Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Hyperbolic relaxation of the chemical potential in the viscous Cahn–Hilliard equation

In memory of Prof. Dr. Wolfgang Dreyer with admiration, sympathy and friendship

Pierluigi Colli[1], Jürgen Sprekels[2]

submitted: September 4, 2024

[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
and Research Associate at the IMATI – C.N.R. Pavia
via Ferrata 5
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: juergen.sprekels@wias-berlin.de

No. 3128
Berlin 2024

2020 *Mathematics Subject Classification.* 35M33, 35M87, 35B40, 37D35.

Key words and phrases. Cahn–Hilliard system, hyperbolic relaxation, partial differential equations, initial-boundary value problem, well-posedness, continuous dependence, regularity, asymptotic convergence.

This research has been supported by the MIUR-PRIN Grant 2020F3NCPX "Mathematics for industry 4.0 (Math4I4)". In addition, PC indicates his affiliation to the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica). During the preparation of this paper, individual visits to the WIAS in Berlin and to the Dipartimento di Matematica "F. Casorati" in Pavia have been exchanged by the two authors: the warm hospitality of the two institutions is gratefully acknowledged.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

In this paper, we study a hyperbolic relaxation of the viscous Cahn–Hilliard system with zero Neumann boundary conditions. In fact, we consider a relaxation term involving the second time derivative of the chemical potential in the first equation of the system. We develop a well-posedness, continuous dependence and regularity theory for the initial-boundary value problem. Moreover, we investigate the asymptotic behavior of the system as the relaxation parameter tends to 0 and prove the convergence to the viscous Cahn–Hilliard system.

1 Introduction

In this paper, we deal with an initial-boundary value problem for a system of partial differential equations of viscous Cahn–Hilliard type, which in particular includes a hyperbolic relaxation term in the first equation.

The system is stated as follows:

$$\alpha\partial_{tt}\mu + \partial_t\varphi - \Delta\mu = 0 \quad \text{in } Q := \Omega \times (0,T), \tag{1.1}$$

$$\tau\partial_t\varphi - \Delta\varphi + f'(\varphi) = \mu + g \quad \text{in } Q, \tag{1.2}$$

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T), \tag{1.3}$$

$$\mu(0) = \mu_0, \;\; (\partial_t\mu)(0) = \nu_0, \;\; \varphi(0) = \varphi_0 \quad \text{in } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$, $N \in \{1,2,3\}$, is a bounded and connected domain with smooth boundary $\partial\Omega$ and T denotes some final time. We denote by $\mathbf{n}$ the unit outward normal to $\partial\Omega$, with the associated outward normal derivative $\partial_{\mathbf{n}}$. Note that $\partial_{\mathbf{n}}$ appears in the homogeneous Neumann boundary conditions stated in (1.3) for both the variables μ and φ.

The equations (1.1)–(1.2) constitute a variation of the Cahn–Hilliard system (introduced in [5] and first approached mathematically in [18])

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{in } Q, \tag{1.5}$$

$$-\Delta\varphi + f'(\varphi) = \mu + g \quad \text{in } Q, \tag{1.6}$$

in which a viscosity term $\tau\partial_t\varphi$ has been included in the second equation and where especially the hyperbolic relaxation term $\alpha\partial_{tt}\mu$ has been added in the first equation. The viscous Cahn–Hilliard system

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{in } Q, \tag{1.7}$$

$$\tau\partial_t\varphi - \Delta\varphi + f'(\varphi) = \mu + g \quad \text{in } Q, \tag{1.8}$$

DOI 10.20347/WIAS.PREPRINT.3128

is well known and was already investigated in a number of papers (see [8, 9, 11–15, 20–23] to quote some recent contributions), while to our knowledge an inertial term like $\alpha\partial_{tt}\mu$ in (1.1) is not common and possibly deserves to be examined. In this class of problems, the unknown functions φ and μ ususally stand for the *order parameter*, which can represent a scaled density of one of the involved phases, and the *chemical potential* associated with the phase separation process, respectively.

Moreover, f' denotes the derivative (if it exists) of a double-well potential f, which in general is split into a (possibly nondifferentiable) convex part f_1 and a smooth and concave perturbation f_2. Typical and physically significant examples for f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{\rm reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{1.9}$$

$$f_{\rm log}(r) := \begin{cases} (1+r)\,\ln(1+r) + (1-r)\,\ln(1-r) - c_1 r^2 & \text{if } r \in (-1,1) \\ 2\ln(2) - c_1 & \text{if } r \in \{-1,1\} \\ +\infty & \text{if } r \notin [-1,1] \end{cases}, \tag{1.10}$$

$$f_{\rm 2obs}(r) := \begin{cases} c_2(1-r^2) & \text{if } r \in [-1,1] \\ +\infty & \text{if } r \notin [-1,1] \end{cases}. \tag{1.11}$$

Here, the constants c_i in (1.10) and (1.11) satisfy $c_1 > 1$ and $c_2 > 0$, so that $f_{\rm log}$ and $f_{\rm 2obs}$ are nonconvex. Notice that for $f = f_{\rm log}$ the term $f'(\varphi)$ occurring in (1.2) becomes singular as $\varphi \searrow -1$ and $\varphi \nearrow 1$, which forces the order parameter φ to attain its values in the physically meaningful range $(-1,1)$. In the nonsmooth case (1.11), the convex part f_1 is given by the indicator function of $[-1,1]$. Accordingly, in such cases one has to replace the derivative of the convex part by the subdifferential ∂f_1 and, consequently, to interpret (1.2) as a differential inclusion or a variational inequality. We also note that τ is a fixed positive parameter (the viscosity coefficient), while for the positive parameter α we will also discuss the asymptotic convergence to 0. We point out that in (1.2) a known forcing term g is present that may be interpreted as a direct or secondary control term which acts on the system. In this connection, we mention that optimal control problems for viscous Cahn–Hilliard systems with a distributed control term involving g have recently been treated in the paper [14].

Some hyperbolic relaxations of the viscous Cahn–Hilliard system have been already considered and studied: let us mention the recent contributions [3,6,7,17,26,27]. However, the available investigations are concerned with systems where the inertial term involves the phase variable φ. In our case, the system (1.1)–(1.4) couples a wave-type equation for μ combined with a source term given by $-\partial_t\varphi$, with a semilinear parabolic equation in which the source term includes μ.

From the energetic viewpoint, there is a change with respect to the viscous (and nonviscous) Cahn–Hilliard equation. To see this, let us for simplicity argue now on the case $g = 0$. Indeed, for (1.7)–(1.8), as well as for (1.5)–(1.6), the basic energy estimate is obtained by testing (1.7) by μ, (1.8) by $\partial_t\varphi$, and then adding and thus producing a cancellation of the terms containing the product $\mu\partial_t\varphi$. Therefore, one has

$$\int_0^t\!\!\int_\Omega |\nabla\mu|^2 + \tau\int_0^t\!\!\int_\Omega |\partial_t\varphi|^2 + \int_\Omega\Big(\frac{1}{2}|\nabla\varphi(t)|^2 + f(\varphi(t))\Big) = \text{constant}$$

for $t \in [0,T]$, where the first two terms are dissipative and the energy term is given by the third one. The second term is missing in the case of the Cahn–Hilliard system (1.5)–(1.6), but the energy is the same and there is only one term for dissipation. On the other hand, it turns out to be more difficult and involved to recover an energy estimate for (1.1)–(1.2): as you will check in the sequel, our main estimate is constructed by testing (1.1) by $\partial_t\mu$ and the time derivative of (1.2) by $\partial_t\varphi$, in order to have a cancellation of the terms containing the product $\partial_t\varphi\,\partial_t\mu$. By integration, we then obtain

$$\int_\Omega\Big(\frac{\alpha}{2}|\partial_t\mu(t)|^2+\frac{1}{2}|\nabla\mu(t)|^2+\frac{\tau}{2}|\partial_t\varphi(t)|^2\Big) + \int_0^t\!\!\int_\Omega|\nabla(\partial_t\varphi)|^2+\int_0^t\!\!\int_\Omega f_1''(\varphi_n)|\partial_t\varphi|^2 = \text{constant } - \int_0^t\!\!\int_\Omega f_2''(\varphi)|\partial_t\varphi|^2 \quad \text{for } t\in[0,T],$$

where the energy is now located in the first integral in which neither the nonlinearity f nor any of its derivatives occur. Moreover, note that the last term on the left-hand side induces dissipation (as f_1'' is nonnegative), but on the right-hand side the complementary term may be positive and grow with respect to t, since f_2 is concave, in general. The viscous contribution in (1.2) is important here to control this term on the right-hand side, since the addendum $\frac{\tau}{2}|\partial_t\varphi(t)|^2$ is part in the energy. However, by our estimate we can proceed in the analysis and not only construct a well-posedness theory but also investigate the asymptotic behavior as the parameter α converges to 0.

This paper is dedicated to the memory of Wolfgang Dreyer, who recently passed away. The authors of this paper were fortunate to have benefited from Wolfgang's friendship, as well as from his exceptional expertise and insight in Thermodynamics and Applied Mathematics. He was a brilliant and generous scientist who truly enjoyed engaging in scientific discussions with friends and colleagues. We both feel enriched by having known him and are grateful for the opportunity to have collaborated with him. Together, along with other colleagues, we co-authored the paper [2] that explored the effects of phase separation driven by mechanical actions in tin/lead alloys, and where the corresponding system of partial differential equations already included equations of Cahn–Hilliard type.

The paper is organized as follows. In the following section, we formulate the general assumptions and state the main results concerning the system (1.1)–(1.4). In Section 3, we then prove the existence of a solution by using a double approximation based on a Yosida regularization of ∂f_1 and on a Faedo–Galerkin scheme. This proof requires the main analytical effort of this paper, since it involves a number of estimates and two passage-to-the-limit processes. In Section 4, we show the results on continuous dependence with respect to data and on the regularity of the solution: actually, three theorems are proved there. The final Section 5.1 then brings the asymptotic results of the convergence of the system to the viscous Cahn–Hilliard system as α tends to 0 and an estimate of the difference of solutions in terms of a precise rate of convergence.

