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Free boundary limits of coupled bulk-surface models for receptor-ligand interactions on evolving domains

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Abstract

We derive various novel free boundary problems as limits of a coupled bulk-surface reaction-diffusion system modelling ligand-receptor dynamics on evolving domains. These limiting free boundary problems may be formulated as Stefan-type problems on an evolving hypersurface. Our results are new even in the setting where there is no domain evolution. The models are of particular relevance to a number of applications in cell biology. The analysis utilises L^∞-estimates in the manner of De Giorgi iterations and other technical tools, all in an evolving setting. We also report on numerical simulations.

1 Introduction

In this paper we obtain a sequence of novel free boundary limits of a system of reaction-diffusion equations holding on time-evolving bulk-surface domains by sending parameters in the equations to zero. The limit problems have unknown moving boundaries on the given evolving surface of the evolving bulk domain. The final example of these problems that we derive can be written as a degenerate parabolic equation similar to a Stefan or Hele-Shaw problem holding on an evolving hypersurface associated with a Dirichlet-to-Neumann map. Indeed, it consists of a function $u \geq 0$ which is harmonic on the evolving domain $\Omega(t)$ coupled through its normal derivative and a monotone inclusion to a surface quantity $v \leq 0$ that satisfies a parabolic equation on a portion $\Gamma(t)$ of the boundary $\partial\Omega(t)$:

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega(t), \\ \partial^\bullet v + v\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) &= -\nabla u \cdot \nu && \text{on } \Gamma(t), \\ v &\in \beta(u) && \text{on } \Gamma(t). \end{aligned}$$

Here the graph β is such that the last line essentially encodes a *complementarity condition* of the form $uv = 0$ on $\Gamma(t)$. This type of problem is related to earlier works by two of the coauthors on degenerate equations posed on an evolving surface [1, 2]. Full details of all of these notations and concepts will, of course, be expounded upon in the course of the paper.

1.1 Mathematical setting

For each $t \in [0, T]$, let $D(t) \subset \mathbb{R}^{d+1}$ be a smooth domain containing a C^2-hypersurface $\Gamma(t)$ which separates $D(t) = I(t) \cup \Omega(t)$ into an interior region $I(t)$ and an exterior Lipschitz domain $\Omega(t)$ (see Figure 1). We suppose that the surface $\Gamma(t)$ and the outer boundary $\partial D(t)$ both evolve in time through prescribed kinematic normal velocity fields $\mathbf{V}$ and $\mathbf{V}_o$ respectively. Motivated by a model of ligand-receptor dynamics we consider a system of reaction advection-diffusion equations. Within the evolving bulk domain $\Omega(t)$ there is a concentration field u subject to the diffusive and advective flux

$$q_u := -\nabla u + u\mathbf{V}_\Omega$$

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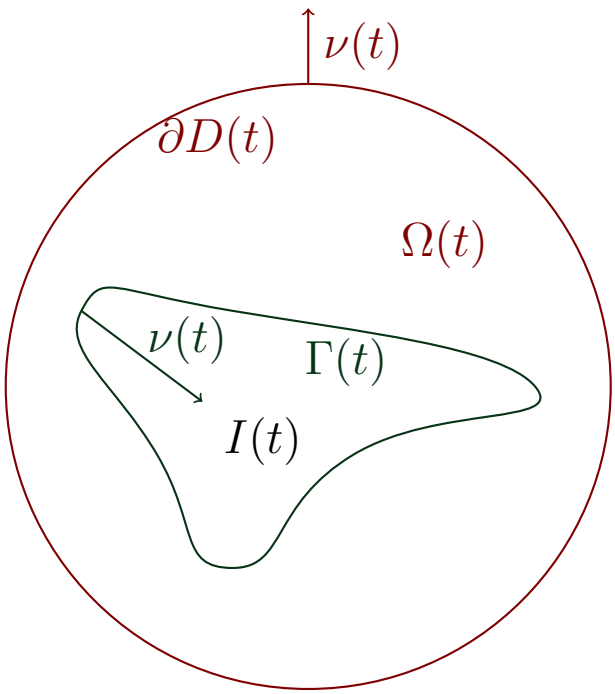


Figure 1: A sketch of the geometry.

and within the evolving surface domain $\Gamma(t)$ there are concentration fields w, z that are subject to the diffusive and advective fluxes

$$q_w := -\delta_\Gamma \nabla_\Gamma w + w\mathbf{V}_\Gamma^\tau, \qquad q_z := -\delta_\Gamma \nabla_\Gamma z + w\mathbf{V}_\Gamma^\tau.$$

Here $\mathbf{V}_\Omega$ and $\mathbf{V}_\Gamma^\tau$ are prescribed advective velocity fields satisfying

$$\mathbf{V}_\Gamma := \mathbf{V}+\mathbf{V}_\Gamma^\tau, \quad (\mathbf{V}_\Gamma(t)\cdot\nu(t))\nu(t) = \mathbf{V}(t) \text{ on } \Gamma(t), \quad \text{and} \quad (\mathbf{V}_\Omega(t)\cdot\nu(t))\nu(t) = \mathbf{V}_o(t) \text{ on } \partial D(t),$$

with $\nu(t)$ denoting the unit outward-pointing normal vector on $\partial\Omega(t)$. Here $\mathbf{V}_\Gamma^\tau$ can be understood as the tangential component of $\mathbf{V}_\Gamma$.

Balances of mass and reaction kinetics on the surface $\Gamma(t)$ lead to the following non-dimensional system:

$$\begin{aligned}
\delta_\Omega(u_t + \nabla\cdot(u\mathbf{V}_\Omega)) - \Delta u &= 0 && \text{in } \Omega(t),\\
\nabla u\cdot\nu - \delta_\Omega u\,(\mathbf{V}_\Omega - \mathbf{V}_\Gamma)\cdot\nu &= \frac{1}{\delta_{k'}}z - \frac{1}{\delta_k}g(u,w) && \text{on } \Gamma(t),\\
\nabla u\cdot\nu &= 0 && \text{on } \partial D(t),\\
\partial^\circ w + w\nabla_\Gamma\cdot\mathbf{V} + \nabla_\Gamma\cdot(w\mathbf{V}_\Gamma^\tau) - \delta_\Gamma\Delta_\Gamma w &= \frac{1}{\delta_{k'}}z - \frac{1}{\delta_k}g(u,w) && \text{on } \Gamma(t),\\
\partial^\circ z + z\nabla_\Gamma\cdot\mathbf{V} + \nabla_\Gamma\cdot(z\mathbf{V}_\Gamma^\tau) - \delta_{\Gamma'}\Delta_\Gamma z &= \frac{1}{\delta_k}g(u,w) - \frac{1}{\delta_{k'}}z && \text{on } \Gamma(t),
\end{aligned} \tag{1}$$

where $\partial^\circ w = w_t + \nabla w\cdot\mathbf{V}$ is the normal time derivative (see [9, 15]), $\nabla_\Gamma = \nabla_{\Gamma(t)}$ and $\Delta_\Gamma = \Delta_{\Gamma(t)}$ denote the tangential gradient and Laplace–Beltrami operator respectively on $\Gamma(t)$ (for ease of reading, we omit the dependence on time and simply write ∇_Γ and Δ_Γ), the dimensionless constants $\delta_\Omega, \delta_\Gamma, \delta_{\Gamma'}, \delta_k$ and $\delta_{k'}$ are positive and $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a given function satisfying Assumption 2.4 below. We endow the system above with non-negative and bounded initial data:

$$(u(0), w(0), z(0)) = (u_0, w_0, z_0) \in L^\infty(\Omega_0)\times L^\infty(\Gamma_0)^2 \quad \text{and} \quad u_0, w_0, z_0 \geq 0,$$

where $\Omega_0 := \Omega(0)$ and $\Gamma_0 := \Gamma(t)$. In terms of the biological application we have in mind, the quantity u represents the concentration of ligands in the domain, w the concentration of receptors on the surface and z the concentration of the complex formed by ligand-receptor binding (see [20, 5] and references therein for more details). The system (1) is a reaction-diffusion system on an evolving domain and details of its derivation can be found in Appendix C. Regarding g, there are two important examples to have in mind:

$$g(u,w) = uw \tag{2}$$

and

$$g(u,w) = \frac{u^n w}{1+u^n} \text{ for } n > 1, \tag{3}$$

representing the biologically relevant cases of quadratic binding and cooperative (Hill function or Michaelis–Menten) binding respectively.

The various parameters appearing in (1) correspond to rates of reaction or diffusion. The purpose of this paper is to derive various limit problems that exhibit free boundary features on the evolving surface $\Gamma(t)$ by sending combinations of the parameters $\delta_\Omega, \delta_k, \delta_{k'}^{-1}, \delta_\Gamma, \delta'_\Gamma$ to zero. Let us highlight some of the difficulties:

- Due in part to the nonlinear nature of g, it become necessary to obtain L^∞-estimates on the solutions of (1). This is non-trivial because of the bulk-surface coupling and the Robin boundary condition for u. We overcome this with De Giorgi arguments along the same vein as in [5].
- As (uniform) estimates on the time derivatives of the solutions is an issue, lack of compactness causes difficulties. We work around this by deriving estimates on difference quotients on pulled back equations (which are essentially parabolic PDEs with time-dependent coefficients).
- In the $\delta_k \to 0$ limit the diffusion coefficients for the two surface quantities w and z are generally different (the system (1) is of *cross-diffusion* type); this means that one cannot simply add the two equations in order to 'cancel out' the right-hand sides to derive estimates. Instead, we utilise a duality approach.
- The geometry involved in the problem is non-standard: $\Omega(t)$ is an annular domain and the interface $\Gamma(t)$ is only one part of $\partial\Omega(t)$, which sometimes leads to complications in the analysis. Furthermore, the domain is evolving in time, calling for the use of special time-evolving Bochner spaces and related theory in the spirit of [3, 6].

It is worth writing the system in the case where there is no movement or evolution of the domain:

$$\delta_\Omega u_t - \Delta u = 0 \qquad \text{in } \Omega, \tag{4a}$$

$$\nabla u \cdot \nu = \frac{1}{\delta_{k'}} z - \frac{1}{\delta_k} g(u,w) \qquad \text{on } \Gamma, \tag{4b}$$

$$\nabla u \cdot \nu = 0 \qquad \text{on } \partial D, \tag{4c}$$

$$w_t - \delta_\Gamma \Delta_\Gamma w = \frac{1}{\delta_{k'}} z - \frac{1}{\delta_k} g(u,w) \qquad \text{on } \Gamma, \tag{4d}$$

$$z_t - \delta_{\Gamma'} \Delta_\Gamma z = \frac{1}{\delta_k} g(u,w) - \frac{1}{\delta_{k'}} z \qquad \text{on } \Gamma, \tag{4e}$$

$$(u(0), w(0), z(0)) = (u_0, w_0, z_0). \tag{4f}$$

We emphasise that the results of this paper are new even in this case where there is no evolution.

1.2 Outline of paper and our contributions

In Section 1.3 we detail some background on the biological motivation behind the system (1). We then set up the function spaces and define some notation in Section 1.4. Our main results are set out in the subsections of Section 2. We present in Section 2.1 the system that arises in the $\delta_k \to 0$ limit, in Section 2.2 the $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit, and in Section 2.3 the $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit. In Section 2.3.1, we replace the Neumann condition on the outer boundary $\partial D(t)$ with a Dirichlet condition and study the same limit as Section 2.3. In Section 2.4, we give free boundary reformulations of the limiting systems that we derived as degenerate parabolic equations on evolving surfaces. Section 2.5 presents our results as applied to the stationary, non-evolving setting (which is the usual setting in much of the literature), i.e., to the system (4). We explore in Section 2.6 the implications of our results in the context of the biological application that motivated this work.

Regarding the rest of the paper, in Section 3 we give some L^2- and L^∞-estimates on solutions of the heat equation on evolving domains and surfaces. Section 4 contains various estimates related to (1) that we shall need in obtaining the existence of limits of the solutions as well as that of other related quantities. From Section 5 to Section 7, we prove the results on each of the limits that were stated in Section 2.1—Section 2.3.1. We conclude the paper proper by presenting some numerical simulations illustrating our theory in Section 8. We collect in Appendix A some useful identities related to integration by parts on evolving spaces, in Appendix B results related to the function g, and in Appendix C we justify our initial model and our motivation for taking limits by nondimensionalisation.

Let us briefly situate our work. This paper is closely related to [5, 20]. In [20] singular limits of a simplified system (where z is neglected) are studied on a non-evolving domain. The paper [5] considers the well posedness of (7) in an evolving domain but not the limiting behaviour. Note that in both works the choice $g(u, w) = uw$ is fixed; here we allow for greater generality in that too. See, as mentioned, [5] for existence results in the case of $g(u, w) = uw$ and [8] for existence results for a larger class of g (but with a different geometric setup).

1.3 Biophysical background

A number of recent theoretical and computational works seek to model receptor-ligand interactions using coupled bulk-surface systems of PDEs in the context of cell biology, employing models which are similar in structure to those considered in this work typically on fixed domains e.g., [23]. Models with similar features arise in the modelling of signalling networks coupling the dynamics of ligands within the cell (e.g., G-proteins) with those on the cell surface [31, 29, 26, 44, 38, 50, 22]; these again have primarily been considered on fixed domains. The rigorous derivation of reduced models based on biologically relevant asymptotic limits of coupled bulk-surface models on fixed domains in the context of cell biology has been the subject of a large number of recent works applied to phenomena such as pattern formation [49, 7], lipid raft formation [24], cell signalling [39, 48], and polarisation [33, 34, 35]. In most applications however, the motion of cells necessitates the consideration of evolving domains. For example, in chemotaxis the motion of cells has been conjectured to act as a barrier to successful gradient sensing that must be overcome [21], possibly via the ability of cells to create their own chemotactic gradients, i.e., to influence the bulk ligand field [40, 42, 41]. We note the recent computational works in this direction [36, 37]. We also mention [43] for a framework and theoretical study of bulk-surface systems (among other problems) and their relationship to Onsager systems. Recently, existence and well posedness theory has been developed for bulk-surface systems in the evolving domain setting [5, 8]. There are however, to our knowledge, no corresponding results on

asymptotic limits of bulk-surface systems on evolving domains. Motivated by this need, in this work we focus on understanding asymptotic limits of theoretical models for receptor-ligand dynamics in cell biology consisting of a coupled system of bulk-surface partial differential equations on evolving domains.

Whilst our focus is on receptor-ligand dynamics in biological cells, problems of a similar structure arise in fields such as semiconductors [25] and, in particular, in ecology where one considers populations consisting of two or more competing species [27]. Such an ecological scenario can be modelled by so-called spatial segregation models and the corresponding asymptotic limits have been the subject of much mathematical study, e.g., [11, 12, 13, 28]. Although domain evolution may be less relevant in this setting, our results are new even on fixed domains, and hence potentially relevant.

A number of Stefan-type free boundary problems have been derived as fast-reaction limits of systems of parabolic equations in fixed domains, for a survey see [46, Chapter 10]. We further note that free boundary problems of Stefan or Hele-Shaw type arise as singular limits of models for tumour growth [47].

1.4 Notation and functional framework

It is convenient to parametrise the evolving geometric domain and function spaces defined on it using a continuously differentiable velocity field $\mathbf{V}_p\colon [0,T]\times\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$ for which there exists a flow $\Phi\colon [0,T]\times\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$ such that

(i) $\Phi_t := \Phi(t,\cdot)\colon \overline{\Omega_0}\to\overline{\Omega(t)}$ is a C^2-diffeomorphism with $\Phi_t(\Gamma_0)=\Gamma(t)$ and $\Phi_t(\partial D_0)=\partial D(t)$,

(ii) Φ_t solves the ODE

$$\frac{d}{dt}\Phi_t(\cdot)=\mathbf{V}_p(t,\Phi_t(\cdot)),$$
$$\Phi_0=\mathrm{Id}.$$

Note that

$$(\mathbf{V}_p\cdot\nu)\nu=\mathbf{V} \text{ on } \Gamma(t) \quad\text{and}\quad (\mathbf{V}_p\cdot\nu)\nu=\mathbf{V}_o \text{ on } \partial D(t). \tag{5}$$

Functions on the evolving domain are paramerised over the initial domain in the following way. We define the mapping ϕ_{-t} acting on a function $u\colon \Omega(t)\to\mathbb{R}$ by $\phi_{-t}u := u\circ\Phi_t$; this is a linear homeomorphism (its inverse will be denoted by ϕ_t) between the spaces $L^p(\Omega(t))$, $H^1(\Omega(t))$ and the reference spaces $L^p(\Omega_0)$, $H^1(\Omega_0)$ respectively [6, 4, 1]. The same holds for the corresponding L^p and Sobolev spaces on Γ and ∂D [1] for ϕ_{-t} given by the same expression. If we also assume

(i) $\Phi_t := \Phi(t,\cdot)\colon \Gamma_0\to\Gamma(t)$ is a C^3-diffeomorphism,

(ii) $\Phi_{(\cdot)}\in C^3([0,T]\times\Gamma_0)$,

then the above property also holds for the fractional Sobolev space $H^{1/2}(\Gamma(t))$ [4, §5.4.1]. We may then define the Banach spaces $L^p_{L^q(\Omega)}$, $L^2_{H^1(\Omega)}$, $L^p_{L^q(\Gamma)}$, $L^2_{H^1(\Gamma)}$, $L^2_{H^{1/2}(\Gamma)}$, which are Hilbert spaces for $p=q=2$; these are the evolving versions of the usual Bochner spaces to handle (sufficiently regular) time-evolving Banach spaces $X\equiv\{X(t)\}_{t\in[0,T]}$. To be precise, the notation L^p_X stands for

$$L^p_X := \left\{ u:[0,T]\to\bigcup_{t\in[0,T]}X(t)\times\{t\},\ \ t\mapsto(\hat{u}(t),t) \ \ \Big|\ \ \phi_{-(\cdot)}\hat{u}(\cdot)\in L^p(0,T;X_0)\right\}$$

where for each $t \in [0,T]$, the map $\phi_{-t}\colon X(t) \to X_0$ is a given linear homeomorphism satisfying certain properties; the corresponding norm is (identifying $\hat{u}$ with u)

$$\|u\|_{L^p_X} := \left(\int_0^T \|u(t)\|^p_{X(t)}\right)^{\frac{1}{p}}$$

(with the obvious modification for $p = \infty$). We also define, for $k \in \mathbb{N} \cup \{0\}$, the space C^k_X of k-times continuously differentiable functions on $[0,T]$:

$$C^k_X = \left\{ \eta\colon [0,T] \to \bigcup_{t\in[0,T]} X(t) \times \{t\}, \quad t \mapsto (\eta(t), t) \quad | \quad \phi_{-(\cdot)}\eta(\cdot) \in C^k([0,T]; X_0) \right\}.$$

For a sufficiently smooth quantity u defined in $\Omega(t)$ its (classical or strong) parametric material derivative is given by

$$\partial^\bullet_\Omega u = u_t + \nabla u \cdot \mathbf{V}_p \tag{6}$$

(see [6, 3, 4] and references therein). This derivative takes into account that spatial points $x = x(t)$ also depend on time and that their trajectory has been parametrised by the velocity field $\mathbf{V}_p$. For a sufficiently smooth quantity w defined on the evolving surface $\Gamma(t)$, the classical parametric material derivative has the expression

$$\partial^\bullet_\Gamma w := \partial^\circ w + \nabla_\Gamma w \cdot \mathbf{V}_p.$$

The paper [3] (see also [6]) also defines a weaker version of the above-introduced parametric material derivative (i.e., a notion of a weak time derivative in an evolving space) which we can call the weak parametric material derivative. This generalises (6): a function $u \in L^2_{H^1(\Omega)}$ has a weak parametric material derivative (or weak time derivative) $\partial^\bullet_\Omega u \in L^2_{H^1(\Omega)^*}$ if and only if

$$\int_0^T \langle \partial^\bullet_\Omega u(t), \eta(t)\rangle_{H^1(\Omega(t))^*, H^1(\Omega(t))} = -\int_0^T \int_{\Omega(t)} u(t)\partial^\bullet_\Omega \eta(t) - \int_0^T \int_{\Omega(t)} u(t)\eta(t)\nabla \cdot \mathbf{V}_p$$

holds for all functions η that are smooth and compactly supported (in time) with values in $H^1(\Omega(t))$, where $\partial^\bullet_\Omega \eta(t)$ is given by the formula (6). A similar definition with the right modifications defines the weak parametric material derivative $\partial^\bullet_\Gamma w \in L^2_{H^{-1}(\Gamma)}$ for a function $w \in L^2_{H^1(\Gamma)}$:

$$\int_0^T \langle \partial^\bullet_\Gamma w(t), \eta(t)\rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))} = -\int_0^T \int_{\Gamma(t)} w(t)\partial^\bullet_\Gamma \eta(t) - \int_0^T \int_{\Gamma(t)} w(t)\eta(t)\nabla_\Gamma \cdot \mathbf{V}_p.$$

We shall not use the notation $\partial^\bullet_\Omega, \partial^\bullet_\Gamma$ further but will simply write the weak parametric material derivative as $\partial^\bullet$. With this in hand, we define the evolving versions of Sobolev–Bochner spaces:

$$\mathbb{W}(H^1(\Omega), H^1(\Omega)^*) := \left\{ u \in L^2_{H^1(\Omega)} \mid \partial^\bullet u \in L^2_{H^1(\Omega)^*} \right\}$$
$$\mathbb{W}(H^1(\Omega), L^2(\Omega)) := \left\{ u \in L^2_{H^1(\Omega)} \mid \partial^\bullet u \in L^2_{L^2(\Omega)} \right\}$$

(and similarly the corresponding spaces on $\{\Gamma(t)\}$) and more generally

$$\mathbb{W}(X,Y) = \left\{ u \in L^2_X \mid \partial^\bullet u \in L^2_Y \right\}$$

given families of time-evolving Banach spaces X and Y. We alternatively sometimes use the notation

$$H^1_Y := \mathbb{W}(Y,Y).$$

We do not give the precise technical details and properties of these spaces here but refer to [6] (and also [3, 4, 1]) for the interested reader. At this point note that in Appendix A we collect some useful integration by parts identities that will be used throughout the paper.

2 Free boundary problems on moving domains as limit systems; main results

From (1), by adding and subtracting the parametrised velocity $\mathbf{V}_p$ (see Section 1.4), we obtain the model

$$\delta_\Omega(\partial^\bullet u + u\nabla\cdot\mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega u)) - \Delta u = 0 \qquad \text{in } \Omega(t), \tag{7a}$$

$$\nabla u\cdot\nu - \delta_\Omega j u = \frac{1}{\delta_{k'}}z - \frac{1}{\delta_k}g(u,w) \qquad \text{on } \Gamma(t), \tag{7b}$$

$$\nabla u\cdot\nu = 0 \qquad \text{on } \partial D(t), \tag{7c}$$

$$\partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_p - \delta_\Gamma\Delta_\Gamma w + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w) = \frac{1}{\delta_{k'}}z - \frac{1}{\delta_k}g(u,w) \qquad \text{on } \Gamma(t), \tag{7d}$$

$$\partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_p - \delta_{\Gamma'}\Delta_\Gamma z + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) = \frac{1}{\delta_k}g(u,w) - \frac{1}{\delta_{k'}}z \qquad \text{on } \Gamma(t), \tag{7e}$$

$$(u(0), w(0), z(0)) = (u_0, w_0, z_0). \tag{7f}$$

In the above we use the notation

$$\mathbf{J}_\Omega := \mathbf{V}_\Omega - \mathbf{V}_p, \qquad \mathbf{J}_\Gamma := \mathbf{V}_\Gamma - \mathbf{V}_p \qquad \text{and} \qquad j := (\mathbf{V}_\Omega - \mathbf{V}_\Gamma)\cdot\nu = \mathbf{J}_\Omega|_\Gamma\cdot\nu. \tag{8}$$

Observe that j is the jump in the normal velocities on Γ and that due to the compatibility conditions, $\mathbf{J}_\Gamma = \mathbf{V}_\Gamma^\tau - \mathbf{V}_p^\tau$ has no normal component, i.e., $\mathbf{J}_\Gamma\cdot\nu = 0$ on Γ, and we also have $\mathbf{J}_\Omega\cdot\nu = 0$ on ∂D.

Remark 2.1. *The system is independent of the choice of $\mathbf{V}_p$ provided* (5) *holds. The analysis uses function spaces (introduced in Section 1.4) that do depend on $\mathbf{V}_p$. Introducing this velocity field may be convenient in applications. For example in numerical simulations it may be used to avoid degenerate meshes [15, 18].*

Remark 2.2. *The mass of $w+z$ is conserved: using the equation satisfied by $w+z$ and the fact that $\mathbf{J}_\Gamma$ is purely tangential, we can show*

$$\frac{d}{dt}\int_{\Gamma(t)} w + z = 0.$$

In a similar way the mass of $\delta_\Omega u + z$ is conserved:

$$\frac{d}{dt}\left(\delta_\Omega\int_{\Omega(t)} u + \int_{\Gamma(t)} z\right) = 0.$$

In the following, we will take various limits in (7) and end up with systems that resemble this system. We do not, in this work, consider the well posedness of the problem (7) but instead we will assume it. In fact, we make the following standing assumption.

Assumption 2.3 (Standing assumption)**.** *Regarding* (7)*, we take non-negative initial data $u_0 \in H^1(\Omega_0)\cap L^\infty(\Omega_0)$ and $w_0, z_0 \in H^1(\Gamma_0)\cap L^\infty(\Gamma_0)$, and we will assume the existence of a unique non-negative solution*

$$u \in L^2_{H^1(\Omega)} \cap H^1_{H^1(\Omega)^*} \quad \textit{and} \quad w, z \in L^2_{H^1(\Gamma)} \cap H^1_{L^2(\Gamma)}$$

with

$$g(u,w) \in L^2_{L^2(\Gamma)}.$$

Note in particular that we assume non-negativity of solutions; this is reasonable for the application we have in mind and can be proved under (the assumed) non-negative initial data, uniqueness of solutions and reasonable assumptions on g (see [8, Theorem 1.1]); in particular

$$g(u,0) \leq 0 \quad \text{and} \quad g(0,w) \leq 0 \qquad \forall u,w \geq 0$$

would be sufficient. Indeed, we may replace the reaction terms in (7) by $\frac{1}{\delta_{k'}} z^+ - \frac{1}{\delta_k} g(u^+, w^+)$ and by testing the resulting system with the negative parts of the solutions, we can show that its solutions are non-negative under the condition on g above. By uniqueness, it follows that the solution of (7) is also non-negative.

We state an assumption on the function g.

Assumption 2.4 (Assumptions on g)**.** *Throughout the paper, the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies*

$$u,w \geq 0 \implies g(u,w) \geq 0.$$

When viewing g as a superposition operator, we will in different theorems use one of the following assumptions as and when needed:

$$u_n \rightharpoonup u \text{ in } L^2_{H^{1/2}(\Gamma)}, \quad v_n \to v \text{ in } L^2_{L^2(\Gamma)}, \quad g(u_n,v_n) \to 0 \text{ in } L^1_{L^1(\Gamma)} \implies g(u,v) = 0 \tag{9}$$

$$u_n \to u \text{ in } L^2_{L^2(\Gamma)}, \quad u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty_{L^\infty(\Gamma)}, \quad v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty_{L^\infty(\Gamma)}, \quad g(u_n,v_n) \to 0 \text{ in } L^1_{L^1(\Gamma)} \implies g(u,v) = 0 \text{ in } L^1_{L^1(\Gamma)} \tag{10}$$

Proposition 2.5. *Both choices of g in* (2) *and* (3) *satisfy assumptions* (9) *and* (10)*.*

In order not to break the flow of the section, the proof of this result has been placed in Appendix B. Now, to write down the limiting systems that we obtain, let us first introduce the space of test functions:

$$\mathcal{V} := \left\{ \eta \in L^2_{H^1(\Omega)} \cap H^1_{L^2(\Omega)} : \eta|_\Gamma \in L^2_{H^1(\Gamma)} \cap H^1_{L^2(\Gamma)} \right\} \tag{11}$$

where by the restriction above we mean in the sense of the Sobolev trace, applied pointwise a.e. in time:

$$(\eta|_\Gamma)(t) := \tau_t \eta(t)$$

with $\tau_t\colon H^1(\Omega(t)) \to H^{1/2}(\Gamma(t))$ the trace operator. We will usually omit explicitly writing the trace operator in what follows.

