathcal{H}_l))$, it is straight forward that the measurability of (69) with respect to t is equivalent to the measurability of the function

$$J \ni t \mapsto \psi(\phi_t(\cdot)) \in L^{p'}(M;\mathcal{H}_l) \tag{70}$$

for every function $\psi \in C^\infty_{\mathfrak{D}}(\Omega)$.
In the next lemma we will give a simple and absolutely natural condition which indeed implies this.

Lemma 5.3. *Let ψ be uniformly continuous on Ω and adopt the above conditions on the mappings ϕ_t. Assume that the mappings $J \ni t \mapsto \phi_t(x) \in \Omega$ are measurable for every $x \in M$. Then*

$$J \ni t \mapsto \psi(\phi_t(\cdot)) =: f_t \in L^r(M;\mathcal{H}_l) \tag{71}$$

is measurable for every $r \in]1,\infty[$.

Proof. First one observes that the system of functions $\{f_t\}_t$ is equicontinuous on M according to the uniform continuity of ψ and the (uniform) Lipschitz properties of the mappings ϕ_t. Let $\{x_j\}_j$ be a countable, dense subset of M. Standard arguments (see [13, Ch. 13.9, 13.9.6]) tell us that, for every $x \in M$, the function $J \ni t \mapsto f_t(x)$ is measurable. Let $\epsilon > 0$ be arbitrary. So, by Lusin's theorem, for every j there is a compact set $\mathcal{K}^j_\epsilon \subset J$, such that $|J \setminus \mathcal{K}^j_\epsilon| \le \epsilon 2^{-j-1}$ and the mapping

$$\mathcal{K}^j_\epsilon \ni t \mapsto f_t(x_j)$$

is continuous (see [13, Ch. 13.9, 13.9.4]). Define $\mathcal{K} = \cap_j \mathcal{K}^j_\epsilon$. We show:
For *every* $x \in M$, the mapping

$$\mathcal{K} \ni t \mapsto f_t(x) \tag{72}$$

is continuous. One has

$$|f_t(x) - f_s(x)| \le |f_t(x) - f_t(x_j)| + |f_t(x_j) - f_s(x_j)| + |f_s(x) - f_s(x_j)|,$$

and all three addends can be made arbitrarily small by taking x_j close enough to x. Let $\varphi \in L^{r'}(M;\mathcal{H}_l)$. Knowing the continuity of (72), Lebesgue dominance tells us that

$$\mathcal{K} \ni t \mapsto \int_M f_t\, \varphi\, d\mathcal{H}_l \tag{73}$$

is continuous (see [13, Ch. 13.8, 13.8.6]) . But the measure of $J \setminus \mathcal{K}$ is at most ϵ. So Lusin's theorem again applies and tells us that

$$J \ni t \mapsto \int_M f_t\, \varphi\, d\mathcal{H}_l \tag{74}$$

is measurable. This shows that (71) is weakly measurable, and, hence, measurable. □

Finally, Assu. 5.1 can be relaxed in a straight forward manner as follows: divide the interval into the intervals $J_1, J_2, \dots$ and demand for every subinterval $J = J_k$ again Assu. 5.1.

6 Concluding remarks

(a) The assignment

$$\varrho : C_0(J \times \Omega) \ni f \mapsto \int_J \int_\Omega f(t,x)\, d\rho_t(x)\, dt \tag{75}$$

defines a measure on $J \times \Omega$, if the mapping $t \mapsto \rho_t \in \mathcal{M}(\Omega)$ is weakly measurable and some integrability condition

$$\int_J \|\rho_t\|_{\mathcal{M}}^s dt < \infty \tag{76}$$

holds.
Conversely, if ϱ is a measure on $J \times \Omega$, then it always admits a disintegration of type

$$C_0(J \times \Omega) \ni f \mapsto \int_J \int_\Omega f(t,x)\, d\varrho_t(x)\, d\varpi(t), \tag{77}$$

each ϱ_t being a measure on Ω and ϖ being a measure on $\overline{J}$, see [27].
Thus, our result is proved for measures on $J \times \Omega$ for which the measure ϖ is the Lebesgue measure on J and the measures ϱ_t are of the form $\sigma_t \mathcal{H}_l|_{M_t}$, satisfying the integrability condition (76). This condition (76) seems reasonable in applications, see [8] and [41].