We fix some notation. For any Banach space X, we let X^* denote its dual space, and $\|\cdot\|_X$ stands for the norm in X and any power of X. For two Banach spaces X and Y that are both continuously embedded in some topological vector space Z, the linear space $X\cap Y$ is the Banach space equipped with its natural norm $\|v\|_{X\cap Y}:=\|v\|_X+\|v\|_Y$ for $v\in X\cap Y$. The standard Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{m,p}(\Omega)$ are defined on Ω for $1\le p\le\infty$ and $m\in\mathbb{N}\cup\{0\}$. For the sake of convenience, we denote the norm of $L^p(\Omega)$ by $\|\cdot\|_p$ for $1\le p\le\infty$. If $p=2$, we employ the usual notation $H^m(\Omega):=W^{m,2}(\Omega)$. We also set

$$H:=L^2(\Omega),\quad V:=H^1(\Omega),\quad W:=\big\{v\in H^2(\Omega):\ \partial_{\mathbf{n}}v=0 \text{ on } \Gamma\big\}.$$

Moreover, V^* is the dual space of V, and $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between V^* and V. We denote by $(\,\cdot\,,\,\cdot\,)$ the natural inner product in H. As usual, H is identified with a subspace of the dual space V^* according to the identity

$$\langle u,v\rangle=(u,v)\quad\text{for every } u\in H \text{ and } v\in V.$$

Note that $W \subset V \subset H \equiv H^* \subset V^*$ with dense and compact embeddings. About the constants used in the sequel for estimates, we adopt the rule that C denotes any positive constant that depends only on the given data. The value of such generic constants C may change from formula to formula or even within the lines of the same formula. Finally, the notation C_δ indicates a positive constant that additionally depends on the quantity δ.

2 Main results

In this section, we formulate the general assumptions for the data of the system (1.1)–(1.4) and state existence, continuous dependence, and regularity results. First, let us remark that the positive parameter α is not listed in the assumptions below, since it is also involved in the related asymptotic analysis, and, consequently, we let

$$0 < \alpha \le 1.$$

On the other hand, throughout the paper we suppose that

$$\tau > 0 \text{ is a fixed constant.} \tag{2.1}$$

For the nonlinearity f we generally assume that

$$\begin{aligned} & f = f_1 + f_2, \quad \text{where} \\ & \quad f_1 : \mathbb{R} \to [0, +\infty] \text{ is convex and lower semicontinuous with } f_1(0) = 0, \\ & \quad f_2 : \mathbb{R} \to \mathbb{R} \text{ has a Lipschitz continuous first derivative } f_2' \text{ on } \mathbb{R}. \end{aligned} \tag{2.2}$$

A consequence of (2.2) is that

$$\text{the subdifferential } \partial f_1 \text{ is maximal monotone in } \mathbb{R} \times \mathbb{R}, \text{ with } 0 \in \partial f_1(0), \tag{2.3}$$

and an important requirement for the sequel is that

$$\text{the domain } D(\partial f_1) \text{ of } \partial f_1 \text{ has a non-empty interior containing } 0. \tag{2.4}$$

Note that these conditions are fulfilled in each of the cases considered in (1.9), (1.10), (1.11) with the domain $D(\partial f_1)$ given by $\mathbb{R}$, $(-1, 1)$, $[-1, 1]$, respectively. From now onwards we use the symbol $\partial f_1^\circ(r)$ for the element of $\partial f_1(r)$ (with $r \in D(\partial f_1)$) having minimum modulus, and we extend the notations f_1, ∂f_1, $D(\partial f_1)$, and ∂f_1° to the corresponding functionals and the operators induced on L^2 spaces.

Also, we assume for the forcing term g and the initial values μ_0, ν_0, φ_0 that

$$g \in H^1(0, T; H), \tag{2.5}$$

$$\mu_0 \in V, \quad \nu_0 \in H, \tag{2.6}$$

$$\varphi_0 \in W \cap D(\partial f_1) \text{ with } \partial f_1^\circ(\varphi_0) \in H. \tag{2.7}$$

Note that the condition $\varphi_0 \in W$ implies that $\varphi_0 \in C^0(\overline{\Omega})$. Moreover, we require that

$$m_0 := \frac{1}{|\Omega|} \int_\Omega \varphi_0 \text{ lies in the interior of } D(\partial f_1). \tag{2.8}$$

Here, $|\Omega|$ denotes the Lebesgue measure of Ω, and m_0 thus represents the mean value of φ_0. In the following, we use the general notation $\overline{v}$ to denote the mean value of a generic function $v \in L^1(\Omega)$. If v is in V^*, then we can set

$$\overline{v} := \frac{1}{|\Omega|} \langle v, 1 \rangle \tag{2.9}$$

as well, noting that the constant function 1 is an element of V. Clearly, $\overline{v}$ is the usual mean value of v if $v \in H$. Note also that $m_0 = \overline{\varphi_0}$.

Let us now specify our notion of solution. We state the problem (1.1)–(1.4) in a variational form. We also introduce an additional variable ξ, which plays the role of $f_1'(\varphi)$ in the case when the derivative of f_1 is replaced by a real subdifferential ∂f_1. Namely, the solution is a triple (μ, φ, ξ) satisfying the regularity requirements

$$\mu \in W^{2,\infty}(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{2.10}$$
$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.11}$$
$$\xi \in L^\infty(0,T;H), \tag{2.12}$$

and the following variational equations and initial conditions:

$$\alpha \langle \partial_{tt}\mu(t), v \rangle + (\partial_t \varphi(t), v) + \int_\Omega \nabla\mu(t) \cdot \nabla v = 0$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{2.13}$$

$$\tau(\partial_t \varphi(t), v) + \int_\Omega \nabla\varphi(t) \cdot \nabla v + (\xi(t) + f_2'(\varphi(t)), v) = (\mu(t) + g(t), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{2.14}$$

$$\xi \in \partial f_1(\varphi) \quad \text{a.e. in } Q, \tag{2.15}$$

$$\mu(0) = \mu_0\,, \quad (\partial_t\mu)(0) = \nu_0\,, \quad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega\,. \tag{2.16}$$

Of course, in view of the regularities in (2.10)–(2.12), the variational equality (2.14) is actually equivalent to an equation (cf. (1.2)) plus the boundary condition for φ which is already encoded in the fact that $\varphi \in L^\infty(0,T;W)$. Therefore, (2.14) can be replaced by

$$\tau\partial_t\varphi - \Delta\varphi + \xi + f_2'(\varphi) = \mu + g \quad \text{a.e. in } Q. \tag{2.17}$$

On the contrary, the analogue equivalence for (2.13) (cf. (1.1)) would be true only if μ were more regular (cf. the first ad third term in (2.13)).

Remark 2.1. Note that, owing to the compactness of the embedding $W \subset C^0(\overline{\Omega})$ for $N \le 3$, it follows from [28, Sect. 8, Cor. 4] and the regularity (2.11) that $\varphi \in C^0(\overline{Q})$. By the same token, we have thanks to (2.10) that $\mu \in C^1([0,T];V^*) \cap C^0([0,T];H)$, and, consequently, $\partial_t\mu$ is at least weakly continuous from $[0,T]$ to H, which gives a meaning to the initial conditions in (2.16).

The next statement yields a well-posedness result for (2.13)–(2.16).

Theorem 2.2. *Assume that* (2.1)–(2.8) *are fulfilled. Then there exists a unique triple (μ, φ, ξ), with the regularity as in* (2.10)–(2.12)*, that solves problem* (2.13)–(2.16) *and satisfies the estimate*

$$\alpha\|\mu\|_{W^{2,\infty}(0,T;V^*)} + \alpha^{1/2}\|\mu\|_{W^{1,\infty}(0,T;H)} + \|\mu\|_{L^\infty(0,T;V)}$$
$$+ \|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\xi\|_{L^\infty(0,T;H)} \le K_1 \tag{2.18}$$

for some constant $K_1 > 0$ that depends only on Ω, T and the data in (2.1)–(2.8)*, but is independent of α.*

The uniqueness property stated above is a consequence of the following continuous dependence result. Here, we use the notation

$$1 * v(t) = \int_0^t v(s)ds, \quad \text{for } v \in L^1(0,T;V^*) \text{ at least.}$$

Theorem 2.3. *Under the assumptions* (2.1)–(2.4)*, let $g_i, \mu_{0,i}, \nu_{0,i}, \varphi_{0,i}, i = 1,2$, be two sets of data satisfying* (2.5)–(2.8)*, and let $(\mu_i, \varphi_i, \xi_i)$, $i = 1,2$, denote any corresponding solutions to problem* (2.13)–(2.16) *with the regularity as in* (2.10)–(2.12)*. Then, the estimate*

$$\begin{aligned} &\alpha^{1/2}\|\mu_1 - \mu_2\|_{L^\infty(0,T;H)} + \|\nabla(1 * (\mu_1 - \mu_2))\|_{L^\infty(0,T;H)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\le K_2 \left(\|g_1 - g_2\|_{L^2(0,T;H)} + \alpha^{1/2}\|\mu_{0,1} - \mu_{0,2}\|_H + \alpha^{1/2}\|\nu_{0,1} - \nu_{0,2}\|_H \right) \\ &\quad + K_2\big(1 + \alpha^{-1/2}\big)\|\varphi_{0,1} - \varphi_{0,2}\|_H \end{aligned} \tag{2.19}$$

holds true with a constant $K_2 > 0$ that depends only on Ω, T, τ, some Lipschitz constant for f_2', and is independent of α.

We observe that, by taking the same data in Theorem 2.3, the estimate (2.19) ensures uniqueness for the solution components μ and φ in the statement of Theorem 2.2, while the uniqueness of ξ results from (2.17) since the other terms in the equality are uniquely determined.