2.1 The $\delta_k \to 0$ limit

We first consider the **fast binding** limit $\delta_k \to 0$.

Definition 2.6. We say that $(u,w,z) \in L^2_{H^1(\Omega)} \times (L^2_{H^1(\Gamma)})^2$ is a *weak solution* of the problem

$$\delta_\Omega(\partial^\bullet u + u\nabla\cdot\mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega u)) - \Delta u = 0 \qquad \text{in } \Omega(t), \tag{12a}$$

$$\nabla u \cdot \nu = 0 \qquad \text{on } \partial D(t), \tag{12b}$$

$$\partial^\bullet w - \delta_\Gamma \Delta_\Gamma w + w\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w) = \nabla u\cdot\nu - \delta_\Omega j u \qquad \text{on } \Gamma(t), \tag{12c}$$

$$\partial^\bullet z - \delta_{\Gamma'} \Delta_\Gamma z + z\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) = -\nabla u\cdot\nu + \delta_\Omega j u \qquad \text{on } \Gamma(t), \tag{12d}$$

$$g(u,w) = 0 \qquad \text{on } \Gamma(t), \tag{12e}$$

$$(u(0),w(0),z(0)) = (u_0,w_0,z_0), \tag{12f}$$

if for every $\eta \in \mathcal{V}$ with $\eta(T) = 0$,

$$
\begin{aligned}
&-\int_{\Omega_0} u_0\eta(0) + \frac{1}{\delta_\Omega}\int_{\Gamma_0} w_0\eta(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} w\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u\cdot\nabla\eta \\
&-\frac{\delta_\Gamma}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma w\cdot\nabla_\Gamma\eta + \int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u)\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w)\eta \\
&-\int_0^T\int_{\Gamma(t)} ju\eta = 0, \qquad (13a)\\
&-\int_{\Omega_0} u_0\eta(0) - \frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\eta(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u\cdot\nabla\eta \\
&+\frac{\delta_{\Gamma'}}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma z\cdot\nabla_\Gamma\eta + \int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u)\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z)\eta \\
&-\int_0^T\int_{\Gamma(t)} ju\eta = 0, \qquad (13b)\\
&g(u,w) = 0 \text{ in } L^1_{L^1(\Gamma)}. \qquad (13c)
\end{aligned}
$$

Our notion of solution is very weak in the sense that we put temporal derivatives onto the test function and the normal derivative of u has to be understood also in a very weak sense (see Remark 2.9).

In Section 5, we will prove the following. The result requires a restriction on the dimension because we need an L^∞-estimate on w (see Corollary 4.6). The said L^∞ estimate can be obtained in a different (easier) way for the other limits, hence no restriction on the dimension is needed in those limiting regimes.

Theorem 2.7 (The δ_k limit)**.** *Assume* (9) *and let $d \le 3$. As $\delta_k \to 0$, the solution (u_k, w_k, z_k) of* (7) *converges to a weak solution $(u, w, z) \in L^2_{H^1(\Omega)} \times (L^2_{H^1(\Gamma)})^2$ of the problem* (12)*.*

Remark 2.8. *Note that z is uncoupled from the other quantities; indeed we can solve for u and define z as the solution of* (12d) *along with $z(0) = z_0$ in some sense. Both u and w are linked through the complementarity condition* (12e)*.*

Remark 2.9. *The normal derivatives appearing in* (12c) *and* (12d) *are to be understood in a formal sense; they are not even weak normal derivatives (we would need Δu in $L^2(\Omega(t))$ for that).*

2.2 The $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit

In this section we look at the **fast binding, vanishing surface diffusion** limit $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$.

Definition 2.10. We say that $(u, w, z) \in L^2_{H^1(\Omega)} \times (L^2_{L^2(\Gamma)})^2$ is a *weak solution* of the problem

$$
\begin{aligned}
\delta_\Omega(\partial^\bullet u + u\nabla_\Gamma\cdot\mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega u)) - \Delta u &= 0 && \text{in } \Omega(t), && (14a)\\
\nabla u\cdot\nu &= 0 && \text{on } \partial D(t), && (14b)\\
\partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w) &= \nabla u\cdot\nu - \delta_\Omega ju && \text{on } \Gamma(t), && (14c)\\
\partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) &= -\nabla u\cdot\nu + \delta_\Omega ju && \text{on } \Gamma(t), && (14d)\\
g(u,w) &= 0 && \text{on } \Gamma(t), && (14e)\\
(u(0), w(0), z(0)) &= (u_0, w_0, z_0), && && (14f)
\end{aligned}
$$

if for every $\eta \in \mathcal{V}$ with $\eta(T) = 0$,

$$-\int_{\Omega_0} u_0\psi(0) + \frac{1}{\delta_\Omega}\int_{\Gamma_0} w_0\psi(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} w\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u\cdot\nabla\eta$$
$$+\int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u)\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} w\mathbf{J}_\Gamma\cdot\nabla_\Gamma\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0, \tag{15a}$$

$$-\int_{\Omega_0} u_0\psi(0) - \frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\psi(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u\cdot\nabla\eta$$
$$+\int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u)\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\mathbf{J}_\Gamma\cdot\nabla_\Gamma\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0, \tag{15b}$$

$$g(u,w) = 0 \text{ in } L^1_{L^1(\Gamma)}. \tag{15c}$$

Note that we have weakened the spatial regularity for w and z from Theorem 2.7 even further: the divergence of the jump terms are satisfied in a weak sense, i.e., the test function carries the derivatives. The next theorem will be shown in Section 6.

Theorem 2.11 (The $\delta_k = \delta_\Gamma = \delta_{\Gamma'}$ limit)**.** *Assume* (10)*. As* $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$*, the solution* (u_k, w_k, z_k) *of* (7) *converges to a weak solution* $(u,w,z) \in L^2_{H^1(\Omega)} \times (L^2_{L^2(\Gamma)})^2$ *of the problem* (14)*.*

2.3 The $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit

Finally, we study the **fast binding, vanishing surface diffusion, quasi-steady bulk, slow unbinding** limit $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} \to 0$.

Definition 2.12. We say that $(u,w,z) \in L^2_{H^1(\Omega)} \times (L^2_{L^2(\Gamma)})^2$ is a *weak solution* of the problem

$$\nabla u \equiv 0 \quad \text{in } \Omega(t), \tag{16a}$$
$$\partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w) = 0 \quad \text{on } \Gamma(t), \tag{16b}$$
$$\partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) = 0 \quad \text{on } \Gamma(t), \tag{16c}$$
$$g(u,w) = 0 \quad \text{on } \Gamma(t), \tag{16d}$$
$$(w(0), z(0)) = (w_0, z_0), \tag{16e}$$

if for every $\eta \in \mathcal{V}$ with $\eta(T) = 0$,

$$\nabla u = 0 \text{ in } L^1_{L^1(\Omega)},$$
$$\int_{\Gamma_0} w_0\eta(0) + \int_0^T\int_{\Gamma(t)} w\partial^\bullet\eta + w\mathbf{J}_\Gamma\cdot\nabla_\Gamma\eta = 0,$$
$$\int_{\Gamma_0} z_0\eta(0) + \int_0^T\int_{\Gamma(t)} z\partial^\bullet\eta + z\mathbf{J}_\Gamma\cdot\nabla_\Gamma\eta = 0,$$
$$g(u,w) = 0 \text{ in } L^1_{L^1(\Gamma)}.$$

The proof of the following theorem can be found in Section 7.

Theorem 2.13 (The $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'}$ limit)**.** *Assume* (10)*. As* $\delta_\Omega = \delta_k = \frac{1}{\delta_{k'}} = \delta_\Gamma = \delta_{\Gamma'} \to 0$*, the solution* (u_k, w_k, z_k) *of* (7) *converges to a weak solution* $(u,w,z) \in L^2_{H^1(\Omega)} \times (L^2_{L^2(\Gamma)})^2$ *of the problem* (16)*.*

We can show that in fact the masses of w and z are conserved:

Lemma 2.14. *We have*

$$\int_{\Gamma(t)} w(t) = \int_{\Gamma_0} w_0 \quad \textit{and} \quad \int_{\Gamma(t)} z(t) = \int_{\Gamma_0} z_0.$$

Proof. Taking $\eta = \xi$ where $\xi \in C_c^\infty(0,T)$ as a test function in the weak form for w:

$$\int_0^T \xi'(t) \int_{\Gamma(t)} w(t) = 0 \qquad \forall \xi \in C_c^\infty(0,T),$$

hence by the du Bois-Reymond lemma we obtain

$$\int_{\Gamma(t)} w(t) = c \quad \text{for a.e. } t \in (0,T)$$

where $c \geq 0$ is a constant. Let us now determine the value of c. If we select $\eta = \xi \in C_c^\infty([0,T))$ in the weak form for w, then we have

$$-\int_{\Gamma_0} w_0 \xi(0) = c \int_0^T \xi'(t) = -c\xi(0)$$

which finally yields $c = \int_{\Gamma_0} w_0$. The argument for z is similar. □

We can also prove that u is in fact the zero function under a reasonable assumption on the initial condition for w, and for a large class of reaction terms g.

Lemma 2.15. *If g is such that $g(u,w) = 0$ implies $uw = 0$, and if*

$$\int_{\Gamma_0} w_0 > 0,$$

then $u \equiv 0$.

Proof. Since $\nabla u(t) \equiv 0$, $u(t)$ is constant in space for a.e. fixed t, so we can write the complementarity condition as

$$\int_0^T u(t) \int_{\Gamma(t)} w(t) = 0.$$

This implies from above that if $\int_{\Gamma_0} w_0 > 0$, then $\int_0^T u(t) = 0$, which given the non-negativity of u yields $u \equiv 0$. □

Since the assumption of the previous lemma is very reasonable and it leads us to the trivial solution $u \equiv 0$, we shall also consider a modification of (7) where we replace the Neumann boundary condition (7c) by a Dirichlet one in the next subsection.

2.3.1 The Dirichlet case of the limit

We consider here the problem (7) with a Dirichlet boundary condition on the outer boundary instead of the Neumann condition:

$$\delta_\Omega(\partial^\bullet u + u\nabla\cdot\mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega u)) - \Delta u = 0 \qquad \text{in } \Omega(t), \tag{18a}$$

$$\nabla u\cdot\nu - \delta_\Omega j u = \frac{1}{\delta_{k'}}z - \frac{1}{\delta_k}g(u,w) \qquad \text{on } \Gamma(t), \tag{18b}$$

$$u = u_D \qquad \text{on } \partial D(t), \tag{18c}$$

$$\partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_p - \delta_\Gamma\Delta_\Gamma w + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w) = \frac{1}{\delta_{k'}}z - \frac{1}{\delta_k}g(u,w) \qquad \text{on } \Gamma(t), \tag{18d}$$

$$\partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_p - \delta_{\Gamma'}\Delta_\Gamma z + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) = \frac{1}{\delta_k}g(u,w) - \frac{1}{\delta_{k'}}z \qquad \text{on } \Gamma(t), \tag{18e}$$

$$(u(0), w(0), z(0)) = (u_0, w_0, z_0), \tag{18f}$$

where $u_D \in C^0_{H^{1/2}(\partial D)} \cap L^\infty_{L^\infty(\partial D)}$, $u_D \geq 0$ is a given non-negative function. Define the space

$$H^1_d(\Omega(t)) := \{v \in H^1(\Omega(t)) : v|_{\partial D(t)} = u_D\}.$$

Like before, we will eschew the details of well posedness for this system and simply assume the existence of a unique non-negative solution

$$u \in L^2_{H^1_d(\Omega)} \cap H^1_{H^1_d(\Omega)^*} \quad \text{and} \quad w, z \in L^2_{H^1(\Gamma)} \cap H^1_{L^2(\Gamma)}$$

with

$$g(u,w) \in L^2_{L^2(\Gamma)}.$$

Note that the weak formulation for (18) is obtained by testing the u equation with a test function from the space $L^2_{H^1_e(\Omega)}$ and asking for the trace of $u(t)$ on $\partial D(t)$ to coincide with $u_D(t)$.

Now, we have to change the space $\mathcal{V}$ to take care of the Dirichlet data and this can be done with first defining

$$H^1_e(\Omega(t)) := \{v \in H^1(\Omega(t)) : v|_{\partial D(t)} = 0\}$$

and then

$$\mathcal{V}_e := \left\{\eta \in L^2_{H^1_e(\Omega)} \cap H^1_{L^2(\Omega)} : \eta|_\Gamma \in L^2_{H^1(\Gamma)} \cap H^1_{L^2(\Gamma)}\right\}.$$

Definition 2.16. We say that $(u,w,z) \in L^2_{H^1(\Omega)} \times (L^2_{L^2(\Gamma)})^2$ is a *weak solution* of the problem

$$\Delta u = 0 \qquad \text{in } \Omega(t), \tag{19a}$$

$$u = u_D \qquad \text{on } \partial D(t), \tag{19b}$$

$$\partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w) = \nabla u\cdot\nu \qquad \text{on } \Gamma(t), \tag{19c}$$

$$\partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) = -\nabla u\cdot\nu \qquad \text{on } \Gamma(t), \tag{19d}$$

$$g(u,w) = 0 \qquad \text{on } \Gamma(t), \tag{19e}$$

$$(w(0), z(0)) = (w_0, z_0), \tag{19f}$$

if for every $\eta \in \mathcal{V}_e$ with $\eta(T) = 0$,

$$\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta + \int_{\Gamma_0} w_0 \eta(0) + \int_0^T \int_{\Gamma(t)} w \partial^\bullet \eta + w \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta = 0, \tag{20a}$$

$$\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z \partial^\bullet \eta + z \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta = 0, \tag{20b}$$

$$u|_{\partial D} = u_D \text{ in } L^2_{H^{1/2}(\partial D)}, \tag{20c}$$

$$g(u, w) = 0 \text{ in } L^1_{L^1(\Gamma)}. \tag{20d}$$

In Section 7.3, we show the following.

Theorem 2.17 (The $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'}$ limit in the Dirichlet case)**.** *Assume*

$$u_n \rightharpoonup u \text{ in } L^2_{H^{1/2}(\Gamma)}, \quad u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty_{L^\infty(\Gamma)}, \quad v_n \to v \text{ in } L^2_{H^{-1/2}(\Gamma)}, \quad g(u_n, v_n) \to 0 \text{ in } L^1_{L^1(\Gamma)} \implies g(u, v) = 0 \text{ in } L^1_{L^1(\Gamma)}. \tag{21}$$

As $\delta_\Omega = \delta_k = \frac{1}{\delta_{k'}} = \delta_\Gamma = \delta_{\Gamma'} \to 0$, the solution (u_k, w_k, z_k) of (18) *converges to a weak solution $(u, w, z) \in L^2_{H^1(\Omega)} \times (L^2_{L^2(\Gamma)})^2$ of the problem* (19)*.*

Proposition 2.18. *The choice $g(u, w) = uw$ satisfies assumption* (21)*.*

The proof is in Appendix B.

2.4 Reformulation as free boundary problems on evolving surfaces.

We shall now formulate the limiting problems we have derived as free boundary problems. We make the following standing assumption (which clearly holds for the two exemplars (2) and (3)) for this section.

Assumption 2.19. *Suppose g has the property that $g(u, w) = 0$ if and only if $uw = 0$.*

Define the maximal monotone graph $\beta \colon \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$\beta(r) := \begin{cases} \emptyset & : r < 0 \\ (-\infty, 0] & : r = 0 \\ 0 & : r > 0. \end{cases}$$

If we define $v := -w$ we can recast the complementarity condition $g(u, w) = 0$ on $\Gamma(t)$ as

$$v \in \beta(u) \text{ on } \Gamma(t).$$

Remark 2.20. *Alternatively, defining*

$$\alpha(r) := \begin{cases} 0 & : r < 0 \\ [0, \infty) & : r = 0 \\ \emptyset & : r > 0, \end{cases}$$

we can rewrite $g(u, w) = 0$ on $\Gamma(t)$ as $u \in \alpha(v)$ on $\Gamma(t)$.

Define a map $P\colon L^2_{H^{1/2}(\Gamma)} \to L^2_{H^1(\Omega)}$ as the solution map $u \mapsto U$ of

$$\begin{aligned} \delta_\Omega(\partial^\bullet U + U\nabla\cdot\mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega U)) - \Delta U &= 0 && \text{in } \Omega(t),\\ \nabla U\cdot\nu &= 0 && \text{on } \partial D(t),\\ U &= u && \text{on } \Gamma(t),\\ U(0) &= u_0 && \text{on } \Omega_0, \end{aligned}$$

and define the associated Dirichlet-to-Neumann map $A\colon L^2_{H^{1/2}(\Gamma)} \to L^2_{H^{-1/2}(\Gamma)}$ by

$$A(u) := \nabla P(u)\cdot\nu$$

as the normal derivative on Γ. Finally, we set $v_0 := -u_0$ as the initial data.

The δ_k limit. We can rewrite the problem (12) as: find $(u,v) \in L^2_{H^{1/2}(\Gamma)} \times \mathbb{W}(H^1(\Gamma), H^1(\Gamma)^*)$ with $u \geq 0$, $v \leq 0$, such that

$$\begin{aligned} \partial^\bullet v - \delta_\Gamma\Delta_\Gamma v + v\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma v) + A(u) &= \delta_\Omega j u && \text{on } \Gamma(t),\\ v &\in \beta(u) && \text{on } \Gamma(t),\\ v(0) &= v_0. \end{aligned}$$

The $\delta_k = \delta_\Gamma = \delta_{\Gamma'}$ limit. Regarding the second limiting problem (14), we can write it as: find $(u,v) \in L^2_{H^{1/2}(\Gamma)} \times \mathbb{W}(H^1(\Gamma), H^1(\Gamma)^*)$ with $u \geq 0$, $v \leq 0$, such that

$$\begin{aligned} \partial^\bullet v + v\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma v) + A(u) &= \delta_\Omega j u && \text{on } \Gamma(t),\\ v &\in \beta(u) && \text{on } \Gamma(t),\\ v(0) &= v_0. \end{aligned}$$

The $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'}$ limit in the Dirichlet case. As done in [20], we shall separate the two Dirichlet conditions for u (coming from u_D and $u|_{\Gamma(t)}$) into two separate equations. To do so, let us first define a map $E\colon L^2_{H^{1/2}(\Gamma)} \to L^2_{H^1(\Omega)}$ as the solution map $u \mapsto U$ of

$$\begin{aligned} \Delta U &= 0 && \text{in } \Omega(t),\\ U &= 0 && \text{on } \partial D(t),\\ U &= u && \text{on } \Gamma(t), \end{aligned}$$

and define the associated Dirichlet-to-Neumann map $A_D\colon L^2_{H^{1/2}(\Gamma)} \to L^2_{H^{-1/2}(\Gamma)}$ by

$$A_D(u) := \nabla E(u)\cdot\nu$$

as the normal derivative on Γ. We refer to [4, §5.4.2] for more rigorous details about these maps in a similar setting.

Define also the element h by

$$\begin{aligned} \Delta h &= 0 && \text{in } \Omega(t),\\ h &= u_D && \text{on } \partial D(t),\\ h &= 0 && \text{on } \Gamma(t). \end{aligned}$$

We can rewrite the problem (19) as: find $(u,v) \in L^2_{H^{1/2}(\Gamma)} \times \mathbb{W}(L^2(\Gamma), H^{-1/2}(\Gamma))$ with $u \geq 0$, $v \leq 0$, such that

$$\begin{aligned} \partial^\bullet v + v\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) + A_D(u) + \nabla h \cdot \nu = 0 \qquad & \text{on } \Gamma(t), \\ v \in \beta(u) \quad & \text{on } \Gamma(t), \\ v(0) = v_0. \end{aligned} \tag{22}$$

This is an example of a Stefan-type free boundary problem on an evolving surface and is related to [1], where the existence and uniqueness of a Stefan problem on an evolving surface was proved.

2.5 The non-evolving setting

For convenience, let us write down the limiting problems we get in case there is no domain evolution.

Theorem 2.21 (The δ_k limit)**.** *Assume*

$$u_n \rightharpoonup u \text{ in } L^2(0,T;H^{1/2}(\Gamma)), \quad v_n \to v \text{ in } L^2(0,T;L^2(\Gamma)), \quad g(u_n,v_n) \to 0 \text{ in } L^1(0,T;L^1(\Gamma)) \implies g(u,v) = 0, \tag{23}$$

and let $d \leq 3$. As $\delta_k \to 0$, the solution (u_k, w_k, z_k) of (4) *converges to a weak solution $(u,w,z) \in L^2(0,T;H^1(\Omega)) \times (L^2(0,T;H^1(\Gamma)))^2$ of the problem*

$$\delta_\Omega \dot{u} - \Delta u = 0 \qquad \text{in } \Omega, \tag{24a}$$
$$\nabla u \cdot \nu = 0 \qquad \text{on } \partial D, \tag{24b}$$
$$\dot{w} - \delta_\Gamma \Delta_\Gamma w = \nabla u \cdot \nu \qquad \text{on } \Gamma, \tag{24c}$$
$$\dot{z} - \delta_{\Gamma'} \Delta_\Gamma z = -\nabla u \cdot \nu \qquad \text{on } \Gamma, \tag{24d}$$
$$g(u,w) = 0 \qquad \text{on } \Gamma, \tag{24e}$$
$$(u(0), w(0), z(0)) = (u_0, w_0, z_0). \tag{24f}$$

When $g(u,w) = 0$ implies $uw = 0$, the (u,w) system above corresponds directly to the limiting system obtained in [20, Theorem 5.3].

Theorem 2.22 (The $\delta_k = \delta_\Gamma = \delta_{\Gamma'}$ limit)**.** *Assume*

$$u_n \to u \text{ in } L^2(0,T;L^2(\Gamma)), \quad u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^\infty(\Gamma)), \quad v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;L^\infty(\Gamma)), \quad g(u_n,v_n) \to 0 \text{ in } L^1(0,T;L^1(\Gamma)) \implies g(u,v) = 0 \text{ in } L^1(0,T;L^1(\Gamma)). \tag{25}$$

As $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$, the solution (u_k, w_k, z_k) of (4) *converges to a weak solution $(u,w,z) \in L^2(0,T;H^1(\Omega)) \times (L^2(0,T;L^2(\Gamma)))^2$ of the problem*

$$\delta_\Omega \dot{u} - \Delta u = 0 \qquad \text{in } \Omega, \tag{26a}$$
$$\nabla u \cdot \nu = 0 \qquad \text{on } \partial D, \tag{26b}$$
$$\dot{w} = \nabla u \cdot \nu \qquad \text{on } \Gamma, \tag{26c}$$
$$\dot{z} = -\nabla u \cdot \nu \qquad \text{on } \Gamma, \tag{26d}$$
$$g(u,w) = 0 \qquad \text{on } \Gamma, \tag{26e}$$
$$(u(0), w(0), z(0)) = (u_0, w_0, z_0). \tag{26f}$$

When $g(u,w)=0$ implies $uw=0$, uniqueness for this system follows from uniqueness for the (u,w) system (as z can be treated as uncoupled), which was shown in [20, Theorem 6.3]. There, the authors proved uniqueness by essentially taking the difference of the weak formulations for two solutions and testing with an integral over time of the difference of the two solutions (note that such an approach would not work directly in the evolving space setting since the spatial points are time-dependent too).

Theorem 2.23 (The $\delta_\Omega=\delta_k=\delta_{k'}^{-1}=\delta_\Gamma=\delta_{\Gamma'}$ limit)**.** *Assume* (25)*. As* $\delta_\Omega=\delta_k=\frac{1}{\delta_{k'}}=\delta_\Gamma=\delta_{\Gamma'}\to 0$*, the solution* (u_k,w_k,z_k) *of* (4) *converges to a weak solution* $(u,w,z)\in L^2(0,T;H^1(\Omega))\times(L^2(0,T;L^2(\Gamma)))^2$ *of the problem*

$$\nabla u=0 \qquad \text{in } \Omega, \tag{27a}$$
$$\dot{w}=0 \qquad \text{on } \Gamma, \tag{27b}$$
$$\dot{z}=0 \qquad \text{on } \Gamma, \tag{27c}$$
$$g(u,w)=0 \qquad \text{on } \Gamma, \tag{27d}$$
$$(w(0),z(0))=(w_0,z_0). \tag{27e}$$

Theorem 2.24 (The $\delta_\Omega=\delta_k=\delta_{k'}^{-1}=\delta_\Gamma=\delta_{\Gamma'}$ limit in the Dirichlet case)**.** *Assume that* $u_D\in C^0([0,T];H^{1/2}(\partial D))\cap L^\infty(0,T;L^\infty(\partial D))$ *and*

$$u_n\rightharpoonup u \text{ in } L^2(0,T;H^{1/2}(\Gamma)),\quad u_n\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^\infty(\Gamma)),\quad v_n\to v \text{ in } L^2(0,T;H^{-1/2}(\Gamma)),$$
$$g(u_n,v_n)\to 0 \text{ in } L^1(0,T;L^1(\Gamma)) \implies g(u,v)=0 \text{ in } L^1(0,T;L^1(\Gamma)). \tag{28}$$

As $\delta_\Omega=\delta_k=\frac{1}{\delta_{k'}}=\delta_\Gamma=\delta_{\Gamma'}\to 0$*, the solution* (u_k,w_k,z_k) *of* (4) *with* (4c) *replaced by*

$$u=u_D \quad \text{on } \partial D$$

converges to a weak solution $(u,w,z)\in L^2(0,T;H^1(\Omega))\times(L^2(0,T;L^2(\Gamma)))^2$ *of the problem*

$$\Delta u=0 \qquad \text{in } \Omega, \tag{29a}$$
$$u=u_D \qquad \text{on } \partial D, \tag{29b}$$
$$\dot{w}=\nabla u\cdot\nu \qquad \text{on } \Gamma, \tag{29c}$$
$$\dot{z}=-\nabla u\cdot\nu \qquad \text{on } \Gamma, \tag{29d}$$
$$g(u,w)=0 \qquad \text{on } \Gamma, \tag{29e}$$
$$(w(0),z(0))=(w_0,z_0). \tag{29f}$$

2.6 Biological implications

For the representative forms of g given in (2) and (3), the reason the fast reaction nature of the limits ($\delta_k\to 0$) leads to interesting free boundary problems is because of the complementarity nature of the resulting limit

$$u\geq 0, \qquad w\geq 0, \qquad uw=0 \qquad \text{on } \Gamma(t).$$

In regions on $\Gamma(t)$ where $w>0$ the upshot is that the boundary condition for the ligand corresponds to a zero Dirichlet condition whilst in regions on $\Gamma(t)$ where $w=0$ the boundary condition for the ligand corresponds to a zero Neumann condition (total flux, i.e., diffusive plus advective is zero).

These two boundary conditions are referred to in the biological literature as *perfectly absorbing* and *perfectly reflecting/monitoring* respectively and have been widely studied as relevant simplifications of receptor-ligand interactions; see e.g., [21] and references therein. This work, therefore, gives a rigorous justification for the consideration of these boundary conditions as singular limits of models for receptor-ligand interactions.

The numerical results of Section 8 illustrate the significance of domain evolution, both in terms of generating spatial heterogeneity and effects that arise due to differences between the material velocity of the cell membrane ($\mathbf{V}_\Gamma$) and the material velocity of the extracellular medium ($\mathbf{V}_\Omega$). Such aspects have received limited computational investigation [36, 37] however; symptotic limits of models in cell biology (which have been derived in a number of contexts) have been focused solely on the fixed domain setting. This work provides a framework for extending these analyses to the evolving domain setting thereby increasing their applicability to understanding biological problems.

Furthermore, this work suggests that models for receptor-ligand dynamics on evolving domains involving fast reaction kinetics can be derived using classical elements of free boundary methodology as components of the modelling.

3 Preliminary results on PDEs on evolving spaces

To study the limiting behaviour of the system (7), we need a number of uniform estimates. In this section we derive various technical results and estimates for equations on evolving domains and surfaces that will be used later on for this purpose.