(b) Basing on the presented results and [34, Prop. 2.2.2], one can – quite analogously – also treat the initial value problem with initial $u_0 \neq 0$. We leave this to the reader.

(c) In fact, Prop. 2.6 remains true even under still weaker assumptions, see [24] for details. Since the suppositions in Assu. 2.1 deliver a frame which is, on one hand, rather general and can be, on the other, be overlooked also by non-specialists in elliptic and parabolic theory, we decided to take this as the general assumption in this paper.

(d) We have restricted to the case where the measures live on subsets of *integer* dimension only for technical simplicity. The basis in geometric measure theory on which our results rest is established in [29] for the general case also. Everything can then be proved quite analogously. Since we are not aware of any applications of this we did not carry out this here but restricted to integral dimensions.

(e) One can consider also right-hand sides where the sets M_t are unions $M_t = M_t^1 \cup M_t^2$, M_t^1 being an l_1-set and M_t^2 being an l_2-set. If p_1, p_2 allow embeddings $L^{p_1}(M_t^1) \hookrightarrow W_{\mathfrak{D}}^{-1,q}$ and $L^{p_2}(M_t^2) \hookrightarrow W_{\mathfrak{D}}^{-1,q}$, then the corresponding parabolic equation can be treated as before.

(f) Generalizations to complex coefficients and even to systems are possible. These can rest on the fact that Prop. 2.6 is available also in these cases (see [24]). Also Prop. 2.16 does not require that the coefficients are real: basing on the novel and pioneering results of Moritz Egert [17] on can prove that the elliptic (system) operators also satisfy maximal parabolic regularity on (certain) spaces $W_{\mathfrak{D}}^{-1,q}(\Omega)$. Having this at hand, one can again exploit [14, Thm. 3.4] in order to extrapolate also *non-autonomous* maximal parabolic regularity – for complex coefficients.
Another matter is it with the things in Ch. 4: the results for the spatial two-dimensional case remain valid, but for the three-dimensional go wrong.

(g) Generalizations to semilinear/quasilinear equations are possible, but highly non-trivial, compare [37]. This is entirely out of scope here. In particular, the existence of the solutions over the whole interval is a delicate matter and wrong in general.

(h) As the title of [32] suggests, it can happen that distributional objects are of interest which are *not* necessarily measures. Consider the following situation: Take $\Omega \subset \mathbb{R}^2$ as a Lipschitz domain which contains a subinterval $]-a,a[\ni 0$ of the x-axis. Define the distribution Ψ on Ω as the PV distribution on $]-a,a[$ as follows:

$$\langle \Psi, v\rangle = \lim_{\epsilon\mapsto 0}\int_{-a}^{-\epsilon}\frac{\overline{v}(x)}{x}dx + \int_{-a}^{-\epsilon}\frac{\overline{v}(x)}{x}dx, \quad v\in W^{1,q}_{\mathfrak{D}}(\Omega),\ q>2. \tag{78}$$

It is not hard to see that the forming in (78) is well-defined and continuous on Hölder spaces, hence on $W^{1,q}_{\mathfrak{D}}(\Omega)$ with $q>2$. Consequently, the so defined Ψ – *not* being a measure – belongs to any $W^{-1,q}_{\mathfrak{D}}(\Omega)$ ($q\in\,]1,2[$) and lives on a one-dimensional manifold. We expect that such distributional objects, entering the parabolic equations as right-hand sides, can be treated entirely the same way as the measures above under our consideration.
We suggest that similar constructions can be found also in higher dimensions, but do not expediate this here further.