On the basis of the estimates (2.18) and (2.19), we are interested to investigate the asymptotic behavior of the problem (2.13)–(2.16) as $\alpha \searrow 0$. This analysis will be developed in Section 5, whereas now we discuss some further results that mostly depend on the values of $\alpha > 0$. A regularity result is stated below, under the additional assumption that

$$\mu_0 \in W, \quad \nu_0 \in V. \tag{2.20}$$

Theorem 2.4. *Assume that* (2.1)–(2.8) *and* (2.20) *are fulfilled. Then the unique solution (μ, φ, ξ) to problem* (2.13)–(2.16) *satisfies*

$$\mu \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V) \cap L^\infty(0,T;W), \tag{2.21}$$

and there exists a constant K_3, independent of α, such that

$$\alpha\|\mu\|_{W^{2,\infty}(0,T;H)} + \alpha^{1/2}\|\mu\|_{W^{1,\infty}(0,T;V)} + \|\mu\|_{L^\infty(0,T;W)} \le K_3\big(1 + \alpha^{-1/2}\big). \tag{2.22}$$

Notice that, due to the compactness of the embedding $W \subset C^0(\overline{\Omega})$, it follows from [28, Sect. 8, Cor. 4] that $\mu \in C^0(\overline{Q})$.

For the next regularity and continuous dependence result we have to assume further regularity for g, that is,

$$g \in L^\infty(Q), \tag{2.23}$$

and for the nonlinearity f. Namely, we suppose that the effective domain of ∂f_1 is an open interval and that the restriction of f_1 to this interval is a smooth function. More precisely, we assume that

$$\begin{aligned} &D(\partial f_1) = (r_-, r_+), \quad \text{with } -\infty \le r_- < 0 < r_+ \le +\infty, \\ &\text{and the restriction of } f_1 \text{ to } (r_-, r_+) \text{ belongs to } C^2(r_-, r_+). \end{aligned} \tag{2.24}$$

Then, for $r \in (r_-, r_+)$, the subdifferential $\partial f_1(r)$ reduces to the singleton $\{f_1'(r)\}$, and we require that

$$\lim_{r \searrow r_-} f_1'(r) = -\infty\,, \quad \lim_{r \nearrow r_+} f_1'(r) = +\infty. \tag{2.25}$$

Please note that both the potentials f_{reg} and $f_{\log}$ in (1.9) and (1.10) fulfill (2.24)–(2.25) with $(r_-, r_+) = \mathbb{R}$ and $(r_-, r_+) = (-1, 1),$ respectively.

The so-called separation property and a refined continuous dependence result are stated as follows.

Theorem 2.5. *Assume that* (2.1)–(2.8) *and* (2.20)–(2.25) *are fulfilled. There exists two real numbers* r_* *and* r^**, depending on* α *and on the structure of the system, such that*

$$r_- < r_* \le \varphi(x,t) \le r^* < r_+ \quad \textit{for every } \ (x,t) \in \overline{Q}. \tag{2.26}$$

Moreover, if for $i = 1, 2$ *we let* $(g_i, \mu_{0,i}, \nu_{0,i}, \varphi_{0,i})$ *be a set of data and* $(\mu_i, \varphi_i, \xi_i)$*, with* $\xi_i = f_1'(\varphi_i)$*, denote the corresponding solution to problem* (2.13)–(2.16)*, the estimate*

$$\begin{aligned} &\|\mu_1 - \mu_2\|_{H^2(0,T;V^*)\cap W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)} + \|\varphi_1 - \varphi_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V) L^2(0,T;W)} \\ &\le K_4 \Big(\|g_1 - g_2\|_{L^2(0,T;H)} + \|\mu_{0,1} - \mu_{0,2}\|_V + \|\nu_{0,1} - \nu_{0,2}\|_H + \|\varphi_{0,1} - \varphi_{0,2}\|_V \Big) \end{aligned} \tag{2.27}$$

holds true for some constant $K_4 > 0$ *that depends on* α *and on the structure of the system.*

Remark 2.6. Please note that in the case when the domain $D(\partial f_1)$ is the entire real line, i.e., if $r_- = -\infty$ and $r_+ = +\infty$, then the property (2.26) is direcly ensured by the estimate (2.18), since φ is bounded in $C^0(\overline{Q})$; therefore, if $D(\partial f_1) = \mathbb{R}$, then the additional regularity assumptions (2.20) and (2.23) are not needed to prove (2.26).

Remark 2.7. It would be interesting to investigate the system (1.1)–(1.4) from the viewpoint of an optimal control problem, with the distributed control located in the source term g in equation (1.2). Thus, in order to discuss differentiability properties and optimality conditions, it could be important to deal with smoother data and a smooth nonlinearity f, and with the control g lying in a control box in $L^\infty(Q)$ (cf. (2.23)). In this framework, stronger stability and continuous depencence estimates can possibly be derived for the system. However, the estimates (2.18), (2.22) and (2.27) are already a good starting point in that direction.

3 Existence of solutions

In this section, we are going to prove the existence result for the problem (2.13)–(2.16), by constructing a solution (μ, φ, ξ) that satisfies (2.10)–(2.12). We adopt two levels of approximation: at first, we replace the subdifferential ∂f_1 in (2.15) by the derivative of the Moreau–Yosida regularization $f_{1,\varepsilon}$ of f_1, depending on a parameter $\varepsilon \in (0,1)$; then, we apply a Faedo–Galerkin scheme to the resulting approximate system.

To begin with, we consider for every $\varepsilon \in (0,1)$ the Moreau–Yosida regularization $f_{1,\varepsilon}$ of f_1, that is (see, e.g., [1, 4]),

$$f_{1,\varepsilon}(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon} |r-s|^2 + f_1(s) \right\} = \frac{1}{2\varepsilon} \big| r - J_\varepsilon(r) \big|^2 + f_1(J_\varepsilon(r)) = \int_0^r f_{1,\varepsilon}'(s) ds,$$

where $f'_{1,\varepsilon} : \mathbb{R} \to \mathbb{R}$ and the associated resolvent operator J_ε are given by

$$f'_{1,\varepsilon}(r) := \frac{1}{\varepsilon}(r - J_\varepsilon(r)), \quad J_\varepsilon(r) := (I + \varepsilon\partial f_1)^{-1}(r), \quad \text{for all } r \in \mathbb{R},$$

with I denoting the identity operator. Note that the derivative $f'_{1,\varepsilon}$ turns out to be a regularization of the graph ∂f_1. Indeed, $f'_{1,\varepsilon}$ and $f_{1,\varepsilon}$ fullill, for all $0 < \varepsilon < 1$ (see, e.g., [4, pp. 28 and 39]),

$$f'_{1,\varepsilon} : \mathbb{R} \to \mathbb{R} \text{ is monotone and Lipschitz continuous}$$
$$\text{with Lipschitz constant } 1/\varepsilon, \text{ and it holds } f'_{1,\varepsilon}(0) = 0, \tag{3.1}$$
$$|f'_{1,\varepsilon}(r)| \le |\partial f_1^\circ(r)| \quad \text{for every } r \in D(\partial f_1), \tag{3.2}$$
$$0 \le f_{1,\varepsilon}(r) \le f_1(r) \quad \text{for every } r \in \mathbb{R}. \tag{3.3}$$

As for the second approximation, we employ a Faedo–Galerkin discrete scheme using a special basis. To this end, we take the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ of the eigenvalue problem

$$-\Delta v = \lambda v \quad \text{in } \Omega, \qquad \partial_{\mathbf{n}} v = 0 \quad \text{on } \partial\Omega,$$

and let $\{e_j\}_{j\in\mathbb{N}} \subset W$ be the associated eigenfunctions, normalized by $\|e_j\|_H = 1$, $j \in \mathbb{N}$. Then, we have that

$$0 = \lambda_1 < \lambda_2 \le \ldots, \qquad \lim_{j\to\infty} \lambda_j = +\infty,$$
$$\int_\Omega e_j e_k = \int_\Omega \nabla e_j \cdot \nabla e_k = 0 \quad \text{for } j \ne k,$$

and we note that e_1 is just the constant function $|\Omega|^{-1/2}$. We then define the n-dimensional spaces $V_n := \operatorname{span}\{e_1, \ldots, e_n\}$ for $n \in \mathbb{N}$, where V_1 is just the space of constant functions on Ω. It is well known that the union of these spaces is dense in both H and V.

The approximating n-dimensional problem is stated as follows: find functions

$$\mu_n(x,t) = \sum_{j=1}^n \mu_{nj}(t) e_j(x), \quad \varphi_n(x,t) = \sum_{j=1}^n \varphi_{nj}(t) e_j(x), \tag{3.4}$$

such that

$$\alpha(\partial_{tt}\mu_n(t), v) + (\partial_t\varphi_n(t), v) + \int_\Omega \nabla\mu_n(t) \cdot \nabla v = 0$$
$$\text{for all } t \in [0,T] \text{ and every } v \in V_n, \tag{3.5}$$
$$\tau(\partial_t\varphi_n(t), v) + \int_\Omega \nabla\varphi_n(t) \cdot \nabla v + (f'_{1,\varepsilon}(\varphi_n(t)) + f'_2(\varphi_n(t)), v) = (\mu_n(t) + g(t), v)$$
$$\text{for all } t \in [0,T] \text{ and every } v \in V_n, \tag{3.6}$$
$$\mu_n(0) = P_n(\mu_0)\,, \quad (\partial_t\mu_n)(0) = P_n(\nu_0)\,, \quad \varphi_n(0) = P_n(\varphi_0) \quad \text{a.e. in } \Omega\,, \tag{3.7}$$

where P_n denotes the H-orthogonal projection onto V_n. Then $P_n(v) = \sum_{j=1}^n (v, e_j) e_j$ for every $v \in H$, and we have (see, e.g., [10, formula (3.14)])

$$\|P_n(v)\|_Y \le C_\Omega \|v\|_Y \quad \text{for every } v \in Y, \text{ where } Y \in \{H, V, W\}, \tag{3.8}$$

for some constant $C_\Omega > 0$ depending only on Ω. By comparing (3.5)–(3.7) with (2.13)–(2.16), note that the inclusion (2.15) present in (2.13)–(2.16) is not reproduced in (3.5)–(3.7), since the role of the ξ variable is now played by $f'_{1,\varepsilon}(\varphi_n)$, written as it is, in (3.6).