3.1 Estimates on the heat equation on evolving surface

Proposition 3.1. *Let $g \in L^2_{L^2(\Gamma)}$ and $b \in L^\infty_{L^\infty(\Gamma)}$ satisfy $m \leq b \leq M$ a.e. for given constants $m \leq M$. The problem*

$$\begin{aligned} \partial^\bullet y(t) - b(t)\Delta_\Gamma y(t) + \mathbf{J}_\Gamma \cdot \nabla_\Gamma y(t) &= g(t) \quad \text{on } \Gamma(t), \\ y(0) &= 0 \quad \text{on } \Gamma_0, \end{aligned} \tag{30}$$

has a unique solution $y \in \mathbb{W}(H^1(\Gamma), L^2(\Gamma))$ with $\Delta_\Gamma y \in L^2_{L^2(\Gamma)}$ satisfying the estimates

$$\|y\|_{L^\infty_{L^2(\Gamma)}} + \|\partial^\bullet y\|_{L^2_{L^2(\Gamma)}} \leq C(M, m^{-1}) \|g\|_{L^2_{L^2(\Gamma)}}$$

and

$$\|\nabla_\Gamma y\|_{L^\infty_{L^2(\Gamma)}} + \sqrt{m}\, \|\Delta_\Gamma y\|_{L^2_{L^2(\Gamma)}} \leq C(m^{-1}) \|g\|_{L^2_{L^2(\Gamma)}}.$$

Proof. We approximate b by $\{b_\rho\}$ where $\phi_{-(\cdot)} b_\rho \in C^2([0,T] \times \Gamma_0)$ satisfies $m \leq b_\rho \leq M$. Consider the problem

$$\begin{aligned} \partial^\bullet y_\rho(t) - b_\rho(t)\Delta_\Gamma y_\rho(t) + \mathbf{J}_\Gamma \cdot \nabla_\Gamma y_\rho(t) &= g(t) \quad \text{on } \Gamma(t), \\ y_\rho(0) &= 0 \quad \text{on } \Gamma_0. \end{aligned} \tag{31}$$

Arguing similarly as in [1, Lemma 2.13] and with the aid of the existence and regularity result of [3, Theorem 3.13], we obtain a unique solution $y_\rho \in \mathbb{W}(H^1(\Gamma), L^2(\Gamma))$ with $\Delta_\Gamma y_\rho \in L^2_{L^2}$ of (31). It

follows that the equation holds pointwise and we also have the weak formulation

$$\int_0^T \int_{\Gamma(t)} \partial^\bullet y_\rho(t)\eta(t) - \int_0^T \int_{\Gamma(t)} b_\rho(t)\Delta_\Gamma y_\rho(t)\eta(t) + \int_0^T \int_{\Gamma(t)} \mathbf{J}_\Gamma \cdot \nabla_\Gamma y_\rho \eta(t) = \int_0^T \int_{\Gamma(t)} g(t)\eta(t) \tag{32}$$

for all $\eta \in L^2_{H^1(\Gamma)}$. Let us obtain some estimates:

(i) Multiply the equation with $\Delta_\Gamma y_\rho(t)$, integrate over time and manipulate (see Lemma A.2) to give

$$\begin{aligned} -\frac{1}{2}\int_{\Gamma(t)} |\nabla_\Gamma y_\rho(s)|^2 + \frac{1}{2}\int_0^t \int_{\Gamma(s)} \nabla_\Gamma y_\rho(s)^T \mathbf{H}(s) \nabla_\Gamma y_\rho(s) - \int_0^t \int_{\Gamma(s)} b_\rho(s)|\Delta_\Gamma y_\rho(s)|^2 \\ + \int_0^t \int_{\Gamma(s)} \mathbf{J}_\Gamma \cdot \nabla_\Gamma y_\rho(s) \Delta_\Gamma y_\rho(s) = \int_0^t \int_{\Gamma(s)} g(s)\Delta_\Gamma y_\rho(s) \end{aligned}$$

which we further manipulate to get

$$\begin{aligned} \int_{\Gamma(t)} |\nabla_\Gamma y_\rho(s)|^2 + 2m \int_0^t \int_{\Gamma(s)} |\Delta_\Gamma y_\rho(s)|^2 \leq (C_1 + \frac{\|\mathbf{J}_\Gamma\|_\infty}{2\varepsilon_2}) \int_0^t \int_{\Gamma(s)} |\nabla_\Gamma y_\rho(s)|^2 \\ + \int_0^t \int_{\Gamma(s)} \frac{1}{2\epsilon_1}|g(s)|^2 + (2\epsilon_1 + 2\epsilon_2)|\Delta_\Gamma y_\rho(s)|^2 \end{aligned}$$

where we used Young's inequality with ϵ_1 and ϵ_2. Choosing $\epsilon_1 = \epsilon_2 = \frac{1}{4}m$,

$$\int_{\Gamma(t)} |\nabla_\Gamma y_\rho(s)|^2 + m \int_0^t \int_{\Gamma(s)} |\Delta_\Gamma y_\rho(s)|^2 \leq (C_1 + \frac{2\,\|\mathbf{J}_\Gamma\|_\infty}{m}) \int_0^t \int_{\Gamma(s)} |\nabla_\Gamma y_\rho(s)|^2 + \frac{2}{m}\int_0^t \int_{\Gamma(s)} |g(s)|^2,$$

and applying Gronwall's inequality gives

$$\int_{\Gamma(t)} |\nabla_\Gamma y_\rho(s)|^2 \leq \frac{C_2(m^{-1})}{m} \int_0^t \int_{\Gamma(s)} |g(s)|^2.$$

Plugging this back above, we get a similar bound on the Laplacian and we have shown

$$\|\nabla_\Gamma y_\rho\|^2_{L^\infty_{L^2(\Gamma)}} + m\,\|\Delta_\Gamma y_\rho\|^2_{L^2_{L^2(\Gamma)}} \leq C_3(m^{-1})\,\|g\|^2_{L^2_{L^2(\Gamma)}}.$$

(ii) By simply rearranging the equation, we have

$$\begin{aligned} \|\partial^\bullet y_\rho\|_{L^2_{L^2(\Gamma)}} &\leq \|g\|_{L^2_{L^2(\Gamma)}} + M\,\|\Delta_\Gamma y_\rho\|_{L^2_{L^2(\Gamma)}} + \|\mathbf{J}_\Gamma\|_\infty \|\nabla_\Gamma y_\rho\|_{L^2_{L^2(\Gamma)}} \\ &\leq \left(1 + MC_4(m^{-1}) + C_5(m^{-1})\right) \|g\|_{L^2_{L^2(\Gamma)}}. \end{aligned}$$

(iii) Testing the equation with y_ρ and integrating leads to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\Gamma(s)} y_\rho(s)^2 = \int_{\Gamma(s)} g(s)y_\rho(s) + \int_{\Gamma(s)} b_\rho(s)\Delta_\Gamma y_\rho(s) y_\rho(s) + \int_{\Gamma(s)} y_\rho(s)^2 \nabla_\Gamma \cdot \mathbf{V}_p \\ - \int_{\Gamma(t)} \mathbf{J}_\Gamma \cdot \nabla_\Gamma y_\rho y_\rho \end{aligned}$$

from where

$$\int_{\Gamma(t)} y_\rho(t)^2 \leq \int_0^t \int_{\Gamma(s)} g(s)^2 + C_1 \int_0^t \int_{\Gamma(s)} y_\rho(t)^2 + M^2 \int_0^t \int_{\Gamma(s)} |\Delta_\Gamma y_\rho(s)|^2 + C_2 \int_0^t \int_{\Gamma(s)} |\nabla_\Gamma y_\rho|^2$$
$$\leq \left(1 + \frac{C_2 M^2}{m^2} + \frac{C_3}{m}\right) \int_0^t \int_{\Gamma(s)} g(s)^2 + C_1 \int_0^t \int_{\Gamma(s)} y_\rho(s)^2,$$

and Gronwall's inequality yields

$$\|y_\rho\|^2_{L^\infty_{L^2(\Gamma)}} = \int_{\Gamma(t)} y_\rho(t)^2 \leq C\left(1 + C_2(m^{-1})M^2 + C_3(m^{-1})\right) \int_0^t \int_{\Gamma(t)} g(s)^2.$$

Due to these estimates being uniform in ρ, we obtain a limit function $y \in L^\infty_{H^1(\Gamma)} \cap L^2_{H^1(\Gamma)}$ with $\Delta_\Gamma y, \partial^\bullet y \in L^2_{L^2(\Gamma)}$ such that

$$y_\rho \overset{*}{\rightharpoonup} y \text{ in } L^\infty_{H^1(\Gamma)}, \quad \Delta_\Gamma y_\rho \rightharpoonup \Delta_\Gamma y \text{ in } L^2_{L^2(\Gamma)}, \quad \partial^\bullet y_\rho \rightharpoonup \partial^\bullet y \text{ in } L^2_{L^2(\Gamma)},$$

and by compactness we have the stronger convergence

$$y_\rho \to y \text{ in } L^2_{L^2(\Gamma)}.$$

It is then immediate to pass to the limit in the weak form (32) (the Laplacian term can be handled by an argument involving the Dominated Convergence Theorem, making use of the uniform boundedness of b_ρ and the pointwise a.e. convergence of $\phi_{-(\cdot)} b_\rho$ to $\phi_{-(\cdot)} b$) to conclude that the limit function y satisfies for every $\eta \in L^2_{H^1(\Gamma)}$,

$$\int_0^T \int_{\Gamma(t)} \partial^\bullet y(t)\eta(t) - \int_0^T \int_{\Gamma(t)} b(t)\Delta_\Gamma y(t)\eta(t) + \int_0^T \int_{\Gamma(t)} \mathbf{J}_\Gamma \cdot \nabla_\Gamma y \eta(t) = \int_0^T \int_{\Gamma(t)} g(t)\eta(t),$$

i.e., it is a weak solution to (30) with $y(0) = 0$. The proof for the initial condition follows as usual by making use of the stronger convergence $y_\rho \to y$ in $L^2_{L^2(\Gamma)}$. □

Consider the following parabolic inequality

$$\begin{aligned} \partial^\bullet y - D\Delta_\Gamma y + y\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma y) &\leq g \quad \text{on } \Gamma(t), \\ y(0) &= y_0 \quad \text{on } \Gamma_0, \end{aligned} \tag{33}$$

for a given constant $D > 0$, initial data $y_0 \in L^\infty(\Gamma_0)$ and source term $g \in L^\infty_{L^2(\Gamma)}$. By arguing directly as in [5, Lemma 3.1], we obtain the next result.

Lemma 3.2. *If $d \leq 3$, the weak solution of the equation* (33) *given $y_0 \in L^\infty(\Gamma_0)$ and $g \in L^\infty_{L^2(\Gamma)}$ satisfies*

$$\|y\|_{L^\infty_{L^\infty(\Gamma)}} \leq e^{(\|\nabla_\Gamma \cdot \mathbf{V}_\Gamma\|_\infty + D)T} \left(\|y_0\|_{L^\infty(\Gamma_0)} + CD^{-1}\|g\|_{L^\infty_{L^2}}\right).$$

3.2 L^∞-estimate on the heat equation on an evolving domain with Robin boundary conditions

Define the two quantities

$$\|u\|_{Q(\Gamma)} := \max_{t\in[0,T]} \|u(t)\|_{L^2(\Gamma(t))} + \|\nabla_\Gamma u\|_{L^2_{L^2(\Gamma)}}$$

and

$$\|u\|_{Q(\Omega)} := \max_{t\in[0,T]} \|u(t)\|_{L^2(\Omega(t))} + \|\nabla u\|_{L^2_{L^2(\Omega)}}.$$

Let us recall the following interpolation inequality (see [5, Lemma B.2]).

Lemma 3.3. *For $r_* \in [2,\infty]$ and $q_* \in [2, 2d/(d-1)]$ satisfying*

$$\frac{1}{r_*} + \frac{d}{2q_*} = \frac{d+1}{4},$$

we have

$$\|u\|_{L^{r_*}_{L^{q_*}(\Gamma)}} \leq \sqrt{C_I}\, \|u\|_{Q(\Omega)}.$$

The proof of the next result is very similar to what is presented in [5, §3.3] (which itself was based on [45, Proposition 3.1]), but we give it here adapted to a slightly more general equation. Note that a similar result is given in [8] but there the geometric set up is different and here we need to keep track of how the bound depends on the coefficients and data of the problem.

Lemma 3.4. *Let $y \in L^\infty_{L^\infty(\Gamma)}$, $a_0 \in L^\infty(\Omega_0)$ and let a be the nonnegative solution to*

$$\begin{aligned} \partial^\bullet a + a\nabla\cdot\mathbf{V}_p - D\Delta a + \nabla\cdot(\mathbf{J}_\Omega a) &= 0 && \text{in } \Omega(t),\\ D\nabla a\cdot\nu - aj &= y && \text{on } \Gamma(t),\\ \nabla a\cdot\nu &= 0 && \text{on } \partial D(t),\\ a(0) &= a_0 && \text{in } \Omega_0. \end{aligned}$$

Then $a \in L^\infty_{L^\infty(\Omega)}$ and

$$\begin{aligned} a(t) \leq\ & C_1 e^{\|\nabla\cdot\mathbf{V}_\Omega\|_\infty T + D^{-1}C_2\left(\frac{1}{2}\|y\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right)^2 T} \max\left(1, \|a_0\|_\infty, \frac{1}{2}C_3\|y\|_{L^\infty_{L^\infty(\Gamma)}}\right)\\ & \times (\min(1,D)^{-\frac{1}{2\kappa}-\frac{1}{2}} + C_4) \end{aligned}$$

where C_1, C_2, C_3 and C_4 are constants independent of all relevant parameters.

Proof. With the transformation $A := ae^{-\lambda t}$ for a λ to be fixed later, we have

$$\begin{aligned} \partial^\bullet A + A(\nabla\cdot\mathbf{V}_p + \lambda) - D\Delta A + \nabla\cdot(\mathbf{J}_\Omega A) &= 0 && \text{in } \Omega(t),\\ D\nabla A\cdot\nu - Aj &= Y && \text{on } \Gamma(t),\\ \nabla A\cdot\nu &= 0 && \text{on } \partial D(t),\\ A(0) &= a_0 && \text{in } \Omega_0, \end{aligned}$$

where $Y(t) := y(t)e^{-\lambda t}$. Testing the equation for A with $A_k := (A-k)^+$ for a constant k, using the boundary condition as well as (57), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} A_k^2 - \frac{1}{2}\int_{\Omega(t)} A_k^2\nabla\cdot\mathbf{V}_\Omega + \int_{\Omega(t)} AA_k(\nabla\cdot\mathbf{V}_\Omega + \lambda) + D|\nabla A_k|^2 + \frac{1}{2}\int_{\Gamma(t)} A_k^2 j\\ &\quad = \int_{\Gamma(t)} (Y + jA)A_k. \end{aligned}$$

Define the set

$$B_k(s) := \{x \in \Gamma(s) : A_k(s) \geq 0\}.$$

Taking $k \geq 1$, using $AA_k = A_k^2 + kA_k \leq \frac{3}{2}A_k^2 + \frac{1}{2}k^2$ and $A_k = \frac{1}{k}kA_k \leq \frac{1}{k}(\frac{k^2}{2} + \frac{A_k^2}{2})$ we get that the difference of the two integrals over $\Gamma(t)$ can be written as

$$\begin{aligned}\int_{\Gamma(t)} (Y + jA)A_k - \frac{1}{2}A_k^2 j &\leq \int_{B_k(t)} \|Y\|_{L^\infty_{L^\infty(\Gamma)}} \frac{1}{k}\frac{k^2}{2} + \|Y\|_{L^\infty_{L^\infty(\Gamma)}} \frac{1}{k}\frac{A_k^2}{2} + \|j\|_\infty \frac{3}{2}A_k^2 + \|j\|_\infty \frac{1}{2}k^2 \\ &\quad + \frac{1}{2}\|j\|_\infty A_k^2 \\ &\leq \int_{B_k(t)} \frac{\frac{1}{2}\|Y\|_{L^\infty_{L^\infty(\Gamma)}}}{k} k^2 + \left(\frac{1}{2}\|Y\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right) A_k^2 + \frac{1}{2}\|j\|_\infty k^2.\end{aligned}$$

(using $k \geq 1$ on the second term above)

Now we apply the interpolated trace inequality to bound the second term above as

$$\int_{\Gamma(t)} \left(\frac{1}{2}\|Y\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right) A_k^2 \leq \frac{D}{2}\int_{\Omega(t)} |\nabla A_k|^2 + D^{-1}K_1 \left(\frac{1}{2}\|Y\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right)^2 \int_{\Omega(t)} A_k^2$$

where K_1 is a constant independent of all relevant quantities. Picking

$$\lambda := \|\nabla \cdot \mathbf{V}_\Omega\|_\infty + D^{-1}K_1 \left(\frac{1}{2}\|y\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right)^2,$$

using $AA_k = A_k^2 + kA_k \geq A_k^2$, supposing that $k \geq \|a_0\|_{L^\infty(\Omega_0)}$ and plugging the last two inequalities back in the equation above, we get with $m := \min(1, D)$,

$$\frac{m}{2}\|A_k\|^2_{Q(\Omega)} \leq \left(\frac{\frac{1}{2}\|Y\|_{L^\infty_{L^\infty(\Gamma)}}}{k} + \frac{1}{2}\|j\|_\infty\right) k^2 \int_0^t \int_{B_k(s)}. \tag{34}$$

Now, take exponents r and q such that

$$\frac{1}{r} + \frac{d}{2q} < \frac{1}{2}$$

and define κ, r_* and q_* by

$$\frac{1}{r} + \frac{d}{2q} = \frac{1}{2} - \frac{\kappa(d+1)}{2}, \qquad r_* = \frac{2(1+\kappa)r}{r-1}, \qquad \text{and} \qquad q_* = \frac{2(1+\kappa)q}{q-1}$$

(note that $\kappa > 0$). By Hölder's inequality, $\int_0^T \int_{B_k(s)} \leq C_1 \|\chi_{B_k}\|^{2(1+\kappa)}_{L^{r_*}_{L^{q_*}(\Gamma)}}$ where we set $C_1 := |\Gamma|^{1/q}T^{1/r}$. Hence (34) becomes

$$\frac{m}{2}\|A_k\|^2_{Q(\Omega)} \leq \left(\frac{\frac{1}{2}C_1\|Y\|_{L^\infty_{L^\infty(\Gamma)}}}{k} + \frac{1}{2}C_1\|j\|_\infty\right) k^2 \|\chi_{B_k}\|^{2(1+\kappa)}_{L^{r_*}_{L^{q_*}(\Gamma)}}.$$

Taking further

$$k \geq \max\left(1, \|a_0\|_\infty, \frac{1}{2}C_1\|Y\|_{L^\infty_{L^\infty(\Gamma)}}\right) \tag{35}$$

and setting $C_2 := \left(1 + \frac{C_1}{2}\|j\|_\infty\right)$, we deduce

$$\frac{m}{2}\|A_k\|^2_{Q(\Omega)} \leq C_2 k^2 \left(\int_0^T |B_k|^{r_*/q_*}\right)^{2(1+\kappa)/r_*}. \tag{36}$$

Define for a sufficiently large N the sequences $\{k_n\}$ and $\{z_n\}$ via

$$k_n := (2-2^{-n})N \quad \text{and} \quad z_n := \left(\int_0^T |B_{k_n}(t)|^{r_*/q_*}\right)^{2/r_*}.$$

Now, we have from the above

$$\|A_{k_n}\|^2_{Q(\Omega)} \leq 2C_2 m^{-1} k_n^2 z_n^{1+\kappa}. \tag{37}$$

On $\Gamma(t)$, $|A_{k_n}|^2 = |(A-k_n)^+|^2 \geq (k_{n+1}-k_n)^2 \chi_{B_{k_{n+1}}}$, which implies that

$$2^{-2(n+1)} N^2 z_{n+1} \leq \left(\int_0^T \left(\int_{\Gamma(t)} |(A(t)-k_n)^+|^{q_*}\right)^{r_*/q_*}\right)^{2/r_*} \leq C_I \left\|(A-k_n)^+\right\|^2_{Q(\Omega)}$$

with the last inequality by the interpolation inequality of Lemma 3.3. Therefore, using (37) and $k_n^2 \leq N^2(2^2+2^{-2n})$,

$$z_{n+1} \leq 2^6 m^{-1} C_2 C_I 4^n z_n^{1+\kappa}.$$

Now if we take $\hat{k}$ such that it satisfies (35), for $N > \hat{k}$, we calculate

$$\begin{aligned}
(N-\hat{k})^2 z_0 &= \left(\int_0^T \left(\int_{\Gamma(t)} (N-\hat{k})^{q_*} \chi_{B_{k_0}}(t)\right)^{r_*/q_*}\right)^{2/r_*} \\
&\leq \left(\int_0^T \left(\int_{\Gamma(t)} |(A(t)-\hat{k})^+|^{q_*}\right)^{r_*/q_*}\right)^{2/r_*} \\
&\qquad\qquad \text{(since } |(A-\hat{k})^+|^2 \geq (N-\hat{k})^2 \chi_{B_{k_0}} \text{ because } k_0 = N) \\
&\leq C_I \left\|(A-\hat{k})^+\right\|^2_{Q(\Omega)} \\
&\leq 2C_2 m^{-1} C_I \hat{k}^2 C_3 \qquad \text{(using (36))}
\end{aligned}$$

where we defined $C_3 := |\Gamma|^{2(1+\kappa)/q_*} T^{2(1+\kappa)/r_*}$. Picking $N = \hat{k}(\sqrt{C_3} 2^{\frac{1}{2}+\frac{3}{\kappa}+\frac{1}{\kappa^2}} (m^{-1}C_2C_I)^{\frac{1}{2\kappa}+\frac{1}{2}} + 1)$, we have

$$z_0 \leq (2^6 m^{-1} C_2 C_I)^{-1/\kappa} 4^{-1/\kappa^2}.$$

By [30, II, Lemma 5.6], we get $z_n \to 0$ as $n \to \infty$. Since $k_n \to 2N$, we obtain from (37) that $A(t) \leq 2N$ almost everywhere on $\Omega(t)$. Putting everything together, we find the stated bound.

□

4 Uniform bounds

In this section, we aim to get a variety of bounds on the solutions of (7) and related quantities that we can use later on. Note that all of these bounds do not *explicitly* depend on δ_k but do depend on L^p norms of z and w.

Lemma 4.1. *We have*

$$\frac{1}{\delta_k} \int_0^T \int_{\Gamma(t)} g(u,w) \leq \frac{1}{\delta_{k'}} \|z\|_{L^1_{L^1(\Gamma)}} + \delta_\Omega \|u_0\|_{L^2(\Omega)}. \tag{38}$$

Proof. Utilising the transport theorem and the equation for u, we derive

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)} u &= \int_{\Omega(t)} \partial^\bullet u + u\nabla_\Gamma \cdot \mathbf{V}_p \\ &= -\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u) + \frac{1}{\delta_\Omega}\int_{\Omega(t)} \Delta u \\ &= -\int_{\Gamma(t)} ju + \frac{1}{\delta_\Omega}\int_{\Gamma(t)} \nabla u\cdot\nu \qquad \text{(using the divergence theorem)} \\ &= \frac{1}{\delta_\Omega\delta_{k'}}\int_{\Gamma(t)} z - \frac{1}{\delta_\Omega\delta_k}\int_{\Gamma(t)} g(u,w),\end{aligned}$$

thus

$$\frac{1}{\delta_\Omega\delta_k}\int_0^T\int_{\Gamma(t)} g(u,w) = \frac{1}{\delta_\Omega\delta_{k'}}\int_0^T\int_{\Gamma(t)} z + \int_{\Omega_0} u_0 - \int_{\Omega(T)} u(T),$$

which implies the result. □

Lemma 4.2 (Energy estimate on w)**.** *We have*

$$\|w\|^2_{L^\infty_{L^2(\Gamma)}} \le e^{T(\frac{1}{\delta_{k'}}+\|\nabla_\Gamma\cdot\mathbf{V}_\Gamma\|_\infty)}\left(\frac{1}{\delta_{k'}}\|z\|^2_{L^2_{L^2(\Gamma)}} + \|w_0\|^2_{L^2(\Gamma_0)}\right),$$

$$\begin{aligned}2\delta_\Gamma\|\nabla_\Gamma w\|^2_{L^2_{L^2(\Gamma)}} + \frac{2}{\delta_k}\int_0^T\int_{\Gamma(t)} g(u,w)w &\le \frac{1}{\delta_{k'}}(\|w\|^2_{L^2_{L^2(\Gamma)}} + \|z\|^2_{L^2_{L^2(\Gamma)}}) + \|\nabla_\Gamma\cdot\mathbf{V}_\Gamma\|_\infty\|w\|^2_{L^2_{L^2(\Gamma)}} \\ &\quad + \|w_0\|^2_{L^2(\Gamma_0)}.\end{aligned}$$

Proof. Test the equation for w by w (and use the integration by parts identity (62)) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} w^2 + \delta_\Gamma\int_{\Gamma(t)}|\nabla_\Gamma w|^2 + \frac{1}{2}\int_{\Gamma(t)} w^2\nabla_\Gamma\cdot\mathbf{V}_\Gamma + \frac{1}{\delta_k}\int_{\Gamma(t)} g(u,w)w &= \frac{1}{\delta_{k'}}\int_{\Gamma(t)} zw \\ &\le \frac{1}{2\delta_{k'}}\int_{\Gamma(t)} z^2 + w^2.\end{aligned}$$

Integrate now over $[0,t]$ and manipulate to obtain

$$\begin{aligned}\int_{\Gamma(t)} w(t)^2 + 2\delta_\Gamma\int_0^t\int_{\Gamma(s)}|\nabla_\Gamma w|^2 + \frac{2}{\delta_k}\int_0^t\int_{\Gamma(s)} g(u,w)w &\le \frac{1}{\delta_{k'}}\int_0^t\int_{\Gamma(s)} z^2 + w^2 \\ &\quad + \int_0^t\int_{\Gamma(s)} w^2|\nabla_\Gamma\cdot\mathbf{V}_\Gamma| + \int_{\Gamma_0} w_0^2\end{aligned}$$

and here we use Gronwall's lemma. □

Lemma 4.3 (Energy estimate on u)**.** *We have*

$$\|u\|^2_{L^\infty_{L^2(\Omega)}} \le \left(\frac{\|z\|^2_{L^2_{L^2(\Gamma)}}}{\delta_\Omega\delta_{k'}} + \|u_0\|^2_{L^2(\Omega_0)}\right) e^{\frac{1}{\delta_\Omega}\left(C\left(\frac{1}{\delta_{k'}}+\delta_\Omega\|j\|_\infty\right)^2+\delta_\Omega\|\nabla\cdot\mathbf{V}_\Omega\|_\infty\right)T}, \tag{39}$$

$$\begin{aligned}&\|\nabla u\|^2_{L^2_{L^2(\Omega)}} + \frac{2}{\delta_k}\int_0^T\int_{\Gamma(t)} g(u,w)u \\ &\le \frac{\|z\|^2_{L^2_{L^2(\Gamma)}}}{\delta_{k'}} + \left(C\left(\frac{1}{\delta_{k'}}+\delta_\Omega\|j\|_\infty\right)^2 + \delta_\Omega\|\nabla\cdot\mathbf{V}_\Omega\|_\infty\right)\|u\|^2_{L^2_{L^2(\Omega)}} + \delta_\Omega\|u_0\|^2_{L^2(\Omega_0)}.\end{aligned}$$

Proof. Testing the equation for u by u (and using the integration by parts formula (59)), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} u^2 + \frac{1}{\delta_\Omega}\int_{\Omega(t)} |\nabla u|^2 + \frac{1}{2}\int_{\Omega(t)} u^2 \nabla\cdot \mathbf{V}_\Omega \leq \frac{1}{\delta_\Omega}\left(\frac{1}{\delta_{k'}}\int_{\Gamma(t)} \frac{1}{2}z^2 + \frac{1}{2}u^2 - \frac{1}{\delta_k}\int_{\Gamma(t)} g(u,w)u\right) + \frac{1}{2}\int_{\Gamma(t)} u^2 j.$$

Re-arranging and integrating over $[0,t]$ leads to, denoting

$$\theta := \frac{1}{\delta_{k'}} + \delta_\Omega \|j\|_\infty ,$$

the expression

$$\delta_\Omega \int_{\Omega(s)} u(t)^2 + 2\int_0^t \int_{\Omega(s)} |\nabla u|^2 + \frac{2}{\delta_k}\int_0^t \int_{\Gamma(s)} g(u,w)u \leq \frac{\|z\|^2_{L^2_{L^2(\Gamma)}}}{\delta_{k'}} + \theta \int_0^t \int_{\Gamma(s)} u^2 + \delta_\Omega \|\nabla\cdot \mathbf{V}_\Omega\|_\infty \int_0^t \int_{\Omega(s)} u^2 + \delta_\Omega \int_{\Omega_0} u_0^2.$$

Now by the interpolated trace theorem we have

$$\theta \int_{\Gamma(s)} u^2 \leq C\theta^2 \int_{\Omega(s)} u^2 + \int_{\Omega(s)} |\nabla u|^2.$$

Inserting this above, we find

$$\delta_\Omega \int_{\Omega(s)} u(t)^2 + \int_0^t \int_{\Omega(s)} |\nabla u|^2 + \frac{2}{\delta_k}\int_0^t \int_{\Gamma(s)} g(u,w)u \leq \frac{\|z\|^2_{L^2_{L^2(\Gamma)}}}{\delta_{k'}} + \left(C\theta^2 + \delta_\Omega \|\nabla\cdot \mathbf{V}_\Omega\|_\infty\right) \int_0^t \int_{\Omega(s)} u^2 + \delta_\Omega \int_{\Omega_0} u_0^2.$$

Gronwall's inequality now finally implies the result. □

The equation for the sum $v = w + z$ is

$$\partial^\bullet v + v\nabla_\Gamma \cdot \mathbf{V}_p - \delta_{\Gamma'}\Delta_\Gamma v + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) = (\delta_\Gamma - \delta_{\Gamma'})\Delta_\Gamma w, \qquad v(0) = w_0 + z_0.$$

Now that we have a bound for $\nabla_\Gamma w$, we can test the above equation with v and use Young's inequality on the right-hand side and the above bound on w to get a bound independent of δ_k. Though we do already have Lemma 4.2, the following is useful because it shows the influence of the diffusion constants and implies in particular the corollary following the result.