7 Appendix

We give the explanations to the proof of **Prop. 3.4**.
The expression in question which one has to estimate is

$$\|G_\alpha \star f\|^p_{L^p(M;\mathcal{H}_l)} = \int_M \Big|\int_{\mathbb{R}^d} G_\alpha(x-y)f(y)\,dy\Big|^p\,d\mathcal{H}_l(x) \tag{79}$$

We follow widely Jonsson/Wallin with the exception to determine the constant a explicitly here – what should allow an easier reading.
We define the number a via

$$\big(d-\frac{d-l}{p}\big)(1-a)p' = \big(\frac{d}{p'}+\frac{l}{p}\big)(1-a)p' = d. \tag{80}$$

Multiplying this by $\frac{p}{p'}$ and re-arranging terms, one obtains

$$\big(d\frac{p}{p'}+l\big)a = \big(d-\frac{d-l}{p}\big)a\,p = l. \tag{81}$$

Clearly, this gives $a = \big(1+\frac{d}{l}\frac{p}{p'}\big)^{-1}\in\,]0,1[$. Evidently, (80) yields

$$\big(d-\alpha\big)(1-a)p' = \big(d-\frac{d-l}{p}\big)(1-a)p' - (\alpha-\frac{d-l}{p}\big)(1-a)p' =$$

$$= d-(\alpha-\frac{d-l}{p}\big)(1-a)p' < d \tag{82}$$

and (81) provides

$$(d-\alpha)ap = (d-\frac{d-l}{p})ap + (\frac{d-l}{p}-\alpha)ap = l-(\alpha-\frac{d-l}{p})ap < l, \tag{83}$$

thanks to the supposition $\alpha > \frac{d-l}{p}$.

One estimates the r.h.s of (79) by

$$\int_M \Big(\int_{\mathbb{R}^d} \big|G_\alpha(x-y)\big|^{1-a} \big|G_\alpha(x-y)\big|^{a} f(y)\, dy \Big)^p \, d\mathcal{H}_l(x)$$

Applying Hölder's inequality, one further estimates

$$\leq \int_M \Big(\int_{\mathbb{R}^d} \big|G_\alpha(x-y)\big|^{ap} |f(y)|^p dy \cdot \big(\int_{\mathbb{R}^d} |G_\alpha(x-y)|^{(1-a)p'} dy \big)^{\frac{p}{p'}} \Big) d\mathcal{H}_l(x).$$

The crucial point is to show that the terms

$$\int_{\mathbb{R}^d} |G_\alpha(x-y)|^{(1-a)p'} dy = \int_{\mathbb{R}^d} |G_\alpha(y)|^{(1-a)p'} dy, \tag{84}$$

and

$$\int_M |G_\alpha(x-y)|^{ap} d\mathcal{H}_l(x), \quad y \in \mathbb{R}^d, \tag{85}$$

may be estimated and this uniformly for sets M admitting the same constant $\mathfrak{c}^\bullet$. Investing the exponential decay of the Bessel kernel at ∞ (see [43, Ch. V.3]) one sees that (84) makes no difficulties at ∞. But around zero (84) also converges, thanks to

$$|G_\alpha(z)| \leq \gamma\, |z|^{\alpha-d}, \tag{86}$$

(see [43, Ch. V.3]) in combination with (82).

(85) can be written as

$$\int_{M \cap \{x : |x-y| > 1\}} |G_\alpha(x-y)|^{ap} d\mathcal{H}_l(x) + \int_{M \cap \{x : |x-y| \leq 1\}} |G_\alpha(x-y)|^{ap} d\mathcal{H}_l(x).$$

According to (86), the first integral is not larger than $\gamma^{ap}\ \mathcal{H}_l(M)$, and $\mathcal{H}_l(M)$ is not larger than $\mathfrak{c}^\bullet \times \tau$ – τ being the number of(shifted) unit balls $B(z,1)$ required for a covering of M. The second integral is estimated by again employing (86) in combination with (83), what yields $|G_\alpha(x-y)|^{ap} \leq \gamma^{ap}|x-y|^{-\sigma}$ with $\sigma < l$. Afterwards one applies [29, Ch. V.1.2 Lemma 1]. This shows, first, that (85) is indeed finite – and may be estimated uniformly with respect to $y \in \mathbb{R}^d$. But even more, one sees that the constant $\mathfrak{c}^\bullet$ enters *linearly* in this estimate.

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