Next, we take $v = e_k$ in all of the equations (3.5)–(3.7), for $k = 1, \ldots, n$, obtaining the system

$$\alpha \frac{d^2}{dt^2}\mu_{nk} + \frac{d}{dt}\varphi_{nk} + \lambda_k\, \mu_{nk} = 0 \quad \text{in } (0,T), \tag{3.9}$$

$$\tau \frac{d}{dt}\varphi_{nk} + \lambda_k\, \varphi_{nk} + (f'_{1,\varepsilon}(\varphi_n) + f_2(\varphi_n), e_k) = \mu_{nk} + (g, e_k) \quad \text{in } (0,T), \tag{3.10}$$

$$\mu_{nk}(0) = (\mu_0, e_k), \quad \frac{d}{dt}\mu_{nk}(0) = (\nu_0, e_k), \quad \varphi_{nk}(0) = (\varphi_0, e_k). \tag{3.11}$$

Then we have to deal with a Cauchy problem for a system of ordinary differential equations, which is of second order in the variables μ_{nk} and of first order in the variables φ_{nk}. This system is set in explicit form and offers Lipschitz continuous nonlinearities and source terms (g, e_k) in $H^1(0,T)$ (cf. (2.5)). By Carathéodory's theorem, the Cauchy problem (3.9)–(3.11) has a unique solution expressed by $\mu_{nk},\ \varphi_{nk}$, with $\mu_{nk} \in H^3(0,T)$ and $\varphi_{nk} \in H^2(0,T)$, for $k = 1, \ldots, n$. On account of (3.9)–(3.10) and (3.4), this solution uniquely determines a pair $(\mu_n, \varphi_n) \in H^3(0,T;V_n) \times H^2(0,T;V_n)$ that solves (3.5)–(3.7).

We now derive a series of a priori estimates for the finite-dimensional approximations. In the following, $C > 0$ denotes constants that may depend on the data of the state system, but are independent of $n \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $\alpha \in (0,1]$.

First estimate. The aim is taking the time derivative of (3.6) and then testing by $\partial_t\varphi_n$. In addition, we add the resulting equality to (3.5) where we choose $v = \partial_t\mu_n$. By this approach we obtain the cancellation of two terms. Next, we integrate with respect to time and, in order to recover our estimate, we have to control the H-norm of the initial value $\partial_t\varphi_n(0)$. Taking $t = 0$ and $v = \partial_t\varphi_n(0)$ in (3.6), by (3.7) we easily infer that

$$\begin{aligned}\tau\|\partial_t\varphi_n(0)\|_H^2 &= (\Delta\varphi_n(0) - f'_{1,\varepsilon}(\varphi_n(0)) - f'_2(\varphi_n(0)) + \mu_n(0) + g(0), \partial_t\varphi_n(0)) \\ &\le \frac{\tau}{2}\|\partial_t\varphi_n(0)\|_H^2 + \frac{1}{2\tau}\|\Delta P_n(\varphi_0) - f'_{1,\varepsilon}(P_n(\varphi_0)) - f'_2(P_n(\varphi_0)) + P_n(\mu_0) + g(0)\|_H^2,\end{aligned}$$

thanks to the Schwarz and Young inequalities. By virtue of the assumptions (2.5)–(2.7) on g and the initial data, and of the property (3.8), it turns out that there is a constant C_ε, depending only on the data and on ε, such that

$$\tau\|\partial_t\varphi_n(0)\|_H \le \|\Delta P_n(\varphi_0) - f'_{1,\varepsilon}(P_n(\varphi_0)) - f'_2(P_n(\varphi_0)) + P_n(\mu_0) + g(0)\|_H \le C_\varepsilon\,. \tag{3.12}$$

The dependence on ε follows from the Lipschitz continuity of $f'_{1,\varepsilon}$ with constant $1/\varepsilon$ (cf. (3.1)), while f'_2 is Lipschitz continuous independently of ε (see (2.2)).

Now, we can perform the computation described above and deduce that

$$\begin{aligned}
&\frac{\alpha}{2}\|\partial_t\mu_n(t)\|_H^2 + \frac{1}{2}\|\nabla\mu_n(t)\|_H^2 + \frac{\tau}{2}\|\partial_t\varphi_n(t)\|_H^2 \\
&\quad + \iint_{Q_t} |\nabla(\partial_t\varphi_n)|^2 + \iint_{Q_t} f_{1,\varepsilon}''(\varphi_n)|\partial_t\varphi_n|^2 \\
&\le \frac{\alpha}{2}\|P_n(\nu_0)\|_H^2 + \frac{1}{2}\|\nabla P_n(\mu_0)\|_H^2 \\
&\quad + \frac{1}{2\tau}\|\Delta P_n(\varphi_0) - f_{1,\varepsilon}'(P_n(\varphi_0)) - f_2'(P_n(\varphi_0)) + P_n(\mu_0) + g(0)\|_H^2 \\
&\quad - \iint_{Q_t} f_2''(\varphi_n))|\partial_t\varphi_n|^2 + \iint_{Q_t} \partial_t g\, \partial_t\varphi_n,
\end{aligned} \tag{3.13}$$

where we have used the notation

$$Q_t := \Omega \times (0,t), \quad t \in (0,T].$$

By the monotonicity of $f_{1,\varepsilon}'$, the last term on the left-hand side of (3.13) is nonnegative. Moreover, owing to (2.2), we have that $|f_2''(\varphi_n)||\partial_t\varphi_n|^2 \le C|\partial_t\varphi_n|^2$ a.e. in Q_t. Then, in view of (2.5), (2.6), (3.8), Young's inequality, and Gronwall's lemma, it is straightforward to infer from (3.13) that

$$\begin{aligned}
&\alpha^{1/2}\|\partial_t\mu_n\|_{L^\infty(0,T;H)} + \|\nabla\mu_n\|_{L^\infty(0,T;H)} + \|\partial_t\varphi_n\|_{L^\infty(0,T;H)} + \|\nabla(\partial_t\varphi_n)\|_{L^2(0,T;H)} \\
&\le C\big(\|\nu_0\|_H + \|\mu_0\|_V + \|\partial_t g\|_{L^2(0,T;H)} \\
&\qquad + \|\Delta P_n(\varphi_0) - f_{1,\varepsilon}'(P_n(\varphi_0)) - f_2'(P_n(\varphi_0)) + P_n(\mu_0) + g(0)\|_H\big),
\end{aligned} \tag{3.14}$$

for some constant C depending only on data, as $0 < \alpha \le 1$. Therefore, recalling the initial conditions in (3.7), since

$$\mu_n(t) = P_n(\mu_0) + \int_0^t \partial_t\mu_n(s)ds, \quad \varphi_n(t) = P_n(\varphi_0) + \int_0^t \partial_t\varphi_n(s)ds \quad \text{for all } t \in [0,T],$$

we easily conclude from (3.12) that

$$\alpha^{1/2}\|\mu_n\|_{W^{1,\infty}(0,T;H)} + \|\nabla\mu_n\|_{L^\infty(0,T;H)} + \|\varphi_n\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le C_\varepsilon\,. \tag{3.15}$$

Complementary estimates. Taking now $v \in V$, and using (3.5) and (3.8), we have that

$$\begin{aligned}
&\alpha\langle\partial_{tt}\mu_n(t), v\rangle = \alpha(\partial_{tt}\mu_n(t), P_n(v)) + \alpha(\partial_{tt}\mu_n(t), v - P_n(v)) \\
&\le |\alpha(\partial_{tt}\mu_n(t), P_n(v))| \le \left|(\partial_t\varphi_n(t), P_n(v)) + \int_\Omega \nabla\mu_n(t)\cdot\nabla P_n(v)\right| \\
&\le C\big(\|\partial_t\varphi_n\|_{L^\infty(0,T;H)} + \|\nabla\mu_n\|_{L^\infty(0,T;H)}\big)\|v\|_V \quad \text{for a.e. } t \in (0,T),
\end{aligned} \tag{3.16}$$

so that from (3.15) it clearly follows that

$$\alpha\|\partial_{tt}\mu_n\|_{L^\infty(0,T;V^*)} \le C_\varepsilon\,. \tag{3.17}$$

In addition, we can take $v = -\Delta(\varphi_n(t))$ in (3.6) and integrate by parts in some term. With the help of Young's inequality and the Lipschitz continuity of f_2', we obtain

$$\begin{aligned}
&\|\Delta\varphi_n(t)\|_H^2 + \int_\Omega f''_{1,\varepsilon}(\varphi_n(t))|\nabla\varphi_n(t)|^2 \\
&= (\tau\partial_t\varphi_n(t) + f_2'(\varphi_n(t)) - g(t), \Delta\varphi_n(t)) + \int_\Omega \nabla\mu_n(t)\cdot\nabla\varphi_n(t) \\
&\leq \frac{1}{2}\|\Delta\varphi_n(t)\|_H^2 + C\big(1 + \|\varphi_n\|^2_{W^{1,\infty}(0,T;H)} + \|g\|^2_{L^\infty(0,T;H)}\big) \\
&\quad + \|\nabla\mu_n\|_{L^\infty(0,T;H)}\|\nabla\varphi_n\|_{L^\infty(0,T;H)} \quad \text{for a.e. } t\in(0,T), \qquad (3.18)
\end{aligned}$$

where the second term in the first line is nonnegative due to (3.1). Consequently, from (3.15) and the elliptic regularity theory, we find that

$$\|\Delta\varphi_n\|_{L^\infty(0,T;H)} + \|\varphi_n\|_{L^\infty(0,T;W)} \leq C_\varepsilon\,. \qquad (3.19)$$

Passage to the limit in the Faedo–Galerkin scheme. By virtue of the uniform estimates shown above, there exists a pair $(\mu_\varepsilon, \varphi_\varepsilon)$ such that (possibly on a subsequence, which is again labeled by $n\in\mathbb{N}$)

$$\mu_n \to \mu_\varepsilon \quad \text{weakly star in } W^{2,\infty}(0,T;V^*)\cap W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V), \qquad (3.20)$$

$$\varphi_n \to \varphi_\varepsilon \quad \text{weakly star in } W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W). \qquad (3.21)$$

By (3.20) and the compact embeddings $V\subset H\subset V^*$, it follows from [28, Sect. 8, Cor. 4] that

$$\mu_n \to \mu_\varepsilon \quad \text{strongly in } C^1([0,T];V^*)\cap C^0([0,T];H)\,. \qquad (3.22)$$