Lemma 4.4 (Energy estimate on v)**.** *We have*

$$\|v\|^2_{L^\infty_{L^2(\Gamma)}} \leq e^{\|\nabla_\Gamma\cdot \mathbf{V}_\Gamma\|_\infty T}\left(\frac{|\delta_\Gamma - \delta_{\Gamma'}|^2}{\delta_{\Gamma'}}\|\nabla_\Gamma w\|^2_{L^2_{L^2(\Gamma)}} + \|v_0\|^2_{L^2(\Gamma_0)}\right),$$

$$\delta'_\Gamma \|\nabla_\Gamma v\|^2_{L^2_{L^2(\Gamma)}} \leq \frac{|\delta_\Gamma - \delta_{\Gamma'}|^2}{\delta_{\Gamma'}}\|\nabla_\Gamma w\|^2_{L^2_{L^2(\Gamma)}} + \|v_0\|^2_{L^2(\Gamma_0)} + \|\nabla_\Gamma\cdot \mathbf{V}_\Gamma\|_\infty \|v\|^2_{L^2_{L^2(\Gamma)}}.$$

Proof. From the equation for v and Young's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} v(t)^2 + \delta_{\Gamma'}\int_{\Gamma(t)} |\nabla_\Gamma v|^2 + \frac{1}{2}\int_{\Gamma(t)} v^2 \nabla_\Gamma \cdot \mathbf{V}_\Gamma \leq \frac{|\delta_\Gamma - \delta_{\Gamma'}|}{2\varepsilon}\int_{\Gamma(t)} |\nabla_\Gamma w|^2 + \frac{\varepsilon|\delta_\Gamma - \delta_{\Gamma'}|}{2}\int_{\Gamma(t)} |\nabla_\Gamma v|^2,$$

and choosing $\varepsilon = \delta_{\Gamma'}/|\delta_\Gamma - \delta_{\Gamma'}|$, we obtain

$$\frac{d}{dt}\int_{\Gamma(t)} v(t)^2 + \delta_{\Gamma'}\int_{\Gamma(t)} |\nabla_\Gamma v|^2 \leq \frac{|\delta_\Gamma - \delta_{\Gamma'}|^2}{\delta_{\Gamma'}}\int_{\Gamma(t)} |\nabla_\Gamma w|^2 + \|\nabla_\Gamma \cdot \mathbf{V}_\Gamma\|_\infty \int_{\Gamma(t)} v^2.$$

Gronwall's inequality gives the claim. □

Corollary 4.5. *We have*

$$\|w\|^2_{L^\infty_{L^2(\Gamma)}} + \|z\|^2_{L^\infty_{L^2(\Gamma)}} \leq C\left(\frac{|\delta_\Gamma - \delta_{\Gamma'}|^2}{\delta_{\Gamma'}}\|\nabla_\Gamma w\|^2_{L^2_{L^2(\Gamma)}} + \|v_0\|^2_{L^2(\Gamma_0)}\right).$$

Proof. This follows directly from Lemma 4.4 and the non-negativity of z and w. □

Now we look for an L^∞-estimate on w.

Corollary 4.6. *If $d \leq 3$ and $w_0 \in L^\infty(\Omega_0)$, we have*

$$\|w\|_{L^\infty_{L^\infty(\Gamma)}} \leq e^{(\|\nabla_\Gamma \cdot \mathbf{V}_\Gamma\|_\infty + \delta_\Gamma)T}\left(\|w_0\|_{L^\infty(\Gamma_0)} + \frac{C}{\delta_\Gamma \delta_{k'}}\|z\|^2_{L^\infty_{L^2(\Gamma)}}\right).$$

Proof. This follows from the De Giorgi estimate in Lemma 3.2. □

Corollary 4.7. *We have*

$$\delta'_\Gamma \|\nabla_\Gamma z\|^2_{L^2_{L^2(\Gamma)}} \leq 2\left(\frac{|\delta_\Gamma - \delta_{\Gamma'}|^2}{\delta_{\Gamma'}}\|\nabla_\Gamma w\|^2_{L^2_{L^2(\Gamma)}} + \|v_0\|^2_{L^2(\Gamma_0)} + \|\nabla_\Gamma \cdot \mathbf{V}_\Gamma\|_\infty \|v\|^2_{L^2_{L^2(\Gamma)}}\right) + \frac{2\delta_{\Gamma'}}{\delta_\Gamma}\left(\frac{1}{\delta_{k'}}(\|w\|^2_{L^2_{L^2(\Gamma)}} + \|z\|^2_{L^2_{L^2(\Gamma)}}) + \|\nabla_\Gamma \cdot \mathbf{V}_\Gamma\|_\infty \|w\|^2_{L^2_{L^2(\Gamma)}} + \|w_0\|^2_{L^2(\Gamma_0)}\right).$$

Proof. This is due to $z = v - w$ and the gradient bounds in Lemma 4.2 and Lemma 4.4.

□

4.1 A more refined bound

The most obvious way to get uniform L^2 bounds on w and z is to test the w and z equations with the respective solutions as done in the previous section. There we saw that for w, we need to control $\frac{1}{\delta_{k'}}\int_{\Gamma(t)} zw$ and so we would need a bound on z. The z equation however would require us to control $\frac{1}{\delta_k}\int_{\Gamma(t)} g(u,w)z$; to do this we need more refined bounds. Hence, let us try a different approach in which we can essentially'cancel' the right-hand side terms.

Recall that $v = w + z$ solves the equation

$$\partial^\bullet v + v\nabla_\Gamma \cdot \mathbf{V}_p - \delta_{\Gamma'}\Delta_\Gamma v + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) = (\delta_\Gamma - \delta_{\Gamma'})\Delta_\Gamma w,$$
$$v(0) = w_0 + z_0.$$

A different form of this equation is useful here because to get a bound on v by testing the above equation with v (as would be the natural first step), we would need control of the gradient of w; but to do that we would (see Lemma 4.2) need to control $\int_{\Gamma(t)} zw$ in an appropriate way.

Define the measurable function

$$a(t,x) := \begin{cases} (\delta_\Gamma w + \delta'_\Gamma z)/(w+z) & : w + z > 0 \\ 1 & : w + z = 0. \end{cases}$$

Note that

$$m \le a(t,x) \le M$$

where

$$m =: \min(1, \delta_\Gamma, \delta'_\Gamma) \qquad \text{and} \qquad M := \max(1, \delta_\Gamma, \delta'_\Gamma),$$

so the coefficient is bounded away from zero and infinity. We see that v solves the equation

$$\begin{aligned} \partial^\bullet v + v\nabla_\Gamma \cdot \mathbf{V}_p - \Delta(a(t)v) + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) &= 0, \\ v(0) &= w_0 + z_0. \end{aligned} \tag{40}$$

We have

$$\|v\|_{L^2_{L^2(\Gamma)}} = \sup_{\substack{\varphi \in L^2_{L^2(\Gamma)} \\ \|\varphi\|_{L^2(L^2)}=1}} \int_0^T \int_{\Gamma(t)} v\varphi.$$

For each such φ, take η to be the solution of the backwards heat equation

$$\begin{aligned} \partial^\bullet \eta(t) + a(t)\Delta_\Gamma \eta + \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta &= \varphi \quad \text{on } \Gamma(t), \\ \eta(T) &= 0 \quad \text{on } \Gamma(T). \end{aligned} \tag{41}$$

Then the above becomes

$$\begin{aligned} \|v\|_{L^2_{L^2(\Gamma)}} &= \sup_{\substack{\varphi \in L^2_{L^2(\Gamma)} \\ \|\varphi\|=1}} \int_0^T \int_{\Gamma(t)} v(\partial^\bullet \eta(t) + a(t)\Delta_\Gamma \eta + \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta) \\ &= \sup_{\substack{\varphi \in L^2_{L^2(\Gamma)} \\ \|\varphi\|=1}} -\int_{\Gamma_0} v(0)\eta(0) + \int_0^T \int_{\Gamma(t)} -\partial^\bullet v\eta - v\eta\nabla_\Gamma \cdot \mathbf{V}_p + \Delta_\Gamma(a(t)v)\eta - \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v)\eta \\ &\qquad \text{(using the divergence theorem identity (60))} \\ &= -\sup_{\substack{\varphi \in L^2_{L^2(\Gamma)} \\ \|\varphi\|=1}} \int_{\Gamma_0} v(0)\eta(0) \qquad \text{(since } v \text{ solves (40))} \\ &\le \|w_0 + z_0\|_{L^2(\Gamma_0)} \sup_{\substack{\varphi \in L^2_{L^2(\Gamma)} \\ \|\varphi\|=1}} \|\eta(0)\|_{L^2(\Gamma_0)} \\ &\le \|w_0 + z_0\|_{L^2(\Gamma_0)} \sup_{\substack{\varphi \in L^2_{L^2(\Gamma)} \\ \|\varphi\|=1}} \|\eta\|_{C^0_{L^2(\Gamma)}}. \end{aligned} \tag{42}$$

Hence we need to estimate the above norm of η. First, let us study (41).

Lemma 4.8. *We have* $\eta \in \mathbb{W}(H^1(\Gamma), L^2(\Gamma))$ *and*

$$\|\eta\|_{L^\infty_{L^2(\Gamma)}} + \|\nabla_\Gamma \eta\|_{L^\infty_{L^2(\Gamma)}} + \|\Delta_\Gamma \eta\|_{L^2_{L^2(\Gamma)}} + \|\partial^\bullet \eta\|_{L^2_{L^2(\Gamma)}} \leq C \|\varphi\|_{L^2_{L^2(\Gamma)}}$$

where $C = C(M, m^{-1})$.

Proof. By reversing time, i.e. setting $\mathbf{J} = -\mathbf{J}_\Gamma(T-t)$, $\vartheta(t) = \eta(T-t)$, $b(t) = a(T-t)$ and $g(t) = -\varphi(T-t)$, and defining the family of surfaces

$$\hat{\Gamma}(t) := \Gamma(T-t)$$

and the parametric material derivative

$$\hat{\partial^\bullet} \vartheta(t) := \vartheta_t(t) + \nabla \vartheta(t) \cdot \hat{\mathbf{V}}(t), \qquad \hat{\mathbf{V}}(t) = -\mathbf{V}(T-t),$$

the problem (41) is equivalent to

$$\begin{aligned} \hat{\partial^\bullet} \vartheta(t) - b(t)\Delta_\Gamma \vartheta(t) + \mathbf{J} \cdot \nabla_\Gamma \vartheta &= g(t) \quad \text{on } \hat{\Gamma}(t), \\ \vartheta(0) &= 0 \qquad \text{on } \hat{\Gamma}_0, \end{aligned}$$

with $b \in L^\infty_{L^\infty(\hat{\Gamma})}$ and $g \in L^2_{L^2(\hat{\Gamma})}$. The claim follows from Proposition 3.1. □

We finally come to the bound, which is uniform in δ_k and $\delta_{k'}$ (but does depend on the diffusion coefficients).

Lemma 4.9. *The following bound holds*

$$\|v\|_{L^2_{L^2(\Gamma)}} + \|w\|_{L^2_{L^2(\Gamma)}} + \|z\|_{L^2_{L^2(\Gamma)}} \leq C(M, m^{-1}) \|w_0 + z_0\|_{L^2(\Gamma_0)}$$

where $m := \min(1, \delta_\Gamma, \delta'_\Gamma)$ *and* $M := \max(1, \delta_\Gamma, \delta'_\Gamma)$.

Proof. Using simply the continuous embedding $\mathbb{W}(H^1(\Gamma), H^1(\Gamma)^*) \hookrightarrow C^0_{L^2(\Gamma)}$ and the estimate of Lemma 4.8, we easily derive

$$\|\eta\|_{C^0_{L^2(\Gamma)}} \leq C \|\varphi\|_{L^2_{L^2(\Gamma)}}$$

where $C = C(M, m^{-1})$. Utilising this and the characterisation in (42),

$$\|v\|_{L^2_{L^2(\Gamma)}} \leq C \|w_0 + z_0\|_{L^2(\Gamma_0)},$$

so that v and hence w and z are bounded in $L^2_{L^2(\Gamma)}$ uniformly in δ_k and δ_k. □

4.2 Bound on the difference quotients of w

In the previous sections we established bounds for w and z in $L^2_{L^2(\Gamma)}$ which are independent of both δ_k and δ'_k, but dependent on the diffusion coefficients δ_Γ and δ'_Γ. These should enable us to obtain bounds on the difference quotients for the pullback of w; this will come in use to obtain strong convergence by application of the Aubin–Lions–Simon compactness theorem, see Theorem 5.1.

We introduce the following positive-definite (with a constant that is uniform in time) matrix and its determinant

$$\mathbf{A}_t := (\mathbf{D}_{\Gamma_0}\Phi_t)^{\mathcal{T}}\mathbf{D}_{\Gamma_0}\Phi_t + \nu_0 \otimes \nu_0, \qquad a_t := \det \mathbf{A}_t.$$

Its inverse has the expression [10, Proposition 4.1])

$$(\mathbf{A}_t)^{-1} = \phi_{-t}((\mathbf{D}_{\Gamma(t)}\Phi_0^t)(\mathbf{D}_{\Gamma(t)}\Phi_0^t)^T) + \nu_0 \otimes \nu_0,$$

which is utilised in the following expression for the pullback of the gradient of a sufficiently smooth function $y\colon \Gamma(t) \to \mathbb{R}$:

$$\phi_{-t}\left(\nabla_{\Gamma(t)} y\right) = \mathbf{D}_{\Gamma_0}\Phi_t(\mathbf{A}_t)^{-1}\nabla_{\Gamma_0}\left(\phi_{-t} y\right).$$

Lemma 4.10. *The pullback $\tilde{w} := \phi_{-(\cdot)}w$ of w satisfies*

$$\int_{\Gamma_0} \tilde{w}'\psi + \delta_\Gamma \int_{\Gamma_0} [\mathbf{B}(t)\nabla_\Gamma \tilde{w}] \cdot \nabla_\Gamma \psi + J_t \psi [\mathbf{B}(t)\nabla_\Gamma \tilde{w}] \cdot \nabla_\Gamma (1/J_t) + \int_{\Gamma_0} \tilde{w}\psi\phi_{-t}(\nabla_\Gamma \cdot \mathbf{V}_\Gamma)$$
$$+ \int_{\Gamma_0} \phi_{-t}(\mathbf{J}_\Gamma) \cdot \mathbf{C}(t)\nabla_\Gamma \tilde{w}\psi = \int_{\Gamma_0} \left(\frac{\tilde{z}}{\delta_{k'}} - \frac{g(\tilde{u},\tilde{w})}{\delta_k}\right)\psi$$

for all $\psi \in H^1(\Gamma_0)$ where

$$\mathbf{B}(t) := (\mathbf{A}_t)^{-T}(\mathbf{D}_{\Gamma_0}\Phi_t)^T\mathbf{D}_{\Gamma_0}\Phi_t(\mathbf{A}_t)^{-1} \qquad \textit{and} \qquad \mathbf{C}(t) := \mathbf{D}_{\Gamma_0}\Phi_t(\mathbf{A}_t)^{-1}.$$

Proof. The weak formulation for w can be written as

$$\int_{\Gamma(t)} \partial^\bullet w\eta + \delta_\Gamma \int_{\Gamma(t)} \nabla_\Gamma w \nabla_\Gamma \eta + \int_{\Gamma(t)} w\eta\nabla_\Gamma \cdot \mathbf{V}_\Gamma + \mathbf{J}_\Gamma \cdot \nabla_\Gamma w\eta = \int_{\Gamma(t)} \left(\frac{z}{\delta_{k'}} - \frac{g(u,w)}{\delta_k}\right)\eta$$

for every $\eta \in L^2_{H^1(\Gamma)}$. Setting $\tilde{w} := \phi_{-(\cdot)}w$, supposing that $\eta = \phi_t\tilde{\varphi}$ for arbitrary $\tilde{\varphi} \in H^1(\Gamma_0)$, and defining $J_t := |\mathbf{D}_{\Gamma_0}\Phi_t|$, we can pull back the integrals onto Γ_0:

$$\int_{\Gamma_0} \tilde{w}'\tilde{\varphi}J_t + \delta_\Gamma \int_{\Gamma_0} \phi_{-t}(\nabla_\Gamma w \nabla_\Gamma \eta)J_t + \int_{\Gamma_0} \tilde{w}\tilde{\varphi}\phi_{-t}(\nabla_\Gamma \cdot \mathbf{V}_\Gamma)J_t + \phi_{-t}(\mathbf{J}_\Gamma \cdot \nabla_\Gamma w)\tilde{\varphi}J_t$$
$$= \int_{\Gamma_0} \left(\frac{\tilde{z}}{\delta_{k'}} - \frac{g(\tilde{u},\tilde{w})}{\delta_k}\right)\tilde{\varphi}J_t.$$

Thus with $\mathbf{C}(t) := \mathbf{D}_{\Gamma_0}\Phi_t(\mathbf{A}_t)^{-1}$ and writing $\nabla_\Gamma\tilde{\varphi} = \nabla_\Gamma(\tilde{\varphi}J_t/J_t)$,

$$\int_{\Gamma_0} \tilde{w}'\tilde{\varphi}J_t + \delta_\Gamma \int_{\Gamma_0} J_t[\mathbf{B}(t)\nabla_\Gamma \tilde{w}] \cdot \nabla_\Gamma(\tilde{\varphi}J_t/J_t) + \int_{\Gamma_0} \tilde{w}\tilde{\varphi}\phi_{-t}(\nabla_\Gamma \cdot \mathbf{V}_\Gamma)J_t + \phi_{-t}(\mathbf{J}_\Gamma) \cdot \mathbf{C}(t)\nabla_\Gamma \tilde{w}\tilde{\varphi}J_t$$
$$= \int_{\Gamma_0} \left(\frac{\tilde{z}}{\delta_{k'}} - \frac{g(\tilde{u},\tilde{w})}{\delta_k}\right)\tilde{\varphi}J_t.$$

Now since $\tilde{\varphi}$ is arbitrary and multiplication by J_t is an isomorphism from $H^1(\Gamma_0)$ to itself, we obtain

$$\int_{\Gamma_0} \tilde{w}'\psi + \delta_\Gamma \int_{\Gamma_0} J_t[\mathbf{B}(t)\nabla_\Gamma \tilde{w}] \cdot \nabla_\Gamma(\psi/J_t) + \int_{\Gamma_0} \tilde{w}\psi\phi_{-t}(\nabla_\Gamma \cdot \mathbf{V}_\Gamma) + \phi_{-t}(\mathbf{J}_\Gamma) \cdot \mathbf{C}(t)\nabla_\Gamma \tilde{w}\psi$$
$$= \int_{\Gamma_0} \left(\frac{\tilde{z}}{\delta_{k'}} - \frac{g(\tilde{u},\tilde{w})}{\delta_k}\right)\psi \qquad \forall \psi \in H^1(\Gamma_0).$$

Expanding the term $\nabla_\Gamma(\psi/J_t) = (1/J_t)\nabla_\Gamma\psi + \psi\nabla_\Gamma(1/J_t)$, we get the desired weak formulation. □

Lemma 4.11. *Set* $\mathbf{J}_0(t) := \phi_{-t}(\mathbf{J}_\Gamma(t))$*. For* $h < T$*, we have*

$$\begin{aligned}
&\frac{1}{h}\int_0^{T-h}\int_{\Gamma_0}|\tilde{w}(t+h)-\tilde{w}(t)|^2\\
&\leq \frac{2}{\delta_{k'}}\|\tilde{z}\|_{L^2(0,T;L^2(\Gamma_0))}\|\tilde{w}\|_{L^2(0,T-h;L^2(\Gamma_0))}+\frac{2}{\delta_k}\|g(\tilde{u},\tilde{w})\|_{L^1(0,T;L^1(\Gamma_0))}\|\tilde{w}\|_{L^\infty(0,T;L^\infty(\Gamma_0))}\\
&\quad + C\delta_\Gamma\|\nabla_\Gamma\tilde{w}\|^2_{L^2(0,T;L^2(\Gamma_0))}+C(\delta_\Gamma+\|\mathbf{J}_0\|_\infty)\|\tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))}\|\nabla_\Gamma\tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))}\\
&\quad + C\|\tilde{w}\|^2_{L^2(0,T;L^2(\Gamma_0))}
\end{aligned}$$

where C *is independent of relevant parameters.*

Proof. We have, for $h > 0$,

$$\begin{aligned}
&\int_{\Gamma_0}|\tilde{w}(t+h)-\tilde{w}(t)|^2\\
&=\int_0^h\frac{d}{ds}\int_{\Gamma_0}(\tilde{w}(t+h)-\tilde{w}(t))(\tilde{w}(t+s)-\tilde{w}(t))\,\mathrm{d}s\\
&=\int_0^h\langle\tilde{w}'(t+s),\tilde{w}(t+h)-\tilde{w}(t)\rangle\,\mathrm{d}s\\
&=\int_0^h\int_{\Gamma_0}\left((\delta_{k'}^{-1}\tilde{z}(t+s)-\delta_k^{-1}g(\tilde{u}(t+s),\tilde{w}(t+s))\right)(\tilde{w}(t+h)-\tilde{w}(t))\,\mathrm{d}s\\
&-\int_0^h\int_{\Gamma_0}\delta_\Gamma\mathbf{B}(t+s)\nabla_\Gamma\tilde{w}(t+s)\nabla_\Gamma(\tilde{w}(t+h)-\tilde{w}(t))\,\mathrm{d}s\\
&-\int_0^h\int_{\Gamma_0}\delta_\Gamma J_{t+s}(\tilde{w}(t+h)-\tilde{w}(t))\mathbf{B}(t+s)\nabla_\Gamma\tilde{w}(t+s)\nabla_\Gamma(1/J_{t+s})\,\mathrm{d}s\\
&-\int_0^h\int_{\Gamma_0}\tilde{w}(t+s)(\tilde{w}(t+h)-\tilde{w}(t))V_0(t+s)\,\mathrm{d}s\\
&\quad-\int_0^h\int_{\Gamma_0}\mathbf{J}_0(t+s)\cdot\mathbf{C}(t+s)\nabla_\Gamma\tilde{w}(t+s)(\tilde{w}(t+h)-\tilde{w}(t))\,\mathrm{d}s,
\end{aligned}$$

where we set $V_0(t) := \phi_{-t}(\nabla_\Gamma\cdot\mathbf{V}_\Gamma(t))$ and $\mathbf{J}_0(t) := \phi_{-t}(\mathbf{J}_\Gamma(t))$. Now, we can integrate this over $t\in(0,T-h)$ and interchange the integrals $\int_0^{T-h}\int_0^h\,\mathrm{d}s\,\mathrm{d}t=\int_0^h\int_0^{T-h}\,\mathrm{d}t\,\mathrm{d}s$ on the right-hand side. Estimating then the resulting right-hand side, we see first of all that the first term becomes

$$\begin{aligned}
&\int_0^h\int_0^{T-h}\int_{\Gamma_0}(\delta_{k'}^{-1}\tilde{z}(t+s)-\delta_k^{-1}g(\tilde{u}(t+s),\tilde{w}(t+s)))(\tilde{w}(t+h)-\tilde{w}(t))\,\mathrm{d}t\,\mathrm{d}s\\
&\leq\frac{h}{\delta_{k'}}\|\tilde{z}(\cdot+s)\|_{L^2(0,T-h;L^2(\Gamma_0))}\|\tilde{w}(\cdot+h)-\tilde{w}(\cdot)\|_{L^2(0,T-h;L^2(\Gamma_0))}\\
&\quad+\frac{h}{\delta_k}\|g(\tilde{u}(\cdot+s),\tilde{w}(\cdot+s))\|_{L^1(0,T-h;L^1(\Gamma_0))}\|\tilde{w}(\cdot+h)-\tilde{w}(\cdot)\|_{L^\infty(0,T-h;L^\infty(\Gamma_0))}\,.
\end{aligned}$$

To conclude we now estimate the remaining integral terms:

$$
\begin{aligned}
&\int_0^h \int_0^{T-h} \int_{\Gamma_0} \delta_\Gamma \mathbf{B}(t+s) \nabla_\Gamma \tilde{w}(t+s) \nabla_\Gamma (\tilde{w}(t+h) - \tilde{w}(t)) \, \mathrm{d}t \, \mathrm{d}s \\
&+ \int_0^h \int_0^{T-h} \int_{\Gamma_0} \delta_\Gamma J_{t+s} (\tilde{w}(t+h) - \tilde{w}(t)) \mathbf{B}(t+s) \nabla_\Gamma \tilde{w}(t+s) \nabla_\Gamma (1/J_{t+s}) \, \mathrm{d}t \, \mathrm{d}s \\
&\quad + \int_0^h \int_0^{T-h} \int_{\Gamma_0} \tilde{w}(t+s)(\tilde{w}(t+h) - \tilde{w}(t)) V_0(t+s) \, \mathrm{d}t \, \mathrm{d}s \\
&\quad + \int_0^h \int_0^{T-h} \int_{\Gamma_0} \mathbf{J}_0(t+s) \cdot \mathbf{C}(t+s) \nabla_\Gamma \tilde{w}(t+s)(\tilde{w}(t+h) - \tilde{w}(t)) \, \mathrm{d}t \, \mathrm{d}s \\
&\leq \int_0^h C\delta_\Gamma \|\nabla_\Gamma \tilde{w}\|^2_{L^2(0,T;L^2(\Gamma_0))} + C\delta_\Gamma \|\tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))} \|\nabla_\Gamma \tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))} + C \|\tilde{w}\|^2_{L^2(0,T;L^2(\Gamma_0))} \\
&\quad + C \|\mathbf{J}_0\|_\infty \|\tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))} \|\nabla_\Gamma \tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))} \, \mathrm{d}s \\
&\leq \Big(C\delta_\Gamma \|\nabla_\Gamma \tilde{w}\|^2_{L^2(0,T;L^2(\Gamma_0))} + C(\delta_\Gamma + \|\mathbf{J}_0\|_\infty) \|\tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))} \|\nabla_\Gamma \tilde{w}\|_{L^2(0,T;L^2(\Gamma_0))} \\
&\quad + C \|\tilde{w}\|^2_{L^2(0,T;L^2(\Gamma_0))} \Big) h.
\end{aligned}
$$

□

5 The $\delta_k \to 0$ limit

We denote the solutions of the problem (7) as (u_k, w_k, z_k) where k is supposed to represent the parameter δ_k. By examining the estimates stated in Lemma 4.2, Lemma 4.3 and Lemma 4.4, we see that we need a bound on $\|z_k\|_{L^2_{L^2(\Gamma)}}$ uniform in δ_k in order to get u_k, w_k and z_k bounded in the energy space $L^\infty_{L^2} \cap L^2_{H^1}$. The duality approach result of Lemma 4.9 provides exactly such a desired bound on z_k. With such an estimate in place, we can then obtain an L^∞ bound on w_k by Corollary 4.6. Hence, we have the existence of functions

$$
u \in L^\infty_{L^2(\Omega)} \cap L^2_{H^1(\Omega)}, \qquad w \in L^\infty_{L^\infty(\Gamma)} \cap L^2_{H^1(\Gamma)}, \qquad z \in L^\infty_{L^2(\Gamma)} \cap L^2_{H^1(\Gamma)},
$$

such that

$$
u_k \rightharpoonup u \quad \text{in } L^2_{H^1(\Omega)} \quad \text{and} \quad u_k \overset{*}{\rightharpoonup} u \text{ in } L^\infty_{L^2(\Omega)} \tag{43}
$$

and

$$
\begin{aligned}
w_k \rightharpoonup w \quad &\text{in } L^2_{H^1(\Gamma)} \quad \text{and} \quad w_k \overset{*}{\rightharpoonup} w \quad \text{in } L^\infty_{L^\infty(\Gamma)}, \\
z_k \rightharpoonup z \quad &\text{in } L^2_{H^1(\Gamma)} \quad \text{and} \quad z_k \overset{*}{\rightharpoonup} z \quad \text{in } L^\infty_{L^2(\Gamma)}.
\end{aligned} \tag{44}
$$

Bearing in mind the assumptions on g in Assumption 2.4, we see that these weak convergences are by themselves not enough to obtain the complementarity condition (12e). We need a strong convergence, for which the following standard result comes in use.

Theorem 5.1 (Aubin–Lions–Simon, [51]). *Let $\{\varphi_k\}$ be a bounded sequence of functions in $L^p(0,T;B)$, where B is a Banach space and $1 \leq p \leq \infty$. If in addition*

(i) the sequence $\{\varphi_k\}$ is bounded in $L^p(0,T;X)$ where $X \subset B$ is compact,

(ii) as $h \to 0$*,*

$$\int_0^{T-h} \|\varphi_k(t+h) - \varphi_k(t)\|_B^2 \,\mathrm{d}t \to 0 \quad \textit{uniformly in } k,$$

then there exists $\varphi \in L^p(0,T;X)$ *such that, up to a subsequence,*

$$\varphi_k \to \varphi \ \ \textit{in } L^p(0,T;B).$$

Making use of the L^1 bound on $\delta_k^{-1} g(u_k, w_k)$ in Lemma 4.1 and the L^∞ bound on w_k, we can then bound the difference quotients of the pullback of w_k uniformly via Lemma 4.11. This estimate on the difference quotients allows us, via the Aubin–Lions–Simon Theorem 5.1, to obtain the stronger convergence

$$w_k \to w \ \text{ in } L^2_{L^2(\Gamma)}.$$

5.1 Passage to the limit

Taking $\eta \in \mathcal{V}$ (recall the definition of this space from (11)) with $\eta(T) \equiv 0$, using the equation for u_k and its boundary condition, we derive

$$\int_0^T \langle \partial^\bullet u_k, \eta\rangle + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u_k \cdot \nabla \eta + \int_0^T\int_{\Omega(t)} u_k \eta \nabla_\Gamma \cdot \mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega u_k)\eta - \int_0^T\int_{\Gamma(t)} j u_k \eta$$
$$= \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \left(\frac{1}{\delta_{k'}} z_k \eta - \frac{1}{\delta_k} g(u_k, w_k)\eta\right). \tag{45}$$

Substituting the equation for z_k on the right-hand side, we obtain

$$\int_0^T \langle \partial^\bullet u_k, \eta\rangle + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u_k \cdot \nabla \eta + \int_0^T\int_{\Omega(t)} u_k \eta \nabla_\Gamma \cdot \mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega u_k)\eta - \int_0^T\int_{\Gamma(t)} j u_k \eta$$
$$+ \frac{1}{\delta_\Omega}\int_0^T \langle \partial^\bullet z_k, \eta\rangle + \frac{\delta_{\Gamma'}}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma z_k \cdot \nabla_\Gamma \eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z_k \eta \nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z_k)\eta = 0.$$

We now integrate by parts on the integrals involving the time derivatives, obtaining

$$-\int_{\Omega_0} u_0 \eta(0) - \int_0^T\int_{\Omega(t)} u_k \partial^\bullet \eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u_k \cdot \nabla \eta + \int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u_k)\eta - \int_0^T\int_{\Gamma(t)} j u_k \eta$$
$$-\frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0 \eta(0) - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z_k \partial^\bullet \eta + \frac{\delta_{\Gamma'}}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma z_k \cdot \nabla_\Gamma \eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z_k)\eta = 0, \tag{46}$$

and using the convergences in (43) and (44), passing to the limit $\delta_k \to 0$ is immediate and leads to

$$-\int_{\Omega_0} u_0 \eta(0) - \int_0^T\int_{\Omega(t)} u \partial^\bullet \eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u \cdot \nabla \eta + \int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u)\eta - \int_0^T\int_{\Gamma(t)} j u \eta$$
$$-\frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0 \eta(0) - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z \partial^\bullet \eta + \frac{\delta_{\Gamma'}}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma z \cdot \nabla_\Gamma \eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z)\eta = 0,$$

which is precisely (13b). We can in a similar way derive the equality (13a) relating u and w if we use the equation for w on the right-hand side of (45).

5.2 Complementarity condition

Note that we have $u_k \rightharpoonup u$ in $L^2_{H^1(\Omega)}$ and thus by continuity of the trace operator we also obtain $u_k \rightharpoonup u$ in $L^2_{H^{1/2}(\Gamma)}$. Rewriting (38) as

$$\int_0^T \int_{\Gamma(t)} g(u_k, w_k) \leq C\delta_k$$

and therefore letting $\delta_k \to 0$ gives $g(u_k, w_k) \to 0$ in $L^1_{L^1(\Gamma)}$. This along with (9) immediately implies that $g(u, w) = 0$ and we conclude the proof of Theorem 2.7.

6 The $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit

In this section we assume $\delta_\Gamma = \delta_{\Gamma'} = \delta_k$, which creates new dependencies on k. As such, the *a priori* estimates we previously established need to be revisited. As before, denote the solutions of the problem as (u_k, w_k, z_k) where k is supposed to represent the parameter δ_k.

(i) First of all, note that we cannot use Lemma 4.9 like we did in the previous section because the bounds of the cited lemma are not uniform in the diffusion coefficients.

On the positive side, because $\delta_\Gamma = \delta_{\Gamma'}$, we can avail ourselves of the estimate in Corollary 4.5, which gives boundedness of z_k and w_k in $L^\infty_{L^2(\Gamma)}$. This we can feed back into Lemma 4.1 and Lemma 4.3 to obtain

$$\|u\|^2_{L^\infty_{L^2(\Omega)}} + \|\nabla u\|^2_{L^2_{L^2(\Omega)}} + \frac{1}{\delta_k}\int_0^T \int_{\Gamma(t)} g(u,w)u + \frac{1}{\delta_k}\int_0^T \int_{\Gamma(t)} g(u,w) \leq C.$$

(ii) From Lemma 4.2 and Corollary 4.7 we obtain the estimate

$$\delta_k \|\nabla_\Gamma z_k\|^2_{L^2_{L^2(\Gamma)}} + \delta_k \|\nabla_\Gamma w_k\|^2_{L^2_{L^2(\Gamma)}} \leq C \tag{47}$$

(note that we no longer get uniform gradient estimates on w_k or z_k), which will come in use later.

(iii) We no longer get a uniform L^∞ bound for w_k from Corollary 4.6 but we can get one easily (thanks to $\delta_\Gamma = \delta_{\Gamma'}$) from Lemma 6.1 below.

(iv) Since we do not have a bound for $\nabla_\Gamma w_k$, we also do not get uniformly bounded difference quotients for w_k from Lemma 4.11 (unless we assume additional conditions, see Remark 7.1; we will not need this in this section). This means that we no longer can get a strong convergence for w_k.

We now establish new estimates to overcome the obstacles described above. What we will do is to get a strong convergence now for u_k like we did for w_k in the previous section. This will require us to get an L^∞-estimate for z, which is dealt with in the next lemma. In it, note that there is no need for the De Giorgi arguments of Lemma 3.2 since v solves the homogeneous problem.

Lemma 6.1. *If $w_0, z_0 \in L^\infty(\Gamma_0)$ and $\delta_\Gamma = \delta_{\Gamma'}$, then*

$$\|w\|_{L^\infty_{L^\infty(\Gamma)}} + \|z\|_{L^\infty_{L^\infty(\Gamma)}} \leq C,$$

where C is independent of all parameters.

Proof. With $v = w+z$ as before, the fact that $\delta_\Gamma = \delta_{\Gamma'}$ implies that v is a solution to the heat equation

$$v_t - \delta_\Gamma \Delta_\Gamma v + v \nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) = 0,$$

and therefore boundedness of v is a consequence of boundedness of v_0 using the standard argument. Since $w, z \geq 0$, the conclusion follows. □

Boundedness for u is more complicated because we are dealing with two sets ($\Omega(t)$ and its boundary) due to the Robin boundary condition. Consider the following

$$\begin{aligned} \partial^\bullet a + a\nabla \cdot \mathbf{V}_p - \delta_\Omega^{-1}\Delta a + \nabla \cdot (\mathbf{J}_\Omega a) &= 0 && \text{in } \Omega(t), \\ \delta_\Omega^{-1}\nabla a \cdot \nu &= \delta_\Omega^{-1}\delta_{k'}^{-1} z + ja && \text{on } \Gamma(t), \\ \nabla a \cdot \nu &= 0 && \text{in } \partial D(t), \\ a(0) &= u_0. \end{aligned} \tag{48}$$

This is almost the same equation for u, but the non-positive term $-g(u,w)$ (which formally speaking can be thought of as entering as a source term) is omitted, thus we expect that $a \geq u$. The next lemma shows that this is indeed the case.

Lemma 6.2. *We have $u \leq a$.*

Proof. Define $U := ue^{-\lambda_u t}$ so that $\partial^\bullet u = e^{\lambda_u t}\partial^\bullet U + \lambda_u u$. The equation for U is

$$\begin{aligned} \partial^\bullet U + U(\nabla \cdot \mathbf{V}_p + \lambda_u) - \delta_\Omega^{-1}\Delta U + \nabla \cdot (\mathbf{J}_\Omega U) &= 0, \\ \delta_\Omega^{-1}\nabla U \cdot \nu &= \delta_\Omega^{-1}(\delta_{k'}^{-1}e^{-\lambda_u t} z - \delta_k^{-1}e^{-\lambda_u t} g(u,w)) + jU, \\ \nabla U \cdot \nu &= 0, \\ U(0) &= u_0. \end{aligned}$$

In a similar way, the equation for a can be transformed via $A := ae^{-\lambda_u t}$ to

$$\begin{aligned} \partial^\bullet A + A(\nabla \cdot \mathbf{V}_p + \lambda_u) - \delta_\Omega^{-1}\Delta A + \nabla \cdot (\mathbf{J}_\Omega A) &= 0, \\ \delta_\Omega^{-1}\nabla A \cdot \nu &= \delta_\Omega^{-1}\delta_{k'}^{-1}e^{-\lambda_u t} z + jA, \\ \nabla A \cdot \nu &= 0, \\ A(0) &= u_0. \end{aligned}$$

Define $Y := U - A$ which satisfies

$$\begin{aligned} \partial^\bullet Y + Y(\nabla \cdot \mathbf{V}_p + \lambda_u) - \delta_\Omega^{-1}\Delta Y + \nabla \cdot (\mathbf{J}_\Omega Y) &= 0 && \text{in } \Omega(t), \\ \delta_\Omega^{-1}\nabla Y \cdot \nu &= -\delta_\Omega^{-1}\delta_k^{-1}e^{-\lambda_u t} g(u,w) + jY && \text{on } \Gamma(t), \\ \nabla Y \cdot \nu &= 0 && \text{on } \partial D(t), \\ Y(0) &= 0. \end{aligned}$$

Test this with Y^+ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)}|Y^+|^2 - \frac{1}{2}\int_{\Omega(t)}|Y^+|^2\nabla\cdot\mathbf{V}_p + \int_{\Omega(t)}(\nabla\cdot\mathbf{V}_p+\lambda_u)|Y^+|^2 + \delta_\Omega^{-1}|\nabla Y^+|^2 + \frac{1}{2}|Y^+|^2\nabla\cdot\mathbf{J}_\Omega \\
&\quad + \frac{1}{2}\int_{\Gamma(t)} j|Y^+|^2 \\
&\quad = \int_{\Gamma(t)} -\delta_\Omega^{-1}\delta_k^{-1}e^{-\lambda_u t}g(u,w)Y^+ + j|Y^+|^2 \\
&\quad \leq \int_{\Gamma(t)} j|Y^+|^2.
\end{aligned}$$

Simplifying,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)}|Y^+|^2 + \int_{\Omega(t)}(\nabla\cdot\mathbf{V}_\Omega+\lambda_u)|Y^+|^2 + \delta_\Omega^{-1}|\nabla Y^+|^2 &\leq \frac{\|j\|_\infty}{2}\int_{\Gamma(t)}|Y^+|^2 \\
&\leq \frac{\|j\|_\infty}{2}\int_{\Omega(t)} C_\epsilon|Y^+|^2 + \epsilon|\nabla Y^+|^2
\end{aligned}$$

where we used the interpolated trace inequality. Rearranging terms, choosing ϵ carefully and applying Gronwall's lemma (and using $Y^+(0)=0$), we get the result. □

Lemma 6.3. *We have*

$$\begin{aligned}
&\|u\|_{L^\infty_{L^\infty(\Omega)}} \\
&\quad \leq C_1 e^{\frac{1}{2}\|\nabla\cdot\mathbf{V}_\Omega\|_\infty T + \delta_\Omega C_2\left(\frac{1}{2}\delta_\Omega^{-1}\delta_{k'}^{-1}\|z\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right)^2 T}\max\left(1, \|u_0\|_\infty, \frac{1}{2}C_3\delta_\Omega^{-1}\delta_{k'}^{-1}\|z\|_{L^\infty_{L^\infty(\Gamma)}}\right) \\
&\qquad \times (\min(1,\delta_\Omega^{-1})^{-\frac{1}{2\kappa}-\frac{1}{2}} + C_4).
\end{aligned}$$

Proof. We can use Lemma 3.4 to obtain an L^∞ bound on the solution a of (48). Using the previous lemma, we get the result via

$$0 \leq u \leq a \leq \|a\|_{L^\infty_{L^\infty(\Omega)}}.$$

□

Using the extra results above we can argue as in the proof of Lemma 4.11 to establish an estimate on the difference quotients now for u.

6.1 Bound on the difference quotients of u

In a similar way to Section 4.2, let us first write down the weak formulation that the pullback of u satisfies. Let $\mathbf{D}\Phi_t$ be the Jacobian matrix of Φ_t and let $J_t := \det\mathbf{D}\Phi_t$ be its determinant. Define $\omega_t := |(\mathbf{D}\Phi_t)^{-T}\nu_0|$, where ν_0 is the outward normal to Γ_0.

Lemma 6.4. *Set* $V_0(t) := \phi_{-t}(\nabla\cdot\mathbf{V}_\Omega)$, $\mathbf{J}_0 := \phi_{-t}(\mathbf{J}_\Omega)$, $j_0 := \phi_{-t}((\mathbf{V}_\Omega - \mathbf{V}_\Gamma)\cdot\tilde{\nu})$, *and*

$$\mathbf{B}(t) := (\mathbf{D}\Phi_t)^{-1}(\mathbf{D}\Phi_t)^{-T}.$$

The pullback $\tilde{u} := \phi_{-(\cdot)}u$ *of* u *satisfies*

$$\langle \tilde{u}_t, \psi\rangle + \int_{\Omega_0} \delta_\Omega^{-1}\mathbf{B}(t)\nabla\tilde{u}\nabla\psi + \delta_\Omega^{-1}\psi J_t\mathbf{B}(t)\nabla\tilde{u}\nabla(1/J_t) + \tilde{u}\psi V_0 + \mathbf{J}_0\cdot(\mathbf{D}\Phi_t)^{-T}\nabla\tilde{u}\psi$$
$$= \int_{\Gamma_0}\frac{1}{\delta_\Omega}\left(\frac{\tilde{z}}{\delta_{k'}} - \frac{g(\tilde{u},\tilde{w})}{\delta_k}\right)\psi\omega_t + j_0\tilde{u}\psi\omega_t \qquad \forall\psi\in H^1(\Omega_0). \tag{49}$$

Proof. Pulling back the weak form of the u equation gives (see [14, Chapter 9, §4.2] for the boundary integral)

$$\langle \tilde{u}_t, J_t\tilde{\varphi}\rangle + \int_{\Omega_0}\delta_\Omega^{-1}J_t\mathbf{B}(t)\nabla\tilde{u}\nabla\tilde{\varphi} + \tilde{u}\tilde{\varphi}V_0J_t + J_t\mathbf{J}_0\cdot(\mathbf{D}\Phi_t)^{-T}\nabla\tilde{u}\tilde{\varphi}$$
$$= \int_{\Gamma_0}\frac{1}{\delta_\Omega}\left(\frac{\tilde{z}}{\delta_{k'}} - \frac{g(\tilde{u},\tilde{w})}{\delta_k}\right)\tilde{\varphi}J_t\omega_t + j_0\tilde{u}\tilde{\varphi}J_t\omega_t.$$

Now, again using that multiplication by J_t is an isomorphism, this becomes

$$\langle \tilde{u}_t, \psi\rangle + \int_{\Omega_0}\delta_\Omega^{-1}J_t\mathbf{B}(t)\nabla\tilde{u}\nabla((J_t^0)^{-1}\psi) + \tilde{u}\psi V_0 + \mathbf{J}_0\cdot(\mathbf{D}\Phi_t)^{-T}\nabla\tilde{u}\psi$$
$$= \int_{\Gamma_0}\frac{1}{\delta_\Omega}\left(\frac{\tilde{z}}{\delta_{k'}} - \frac{g(\tilde{u},\tilde{w})}{\delta_k}\right)\psi\omega_t + j_0\tilde{u}\psi\omega_t.$$

Expanding $J_t\mathbf{B}(t)\nabla\tilde{u}\nabla((1/J_t)\psi) = \mathbf{B}(t)\nabla\tilde{u}\nabla\psi + \psi J_t\mathbf{B}(t)\nabla\tilde{u}\nabla(1/J_t)$, integrating by parts in time and relabelling, we obtain (49). □

Lemma 6.5. *We have*

$$\frac{1}{h}\int_0^{T-h}\int_{\Omega_0}|\tilde{u}(t+h)-\tilde{u}(t)|^2 \le C\left(\frac{1}{\delta_\Omega}+1\right)\|\nabla\tilde{u}\|^2_{L^2(0,T;\Omega_0)} + C\|\tilde{u}\|^2_{L^2(0,T;\Omega_0)}$$
$$+ 2\delta_\Omega^{-1}\delta_{k'}^{-1}\|\tilde{z}\|_{L^2(0,T;L^2(\Gamma_0))}\|\tilde{u}\|_{L^2(0,T;L^2(\Gamma_0))}$$
$$+ \frac{2}{\delta_\Omega\delta_k}\|g(\tilde{u},\tilde{w})\|_{L^1(0,T;L^1(\Gamma_0))}\|\tilde{u}\|_{L^\infty(0,T;L^\infty(\Gamma_0))} + 2\|\tilde{u}\|^2_{L^2(0,T;L^2(\Gamma_0))}.$$

Proof. Beginning as in the proof of Lemma 4.11, we have

$$
\begin{aligned}
&\int_{\Omega_0} |\tilde{u}(t+h) - \tilde{u}(t)|^2 \\
&= \int_0^h \langle \tilde{u}_t(t+s), \tilde{u}(t+h) - \tilde{u}(t) \rangle \\
&= -\frac{1}{\delta_\Omega} \int_0^h \int_{\Omega_0} \mathbf{B}(t+s) \nabla \tilde{u}(t+s) \cdot \nabla(\tilde{u}(t+h) - \tilde{u}(t)) \\
&\quad - \frac{1}{\delta_\Omega} \int_0^h \int_{\Omega_0} J_{t+s}(\tilde{u}(t+h) - \tilde{u}(t)) \mathbf{B}(t+s) \nabla \tilde{u}(t+s) \nabla(1/J_{t+s}) \\
&\quad - \int_0^h \int_{\Omega_0} \tilde{u}(t+s)(\tilde{u}(t+h) - \tilde{u}(t)) V_0(t+s) \\
&\quad - \int_0^h \int_{\Omega_0} \mathbf{J}_0(t+s) \cdot (\mathbf{D}\Phi_{t+s})^{-T} \nabla \tilde{u}(t+s)(\tilde{u}(t+h) - \tilde{u}(t)) \\
&\quad + \int_0^h \int_{\Gamma_0} \frac{1}{\delta_\Omega} \left(\frac{\tilde{z}(t+s)}{\delta_{k'}} - \frac{g(\tilde{u}(t+s), \tilde{w}(t+s))}{\delta_k} \right) (\tilde{u}(t+h) - \tilde{u}(t)) \omega_t \\
&\quad + \int_0^h \int_{\Gamma_0} j_0(t+s) \tilde{u}(t+s)(\tilde{u}(t+h) - \tilde{u}(t)) \omega_t.
\end{aligned}
$$

We now integrate over $(0, T-h)$ and switch integrals like in Lemma 4.11 and use (39) to obtain for the inner integrals

$$
\begin{aligned}
&\frac{1}{\delta_\Omega} \int_0^{T-h} \int_{\Omega_0} \mathbf{B}(t+s) \nabla \tilde{u}(t+s) \cdot \nabla(\tilde{u}(t+h) - \tilde{u}(t)) \\
&\quad + \frac{1}{\delta_\Omega} \int_0^{T-h} \int_{\Omega_0} J_{t+s}(\tilde{u}(t+h) - \tilde{u}(t)) \mathbf{B}(t+s) \nabla \tilde{u}(t+s) \nabla(1/J_{t+s}) \\
&\quad + \int_0^{T-h} \int_{\Omega_0} \tilde{u}(t+s)(\tilde{u}(t+h) - \tilde{u}(t)) V_0(t+s) \\
&\quad + \int_0^{T-h} \int_{\Omega_0} \mathbf{J}_0(t+s) \cdot (\mathbf{D}\Phi_{t+s})^{-T} \nabla \tilde{u}(t+s)(\tilde{u}(t+h) - \tilde{u}(t)) \\
&\leq \frac{C}{\delta_\Omega} \|\nabla \tilde{u}\|^2_{L^2(0,T;L^2(\Omega_0))} + \|\tilde{u}\|_{L^2(0,T;L^2(\Omega_0))} \|\nabla \tilde{u}\|_{L^2(0,T;L^2(\Omega_0))} + \|\nabla \tilde{u}\|^2_{L^2(0,T;L^2(\Omega_0))} \\
&\quad + C \|\tilde{u}\|^2_{L^2(0,T;L^2(\Omega_0))} \\
&\leq C \left(\frac{1}{\delta_\Omega} + 1 \right) \|\nabla \tilde{u}\|^2_{L^2(0,T;\Omega_0)} + C \|\tilde{u}\|^2_{L^2(0,T;L^2(\Omega_0))}
\end{aligned}
$$

and

$$\begin{aligned}
&\int_0^{T-h}\int_{\Gamma_0}\frac{1}{\delta_\Omega}\left(\frac{\tilde z(t+s)}{\delta_{k'}}-\frac{g(\tilde u(t+s),\tilde w(t+s))}{\delta_k}\right)(\tilde u(t+h)-\tilde u(t))\omega_t\\
&\quad+\int_0^{T-h}\int_{\Gamma_0} j_0\tilde u(t+s)(\tilde u(t+h)-\tilde u(t))\omega_t\\
&\le \delta_\Omega^{-1}\delta_{k'}^{-1}\,\|\tilde z\|_{L^2(0,T-h;L^2(\Gamma_0))}\,\|\tilde u(\cdot+h)-\tilde u\|_{L^2(0,T-h;L^2(\Gamma_0))}\\
&\quad+\frac{1}{\delta_\Omega\delta_k}\,\|g(\tilde u(t+\cdot),\tilde w(t+\cdot))\|_{L^1(0,T-h;L^1(\Gamma_0))}\,\|\tilde u(\cdot+h)-\tilde u\|_{L^\infty(0,T-h;L^\infty(\Gamma_0))}\\
&\quad+\|\tilde u(t+\cdot)\|_{L^2(0,T-h;L^2(\Gamma_0))}\,\|\tilde u(\cdot+h)-\tilde u\|_{L^2(0,T-h;L^2(\Gamma_0))}\\
&\le 2\delta_\Omega^{-1}\delta_{k'}^{-1}\,\|\tilde z\|_{L^2(0,T;L^2(\Gamma_0))}\,\|\tilde u\|_{L^2(0,T;L^2(\Gamma_0))}\\
&\quad+\frac{2}{\delta_\Omega\delta_k}\,\|g(\tilde u,\tilde w)\|_{L^1(0,T;L^1(\Gamma_0))}\,\|\tilde u\|_{L^\infty(0,T;L^\infty(\Gamma_0))}+2\,\|\tilde u\|^2_{L^2(0,T;L^2(\Gamma_0))}\,.
\end{aligned}$$

□

Using the previously established uniform bounds (including the $L^1_{L^1(\Gamma)}$ bound for $\delta_k^{-1}g(u,w)$ and the $L^\infty_{L^\infty(\Omega)}$ bound for u from Lemma 6.3), we are led to

$$\int_0^{T-h}\int_{\Omega_0}|\tilde u(t+h)-\tilde u(t)|^2\le C\left(\frac{1}{\delta_\Omega}+1+\frac{1}{\delta_\Omega\delta_{k'}}\right)h$$

where C is independent of δ_k.

6.2 Passage to the limit

Using the estimates in the previous section we have limit functions

$$u\in L^\infty_{L^\infty(\Omega)}\cap L^2_{H^1(\Omega)},\qquad w\in L^\infty_{L^\infty(\Gamma)},\qquad z\in L^\infty_{L^\infty(\Gamma)},$$

such that

$$\begin{aligned}
u_k &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty_{L^\infty(\Omega)} \quad\text{and}\quad u_k\rightharpoonup u \quad\text{in } L^2_{H^1(\Omega)},\\
w_k &\overset{*}{\rightharpoonup} w \quad \text{in } L^\infty_{L^\infty(\Gamma)} \quad\text{and}\quad z_k\overset{*}{\rightharpoonup} z \quad\text{in } L^\infty_{L^\infty(\Gamma)}.
\end{aligned}$$

As a result we also get

$$u_k\overset{*}{\rightharpoonup} u \quad\text{in } L^\infty_{L^\infty(\Gamma)}.$$

The estimate on the difference quotients for u_k allows us by Theorem 5.1 to obtain the stronger convergence

$$u_k\to u \ \text{ in } L^2_{L^2(\Omega)},$$

and by [20, Lemma 3.9] we deduce the same for the trace of u:

$$u_k\to u \ \text{ in } L^2_{L^2(\Gamma)}.$$

Taking again $\eta\in\mathcal{V}$ with $\eta(T)=0$ and recalling the weak formulation relating u_k and w_k in (46), bearing in mind now that $\delta_k=\delta_\Gamma=\delta_{\Gamma'}$, we have

$$\begin{aligned}
&-\int_{\Omega_0}u_0\eta(0)-\int_0^T\int_{\Omega(t)}u_k\partial^\bullet\eta+\frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)}\nabla u_k\cdot\nabla\eta+\int_0^T\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega u_k)\eta-\int_0^T\int_{\Gamma(t)}ju_k\eta\\
&-\frac{1}{\delta_\Omega}\int_{\Gamma_0}z_0\eta(0)-\frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)}z_k\partial^\bullet\eta+\frac{\delta_k}{\delta_\Omega}\int_0^T\int_{\Gamma(t)}\nabla_\Gamma z_k\cdot\nabla_\Gamma\eta+\frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)}\nabla_\Gamma\cdot(\mathbf{J}_\Gamma z_k)\eta=0.
\end{aligned}$$

Now, we must use (60) and integrate by parts on the divergence of the jump term since we cannot control the gradient of z_k; doing so yields

$$-\int_{\Omega_0} u_0\eta(0) - \int_0^T\int_{\Omega(t)} u_k\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u_k\cdot\nabla\eta + \int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u_k)\eta - \int_0^T\int_{\Gamma(t)} ju_k\eta$$
$$-\frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\eta(0) - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z_k\partial^\bullet\eta + \frac{\delta_k}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} \nabla_\Gamma z_k\cdot\nabla_\Gamma\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z_k\mathbf{J}_\Gamma\cdot\nabla_\Gamma\eta = 0. \tag{50}$$

The estimate (47) implies

$$\delta_k\|\nabla_\Gamma w_k\|_{L^2_{L^2(\Gamma)}} \le C\sqrt{\delta_k} \quad \text{and} \quad \delta_k\|\nabla_\Gamma z_k\|_{L^2_{L^2(\Gamma)}} \le C\sqrt{\delta_k},$$

so that $\delta_k\nabla_\Gamma z_k$ (and $\delta_k\nabla_\Gamma w_k$) converges to 0 as $\delta_k \to 0$. Passing to the limit using the convergences obtained above yields

$$-\int_{\Omega_0} u_0\eta(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u\cdot\nabla\eta + \int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u)\eta - \int_0^T\int_{\Gamma(t)} ju\eta$$
$$-\frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\eta(0) - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\partial^\bullet\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\mathbf{J}_\Gamma\cdot\nabla_\Gamma\eta = 0.$$

The w case follows similarly.