Moreover, owing to the compactness of the embeddings $W\subset C^0(\overline{\Omega})$ and $W\subset V$, it turns out that

$$\varphi_n \to \varphi_\varepsilon \quad \text{strongly in } C^0(\overline{Q})\cap C^0([0,T];V)\,, \qquad (3.23)$$

whence, by the Lipschitz continuity of $f'_{1,\varepsilon}$ and f_2', we deduce that

$$f'_{1,\varepsilon}(\varphi_n) + f_2'(\varphi_n) \to f'_{1,\varepsilon}(\varphi_\varepsilon) + f_2'(\varphi_\varepsilon) \quad \text{strongly in } C^0(\overline{Q})\cap C^0([0,T];H)\,, \qquad (3.24)$$

at least. Then, taking first $v\in V_k$ with $k\leq n$ in (3.5)–(3.6), and passing to the limit as $n\to\infty$, it is not difficult to infer that

$$\alpha\langle\partial_{tt}\mu_\varepsilon, v\rangle + (\partial_t\varphi_\varepsilon, v) + \int_\Omega \nabla\mu_\varepsilon\cdot\nabla v = 0 \quad \text{a.e. in } (0,T), \qquad (3.25)$$

$$\tau(\partial_t\varphi_\varepsilon, v) + \int_\Omega \nabla\varphi_\varepsilon\cdot\nabla v + (f'_{1,\varepsilon}(\varphi_\varepsilon) + f_2'(\varphi_\varepsilon), v) = (\mu_\varepsilon + g, v) \quad \text{a.e. in } (0,T), \qquad (3.26)$$

at first for all $v\in\cup_{k\in\mathbb{N}}V_k$, and then, by density, for all $v\in V$. Note that, due to the regularity of φ_ε, the second variational equality can be equivalently rewritten as

$$\tau\partial_t\varphi_\varepsilon - \Delta\varphi_\varepsilon + f'_{1,\varepsilon}(\varphi_\varepsilon) + f_2'(\varphi_\varepsilon) = \mu_\varepsilon + g \quad \text{a.e. in } Q, \qquad (3.27)$$

where it is understood that φ_ε satisfies the boundary condition $\partial_{\mathbf{n}}\varphi_\varepsilon = 0$ a.e. on Σ, on account of $\varphi_\varepsilon\in L^\infty(0,T;W)$.

Thanks to (3.22) and (3.23), we can pass to the limit as $n \to \infty$ also in the initial conditions (3.7) and find that

$$\mu_\varepsilon(0) = \mu_0\,, \quad (\partial_t\mu_\varepsilon)(0) = \nu_0\,, \quad \varphi_\varepsilon(0) = \varphi_0\,, \quad \text{a.e. in } \Omega, \tag{3.28}$$

since (see (2.6)–(2.7))

$$P_n(\mu_0) \to \mu_0, \quad P_n(\nu_0) \to \nu_0, \quad P_n(\varphi_0) \to \varphi_0, \quad \text{strongly in } V,\ H,\ W, \ \text{respectively,}$$

and $\partial_t\mu_\varepsilon$ is weakly continuous from $[0,T]$ to H. Also, we can invoke the weak star lower semicontinuity of norms and pass to the limit in (3.14) to derive the inequality

$$\begin{aligned}
&\alpha^{1/2}\|\partial_t\mu_\varepsilon\|_{L^\infty(0,T;H)} + \|\nabla\mu_\varepsilon\|_{L^\infty(0,T;H)} + \|\partial_t\varphi_\varepsilon\|_{L^\infty(0,T;H)} + \|\nabla(\partial_t\varphi_\varepsilon)\|_{L^2(0,T;H)}\\
&\quad\le C\big(\|\nu_0\|_H + \|\mu_0\|_V + \|\partial_t g\|_{L^2(0,T;H)}\\
&\qquad + \|\Delta\varphi_0 - f'_{1,\varepsilon}(\varphi_0) - f'_2(\varphi_0) + \mu_0 + g(0)\|_H\big).
\end{aligned} \tag{3.29}$$

Hence, recalling (2.7) and (3.2), it turns out especially that $\|f'_{1,\varepsilon}(\varphi_0)\|_H$ is bounded independently of ε, which implies that the complete right-hand side of (3.29) is uniformly bounded. Then, arguing as before, we can improve (3.15) for $(\mu_\varepsilon,\varphi_\varepsilon)$ and recover that

$$\alpha^{1/2}\|\mu_\varepsilon\|_{W^{1,\infty}(0,T;H)} + \|\nabla\mu_\varepsilon\|_{L^\infty(0,T;H)} + \|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le C\,. \tag{3.30}$$

As a consequence, by repeating the arguments in (3.16) and (3.18) for μ_ε and φ_ε, we easily find out that

$$\alpha\|\partial_{tt}\mu_\varepsilon\|_{L^\infty(0,T;V^*)} + \|\Delta\varphi_\varepsilon\|_{L^\infty(0,T;H)} + \|\varphi_\varepsilon\|_{L^\infty(0,T;W)} \le C \tag{3.31}$$

for some constant C which is independent of both $\varepsilon \in (0,1)$ and $\alpha \in (0,1]$.

Further estimates. We insert the constant function $v = 1/|\Omega|$ in (3.25) and deduce that $\overline{\partial_t(\alpha\partial_t\mu_\varepsilon + \varphi_\varepsilon)} = 0$ a.e. in $(0,T)$. Hence, in view of (3.28), (2.8) and (2.9) it is straightforward to obtain

$$\alpha\,\overline{\partial_t\mu_\varepsilon}(t) + \overline{\varphi_\varepsilon}(t) = \alpha\,\overline{\nu_0} + m_0 \quad \text{for all } t \in [0,T]. \tag{3.32}$$

Now, we take $v = \varphi_\varepsilon(t) - m_0$ in (3.26) and, without integrating with respect to time, we have that

$$\begin{aligned}
&\int_\Omega |\nabla(\varphi_\varepsilon(t) - m_0)|^2 + (f'_{1,\varepsilon}(\varphi_\varepsilon(t)), \varphi_\varepsilon(t) - m_0)\\
&= -(\tau\partial_t\varphi_\varepsilon(t) + f'_2(\varphi_\varepsilon(t)) - g(t), \varphi_\varepsilon(t) - m_0) + (\mu_\varepsilon(t), \varphi_\varepsilon(t) - m_0).
\end{aligned} \tag{3.33}$$

Now, in view of the properties (2.2)–(2.4) of f_1 and ∂f_1, and on account of (2.8), it turns out that there exist two positive constants δ_0 and C_0, independent of ε, such that

$$f'_{1,\varepsilon}(r)(r - m_0) \ge \delta_0|f'_{1,\varepsilon}(r)| - C_0 \quad \text{for every } r \in \mathbb{R}. \tag{3.34}$$

For this property we refer to [25, Appendix, Prop. A.1] and also to the detailed proof given in [19, p. 908]. Applying (3.34) to the second term in the left-hand side of (3.33), due to (2.5) on g and to the Lipschitz continuity of f'_2, we infer that

$$\delta_0|f'_{1,\varepsilon}(\varphi_\varepsilon(t))| \le C\big(1 + \|\varphi_\varepsilon\|^2_{W^{1,\infty}(0,T;H)}\big) + (\mu_\varepsilon(t), \varphi_\varepsilon(t) - m_0). \tag{3.35}$$

As for the last term in (3.35), thanks to (3.32) and the Poincaré–Wirtinger inequality, we can argue as follows:

$$
\begin{aligned}
&(\mu_\varepsilon(t), \varphi_\varepsilon(t) - m_0) \\
&= (\mu_\varepsilon(t), \alpha\,\partial_t\mu_\varepsilon(t) + \varphi_\varepsilon(t) - \alpha\,\overline{\nu_0} - m_0) + (\mu_\varepsilon(t), \alpha\,\overline{\nu_0} - \alpha\,\partial_t\mu_\varepsilon(t)) \\
&= (\mu_\varepsilon(t) - \overline{\mu_\varepsilon}(t), \alpha\,\partial_t\mu_\varepsilon(t) + \varphi_\varepsilon(t) - \alpha\,\overline{\nu_0} - m_0) + \alpha(\mu_\varepsilon(t), \overline{\nu_0} - \,\partial_t\mu_\varepsilon(t)) \\
&\le C\|\nabla\mu_\varepsilon\|_{L^\infty(0,T;H)}\big(\alpha^{1/2}\|\partial_t\mu_\varepsilon\|_{L^\infty(0,T;H)} + \|\varphi_\varepsilon\|_{L^\infty(0,T;H)} + 1\big) \\
&\quad + C\big(\alpha\|\mu_\varepsilon\|^2_{W^{1,\infty}(0,T;H)} + 1\big)
\end{aligned}
\tag{3.36}
$$

since $0 < \alpha \le 1$. Hence, by (3.35), (3.36), and (3.30), we can conclude that

$$
\|f'_{1,\varepsilon}(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \;\le\; C\,. \tag{3.37}
$$

Next, we choose the constant function $v = 1$ in (3.26). We obtain, for a.e. $t \in (0,T)$,

$$
\int_\Omega f'_{1,\varepsilon}(\varphi_\varepsilon(t)) + \int_\Omega (\tau\partial_t\varphi_\varepsilon + f'_2(\varphi_\varepsilon(t)) - g(t)) \;=\; |\Omega|\,\overline{\mu_\varepsilon}(t). \tag{3.38}
$$

Owing to (3.37) and (3.30), both summands on the left-hand side are bounded in $L^\infty(0,T)$. Then we infer that

$$
\|\overline{\mu_\varepsilon}\|_{L^\infty(0,T)} \le C. \tag{3.39}
$$

Moreover, using again the Poincarè–Wirtinger inequality, we have that

$$
\begin{aligned}
\|\mu_\varepsilon\|_{L^\infty(0,T;H)} &\le \|\mu_\varepsilon - \overline{\mu_\varepsilon}\|_{L^\infty(0,T;H)} + C\|\overline{\mu_\varepsilon}\|_{L^\infty(0,T)} \\
&\le C\|\nabla\mu_\varepsilon\|_{L^\infty(0,T;H)} + C \le C.
\end{aligned}
\tag{3.40}
$$