6.3 Complementarity condition

The complementarity condition in this case can be obtained in the same way as before, first by obtaining that $g(u_n, w_n) \to 0$ as previously and now by using (10).

7 The $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta'_\Gamma \to 0$ limit

Now, we have the following:

(i) Since $\delta_\Gamma = \delta'_\Gamma$, Lemma 6.1 is still useful as it gives uniform bounds for w_k and z_k in $L^\infty_{L^\infty(\Gamma)}$.

(ii) For the gradients, we have (47) still:

$$\delta_k\|\nabla_\Gamma w\|^2_{L^2(L^2)} + \delta_k\|\nabla_\Gamma z\|^2_{L^2(L^2)} \le C, \tag{47}$$

and for $g(u, w)$, note that the estimate (38) implies

$$\frac{1}{\delta_\Omega\delta_k}\int_0^T\int_{\Gamma(t)} g(u,w) \le \|z\|_{L^1_{L^1(\Gamma)}} + \|u_0\|_{L^2(\Omega)}. \tag{51}$$

(iii) Examining Lemma 4.3, we obtain bounds on u in $L^\infty_{L^2(\Omega)}$ and its gradient in $L^2_{L^2(\Omega)}$ because $\delta_{k'}^{-1} = \delta_k$ and due to the boundedness of z. In fact, the gradient estimate in Lemma 4.3 implies

$$\frac{1}{\delta_\Omega}\|\nabla u\|^2_{L^2_{L^2(\Omega)}} \le \frac{\|z\|^2_{L^2_{L^2(\Gamma)}}}{\delta_\Omega\delta_{k'}} + \left(\frac{C}{\delta_\Omega}\left(\frac{1}{\delta_{k'}} + \delta_\Omega\|j\|_\infty\right)^2 + \|\nabla\cdot\mathbf{V}_\Omega\|_\infty\right)\|u\|^2_{L^2_{L^2(\Omega)}}$$
$$+ \|u_0\|^2_{L^2(\Omega_0)} \tag{52}$$

and the right-hand side is bounded uniformly.

Looking at Lemma 6.3, we also get a uniform bound for u in L^∞.

(iv) Lemma 6.5, after making use of the estimates (51) and (52) in combination with the L^∞-estimate, provides a bound on the difference quotients of u.

In summary, we obtain

$$u_k \rightharpoonup u \text{ in } L^2_{H^1(\Omega)} \cap L^\infty_{L^\infty(\Omega)}, \quad w_k \overset{*}{\rightharpoonup} w \text{ in } L^\infty_{L^\infty(\Gamma)}, \quad z_k \overset{*}{\rightharpoonup} z \text{ in } L^\infty_{L^\infty(\Gamma)}.$$

By (52), we have

$$\nabla u_k \to \nabla u \equiv 0 \quad \text{in } L^2_{L^2(\Omega)}.$$

By continuity of the trace operator we also have

$$u_k \rightharpoonup u \text{ in } L^2_{H^{1/2}(\Gamma)},$$

and by the compactness result of Theorem 5.1 (with $X = H^1(\Omega_0)$ and $B = L^2(\Omega_0)$), we conclude the stronger convergence

$$u_k \to u \text{ in } L^2_{L^2(\Omega)},$$

and we again have

$$u_k \to u \text{ in } L^2_{L^2(\Gamma)}.$$

Note that these are exactly the same convergences we obtained in Section 6 but here we additionally have $\nabla u_k \to 0$.

Remark 7.1. *If $\mathbf{J}_\Gamma \equiv 0$, the difference quotient estimate for w_k remains uniform: in the estimate given in Lemma 4.11, we see that the term $\mathbf{J}_0$ would vanish. Then the estimate* (47) *is sufficient to bound the difference quotient uniformly:*

$$\int_0^{T-h} \|w(t+h) - w(t)\|^2_{L^2(\Gamma)} \leq Ch.$$

By Theorem 5.1 with $X = L^2(\Gamma_0)$, $B = H^{-1/2}(\Gamma_0)$ we would get $w_k \to w$ in $L^2_{H^{-1/2}(\Gamma)}$.

7.1 Passage to the limit

Multiplying through (50) by δ_Ω, we get

$$\begin{aligned}
&-\delta_\Omega \int_{\Omega_0} u_0 \eta(0) - \delta_\Omega \int_0^T \int_{\Omega(t)} u_k \partial^\bullet \eta + \int_0^T \int_{\Omega(t)} \nabla u_k \cdot \nabla \eta + \delta_\Omega \int_0^T \int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega u_k) \eta \\
&-\delta_\Omega \int_0^T \int_{\Gamma(t)} j u_k \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z_k \partial^\bullet \eta + \delta_\Omega \int_0^T \int_{\Gamma(t)} \nabla_\Gamma z_k \cdot \nabla_\Gamma \eta \\
&-\int_0^T \int_{\Gamma(t)} z_k \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta = 0.
\end{aligned}$$

As in the previous limit, due to (47), the integral involving $\nabla_\Gamma z_k$ converges to 0 as $\delta_k \to 0$. In the limit, we obtain

$$-\int_{\Gamma_0} z_0\eta(0) - \int_0^T \int_{\Gamma(t)} \partial^\bullet \eta z - \int_0^T \int_{\Gamma(t)} z\mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta = 0.$$

A similar argument gives

$$-\int_{\Gamma_0} w_0\eta(0) - \int_0^T \int_{\Gamma(t)} \partial^\bullet \eta w - \int_0^T \int_{\Gamma(t)} w\mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta = 0.$$

7.2 Complementarity condition

Using (51), we have $g(u_k, w_k) \to 0$ in $L^1_{L^1(\Gamma)}$ and assumption (10) implies that $g(u, w) = 0$.

7.3 The Dirichlet limit

Let us finally address Theorem 2.17. We still obtain the same estimates on w_k and z_k as in Section 7. We now look for an L^∞-estimate on u, which we recall satisfies

$$\begin{aligned}
\partial^\bullet u + u\nabla \cdot \mathbf{V}_p - \frac{1}{\delta_\Omega}\Delta u + \nabla \cdot (\mathbf{J}_\Omega u) &= 0 && \text{in } \Omega(t),\\
\nabla u \cdot \nu &= \frac{1}{\delta_{k'}} z - \frac{1}{\delta_k} g(u, w) + \delta_\Omega j u && \text{on } \Gamma(t),\\
u &= u_D && \text{on } \partial D(t),\\
u(0) &= u_0.
\end{aligned}$$

We need a preliminary result for a related equation. The following lemma essentially modifies the L^∞-estimate of Lemma 3.4 to allow for the Dirichlet boundary condition.

Lemma 7.2. *Let $y \in L^\infty_{L^\infty(\Gamma)}$, $h \in L^\infty_{L^\infty(\partial D)} \cap L^2_{H^{1/2}(\Gamma)}$ with $h \geq 0$, $a_0 \in L^\infty(\Omega_0)$ and let a be the nonnegative solution to*

$$\begin{aligned}
\partial^\bullet a + a\nabla \cdot \mathbf{V}_p - D\Delta a + \nabla \cdot (\mathbf{J}_\Omega a) &= 0 && \textit{in } \Omega(t),\\
D\nabla a \cdot \nu - aj &= y && \textit{on } \Gamma(t),\\
a &= h && \textit{on } \partial D(t),\\
a(0) &= a_0 && \textit{in } \Omega_0.
\end{aligned}$$

Then $a \in L^\infty_{L^\infty(\Omega)}$ and

$$a(t) \leq C_1 e^{\|\nabla\cdot\mathbf{V}_\Omega\|_\infty T + D^{-1}C_2\left(\frac{1}{2}\|y\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right)^2 T} \max\left(1, \|h\|_{L^\infty_{L^\infty(\partial D)}}, \|a_0\|_\infty, \frac{1}{2}C_3\|y\|_{L^\infty_{L^\infty(\Gamma)}}\right)$$
$$\times \left(\min(1, D)^{-\frac{1}{2\kappa}-\frac{1}{2}} + C_4\right)$$

where C_1, C_2, C_3 and C_4 are constants independent of all relevant parameters.

Proof. With the transformation $A := ae^{-\lambda t}$ for a λ to be fixed later, we have

$$\begin{aligned} \partial^\bullet A + A(\nabla\cdot\mathbf{V}_p + \lambda) - D\Delta A + \nabla\cdot(\mathbf{J}_\Omega A) &= 0 && \text{in } \Omega(t),\\ D\nabla A\cdot\nu - Aj &= Y && \text{on } \Gamma(t),\\ A &= H && \text{on } \partial D(t),\\ A(0) &= a_0 && \text{in } \Omega_0, \end{aligned}$$

where $Y(t) := y(t)e^{-\lambda t}$ and $H(t) := he^{-\lambda t}$. Take a constant $k \geq \|h\|_{L^\infty_{L^\infty(\partial D)}}$, then defining (like in Lemma 3.4) $A_k(t) := (A(t)-k)^+$, we see that

$$A_k(t)|_{\partial D(t)} = (A(t)-k)^+|_{\partial D(t)} \leq \left(H - \|h\|_{L^\infty_{L^\infty(\partial D)}}\right)^+ = 0$$

and hence A_k is a valid test function because $A_k(t) \in H^1_e(\Omega(t))$. Testing the equation for A with $A_k := (A-k)^+$, we obtain that A_k satisfies exactly the same equality as in the proof of Lemma 3.4 and the proof carries through if we adjust (35) in the obvious way. □

Lemma 7.3. *We have*

$$\begin{aligned} &\|u\|_{L^\infty_{L^\infty(\Omega)}}\\ &\leq C_1 e^{\frac{1}{2}\|\nabla\cdot\mathbf{V}_\Omega\|_\infty T + \delta_\Omega C_2\left(\frac{1}{2}\delta_\Omega^{-1}\delta_{k'}^{-1}\|z\|_{L^\infty_{L^\infty(\Gamma)}} + 2\|j\|_\infty\right)^2 T}\\ &\quad\times \max\left(1, \|u_D\|_{L^\infty_{L^\infty(\partial D)}}, \|u_0\|_\infty, \frac{1}{2}C_3\delta_\Omega^{-1}\delta_{k'}^{-1}\|z\|_{L^\infty_{L^\infty(\Gamma)}}\right)\\ &\quad\times (\min(1,\delta_\Omega^{-1})^{-\frac{1}{2\kappa}-\frac{1}{2}} + C_4). \end{aligned}$$

Proof. Consider the following

$$\begin{aligned} \partial^\bullet a + a\nabla\cdot\mathbf{V}_p - \delta_\Omega^{-1}\Delta a + \nabla\cdot(\mathbf{J}_\Omega a) &= 0 && \text{in } \Omega(t),\\ \delta_\Omega^{-1}\nabla a\cdot\nu &= \delta_\Omega^{-1}\delta_{k'}^{-1}z + ja && \text{on } \Gamma(t),\\ a &= u_D && \text{on } \partial D(t),\\ a(0) &= u_0 \end{aligned}$$

By arguing almost identically as in Lemma 6.2, we get that $a \geq u$. Using this and the non-negativity of u, the claim follows from Lemma 7.2. □

We need the following version of Lemma 4.1 to bound $\delta_k^{-1}g(u,w)$ in $L^1_{L^1(\Gamma)}$. There, we used the equation for u; here we replace it by using the w equation.

Lemma 7.4. *We have*

$$\frac{1}{\delta_k}\int_0^T\int_{\Gamma(t)} g(u,w) \leq \frac{1}{\delta_{k'}}\|z\|_{L^1_{L^1(\Gamma)}} + \|w_0\|_{L^2(\Omega)}.$$

Proof. This follows from

$$\begin{aligned} \frac{d}{dt}\int_{\Gamma(t)} w &= \int_{\Gamma(t)} \partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_p\\ &= -\int_{\Gamma(t)}\nabla\cdot(\mathbf{J}_\Gamma w) + \delta_\Gamma\int_{\Gamma(t)}\Delta_\Gamma w + \frac{1}{\delta_{k'}}\int_{\Gamma(t)} z - \frac{1}{\delta_k}\int_{\Gamma(t)} g(u,w)\\ &= \frac{1}{\delta_{k'}}\int_{\Gamma(t)} z - \frac{1}{\delta_k}\int_{\Gamma(t)} g(u,w). \end{aligned}$$

□

The energy estimate for u given in Lemma 4.3 no longer applies because we cannot test the u equation with itself, as the space of test functions is different. We modify it as follows.

Lemma 7.5. *Let* $u_D \in L^\infty_{L^\infty(\partial D)} \cap C^0_{H^{1/2}(\partial D)}$. *With*

$$\begin{aligned}
L_1 &:= \|\nabla \cdot \mathbf{V}_\Omega\|_\infty + \|\nabla_\Gamma \cdot \mathbf{V}_p\|_\infty + \|\nabla \cdot \mathbf{J}_\Omega\|_\infty + \|\mathbf{J}_\Omega\|_\infty \,, \\
L_2 &:= (\|\nabla_\Gamma \cdot \mathbf{V}_p\|_\infty + \|\nabla \cdot \mathbf{J}_\Omega\|_\infty) \|\hat{u}\|^2_{L^2_{L^2(\Omega)}} + \|\mathbf{J}_\Omega\|_\infty \|\nabla \hat{u}\|^2_{L^2_{L^2(\Omega)}} \,, \\
K &:= \delta_\Omega L_2 + C(\delta_{k'}^{-1} + \delta_\Omega \|j\|_\infty)^2 \|\hat{u}\|^2_{L^2_{L^2(\Omega)}} + \|\nabla \hat{u}\|^2_{L^2_{L^2(\Omega}} + \delta_\Omega \|u_0 - \tilde{u}\|^2_{L^2(\Omega_0)} \,,
\end{aligned}$$

we have

$$\|u - \hat{u}\|^2_{L^\infty_{L^2(\Omega)}} \leq \left(\frac{\|z\|^2_{L^2_{L^2(\Gamma)}}}{\delta_\Omega \delta_{k'}} + \frac{K}{\delta_\Omega} \right) e^{\frac{1}{\delta_\Omega}\left(\delta_\Omega L_1 + 2C(\delta_{k'}^{-1} + \delta_\Omega \|j\|_\infty)^2\right)T},$$

$$\begin{aligned}
\frac{1}{2} \|\nabla u\|^2_{L^2_{L^2(\Omega)}} + \frac{2}{\delta_k} \int_0^t \int_{\Gamma(t)} g(u,w)u \leq{}& \frac{1}{\delta_{k'}} \|z\|^2_{L^2_{L^2(\Gamma)}} + \left(\delta_\Omega L_1 + C(\delta_{k'}^{-1} + \delta_\Omega \|j\|_\infty)^2\right) \|u - \hat{u}\|^2_{L^2_{L^2(\Omega)}} \\
&+ \delta_\Omega L_2 + C(\delta_{k'}^{-1} + \delta_\Omega \|j\|_\infty)^2 \|\hat{u}\|^2_{L^2_{L^2(\Omega)}} + \|\nabla \hat{u}\|^2_{L^2_{L^2(\Omega)}} \\
&+ \delta_\Omega \|u_0 - \tilde{u}\|^2_{L^2(\Omega_0)} \,.
\end{aligned}$$

Proof. Define $\tilde{u} \in H^1(\Omega_0)$ as the solution of the harmonic extension problem

$$\begin{aligned}
\Delta \tilde{u} &= 0 && \text{in } \Omega_0, \\
\tilde{u} &= 0 && \text{on } \Gamma_0, \\
\tilde{u} &= u_D(0) && \text{on } \partial D_0.
\end{aligned}$$

From this, we construct

$$\hat{u}(t) := \phi_t \tilde{u}$$

which satisfies

$$\begin{aligned}
\partial^\bullet \hat{u} &= 0 && \text{in } \Omega(t), \\
\hat{u} &= 0 && \text{on } \Gamma(t), \\
\hat{u} &= u_D && \text{on } \partial D(t),
\end{aligned}$$

with the first line because $\hat{u}$ is the pushforward of a constant-in-time object and the last because trace operators and the pushforward maps commute. It follows that $\hat{u} \in \mathbb{W}(H^1(\partial D), H^1(\partial D)^*)$ because $\phi_{-t}\hat{u}$ belongs to the associated reference space corresponding to ∂D_0. Test the equation for u with $u - \hat{u}$ (note that $u(t) - \hat{u}(t) \in H^1_e(\Omega(t))$):

$$\begin{aligned}
&\langle \partial^\bullet u, (u - \hat{u}) \rangle + \int_{\Omega(t)} u(u - \hat{u}) \nabla_\Gamma \cdot \mathbf{V}_p + \frac{1}{\delta_\Omega} \int_{\Omega(t)} \nabla u \nabla (u - \hat{u}) + \int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega u)(u - \hat{u}) \\
&\quad = \frac{1}{\delta_\Omega} \int_{\Gamma(t)} u \nabla u \cdot \nu
\end{aligned}$$

using that $\hat{u} = 0$ on $\Gamma(t)$. This is

$$\begin{aligned}
&\langle \partial^\bullet (u - \hat{u}), (u - \hat{u}) \rangle + \int_{\Omega(t)} (u - \hat{u})^2 \nabla_\Gamma \cdot \mathbf{V}_p + \hat{u}(u - \hat{u}) \nabla_\Gamma \cdot \mathbf{V}_p + \frac{1}{\delta_\Omega} \int_{\Omega(t)} \nabla u \nabla (u - \hat{u}) \\
&+ \int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega (u - \hat{u}))(u - \hat{u}) + \nabla \cdot (\mathbf{J}_\Omega \hat{u})(u - \hat{u}) = \frac{1}{\delta_\Omega} \int_{\Gamma(t)} u \nabla u \cdot \nu,
\end{aligned}$$

giving, using the identity (59),

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)}(u-\hat{u})^2+\frac{1}{2}\int_{\Omega(t)}(u-\hat{u})^2\nabla\cdot\mathbf{V}_\Omega+\int_{\Omega(t)}\hat{u}(u-\hat{u})\nabla_\Gamma\cdot\mathbf{V}_p+\frac{1}{\delta_\Omega}\int_{\Omega(t)}\nabla u\nabla(u-\hat{u})$$
$$+\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega\hat{u})(u-\hat{u})=\frac{1}{\delta_\Omega}\int_{\Gamma(t)}u\nabla u\cdot\nu-\frac{1}{2}\int_{\Gamma(t)}u^2 j.$$

Multiplying by $2\delta_\Omega$ and with the same manipulations as in the proof of Lemma 4.3 to deal with the boundary terms, we can write this as

$$\delta_\Omega\frac{d}{dt}\int_{\Omega(t)}(u-\hat{u})^2+\delta_\Omega\int_{\Omega(t)}(u-\hat{u})^2\nabla\cdot\mathbf{V}_\Omega+2\delta_\Omega\int_{\Omega(t)}\hat{u}(u-\hat{u})\nabla_\Gamma\cdot\mathbf{V}_p+2\int_{\Omega(t)}\nabla u\nabla(u-\hat{u})$$
$$+2\delta_\Omega\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega\hat{u})(u-\hat{u})\leq\int_{\Gamma(t)}\frac{1}{\delta_{k'}}z^2+\theta u^2-\frac{2}{\delta_k}g(u,w)u$$

where

$$\theta:=\frac{1}{\delta_{k'}}+\delta_\Omega\left\|j\right\|_\infty.$$

Using Young's inequality we can bound

$$2\delta_\Omega\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega\hat{u})(u-\hat{u})=2\delta_\Omega\int_{\Omega(t)}\nabla\cdot\mathbf{J}_\Omega\hat{u}(u-\hat{u})+\mathbf{J}_\Omega\cdot\nabla\hat{u}(u-\hat{u})$$
$$\leq\delta_\Omega\left\|\nabla\cdot\mathbf{J}_\Omega\right\|_\infty\int_{\Omega(t)}|\hat{u}|^2+(u-\hat{u})^2+\delta_\Omega\left\|\mathbf{J}_\Omega\right\|_\infty\int_{\Omega(t)}|\nabla\hat{u}|^2+(u-\hat{u})^2$$

where we have taken $\left\|\mathbf{J}_\Omega\right\|_\infty=\sup_i\left\|[\mathbf{J}_\Omega]_i\right\|_\infty$. Thus the lower order terms can be estimated by

$$\delta_\Omega(\left\|\nabla\cdot\mathbf{V}_\Omega\right\|_\infty+\left\|\nabla_\Gamma\cdot\mathbf{V}_p\right\|_\infty+\left\|\nabla\cdot\mathbf{J}_\Omega\right\|_\infty+\left\|\mathbf{J}_\Omega\right\|_\infty)\int_{\Omega(t)}(u-\hat{u})^2$$
$$+\delta_\Omega(\left\|\nabla_\Gamma\cdot\mathbf{V}_p\right\|_\infty+\left\|\nabla\cdot\mathbf{J}_\Omega\right\|_\infty)\left\|\hat{u}\right\|^2_{L^2(\Omega(t))}+\delta_\Omega\left\|\mathbf{J}_\Omega\right\|_\infty\left\|\nabla\hat{u}\right\|^2_{L^2(\Omega(t))}.$$

Let us define

$$L_1:=\left\|\nabla\cdot\mathbf{V}_\Omega\right\|_\infty+\left\|\nabla_\Gamma\cdot\mathbf{V}_p\right\|_\infty+\left\|\nabla\cdot\mathbf{J}_\Omega\right\|_\infty+\left\|\mathbf{J}_\Omega\right\|_\infty$$

(as above) and

$$\hat{L}_2(t):=(\left\|\nabla_\Gamma\cdot\mathbf{V}_p\right\|_\infty+\left\|\nabla\cdot\mathbf{J}_\Omega\right\|_\infty)\left\|\hat{u}\right\|^2_{L^2(\Omega(t))}+\left\|\mathbf{J}_\Omega\right\|_\infty\left\|\nabla\hat{u}\right\|^2_{L^2(\Omega(t))}.$$

For the gradient term, we can estimate simply by Young's inequality:

$$\int_{\Omega(t)}\nabla u\nabla(u-\hat{u})=\int_{\Omega(t)}|\nabla u|^2-\nabla u\nabla\hat{u}\geq\frac{1}{2}\int_{\Omega(t)}|\nabla u|^2-|\nabla\hat{u}|^2.$$

We have then

$$\delta_\Omega\frac{d}{dt}\int_{\Omega(t)}(u-\hat{u})^2+\int_{\Omega(t)}|\nabla u|^2\leq\int_{\Gamma(t)}\frac{1}{\delta_{k'}}z^2+\theta u^2-\frac{2}{\delta_k}g(u,w)u+\delta_\Omega L_1\int_{\Omega(t)}(u-\hat{u})^2$$
$$+\delta_\Omega\hat{L}_2(t)+\frac{1}{2}\left\|\nabla\hat{u}\right\|^2_{L^2(\Omega(t))}.$$

Now by the interpolated trace theorem we have, for $\varepsilon>0$,

$$\theta\int_{\Gamma(s)}u^2\leq\int_{\Omega(s)}\frac{C}{2}\theta^2u^2+\frac{1}{2}|\nabla u|^2\leq C\theta^2\int_{\Omega(s)}(u-\hat{u})^2+|\hat{u}|^2+\int_{\Omega(s)}\frac{1}{2}|\nabla u|^2.$$

Inserting this above, defining $L_2(t) := \int_0^t \hat{L}_2(s)$

$$\begin{aligned}
&\delta_\Omega \int_{\Omega(t)} (u-\hat{u})^2 + \frac{1}{2}\int_0^t \int_{\Omega(t)} |\nabla u|^2 + \frac{2}{\delta_k}\int_0^t \int_{\Gamma(t)} g(u,w)u \leq \frac{1}{\delta_{k'}} \|z\|^2_{L^2_{L^2(\Gamma)}} \\
&+ \left(\delta_\Omega L_1 + C\theta^2\right) \int_0^t \int_{\Omega(t)} (u-\hat{u})^2 + \hat{\delta}_\Omega L_2(t) + C\theta^2 \|\hat{u}\|^2_{L^2_{L^2(\Omega)}} + \|\nabla \hat{u}\|^2_{L^2_{L^2(\Omega}} \\
&+ \delta_\Omega \|u_0 - \tilde{u}\|^2_{L^2(\Omega_0)} .
\end{aligned}$$

Thus applying Gronwall's inequality, we end up with, denoting $L_2 = L_2(T)$ and setting

$$K := \delta_\Omega L_2 + C\theta^2 \|\hat{u}\|^2_{L^2_{L^2(\Omega)}} + \|\nabla \hat{u}\|^2_{L^2_{L^2(\Omega}} + \delta_\Omega \|u_0 - \tilde{u}\|^2_{L^2(\Omega_0)} ,$$

the estimate

$$\|u - \hat{u}\|^2_{L^\infty_{L^2(\Omega)}} \leq \left(\frac{\|z\|^2_{L^2_{L^2(\Gamma)}}}{\delta_\Omega \delta_{k'}} + \frac{K}{\delta_\Omega} \right) e^{\frac{1}{\delta_\Omega}\left(\delta_\Omega L_1 + 2C\theta^2\right)T}.$$

□

Note that we no longer get a uniform bound on u in $L^\infty_{L^2(\Omega)}$ (fortunately Lemma 7.3 takes care of this) because of the presence of the $\nabla \hat{u}$ term in the definition of K, but we do get one on its gradient. In summary, we obtain

$$u_k \rightharpoonup u \text{ in } L^2_{H^1(\Omega)} \cap L^\infty_{L^\infty(\Omega)}, \quad w_k \overset{*}{\rightharpoonup} w \text{ in } L^\infty_{L^\infty(\Gamma)}, \quad z_k \overset{*}{\rightharpoonup} z \text{ in } L^\infty_{L^\infty(\Gamma)}.$$

By continuity of the trace operator we have

$$u_k \rightharpoonup u \text{ in } L^2_{H^{1/2}(\Gamma)} \quad \text{and} \quad u_k \rightharpoonup u \text{ in } L^2_{H^{1/2}(\partial D)},$$

the latter of which immediately gives $u|_{\partial D} = u_D$.