Now we can go back to (3.26) or, better, to (3.27) and compare the terms in the equation: from (3.30), (3.31) and (2.5) it follows that $\partial_t\varphi_\varepsilon,\ \Delta\varphi_\varepsilon,\ f'_2(\varphi_\varepsilon),\ \mu_\varepsilon,\ g$ are all uniformly bounded in $L^\infty(0,T;H)$, whence

$$
\|f'_{1,\varepsilon}(\varphi_\varepsilon)\|_{L^\infty(0,T;H)} \;\le\; C\,. \tag{3.41}
$$

Passage to the limit as $\varepsilon \to 0$. Thanks to the uniform estimates (3.30), (3.31), (3.40), and (3.41), it follows that there is a triple (μ, φ, ξ) such that, for some subsequence ε_k tending to 0, it holds

$$
\mu_{\varepsilon_k} \to \mu \quad \text{weakly star in } W^{2,\infty}(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{3.42}
$$

$$
\varphi_{\varepsilon_k} \to \varphi \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{3.43}
$$

$$
f'_{1,\varepsilon_k}(\varphi_{\varepsilon_k}) \to \xi \quad \text{weakly star in } L^\infty(0,T;H). \tag{3.44}
$$

As argued in the previous limit procedure (cf. (3.22)–(3.24)), by compactness, in particular exploiting [28, Sect. 8, Cor. 4], and by the Lipschitz continuity of f'_2, we have that

$$
\mu_{\varepsilon_k} \to \mu \quad \text{strongly in } C^1([0,T];V^*) \cap C^0([0,T];H)\,, \tag{3.45}
$$

$$
\varphi_{\varepsilon_k} \to \varphi \quad \text{strongly in } C^0(\overline{Q}) \cap C^0([0,T];V)\,, \tag{3.46}
$$

$$
f'_2(\varphi_{\varepsilon_k}) \to f'_2(\varphi) \quad \text{strongly in } C^0(\overline{Q}) \cap C^0([0,T];H)\,. \tag{3.47}
$$

Then we can pass to the limit as $\varepsilon_k \to 0$ in (3.25) and (3.26) by finding (2.13) and (2.14), respectively. Moreover, the initial conditions (2.16) follow from (3.28). It remains to check (2.15): but, since the extension of ∂f_1 to $L^2(0,T;H)$ is a maximal monotone operator and $f'_{1,\varepsilon}$ denotes its Yosida approximation, and since we have that

$$\limsup_{k,n\to\infty} \int_0^T (f'_{1,\varepsilon_k}(\varphi_{\varepsilon_k}(t)) - f'_{1,\varepsilon_n}(\varphi_{\varepsilon_n}(t)), \varphi_{\varepsilon_k}(t) - \varphi_{\varepsilon_n}(t))dt = 0$$

due to the weak convergence of $f'_{1,\varepsilon_k}(\varphi_{\varepsilon_k})$ to ξ in $L^2(0,T;H)$ and the strong convergence of φ_{ε_k} to φ in $L^2(0,T;H)$, we can apply [1, Prop. 2.2, p. 38] and recover the inclusion $\xi \in \partial f_1(\varphi)$ in $L^2(0,T;H)$ and almost everywhere in Q.

In conclusion, we note that the triple (μ,φ,ξ) found by the limit procedure is actually the unique solution of the problem (2.13)–(2.16), on account of the continuous dependence result, and that the estimate (2.18) follows easily from the uniform bounds in (3.30), (3.31), (3.40), (3.41) and the weak star lower semicontinuity of norms. Therefore, Theorem 2.2 is completely proved. □

4 Continuous dependence and regularity

In this section, we show report the proofs of the continuous dependence and regularity results.

Proof of Theorem 2.3. We just have to prove the inequality (2.19) by letting (μ_i,φ_i,ξ_i) be any solution of problem (2.13)–(2.16) with the corresponding data $g_i, \mu_{0,i}, \nu_{0,i}, \varphi_{0,i}$, satisfying (2.5)–(2.8) for $i=1,2$. For convenience, within this proof we set

$$g = g_1 - g_2, \quad \mu_0 = \mu_{0,1} - \mu_{0,2}, \quad \nu_0 = \nu_{0,1} - \nu_{0,2}, \quad \varphi_0 = \varphi_{0,1} - \varphi_{0,2},$$

as well as

$$\mu = \mu_1 - \mu_2\,, \quad \varphi = \varphi_1 - \varphi_2\,, \quad \xi = \xi_1 - \xi_2.$$

Then, taking the differences of the respective equalities (2.13)–(2.14) and integrating the one resulting from (2.13) with respect to time, we obtain

$$(\alpha\partial_t\mu(t) + \varphi(t), v) + \int_\Omega \nabla(1*\mu)(t)\cdot\nabla v = (\alpha\nu_0 + \varphi_0, v)$$
$$\text{for a.e. } t\in(0,T) \text{ and every } v\in V, \tag{4.1}$$

$$\tau(\partial_t\varphi(t), v) + \int_\Omega \nabla\varphi(t)\cdot\nabla v + (\xi(t), v) = (\mu(t) + g(t) - f'_2(\varphi_1(t)) + f'_2(\varphi_2(t)), v)$$
$$\text{for a.e. } t\in(0,T) \text{ and every } v\in V, \tag{4.2}$$

$$\xi_i \in \partial f_1(\varphi_i),\ i=1,2, \quad \text{a.e. in } Q, \tag{4.3}$$

$$\mu(0) = \mu_0\,, \quad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega\,. \tag{4.4}$$

Next, we take $v=\mu(t)$ in (4.1) and $v=\varphi(t)$ in (4.2), then we add them noting a cancellation of terms and integrate once more with respect to t. With the help of (4.4), we infer that

$$\begin{aligned}
&\frac{\alpha}{2}\|\mu(t)\|_H^2 + \frac12\|\nabla(1*\mu)(t)\|_H^2 + \frac{\tau}{2}\|\varphi(t)\|_H^2 + \iint_{Q_t}|\nabla\varphi|^2 + \iint_{Q_t}\xi\varphi \\
&\le \frac{\alpha}{2}\|\mu_0\|_H^2 + \frac{\tau}{2}\|\varphi_0\|_H^2 + \iint_{Q_t}(\alpha\nu_0+\varphi_0)\mu \\
&\quad + \iint_{Q_t} g\,\varphi - \iint_{Q_t}(f'_2(\varphi_1) - f'_2(\varphi_2))\varphi
\end{aligned} \tag{4.5}$$

for all $t \in (0,T]$. Now, thanks to (4.3) the last term on the left-hand side is nonnegative. On the right-hand side, by the Young inequality, we have that

$$\iint_{Q_t} (\alpha\nu_0 + \varphi_0)\mu \le \alpha \int_0^t \|\mu(s)\|_H^2 ds + C\big(\alpha\|\nu_0\|_H^2 + \alpha^{-1}\|\varphi_0\|_H^2\big),$$

and, using also the Lipschitz continuity of φ, it follows that

$$\iint_{Q_t} g\,\varphi - \iint_{Q_t} (f_2'(\varphi_1) - f_2'(\varphi_2))\varphi \le C \int_0^t \|\varphi(s)\|_H^2 ds + \frac{1}{2}\|g\|_{L^2(0,T;H)}^2.$$

Therefore, we can collect these inequalities and apply the Gronwall lemma to the resultant from (4.5) in order to plainly obtain the estimate (2.19). □

Proof of Theorem 2.4. We already know from (2.11) that $\partial_t\varphi$ is in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Then, in view of (2.20), the regularity in (2.21) follows from the variational theory for linear evolution problems of second order in time (see, e.g., [16]). Then, in order to reproduce the estimate in (2.22), let us proceed formally and test equation (2.13) by $-\Delta(\partial_t\mu(t))$. By this, we can easily integrate by parts and also with respect to time. With the help of Young's inequality we obtain

$$\begin{aligned}
&\frac{\alpha}{2}\int_\Omega |\nabla(\partial_t\mu(t))|^2 + \frac{1}{2}\int_\Omega |\Delta\mu(t)|^2 \\
&= \frac{\alpha}{2}\int_\Omega |\nabla\nu_0|^2 + \frac{1}{2}\int_\Omega |\Delta\mu_0|^2 - \iint_{Q_t} \nabla(\partial_t\varphi)\cdot\nabla(\partial_t\mu) \\
&\le \frac{\alpha}{2}\|\nu_0\|_V^2 + \frac{1}{2}\|\Delta\mu_0\|_H^2 + \frac{1}{2\alpha}\iint_{Q_t} |\nabla(\partial_t\varphi)|^2 + \frac{\alpha}{2}\iint_{Q_t} |\nabla(\partial_t\mu)|^2,
\end{aligned} \tag{4.6}$$

whence the estimate

$$\alpha^{1/2}\|\mu\|_{W^{1,\infty}(0,T;V)} + \|\mu\|_{L^\infty(0,T;W)} \le C\big(1 + \alpha^{-1/2}\big) \tag{4.7}$$

follows from an application of Gronwall's lemma, along with (2.20) and (2.18). Having shown (4.7), it is now straightforward to compare the terms in (2.13) and to deduce that $\alpha\|\partial_{tt}\varphi\|_{L^\infty(0,T;H)}$ is bounded by a quantity like the right-hand side of (4.7). Thus, we complete the proof of (2.22). □

Proof of Theorem 2.5. We first show (2.26). It is already known from (2.7) that the initial value of φ, i.e. φ_0, belongs to a compact subset of $D(\partial f_1) = (r_-, r_+)$. By the previous proof, we have checked that μ is bounded in $L^\infty(0,T;W)$, hence in $L^\infty(Q)$, as follows from the above estimate (4.7). Then, let us rewrite equation (2.17) as

$$\tau\partial_t\varphi - \Delta\varphi + f_1'(\varphi) = h, \quad \text{with } h = \mu + g - f_2'(\varphi), \quad \text{a.e. in } Q. \tag{4.8}$$

The term ξ in (2.17) has been expressed here as $f_1'(\varphi)$, as it is allowed by the assumption (2.24). Note that the right-hand side h of (4.8) is actually bounded in $L^\infty(Q)$, thanks to (2.23) and the bound for φ ensured by (2.18), along with the Lipschitz continuity of f_2'.