We lack a uniform estimate on $(1/\delta_k)\nabla u_k$ (akin to (52)) and therefore we do not obtain uniform estimates on the difference quotients of u_k via Lemma 6.5. However, if we assume that $\mathbf{J}_\Gamma \equiv 0$, as explained in Remark 7.1, we do get an estimate on the difference quotients of w_k, and so

$$w_k \to w \text{ in } L^2_{H^{-1/2}(\Gamma)}.$$

We also retain the estimate (51) on the gradients of w_k and z_k. To pass to the limit, we begin with taking $\eta \in \mathcal{V}_e$ with $\eta(T) = 0$ and recalling the weak formulation relating u_k and z_k from above:

$$\begin{aligned}
&-\delta_\Omega \int_{\Omega_0} u_0 \eta(0) - \delta_\Omega \int_0^T \int_{\Omega(t)} u_k \partial^\bullet \eta + \int_0^T \int_{\Omega(t)} \nabla u_k \cdot \nabla \eta + \delta_\Omega \int_0^T \int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega u_k)\eta \\
&-\delta_\Omega \int_0^T \int_{\Gamma(t)} j u_k \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z_k \partial^\bullet \eta + \delta_\Omega \int_0^T \int_{\Gamma(t)} \nabla_\Gamma z_k \cdot \nabla_\Gamma \eta \\
&- \int_0^T \int_{\Gamma(t)} z_k \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta = 0.
\end{aligned}$$

Sending $\delta_\Omega \to 0$ yields

$$\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z \partial^\bullet \eta - \int_0^T \int_{\Gamma(t)} z \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta = 0.$$

Recall that we actually assumed that $\mathbf{J}_\Gamma \equiv 0$; we now argue to remove this assumption. Take an arbitrary parametrisation velocity $\widehat{\mathbf{V}_p}$ satisfying the same properties as $\mathbf{V}_p$ in Section 1.4, with corresponding parametrised material derivative $\widehat{\partial^\bullet}$. We have, by adding and subtracting the same terms to the above equality,

$$\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z \widehat{\partial^\bullet} \eta - \int_0^T \int_{\Gamma(t)} z(\mathbf{V}_\Gamma - \widehat{\mathbf{V}_p}) \cdot \nabla_\Gamma \eta$$
$$+ \int_0^T \int_{\Gamma(t)} z \widehat{\partial^\bullet} \eta + \int_0^T \int_{\Gamma(t)} z(\mathbf{V}_\Gamma - \widehat{\mathbf{V}_p}) \cdot \nabla_\Gamma \eta - \int_0^T \int_{\Gamma(t)} z \partial^\bullet \eta = 0.$$

Note that we can write $\widehat{\partial^\bullet}\eta = \partial^\circ \eta + \nabla_\Gamma \eta \cdot \widehat{\mathbf{V}_p}$ and similarly $\partial^\bullet \eta = \partial^\circ \eta + \nabla_\Gamma \eta \cdot \mathbf{V}_\Gamma$, so the terms on the left-hand side of the last line above become

$$\begin{aligned}
&\int_0^T \int_{\Gamma(t)} z \widehat{\partial^\bullet} \eta + \int_0^T \int_{\Gamma(t)} z(\mathbf{V}_\Gamma - \widehat{\mathbf{V}_p}) \cdot \nabla_\Gamma \eta - \int_0^T \int_{\Gamma(t)} z \partial^\bullet \eta \\
&= \int_0^T \int_{\Gamma(t)} z(\widehat{\partial^\bullet} \eta - \partial^\bullet \eta) + \int_0^T \int_{\Gamma(t)} z(\mathbf{V}_\Gamma - \widehat{\mathbf{V}_p}) \cdot \nabla_\Gamma \eta \\
&= \int_0^T \int_{\Gamma(t)} z \nabla_\Gamma \eta \cdot (\widehat{\mathbf{V}_p} - \mathbf{V}_\Gamma) + \int_0^T \int_{\Gamma(t)} z(\mathbf{V}_\Gamma - \widehat{\mathbf{V}_p}) \cdot \nabla_\Gamma \eta \\
&= 0.
\end{aligned}$$

Setting $\widehat{\mathbf{J}_\Gamma} := \mathbf{V}_\Gamma - \widehat{\mathbf{V}_p}$, we have shown that

$$\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z \widehat{\partial^\bullet} \eta - \int_0^T \int_{\Gamma(t)} z \widehat{\mathbf{J}_\Gamma} \cdot \nabla_\Gamma \eta = 0$$

for an arbitrary parametrised velocity $\widehat{\mathbf{V}_p}$. It remains to show that u, w, z belong to the corresponding function spaces defined using $\widehat{\mathbf{V}_p}$ and the associated maps $\widehat{\Phi}$ and $\hat{\phi}$. Writing $\hat{\phi}_{-t} z = z \circ \widehat{\Phi_t} = z \circ \Phi_t \circ \Phi^t \circ \widehat{\Phi_t} = (\phi_{-t} z) \circ \Phi^t \circ \widehat{\Phi_t}$ and using the fact that the composition of diffeomorphisms is a diffeomorphism,

$$\begin{aligned}
\int_0^T \int_{\Gamma_0} |\hat{\phi}_{-t} z(x)|^2 = \int_0^T \int_{\Gamma_0} |(\phi_{-t} z) \circ \Phi^t \circ \widehat{\Phi_t}(x)|^2 &= \int_0^T \int_{\Gamma_0} |(\phi_{-t} z)(y)|^2 \hat{\phi}_{-t}(J^t) \hat{J}_t \\
&\leq C \left\| \phi_{-(\cdot)} z \right\|_{L^2(0,T;L^2(\Gamma_0))}
\end{aligned}$$

and hence $\hat{\phi}_{-(\cdot)} z \in L^2(0,T;L^2(\Gamma_0))$, giving, by compatibility, that $z \in \widehat{L^2_{L^2(\Gamma)}}$. The other quantities can be tackled by a similar argument.

7.3.1 Complementarity condition

The complementarity condition follows by the assumption (21) and Lemma 7.4 (which implies that $g(u_k, w_k) \to 0$ in $L^1_{L^1(\Gamma)}$. Thus we have shown existence for (19).

8 Numerical experiments

We now present some numerical simulations of (1) that support the theoretical results of the previous sections and illustrate a robust numerical method for the simulation of coupled bulk-surface systems

of equations on evolving domains. Our approach is based on a parametric finite element method on moving triangulations that approximate the evolving domain. We employ a piecewise linear coupled bulk-surface finite element method for the approximation. The method is based on the coupled bulk-surface finite element method with an evolving surface finite element method for the approximation of the surface PDEs and an ALE finite element method for the bulk equation c.f., [15, 16, 19, 17] for details on the design and analysis of the numerical methods.

8.1 Domain evolution and discretisation

For all the simulations of this section, we assume the same continuous geometry and use the same discretisation of the geometry. We assume $D(t)$ is a sphere of radius 2 centred at the origin and that the outer boundary of Ω satisfies $\partial D(t) = \partial D(0)$, i.e., the outer boundary is stationary (the method employed below can be straightforwardly extended to the case of a moving outer boundary). Furthermore, for all the simulations we assume the interior surface $\Gamma(t)$ that corresponds to the cell membrane is given by the zero level set of the function

$$\phi(\boldsymbol{x}, t) = \left(x_1 + \tanh(5t)\left(0.7 - x_2^2\right)\right)^2 + x_2^2 + x_3^2 - 1,$$

then

$$\nu(\boldsymbol{x}, t) = -\frac{\nabla\phi(\boldsymbol{x}, t)}{|\nabla\phi(\boldsymbol{x}, t)|} \quad \text{and } V(\boldsymbol{x}, t) = -\frac{\phi_t(\boldsymbol{x}, t)}{|\nabla\phi(\boldsymbol{x}, t)|} \tag{53}$$

define the normal to $\Gamma(t)$ (pointing out of $\Omega(t)$) and the normal velocity. For simplicity we assume the material velocity of Γ, $\mathbf{V}_\Gamma$=$\mathbf{V}$, i.e., it has no tangential component. For the material velocity $\mathbf{V}_\Omega$ of Ω, we will either assume $\mathbf{V}_\Omega = \mathbf{0}$ or that $\mathbf{V}_\Omega = E(\boldsymbol{V}_\Gamma)$ where $E(\mathbf{V}_\Gamma)$ is the harmonic extension of the velocity of the boundary into the interior of Ω, where we recall that the velocity of the outer boundary is zero.

We partition the time interval $[0, T]$ as $0 = t_0 < t_1 < \cdots < t_N = T$, with uniform (for simplicity) timestep $\tau = t_{n+1} - t_n$, $n = 0, 1, \ldots, N-1$. We define initial computational domains Ω_h^0 and Γ_h^0 by requiring that Ω_h^0 is a polyhedral approximation to $\Omega(0)$ and we set $\Gamma_h^0 = \partial\Omega_h^0 \setminus \partial D_h$, i.e., Γ_h^0 is the interior boundary of the polyhedral domain Ω_h^0. We assume that Ω_h^0 is the union of tetrahedra and hence the faces of Γ_h^0 are triangles. We define $\mathcal{T}_h^0$ to be a triangulation of Ω_h^0 consisting of closed simplices. Furthermore, we assume the triangulation is such that for every $k \in \mathcal{T}_h^0$, $k \cap \Gamma_h^0$ consists of at most one face of k. We construct an initial triangulation Ω_h^0 which consists of 19642 tetrahedra with 4135 vertices with a higher resolution in the neighbourhood of Γ_h^0; the induced surface triangulation of Γ_h^0 consists of 1948 triangles with $N_\Gamma = 976$ vertices. As we shall consider Dirichlet conditions on the outer boundary, we denote by N_Ω the number of vertices of $\mathcal{T}_h^0$ that do not lie on the outer boundary ∂D_h^0. We evolve the nodes of the surface triangulation with the normal velocity defined in (53). The nodes of the bulk triangulation are evolved with the harmonic extension of this velocity into the interior. Figure 2 shows the resulting surface triangulation at a series of time points shaded by $V = \boldsymbol{V}_\Gamma \cdot \nu$. To

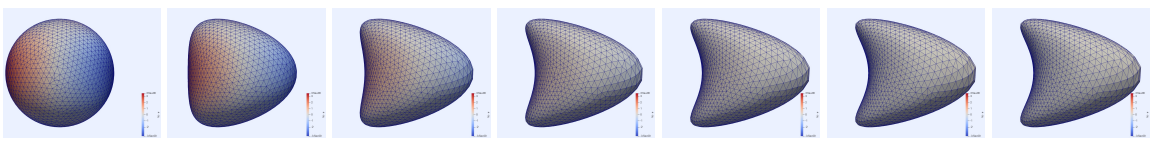


Figure 2: Snapshots of the surface triangulation shaded by $V = \boldsymbol{V}_\Gamma \cdot \nu$ at times $0, 0.1, 0.2, 0.3, 0.4, 0.8$ and 1.0 reading from left to right.

define the triangulations $\mathcal{T}_h^n$ of Ω at each time, $t_n, n \in [1, N]$, we compute the harmonic extension of the velocity of Γ_h^n into Ω_h^n and use this velocity to evolve the nodes of the triangulation of Ω_h^n.

8.2 Evolving coupled bulk-surface finite element method

For the approximation we define bulk and surface finite element spaces

$$\mathbb{V}_h^n := \{\Phi_h \in C(\Omega_h^n) : \Phi_h|\partial_{D_h} = 0, \Phi_h|_k \in \mathbb{P}_1(k) \text{ for each } k \in \mathcal{T}_h^n\}\,, \tag{54a}$$

$$\mathbb{V}_{h,U_D}^n := \{\Phi_h \in C(\Omega_h^n) : \Phi_h|\partial_{D_h} = U_D, \Phi_h|_k \in \mathbb{P}_1(k) \text{ for each } k \in \mathcal{T}_h^n\}\,, \tag{54b}$$

$$\mathbb{V}_{\Gamma,h}^n := \{\Phi_h \in C(\Gamma_h^n) : \Phi_h|_s \in \mathbb{P}_1(s) \text{ for all } k \in \mathcal{T}_h^n \text{ with } s = k \cup \Gamma_h^n \neq \emptyset\}\,. \tag{54c}$$

For $n \in [0, N]$ we denote by $\chi_j^n, \chi_{\Gamma,j_\Gamma}^n$, $j = 1, \dots, N_\Omega$, $j_\Gamma = 1 \dots, N_\Gamma$ the nodal basis of $\mathbb{V}_h^n$ and $\mathbb{V}_{\Gamma,h}^n$ respectively.

The numerical scheme we employ to approximate the solution of (7) reads as follows: for $n = 1, \dots, N$, given $(U_h^{n-1}, W_h^{n-1}, Z_h^{n-1}) \in \left(\mathbb{V}_{h,U_D}^{n-1} \times \left(\mathbb{V}_{\Gamma,h}^{n-1}\right)^2\right)$ find $(U_h^n, W_h^n, Z_h^n) \in \left(\mathbb{V}_{h,U_D}^n \times \left(\mathbb{V}_{\Gamma,h}^n\right)^2\right)$ such that for $j = 1 \dots, N_\Omega$ and for $j_\Gamma = 1 \dots, N_\Gamma$,

$$\begin{aligned}
&\int_{\Omega_h^n} \delta_\Omega \left(U_h^n \chi_j^n - \tau U_h^n \boldsymbol{J}_{\Omega,h}^n \cdot \nabla \chi_j^n\right) + \tau \nabla U_h^n \cdot \nabla \chi_j^n dx + \delta_\Omega \tau \int_{\Gamma_h^n} J_h^n U_h^n \chi_j^n ds \\
&\quad = \delta_\Omega \int_{\Omega_h^{n-1}} U_h^{n-1} \chi_j^{n-1} dx + \tau \int_{\Gamma_h^{n-1}} \frac{1}{\delta_{k'}} Z_h^{n-1} \chi_j^{n-1} - \frac{1}{\delta_k} g(U_h^{n-1}, W_h^{n-1}) \chi_j^{n-1} ds
\end{aligned} \tag{55a}$$

$$\begin{aligned}
&\int_{\Gamma_h^n} W_h^n \chi_{\Gamma,j}^n + \tau \delta_\Gamma \nabla_\Gamma W_h^n \cdot \nabla_\Gamma \chi_{\Gamma,j}^n ds \\
&= \int_{\Gamma_h^{n-1}} \left(W_h^{n-1} + \tau \left(\frac{1}{\delta_{k'}} Z_h^{n-1} \chi_{\Gamma,j}^{n-1} - \frac{1}{\delta_k} g(U_h^{n-1}, W_h^{n-1})\right)\right) \chi_{\Gamma,j}^{n-1} ds
\end{aligned} \tag{55b}$$

$$\begin{aligned}
&\int_{\Gamma_h^n} Z_h^n \chi_{\Gamma,j}^n + \tau \delta_{\Gamma'} \nabla_\Gamma Z_h^n \cdot \nabla_\Gamma \chi_{\Gamma,j}^n ds \\
&= \int_{\Gamma_h^{n-1}} \left(Z_h^{n-1} + \tau \left(\frac{1}{\delta_k} g(U_h^{n-1}, W_h^{n-1}) - \frac{1}{\delta_{k'}} Z_h^{n-1} \chi_{\Gamma,j}^{n-1}\right)\right) \chi_{\Gamma,j}^{n-1} ds
\end{aligned} \tag{55c}$$

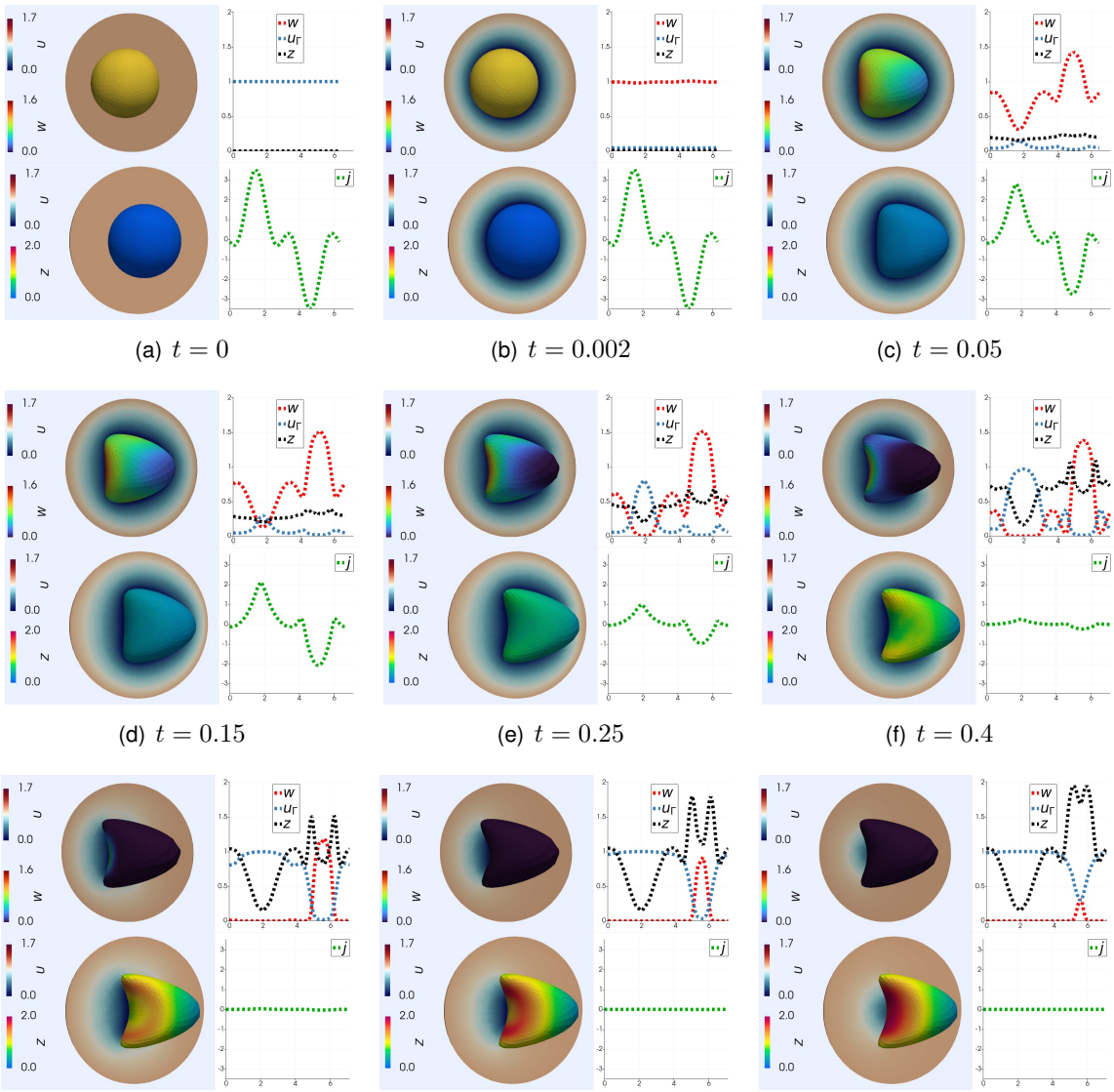
where $\boldsymbol{J}_{\Omega,h}^n$ is an approximation to $\mathbf{J}_\Omega$ and J_h^n an approximation to $\mathbf{J}_\Omega|_\Gamma \cdot \nu$. For the approximation of the initial condition we take the interpolant of the initial data into the respective finite element space.

8.3 Simulations approximating the $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit.

For the simulation results reported on in this subsection, we take $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} = 0.01$ and hence the results can be interpreted as an approximation of the limiting problem stated in Section 2.3. We set $g(u, w) = uw$ to contrast the results we present here in the evolving domain setting with those of [20] for fixed domains. We take constant initial and Dirichlet boundary data for u with $u_0 = u_D = 1$ and constant initial conditions for w and z with $w^0 = 1$ and $z^0 = 0$. We consider the domain evolution described in Section 8.1 with $\mathbf{V}_\Gamma$ as defined therein and $\mathbf{V}_\Omega = \mathbf{0}$. We take $\boldsymbol{J}_{\Omega,h}^n \in \mathbb{V}_h^n$ and define its nodal values such that $J_{\Omega,h}^n(\boldsymbol{X}_j^n) = -(\boldsymbol{X}^n - \boldsymbol{X}_j^{n-1})/\tau, j = 1, \dots, N_\Omega$. For the windshield effect term we take $J_h^n = I_{\Gamma,h}^n \left(-\frac{\phi_t(\boldsymbol{x},t)}{|\nabla\phi(\boldsymbol{x},t)|}\right)$ with $I_{\Gamma,h}$ the linear Lagrange interpolant.

Figure 3 shows the results of a simulation. We observe an initial rapid transition in which the trace of u on Γ vanishes to satisfy the condition $g(u, w) = 0$. The evolution of the domain causes the concentration of w to become spatially heterogenous and by $t = 0.25$ we have a region on Γ where w is zero

which corresponds to the region where the most protrusion has occurred and hence the concentration of w has been reduced. The region where w is zero grows over time and in this region the trace of u on Γ approaches 1 exhibiting the free boundary problem satisfied by the limiting equation. We note that the simulations reported on in this subsection can be interpreted as approximations of the free boundary problem given in (22). In Figure 4 we show corresponding approximations to the position of the free boundary at various times which we approximate as the level set given by $w = 0.1$.



(a) $t = 0$ (b) $t = 0.002$ (c) $t = 0.05$

(d) $t = 0.15$ (e) $t = 0.25$ (f) $t = 0.4$

(g) $t = 0.6$ (h) $t = 0.8$ (i) $t = 1$

Figure 3: Simulation results of Section 8.3 approximating the $\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit of Section 2.3 (see Section 8.3 for details of the parameters). In each subfigure the top left-hand panel indicates the surface shaded by the concentration of w and the bulk domain shaded by the concentration of u with half of the bulk domain made transparent. The bottom left-hand panel indicates the surface shaded by the concentration of z and the bulk domain shaded by the concentration of u with half of the bulk domain made transparent. The top right-hand panel indicates the values of the trace of u (blue), w (red) and z (black) on the curve on Γ_h with $x_3 = 0$ and the bottom right-hand panel indicates the values of the approximation $j = -\mathbf{V}_\Gamma \cdot \nu$ on the curve on Γ_h with $x_3 = 0$.

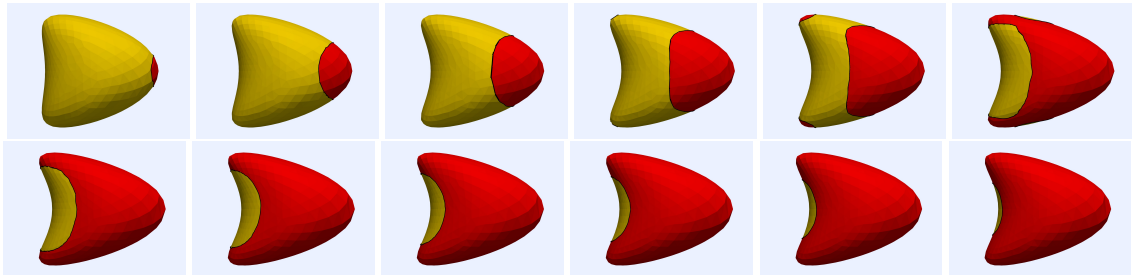


Figure 4: Simulation results of Section 8.3 approximating the evolution of the free boundary in (22) (see Section 8.3 for details). In each snapshot the level curve on which $w = 0.1$ (black line) is shown with the surface coloured yellow for $w > 0.1$ and red for $w < 0.1$. The snapshots are shown from $t = 0.12$ to $t = 0.84$ at uniform steps of 0.06 reading from left to right and top to bottom.

8.4 Simulations approximating the $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit.

For the simulation results reported on in this subsection, we take $\delta_k = \delta_\Gamma = \delta_{\Gamma'} = 0.001$ and take $\delta_\Omega = \delta_{k'} = 1$, hence the results can be interpreted as an approximation of the limiting problem stated in Section 2.2. We set $g(u, w) = \frac{u^2 w}{1+u^2}$. We take constant initial and Dirichlet boundary data for u with $u_0 = u_D = 1$ and for the initial condition for the surface species we set $w^0(\boldsymbol{x}) = e^{-6\left(1-x_1^2\right)}$ and $z^0(\boldsymbol{x}) = 0$.

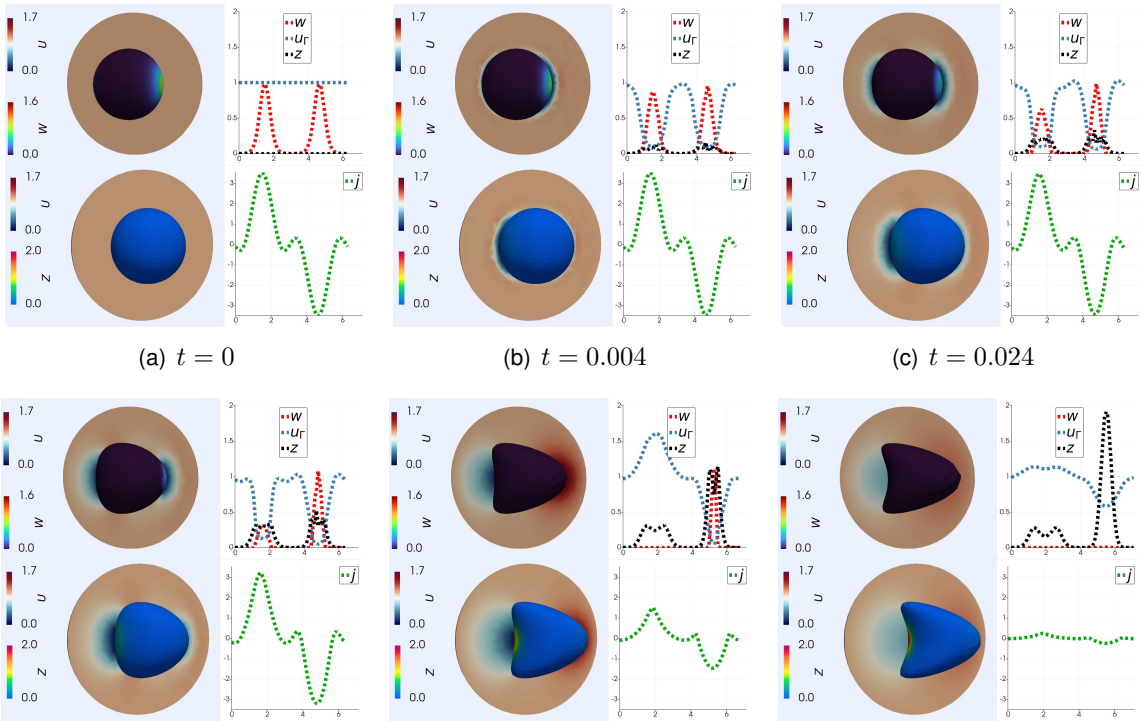
To illustrate the influence of the so called windshield effect, we consider the domain evolution described in Section 8.1 with the windshield effect being present in the case $\mathbf{V}_\Omega = \mathbf{0}$ and absent in the case $\mathbf{V}_\Omega = E(\mathbf{V}_\Gamma)$ as in the latter case the (inner) boundary of Ω moves with the same velocity as the surface Γ. We take the timestep $\tau = 10^{-6}$.

Figure 5 shows the results of the simulation with $\mathbf{V}_\Omega = \mathbf{0}$. We take $\boldsymbol{J}^n_{\Omega,h} \in \mathbb{V}^n_h$ and define its nodal values such that $\boldsymbol{J}^n_{\Omega,h}(\boldsymbol{X}^n_j) = -(\boldsymbol{X}^n - \boldsymbol{X}^{n-1}_j)/\tau, j = 1, \ldots, N_\Omega$. For the windshield effect term we take $J^n_h = I^n_{\Gamma,h}\left(-\frac{\phi_t(\boldsymbol{x},t)}{|\nabla\phi(\boldsymbol{x},t)|}\right)$ with $I_{\Gamma,h}$ the linear Lagrange interpolant. We see a rapid initial evolution such that the supports of the trace of u on Γ and w become close to disjoint and then a slower evolution as w is depleted an u increases. We see u is larger near regions of the surface where j is largest with the maximum value of u exceeding 1 and that u is smaller near regions where j is smallest, thus, demonstrating the windshield effect.

Figure 6 shows the results of the simulation with $\mathbf{V}_\Omega = E(\mathbf{V}_\Gamma)$, i.e., the harmonic extension of the velocity of the surface. In this case $\boldsymbol{J}^n_{\Omega,h} = \mathbf{0}$ and and $J^n_h = 0$, i.e., the scheme is Lagrangian. We see a rapid initial evolution such that the supports of the trace of u on Γ and w become close to disjoint and then a slower evolution as w is depleted and u increases. The absence of the windshield effect results in u not exceeding 1 throughout the evolution and the profiles for w and z remain broadly similar to the case considered in Figure 5. The fact that the surface species exhibit similar dynamics with or without the windshield effect may have implications to biological phenomena such as chemotaxis [21, 36].

A Integration by parts identities

We collect some technical facts that are used in the paper. Here and below, $\Omega \subset \mathbb{R}^{d+1}$ is a sufficiently smooth bounded domain with $\partial\Omega = \Gamma \cup \partial D$.



(a) $t = 0$ (b) $t = 0.004$ (c) $t = 0.024$

(d) $t = 0.06$ (e) $t = 0.2$ (f) $t = 0.4$

Figure 5: Simulation results of Section 8.4 approximating the $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit of Section 2.2 with $\mathbf{V}_\Omega = \mathbf{0}$ illustrating the windshield effect (see Section 8.4 for details of the parameters). In each subfigure the top left-hand panel indicates the surface shaded by the concentration of w and the bulk domain shaded by the concentration of u with half of the bulk domain made transparent. The bottom left-hand panel indicates the surface shaded by the concentration of z and the bulk domain shaded by the concentration of u with half of the bulk domain made transparent. The top right-hand panel indicates the values of the trace of u (blue), w (red) and z (black) on the curve on Γ_h with $x_3 = 0$ and the bottom right-hand panel indicates the values of the approximation $j = -\mathbf{V}_\Gamma \cdot \nu$ on the curve on Γ_h with $x_3 = 0$.