To prove (2.26), it is enough to derive an $L^\infty(Q)$-bound for $f_1'(\varphi)$. Let us outline the argument by proceeding formally and pointing out that just a truncation of the test functions would be needed for a rigorous proof. We take any $p > 2$ and test (4.8) by $|f_1'(\varphi)|^{p-2} f_1'(\varphi)$, a function of φ which is increasing and attains the value 0 at 0 (cf. (2.2)–(2.4)). Then, we integrate from 0 to $t \in (0,T]$, obtaining

$$\int_\Omega \Big(\int_0^{\varphi(t)} |f_1'(s)|^{p-2} f_1'(s) ds \Big) + (p-1) \iint_{Q_t} |f_1'(\varphi)|^{p-2} f_1''(\varphi) |\nabla\varphi|^2 + \iint_{Q_t} |f_1'(\varphi)|^p$$
$$= \int_\Omega \Big(\int_0^{\varphi_0} |f_1'(s)|^{p-2} f_1'(s) ds \Big) + \iint_{Q_t} h |f_1'(\varphi)|^{p-2} f_1'(\varphi) \,. \tag{4.9}$$

Note that the first term and the second term on the left-hand side are nonnegative, in particular, since the derivative f_1'' is nonnegative everywhere in (r_-, r_+). About the right-hand side we may observe that

$$\int_\Omega \Big(\int_0^{\varphi_0} |f_1'(s)|^{p-2} f_1'(s) ds \Big) \le \|f_1'(\varphi_0)\|_\infty^{p-1} \|\varphi_0\|_\infty |\Omega| \,,$$

and, with $p' = p/(p-1)$ and the help of the Young inequality, that

$$\iint_{Q_t} h |f_1'(\varphi)|^{p-2} f_1'(\varphi) \le \|h\|_{L^p(Q_t)} \, \| \, |f_1'(\varphi)|^{p-1} \|_{L^{p'}(Q_t)}$$
$$= \|h\|_{L^p(Q_t)} \, \|f_1'(\varphi)\|_{L^p(Q_t)}^{p/p'} \le \frac{1}{p} \, \|h\|_{L^p(Q_t)}^p + \frac{1}{p'} \, \|f_1'(\varphi)\|_{L^p(Q_t)}^p \,.$$

By rearranging from (4.9), and taking $t = T$, we infer that

$$\|f_1'(\varphi)\|_{L^p(Q)} \le \Big(p \|f_1'(\varphi_0)\|_\infty^{p-1} \|\varphi_0\|_\infty |\Omega| + \|h\|_{L^p(Q)}^p \Big)^{1/p}$$
$$\le \big(p \|f_1'(\varphi_0)\|_\infty^{p-1} \|\varphi_0\|_\infty |\Omega| \big)^{1/p} + \|h\|_{L^p(Q)}.$$

Then, letting p tend to $+\infty$, we conclude that

$$\|f_1'(\varphi)\|_{L^\infty(Q)} \le \|f_1'(\varphi_0)\|_\infty + \|h\|_{L^\infty(Q)},$$

which ensures the validity of (2.26), for some constants r_*, r^* as in the statement.

Next, we argue in order to prove the continuous dependence estimate (2.27). We use the same notation as in the proof of Theorem 2.3, so that

$$g = g_1 - g_2, \quad \mu_0 = \mu_{0,1} - \mu_{0,2}, \quad \nu_0 = \nu_{0,1} - \nu_{0,2}, \quad \varphi_0 = \varphi_{0,1} - \varphi_{0,2},$$

and

$$\mu = \mu_1 - \mu_2 \,, \quad \varphi = \varphi_1 - \varphi_2 \,,$$

where $(\mu_i, \varphi_i, \xi_i)$, with $\xi_i = f_1'(\varphi_i)$, is the solution to problem (2.13)–(2.16) corresponding to $g_i, \mu_{0,i}, \nu_{0,i}, \varphi_{0,i}$, $i = 1, 2$, these data satisfying (2.5)–(2.8). Then, taking the differences of the respective equalities (2.13)–(2.14), we obtain

$$\alpha \langle \partial_{tt}\mu(t), v \rangle + \int_\Omega \nabla\mu(t) \cdot \nabla v = -(\partial_t\varphi(t), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{4.10}$$

$$\tau(\partial_t\varphi(t), v) + \int_\Omega \nabla\varphi(t) \cdot \nabla v = (\mu(t) + g(t) - f'(\varphi_1(t)) + f'(\varphi_2(t)), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{4.11}$$

$$\mu(0) = \mu_0 \,, \quad (\partial_t\mu)(0) = \nu_0 \,, \quad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega \,, \tag{4.12}$$

where we have used $f' = f_1' + f_2'$ in equation (4.11) noting that the estimate (2.26) and the assumptions (2.24) on f_1 now allow us to take f' as a global Lipschitz continuous function (with Lipschitz constant depending on α). In view of (2.19), we have for the term on the right-hand side of (4.11) that

$$\begin{aligned}\|\mu + g - f'(\varphi_1) + f'(\varphi_2)\|_{L^2(0,T;H)} &\leq \|\mu\|_{L^2(0,T;H)} + \|g\|_{L^2(0,T;H)} + C_\alpha\|\varphi\|_{L^2(0,T;H)} \\ &\leq C_\alpha\Big(\|g\|_{L^2(0,T;H)} + \|\mu_0\|_H + \|\nu_0\|_H + \|\varphi_0\|_H\Big),\end{aligned} \tag{4.13}$$

where the constants are denoted by C_α since they depend on α as well. Then, using the standard parabolic regularity estimate (see, e.g., [24] or [16])

$$\begin{aligned}&\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)L^2(0,T;W)} \\ &\leq C\big(\|\mu + g - f'(\varphi_1) + f'(\varphi_2)\|_{L^2(0,T;H)} + \|\varphi_0\|_V\big)\end{aligned}$$

for the φ solution to (4.11) with the respective initial condition, it is straightforward to deduce that

$$\begin{aligned}&\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)L^2(0,T;W)} \\ &\leq C_\alpha\big(\|g\|_{L^2(0,T;H)} + \|\mu_0\|_H + \|\nu_0\|_H + \|\varphi_0\|_V\big).\end{aligned} \tag{4.14}$$

Next, we can choose $v = \partial_t\mu$ in (4.10), integrate with respect to time, and infer that

$$\begin{aligned}&\frac{\alpha}{2}\|\partial_t\mu(t)\|_H^2 + \frac{1}{2}\|\nabla\mu(t)\|_H^2 \\ \leq\ &\frac{\alpha}{2}\|\nu_0\|_H^2 + \frac{1}{2}\|\nabla\mu_0\|_H + \int_0^t \|\partial_t\varphi(s)\|_H\|\partial_t\mu(s)\|_H ds.\end{aligned} \tag{4.15}$$

Then, first applying Young's inequality to the last term and then Gronwall's lemma, we arrive at the estimate

$$\|\mu\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)} \leq C_\alpha\big(\|\mu_0\|_V + \|\nu_0\|_H + \|\partial_t\varphi\|_{L^2(0,T;H)}\big),$$

whence from (4.14) it is clear that

$$\|\mu\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)} \leq C_\alpha\big(\|g\|_{L^2(0,T;H)} + \|\mu_0\|_V + \|\nu_0\|_H + \|\varphi_0\|_V\big).$$

By this estimate, comparison of the terms in (4.10) yields

$$\begin{aligned}\alpha\|\partial_{tt}\mu\|_{L^2(0,T;V^*)} &\leq \|\nabla\mu\|_{L^2(0,T;H)} + \|\partial_t\varphi\|_{L^2(0,T;H)} \\ &\leq C_\alpha\big(\|g\|_{L^2(0,T;H)} + \|\mu_0\|_V + \|\nu_0\|_H + \|\varphi_0\|_V\big),\end{aligned} \tag{4.16}$$

so that (2.27) is completely proved. □

5 Asymptotic analysis

This section is devoted to the study of the asymptotic behavior of the problem (1.1)–(1.4) as α approaches 0. We allow the initial data for μ and $\partial_t\mu$, as well as the source term g, to depend on α, while we keep fixed φ_0, the initial value of φ, for reasons of simplicity in front of restrictions like (2.7) and (2.8) for φ_0.

Thus, for $0 < \alpha \leq 1$, we consider families of data $g_\alpha, \mu_{0,\alpha}, \nu_{0,\alpha}$ such that

$$\{g_\alpha\} \text{ is uniformly bounded in } H^1(0,T;H) \text{ and strongly converges to } g \text{ in } L^2(0,T;H) \text{ as } \alpha \searrow 0, \tag{5.1}$$

$$\{\mu_{0,\alpha}\} \text{ is uniformly bounded in } V, \tag{5.2}$$

$$\{\nu_{0,\alpha}\} \text{ is uniformly bounded in } H. \tag{5.3}$$

Of course, it follows from (5.1) that $g \in H^1(0,T;H)$ and $\partial_t g_\alpha \to \partial_t g$ weakly in $L^2(0,T;H)$. We can state the following convergence result.

Theorem 5.1. *Assume that* (2.1)–(2.4)*,* (2.7)–(2.8)*,* (5.1)–(5.3) *are fulfilled. For all $\alpha \in (0,1]$, let the triple $(\mu_\alpha, \varphi_\alpha, \xi_\alpha)$, with*

$$\mu_\alpha \in W^{2,\infty}(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{5.4}$$

$$\varphi_\alpha \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{5.5}$$

$$\xi_\alpha \in L^\infty(0,T;H), \tag{5.6}$$

be the solution to the initial value problem

$$\alpha\langle\partial_{tt}\mu_\alpha(t), v\rangle + (\partial_t\varphi_\alpha(t), v) + \int_\Omega \nabla\mu_\alpha(t)\cdot\nabla v = 0 \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{5.7}$$

$$\tau(\partial_t\varphi_\alpha(t), v) + \int_\Omega \nabla\varphi_\alpha(t)\cdot\nabla v + (\xi_\alpha(t) + f_2'(\varphi_\alpha(t)), v) = (\mu_\alpha(t) + g_\alpha(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{5.8}$$

$$\xi_\alpha \in \partial f_1(\varphi_\alpha) \quad \text{a.e. in } Q, \tag{5.9}$$

$$\mu_\alpha(0) = \mu_{0,\alpha}\,, \quad (\partial_t\mu_\alpha)(0) = \nu_{0,\alpha}\,, \quad \varphi_\alpha(0) = \varphi_0 \quad \text{a.e. in } \Omega\,. \tag{5.10}$$

Then there exists a triple (μ, φ, ξ) such that, for some subsequence α_k tending to 0, there holds

$$\mu_{\alpha_k} \to \mu \quad \text{weakly star in } L^\infty(0,T;V), \tag{5.11}$$

$$\alpha_k\mu_{\alpha_k} \to 0 \quad \text{weakly star in } W^{2,\infty}(0,T;V^*) \text{ and strongly in } W^{1,\infty}(0,T;H), \tag{5.12}$$

$$\varphi_{\alpha_k} \to \varphi \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \text{ and strongly in } C^0([0,T];V) \cap C^0(\overline{Q}), \tag{5.13}$$

$$\xi_{\alpha_k} \to \xi \quad \text{weakly star in } L^\infty(0,T;H). \tag{5.14}$$

Moreover, (μ, φ, ξ) is a solution to the viscous Cahn–Hilliard system

$$(\partial_t\varphi(t), v) + \int_\Omega \nabla\mu(t)\cdot\nabla v = 0 \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{5.15}$$

$$\tau(\partial_t\varphi(t), v) + \int_\Omega \nabla\varphi(t)\cdot\nabla v + (\xi(t) + f_2'(\varphi(t)), v) = (\mu(t) + g(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V, \tag{5.16}$$

$$\xi \in \partial f_1(\varphi) \quad \text{a.e. in } Q, \tag{5.17}$$

$$\varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega\,. \tag{5.18}$$

Proof. A closer inspection of the proof of Theorem 2.2 in Section 3 reveals that the estimate (2.18) still holds under the conditions (5.1)–(5.3). Then, by a standard weak star compactness argument, we deduce the existence of a subsequence $\alpha_k \searrow 0$ and a triple (μ, φ, ξ) such that (5.11)–(5.14) hold. In fact, the strong convergence property in (5.13) is a consequence of the compactness result reported in [28, Sect. 8, Cor. 4]. Moreover, by the Lipschitz continuity of f_2', we also have that

$$f_2'(\varphi_{\alpha_k}) \to f_2'(\varphi) \quad \text{strongly in } C^0(\overline{Q}) \cap C^0([0,T];H). \tag{5.19}$$

Then, we can pass to the limit in (5.7), (5.8), and the third condition in (5.10), all written for α_k, and easily obtain (5.15), (5.16), (5.18). Recovering (5.17) from (5.9) is straightforward, due to the weak convergence of ξ_{α_k} and the strong convergence of φ_{α_k} in $L^2(0,T;H)$, along with the maximal monotonicity of ∂f_1 (see, e.g., [1, Cor. 2.4, p. 41]). This concludes the proof. □

Remark 5.2. Note that Theorem 5.1 implicitly yields an existence result for solutions to the viscous Cahn–Hilliard system (5.15)–(5.18). The found solution is already a regular and strong solution: indeed, the component φ is in $W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)$ and therefore also in $C^0(\overline{Q})$, while from (5.11), a comparison in (5.15), and the elliptic regularity theory, it turns out that $\mu \in L^\infty(0,T;W) \cap L^2(0,T;H^3(\Omega))$ in addition, so that $\mu \in C^0(\overline{Q})$, in particular. Both the equalities (5.15) and (5.16) can be equivalently rewritten as the equations

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{5.20}$$

$$\tau\partial_t\varphi - \Delta\varphi + \xi + f_2'(\varphi) = \mu + g \quad \text{a.e. in } Q, \tag{5.21}$$

plus the homogeneous boundary conditions

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = 0 \quad \text{a.e. on } \Sigma. \tag{5.22}$$

The mentioned regularity for (μ, φ, ξ) is exactly the same as in [14, Thm. 2.2], where a slightly more general system is investigated. However, the existence of a less regular solution can also be proved, along with the uniqueness of the component φ of the solution, as it results for instance from [9, Thm. 2.5]. Please note that in general uniqueness cannot be expected for ξ and μ unless ∂f_1 is single-valued (like e.g. the case considered in (2.24)); otherwise, only the difference $\xi - \mu$ is uniquely determined from (5.21).

Remark 5.3. By the uniqueness property for the component φ, which is pointed out in the previous remark, we infer that not only a subsequence $\{\varphi_{\alpha_k}\}$ but the entire family $\{\varphi_\alpha\}_{\alpha\in(0,1]}$ converges to φ in the sense of (5.13) as $\alpha \searrow 0$.

The next result is devoted to an error estimate of the difference $\varphi_\alpha - \varphi$ in certain norms and in terms of the parameter α.

Theorem 5.4. *Under the same assumptions as in Theorem 5.1, we let $(\mu_\alpha, \varphi_\alpha, \xi_\alpha)$ denote the solution to* (5.7)–(5.10)*, for $\alpha \in (0,1]$, and (μ, φ, ξ) be the solution to* (5.15)–(5.18) *found by the asymptotic limit in* (5.11)–(5.14)*. Then there is a constant $K_5 > 0$, which depends on the structure of the system but is independent of α, such that*

$$\alpha^{1/2}\|\mu_a\|_{L^\infty(0,T;H)} + \|\nabla(1 * (\mu_\alpha - \mu))\|_{L^\infty(0,T;H)} + \|\varphi_\alpha - \varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$
$$\le K_5 \Big(\alpha^{1/4} + \|g_\alpha - g\|_{L^2(0,T;H)}\Big). \tag{5.23}$$

Proof. We argue similarly as in the proof of Theorem 2.3. We take the difference of (5.7) and (5.15), then we integrate with respect to time with the help of (5.10) and (5.18). We obtain

$$((\varphi_\alpha - \varphi)(t), v) + \int_\Omega \nabla(1 * (\mu_\alpha - \mu))(t) \cdot \nabla v = (\alpha\, \nu_{0,\alpha} - \alpha\, \partial_t \mu_\alpha(t), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in V. \tag{5.24}$$

At the same time, we subtract (5.16) from (5.8) and have that

$$\tau(\partial_t(\varphi_\alpha - \varphi)(t), v) + \int_\Omega \nabla(\varphi_\alpha - \varphi)(t) \cdot \nabla v + ((\xi_\alpha - \xi)(t), v)$$
$$= ((\mu_\alpha - \mu)(t) + (g_\alpha - g)(t) - f_2'(\varphi_\alpha(t)) + f_2'(\varphi(t)), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in V\,. \tag{5.25}$$

Then we take $v = (\mu_\alpha - \mu)(t)$ in (5.24) and $v = (\varphi_\alpha - \varphi)(t)$ in (5.25), sum up noting that a cancellation occurs, and integrate with respect to t. Since the product $(\xi_\alpha - \xi)(\varphi_\alpha - \varphi)$ is nonnegative due to (5.9), (5.17) and the monotonicity of ∂f_1, we easily derive the inequality

$$\frac{\alpha}{2}\|\mu_\alpha(t)\|_H^2 + \frac{1}{2}\|\nabla(1 * (\mu_\alpha - \mu))(t)\|_H^2 + \frac{\tau}{2}\|(\varphi_\alpha - \varphi)(t)\|_H^2 + \iint_{Q_t} |\nabla(\varphi_\alpha - \varphi)|^2$$
$$\leq \frac{\alpha}{2}\|\mu_{0,\alpha}\|_H^2 + \iint_{Q_t} \alpha\, \nu_{0,\alpha}(\mu_\alpha - \mu) + \iint_{Q_t} \alpha\, \partial_t \mu_\alpha\, \mu$$
$$+ \iint_{Q_t} (g_\alpha - g)(\varphi_\alpha - \varphi) - \iint_{Q_t} (f_2'(\varphi_\alpha) - f_2'(\varphi))(\varphi_\alpha - \varphi) \tag{5.26}$$

for all $t \in (0,T]$. Now, we recall the boundedness properties (5.2) and (5.3), the estimate (2.18) for $\|\mu_\alpha\|_{L^\infty(0,T;H)}$ and $\alpha^{1/2}\|\partial_t \mu_\alpha\|_{L^\infty(0,T;H)}$, as well as the regularity $\mu \in L^\infty(0,T;H)$, in order to deduce that

$$\frac{\alpha}{2}\|\mu_{0,\alpha}\|_H^2 + \iint_{Q_t} \alpha\, \nu_{0,\alpha}(\mu_\alpha - \mu) + \iint_{Q_t} \alpha\, \partial_t \mu_\alpha\, \mu$$
$$\leq \alpha\|\mu_{0,\alpha}\|_H^2 + \alpha\|\nu_{0,\alpha}\|_H + C\alpha^{1/2} \leq C\alpha^{1/2}.$$

In addition, by virtue of the Lipschitz continuity of f_2' and the Young inequality, we have that

$$\iint_{Q_t} (g_\alpha - g)(\varphi_\alpha - \varphi) - \iint_{Q_t} (f_2'(\varphi_\alpha) - f_2'(\varphi))(\varphi_\alpha - \varphi)$$
$$\leq \|g_\alpha - g\|_{L^2(0,T;H)}^2 + C \int_0^t \|(\varphi_\alpha - \varphi)(s)\|^2 ds.$$

Then, collecting the above computations in (5.26) and applying Gronwall's lemma, the estimate (5.23) follows. □

We finally notice that (5.23) gives an error estimate, in particular, for

$$\|\varphi_\alpha - \varphi\|_{L^\infty(0,T;H) \cap L^2(0,T;V)}$$

of order $1/4$, provided that the convergence of $\|g_\alpha - g\|_{L^2(0,T;H)}$ to 0 is at least of this order.

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