A.1 Bulk identities

Let $\mathbf{J} \colon \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be a vector field satisfying

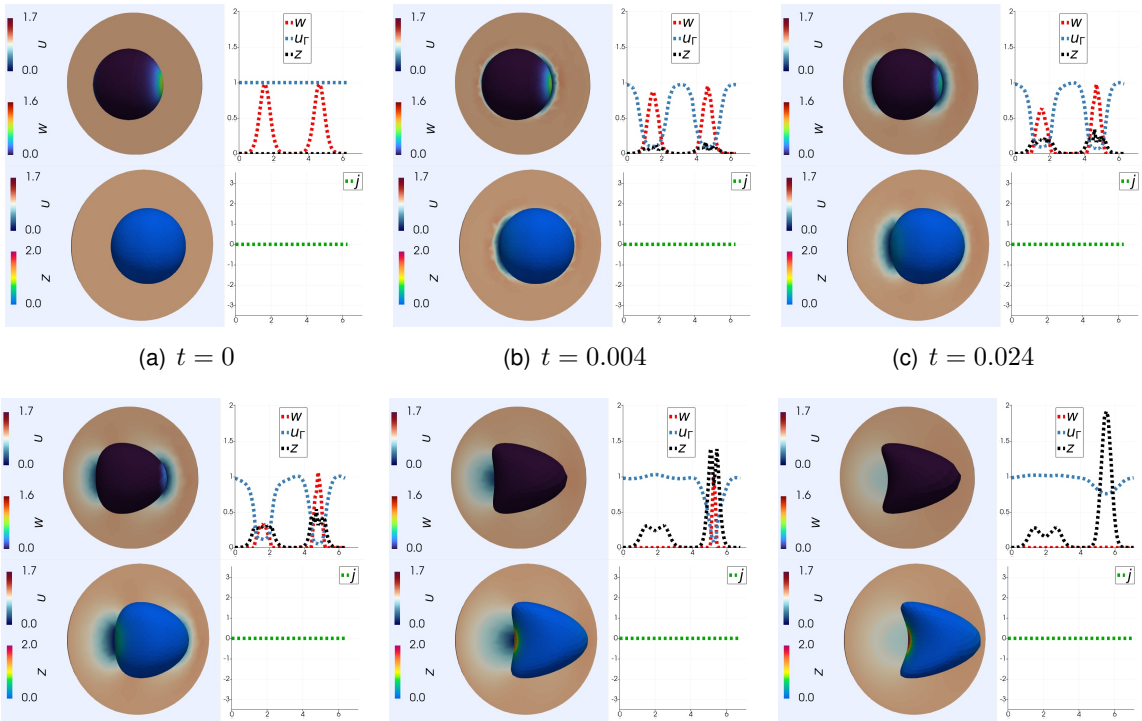
$$\mathbf{J} \cdot \nu = 0 \quad \text{on } \partial D.$$

For sufficiently regular functions $a \colon \Omega \to \mathbb{R}$ and $b \colon \Omega \to \mathbb{R}$, we have from the product rule and the divergence theorem

$$\int_\Omega (\mathbf{J} \cdot \nabla a) b + (\nabla \cdot \mathbf{J}) ab = \int_\Omega \nabla \cdot (a\mathbf{J}) b = \int_\Omega \nabla \cdot (a\mathbf{J} b) - a\mathbf{J} \cdot \nabla b = \int_\Gamma ab\mathbf{J} \cdot \nu - \int_\Omega a(\mathbf{J} \cdot \nabla b).$$

From this we can deduce several expressions that will be useful throughout the paper:

$$\begin{aligned} \int_\Omega \nabla \cdot (a\mathbf{J}) b &= \int_\Gamma ab\mathbf{J} \cdot \nu - \int_\Omega a\mathbf{J} \cdot \nabla b, \\ \int_\Omega (\mathbf{J} \cdot \nabla a) a &= \frac{1}{2} \int_\Gamma a^2 \mathbf{J} \cdot \nu - \frac{1}{2} \int_\Omega a^2 \nabla \cdot \mathbf{J}, \end{aligned} \tag{56}$$



(a) $t = 0$ (b) $t = 0.004$ (c) $t = 0.024$

(d) $t = 0.06$ (e) $t = 0.2$ (f) $t = 0.4$

Figure 6: Simulation results of Section 8.4 approximating the $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit of Section 2.2 with $\mathbf{V}_\Omega = E(\mathbf{V}_\Gamma)$, resulting in no windshield effect, and everything else as in Figure 5 (see Section 8.4 for details of the parameters). In each subfigure the top left-hand panel indicates the surface shaded by the concentration of w and the bulk domain shaded by the concentration of u with half of the bulk domain made transparent. The bottom left-hand panel indicates the surface shaded by the concentration of z and the bulk domain shaded by the concentration of u with half of the bulk domain made transparent. The top right-hand panel indicates the values of the trace of u (blue), w (red) and z (black) on the curve on Γ_h with $x_3 = 0$ and the bottom right-hand panel indicates the values of the approximation $j = 0$ on the curve on Γ_h with $x_3 = 0$ (included simply for comparison with Figure 5).

and with $a_k := (a - k)^+$ for a number $k \in \mathbb{R}$,

$$\int_\Omega \nabla \cdot (\mathbf{J}a)a_k = \int_\Omega (\nabla \cdot \mathbf{J})aa_k + \frac{1}{2}\int_\Gamma a_k^2 \mathbf{J} \cdot \nu - \frac{1}{2}\int_\Omega a_k^2 \nabla \cdot \mathbf{J}. \tag{57}$$

Here, we used that $\nabla a = \nabla a_k$ in $\operatorname{supp}\{a_k\}$ to write $\nabla \cdot (\mathbf{J}a)a_k = (\nabla \cdot \mathbf{J})aa_k + (\mathbf{J} \cdot \nabla a_k)a_k$. Now, using (57), we can derive

$$\int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega a)a^+ = \frac{1}{2}\int_{\Omega(t)} |a^+|^2 \nabla \cdot \mathbf{J}_\Omega + \frac{1}{2}\int_{\Gamma(t)} j|a^+|^2 \tag{58}$$

if we recall that j is the jump of the velocities as defined in (8).

Now if $a, b \in \mathbb{W}(H^1(\Omega), H^1(\Omega)^*)$, we have the transport formula

$$\frac{d}{dt}\int_{\Omega(t)} ab = \langle \partial^\bullet a, b\rangle_{H^1(\Omega(t))^*, H^1(\Omega(t))} + \langle \partial^\bullet b, a\rangle_{H^1(\Omega(t))^*, H^1(\Gamma(t))} + \int_{\Omega(t)} ab\nabla \cdot \mathbf{V}_p$$

and thus

$$\langle \partial^\bullet a, a\rangle = \frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} a^2 - \frac{1}{2}\int_{\Omega(t)} a^2 \nabla \cdot \mathbf{V}_p.$$

Using the fact that the identity (56) implies

$$\int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega a) a = \frac{1}{2} \int_{\Omega(t)} a^2 \nabla \cdot \mathbf{J}_\Omega + \frac{1}{2} \int_{\Gamma(t)} a^2 j,$$

we may deduce

$$\begin{aligned} \langle \partial^\bullet a, a \rangle + \int_{\Omega(t)} a^2 \nabla \cdot \mathbf{V}_p + \int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega a) a &= \frac{1}{2} \frac{d}{dt} \int_{\Omega(t)} a^2 + \frac{1}{2} \int_{\Omega(t)} a^2 \nabla \cdot \mathbf{V}_p \\ &\quad + \frac{1}{2} \int_{\Omega(t)} a^2 \nabla \cdot \mathbf{J}_\Omega + \frac{1}{2} \int_{\Gamma(t)} a^2 j \\ &= \frac{1}{2} \frac{d}{dt} \int_{\Omega(t)} a^2 + \frac{1}{2} \int_{\Omega(t)} a^2 \nabla \cdot \mathbf{V}_\Omega + \frac{1}{2} \int_{\Gamma(t)} a^2 j. \end{aligned} \tag{59}$$

A.2 Surface identities

Recall the divergence theorem[1]

$$\int_\Gamma \nabla_\Gamma \cdot \mathbf{J} = - \int_\Gamma H \mathbf{J} \cdot \nu$$

on closed surfaces [15, Theorem 2.10] where H denotes the mean curvature. We derive for a sufficiently smooth function $a \colon \Gamma \to \mathbb{R}$ the identities

$$\begin{aligned} \int_\Gamma (\mathbf{J} \cdot \nabla_\Gamma a) a &= -\frac{1}{2} \int_\Gamma a^2 H \mathbf{J} \cdot \nu - \frac{1}{2} \int_\Gamma (\nabla_\Gamma \cdot \mathbf{J}) a^2, \\ \int_\Gamma \nabla_\Gamma \cdot (\mathbf{J} a) a_k &= \int_\Gamma (\nabla_\Gamma \cdot \mathbf{J}) a a_k - \frac{1}{2} \int_\Gamma a_k^2 H (\mathbf{J} \cdot \nu) - \frac{1}{2} \int_\Gamma a_k^2 \nabla_\Gamma \cdot \mathbf{J}. \end{aligned}$$

We note the identity

$$\int_\Gamma \nabla_\Gamma \cdot (\mathbf{J}_\Gamma a) \eta = - \int_\Gamma a \mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta, \tag{60}$$

which follows by the divergence theorem and the expansion formula

$$\nabla_\Gamma \cdot (\mathbf{J}_\Gamma a \eta) = \nabla_\Gamma \cdot (\mathbf{J}_\Gamma a) \eta + (\mathbf{J}_\Gamma \cdot \nabla_\Gamma \eta) a$$

bearing in mind $\mathbf{J}_\Gamma \cdot \nu = 0$ (since $\mathbf{V}_p$ and $\mathbf{V}_\Gamma$ have the same normal components). Hence

$$\int_{\Gamma(t)} \nabla_\Gamma \cdot (\mathbf{J}_\Gamma a) a^+ = \frac{1}{2} \int_{\Gamma(t)} |a^+|^2 \nabla_\Gamma \cdot \mathbf{J}_\Gamma. \tag{61}$$

Remark A.1. *The equalities* (58) *and* (61) *also hold when* a^+ *is replaced with* a^-*.*

[1] Bear in mind that we have defined $\nu(t)$ as the unit normal pointing *into* $\Gamma(t)$, hence the minus sign in the formula.

Note that, using again the divergence theorem and remembering $\mathbf{J}_\Gamma \cdot \nu = 0$,

$$\begin{aligned}\int_\Gamma \nabla_\Gamma \cdot (\mathbf{J}_\Gamma a) a &= \int_\Gamma a^2 \nabla_\Gamma \cdot \mathbf{J}_\Gamma + a \mathbf{J}_\Gamma \cdot \nabla_\Gamma a \\ &= \int_\Gamma a^2 \nabla_\Gamma \cdot \mathbf{J}_\Gamma + \frac{1}{2} \mathbf{J}_\Gamma \cdot \nabla_\Gamma (a^2) \\ &= \int_\Gamma a^2 \nabla_\Gamma \cdot \mathbf{J}_\Gamma - \frac{1}{2} \nabla_\Gamma \cdot \mathbf{J}_\Gamma a^2 \\ &= \frac{1}{2} \int_\Gamma a^2 \nabla_\Gamma \cdot \mathbf{J}_\Gamma.\end{aligned}$$

For $a, b \in \mathbb{W}(H^1(\Gamma), H^1(\Gamma)^*)$, we have the formula

$$\frac{d}{dt} \int_{\Gamma(t)} ab = \langle \partial^\bullet a, b \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))} + \langle \partial^\bullet b, a \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))} + \int_{\Gamma(t)} ab \nabla_\Gamma \cdot \mathbf{V}_p$$

and thus we can conclude like before that

$$\langle \partial^\bullet a, a \rangle + \int_{\Gamma(t)} a^2 \nabla_\Gamma \cdot \mathbf{V}_p + \int_{\Gamma(t)} \nabla_\Gamma \cdot (\mathbf{J}_\Gamma a) a = \frac{1}{2} \frac{d}{dt} \int_{\Gamma(t)} a^2 + \frac{1}{2} \int_{\Gamma(t)} a^2 \nabla_\Gamma \cdot \mathbf{V}_\Gamma. \tag{62}$$

Lemma A.2. *For all $a \in \mathbb{W}(H^2(\Gamma), L^2(\Gamma))$, we have*

$$\int_0^t \int_{\Gamma(s)} \partial^\bullet a(s) \Delta_\Gamma a(s) = \frac{1}{2} \int_{\Gamma_0} |\nabla_\Gamma a(0)|^2 - \frac{1}{2} \int_{\Gamma(t)} |\nabla_\Gamma a(t)|^2 + \frac{1}{2} \int_0^t \int_{\Gamma(s)} \nabla_\Gamma a(s)^T \mathbf{H}(s) \nabla_\Gamma a(s).$$

Proof. Take functions $a, b \in C^\infty_{H^2(\Gamma)}$. By integrating by parts and using the formula (see [15, Equation (5.8)])

$$\frac{d}{dt} \int_{\Gamma(t)} \nabla_\Gamma a \cdot \nabla_\Gamma b = \int_{\Gamma(t)} \nabla_\Gamma \partial^\bullet a \cdot \nabla_\Gamma b + \nabla_\Gamma a \cdot \nabla_\Gamma \partial^\bullet b + \nabla_\Gamma a^T \mathbf{H}(t) \nabla_\Gamma b,$$

we get

$$\begin{aligned}\int_{\Gamma(t)} \partial^\bullet a(t) \Delta_\Gamma a(t) = - \int_{\Gamma(t)} \nabla_\Gamma a(t) \nabla_\Gamma \partial^\bullet a(t) = &-\frac{1}{2} \frac{d}{dt} \int_{\Gamma(t)} |\nabla_\Gamma a(t)|^2 \\ &+ \frac{1}{2} \int_{\Gamma(t)} \nabla_\Gamma a(t)^T \mathbf{H}(t) \nabla_\Gamma a(t),\end{aligned}$$

giving

$$\int_0^t \int_{\Gamma(s)} \partial^\bullet a(s) \Delta_\Gamma a(s) = \frac{1}{2} \int_{\Gamma_0} |\nabla_\Gamma a(0)|^2 - \frac{1}{2} \int_{\Gamma(t)} |\nabla_\Gamma a(t)|^2 + \frac{1}{2} \int_0^t \int_{\Gamma(s)} \nabla_\Gamma a(s)^T \mathbf{H}(s) \nabla_\Gamma a(s).$$

Since $C^\infty_{H^2(\Gamma)} \subset \mathbb{W}(H^2(\Gamma), L^2(\Gamma))$ is dense (see the proof of Lemma 2.14 in [1]) and using $\mathbb{W}(H^2(\Gamma), L^2(\Gamma)) \hookrightarrow C^0_{H^1(\Gamma)}$, it follows that the above equality also holds for $a \in \mathbb{W}(H^2(\Gamma), L^2(\Gamma))$.

□

B Verification of assumptions related to g

Proof of Proposition 2.5. We split the proof by the two choices of g.

(i) Let us first address the case (2) where $g(u,w) = uw$. Since at least one of the sequences converges strongly and the other converges weakly in $L^2_{L^2(\Gamma)}$ (from continuous embeddings) by hypothesis, (9) and (10) are trivially met.

(ii) Now we focus on the case of (3) where we have $g(u,w) = u^n w/(1+u^n)$ for $n > 1$. Let us consider the condition (9) first. By pulling back onto the initial surface, we have (by hypothesis) that

$$\int_0^T \int_{\Gamma_0} g(\tilde{u}_k, \tilde{w}_k) J_t^0 \to 0$$

where J_t^0 is the determinant of the transformation. Instead of writing the tildes, we will abuse notation and write simply u_k and w_k for simplicity. Since J_t^0 is strictly positive and g and u_k, w_k are non-negative, the above implies that for a subsequence (which we relabel), we have

$$g(u_k(y), w_k(y)) \to 0 \text{ pointwise a.e. } y \in (0,T) \times \Gamma_0,$$

i.e., the convergence holds for all $y \in (0,T) \times \Gamma_0 \setminus N_1$ where N_1 is a null set. By the assumed strong convergence, we have (again for a subsequence that has been relabelled)

$$w_k \to w \text{ pointwise a.e } y \in (0,T) \times \Gamma_0,$$

i.e., the convergence holds for all $y \in (0,T) \times \Gamma_0 \setminus N_2$ where N_2 is a null set.

We have the identity $B = B \cap A \cup B \cap A^c$ for any two sets A and B, hence

$$\{w \neq 0\} = \{w \neq 0\} \cap (N_1 \cup N_2) \cup \{w \neq 0\} \cap (N_1^c \cap N_2^c).$$

The first intersection on the right-hand side is a null set since it is an intersection with null sets, hence, we have

$$\{w \neq 0\} = \{w \neq 0\} \cap (N_1^c \cap N_2^c)$$

up to sets of measure zero.

Take $y^* \in \{w \neq 0\}$. Then recalling the definition of g and the above decomposition,

$$\frac{u_k(y^*)^n w_k(y^*)}{1 + u_k(y^*)^n} \to 0 \qquad \text{and} \qquad w_k(y^*) \to w(y^*).$$

Since $w(y^*) \neq 0$ and $w \geq 0$, we can find a $K(y^*)$ such that $k \geq K(y^*)$ implies that $w_k(y^*) > 0$ for all $k \geq K(y^*)$. It follows from the above displayed equation (by dividing by $w_k(y^*)$ and taking the limit) that

$$\frac{u_k(y^*)^n}{1 + u_k(y^*)^n} \to 0$$

and hence $u_k(y^*) \to 0$. This holds for all $y^* \in \{w \neq 0\}$. It follows that

$$u_k \chi_{\{w \neq 0\}} \to 0 \text{ pointwise a.e. in } (0,T) \times \Gamma_0.$$

By the continuous embedding, $u_k \rightharpoonup u$ in $L^2((0,T)\times\Gamma_0)$, and because the characteristic function is bounded, $u_k\chi_{\{w\neq 0\}} \rightharpoonup u\chi_{\{w\neq 0\}}$ in the same space. Because the pointwise a.e. and weak limit coincide, we must then have

$$u\chi_{\{w\neq 0\}} = 0,$$

and hence $u = 0$ on $\{w \neq 0\}$. It follows that $uw = 0$, which implies $g(u,w) = 0$. This shows that (9) holds.

For (10), this is much simpler because we have an L^∞ bound and strong convergence for u (and hence strong convergence of the nonlinear part of $g(u,w)$).

□

Proof of Proposition 2.18. If $g(u,w) = uw$, we can manipulate and use the strong and weak convergences:

$$\begin{aligned}\int_0^T\int_{\Gamma(t)} g(u_k,w_k) = \int_0^T \langle w_k,u_k\rangle_{H^{-1/2}(\Gamma(t)),H^{1/2}(\Gamma(t))} &\to \int_0^T \langle w,u\rangle_{H^{-1/2}(\Gamma(t)),H^{1/2}(\Gamma(t))}\\ &= \int_0^T\int_{\Gamma(t)} uw.\end{aligned}$$

□

C Nondimensionalisation

Inspired by [23] as in [20, 5], consider the system

$$\begin{aligned}\partial^\bullet u + u\nabla\cdot\mathbf{V}_\Omega - D_\Omega\Delta u &= 0 && \text{in } \Omega(t),\\ D_\Omega\nabla u\cdot\nu + u(\mathbf{V}_\Gamma - \mathbf{V}_\Omega)\cdot\nu &= k_{\text{off}}z - k_{\text{on}}uw && \text{on } \Gamma(t),\\ \partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_\Gamma - D_\Gamma\Delta w &= k_{\text{off}}z - k_{\text{on}}uw && \text{on } \Gamma(t),\\ \partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_\Gamma - D_{\Gamma'}\Delta z &= -k_{\text{off}}z + k_{\text{on}}uw && \text{on } \Gamma(t).\end{aligned} \tag{63}$$

Introducing suitable scaling parameters, we define the dimensionless variables

$$\overline{u} = \frac{u}{U},\ \overline{w} = \frac{w}{W},\ \overline{z} = \frac{z}{Z},\ \overline{x} = \frac{x}{L},\ \overline{t} = \frac{t}{S},\ \overline{\mathbf{V}}_i = \frac{s\mathbf{V}_i}{L}, \qquad i = \Omega \text{ or } \Gamma,$$

we obtain the following dimensionless system (where we drop the bars for notational simplicity)

$$\begin{aligned}\partial^\bullet u + u\nabla\cdot\mathbf{V}_\Omega - \delta_\Omega^{-1}\Delta u &= 0 && \text{in } \Omega(t),\\ \nabla u\cdot\nu + \delta_\Omega u(\mathbf{V}_\Gamma - \mathbf{V}_\Omega)\cdot\nu &= \delta_{k'}^{-1}z - \delta_k^{-1}uw && \text{on } \Gamma(t),\\ \partial^\bullet w + w\nabla_\Gamma\cdot\mathbf{V}_\Gamma - \delta_\Gamma\Delta w &= \mu(\delta_{k'}^{-1}z - \delta_k^{-1}uw) && \text{on } \Gamma(t),\\ \partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_\Gamma - \delta_{\Gamma'}\Delta z &= \mu'(-\delta_{k'}^{-1}z + \delta_k^{-1}uw) && \text{on } \Gamma(t),\end{aligned} \tag{64}$$

where we have introduced the dimensionless variables

$$\delta_\Omega = \frac{L^2}{D_\Omega S},\ \ \delta_\Gamma = \frac{D_\Gamma S}{L^2},\ \ \delta_{\Gamma'} = \frac{D_{\Gamma'}S}{L^2},\ \ \delta_k = \frac{D_\Omega}{k_{\text{on}}LW},\ \ \delta_{k'}^{-1} = \frac{k_{\text{off}}ZL}{D_\Omega U},\ \ \mu = \frac{SUD_\Omega}{LW},$$
$$\mu' = \frac{SUD_\Omega}{LZ}.$$

Parameter	Value	Source
L	$7.5 \cdot 10^{-6}\,\mathrm{m}$	[23]
U	$1.0 \cdot 10^{-3}\,\mathrm{mol\,m^{-3}}$	[23]
W	$2.3 \cdot 10^{-8}\,\mathrm{mol\,m^{-2}}$	[23]
Z	$2.3 \cdot 10^{-8}\,\mathrm{mol\,m^{-2}}$	limited by total receptor concentration
D_Ω	$1.0 \cdot 10^{-11}\,\mathrm{m^2\,s^{-1}}$	[32]
D_Γ	$1.0 \cdot 10^{-15}\,\mathrm{m^2\,s^{-1}}$	[32]
$D_{\Gamma'}$	$1.0 \cdot 10^{-15}\,\mathrm{m^2\,s^{-1}}$	[32]
k_{on}	$1.0 \cdot 10^{3}\,\mathrm{m^3\,mol^{-1}\,s^{-1}}$	[23]
k_{off}	$5.0 \cdot 10^{-3}\mathrm{s^{-1}}$	[23]

Table 1: Parameters used for rescaling equations. The values for U and W are extreme values taken from within a physical range from [23].

Taking values from Table 1, we infer that

$$\delta_\Omega = (5.6\,\mathrm{s}) \cdot S^{-1}, \quad \delta_k = 5.7 \cdot 10^{-2}, \quad \delta_{k'}^{-1} = 8.7 \cdot 10^{-2},$$
$$\delta_\Gamma = \delta_{\Gamma'} = (1.8 \cdot 10^{-4}\,\mathrm{s}^{-1}) \cdot S, \quad \mu = \mu' = (5.7 \cdot 10^{-2}\,\mathrm{s}^{-1}) \cdot S.$$

We see $\delta_k \ll 1$ motivating the consideration of the limit $\delta_k \to 0$. For $S = L^2/D_\Omega = 5.6\,\mathrm{s}$, i.e., the timescale such that $\delta_\Omega = 1$ we have

$$\delta_\Omega = 1, \qquad \delta_\Gamma = \delta_{\Gamma'} = 1.0 \cdot 10^{-3} \ll 1, \qquad \mu = \mu' = 3.2 \cdot 10^{-1} \approx 1,$$

motivating the limit $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$. Alternatively, taking $S = 10^2\,\mathrm{s}$, i.e., a timescale such that $\mu, \mu' \approx 1$, we have

$$\delta_\Omega = 5.7 \cdot 10^{-2} \ll 1, \qquad \delta_\Gamma = \delta_{\Gamma'} = 1.8 \cdot 10^{-2} \ll 1, \qquad \mu = \mu' = 5.7 \approx 1,$$

motivating the limit $\delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} = \delta_\Omega \to 0$.

References

[1] A. Alphonse and C. M. Elliott. A Stefan problem on an evolving surface. *Philos. Trans. Roy. Soc. A*, 373(2050):20140279, 16, 2015. ISSN 1364-503X,1471-2962. doi: 10.1098/rsta.2014.0279. URL `https://doi.org/10.1098/rsta.2014.0279`.

[2] A. Alphonse and C. M. Elliott. Well-posedness of a fractional porous medium equation on an evolving surface. *Nonlinear Analysis*, 137:3–42, 2016.

[3] A. Alphonse, C. M. Elliott, and B. Stinner. An abstract framework for parabolic PDEs on evolving spaces. *Port. Math.*, 72(1):1–46, 2015. ISSN 0032-5155,1662-2758. doi: 10.4171/PM/1955. URL `https://doi.org/10.4171/PM/1955`.

[4] A. Alphonse, C. M. Elliott, and B. Stinner. On some linear parabolic PDEs on moving hypersurfaces. *Interfaces Free Bound.*, 17(2):157–187, 2015. ISSN 1463-9963,1463-9971. doi: 10.4171/IFB/338. URL `https://doi.org/10.4171/IFB/338`.

[5] A. Alphonse, C. M. Elliott, and J. Terra. A coupled ligand-receptor bulk-surface system on a moving domain: well posedness, regularity, and convergence to equilibrium. *SIAM J. Math. Anal.*, 50(2):1544–1592, 2018. ISSN 0036-1410,1095-7154. doi: 10.1137/16M110808X. URL https://doi.org/10.1137/16M110808X.

[6] A. Alphonse, D. Caetano, A. Djurdjevac, and C. M. Elliott. Function spaces, time derivatives and compactness for evolving families of Banach spaces with applications to PDEs. *J. Differential Equations*, 353:268–338, 2023. ISSN 0022-0396,1090-2732. doi: 10.1016/j.jde.2022.12.032. URL https://doi.org/10.1016/j.jde.2022.12.032.

[7] J. Borgqvist, A. Malik, C. Lundholm, A. Logg, P. Gerlee, and M. Cvijovic. Cell polarisation in a bulk-surface model can be driven by both classic and non-classic Turing instability. *NPJ systems biology and applications*, 7(1):13, 2021.

[8] D. Caetano, C. M. Elliott, and B. Q. Tang. Bulk-surface systems on evolving domains. *arXiv e-prints*, art. arXiv:2212.13288, Dec. 2022. doi: 10.48550/arXiv.2212.13288.

[9] P. Cermelli, E. Fried, and M. E. Gurtin. Transport relations for surface integrals arising in the formulation of balance laws for evolving fluid interfaces. *J. Fluid Mech.*, 544:339–351, 2005. ISSN 0022-1120. doi: 10.1017/S0022112005006695. URL http://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1017/S0022112005006695.

[10] L. Church, A. Djurdjevac, and C. M. Elliott. A domain mapping approach for elliptic equations posed on random bulk and surface domains. *Numer. Math.*, 146(1):1–49, 2020. ISSN 0029-599X. doi: 10.1007/s00211-020-01139-7. URL https://0-doi-org.pugwash.lib.warwick.ac.uk/10.1007/s00211-020-01139-7.

[11] M. Conti, S. Terracini, and G. Verzini. Asymptotic estimates for the spatial segregation of competitive systems. *Advances in Mathematics*, 195(2):524–560, 2005.

[12] E. Crooks, E. Dancer, D. Hilhorst, M. Mimura, and H. Ninomiya. Spatial segregation limit of a competition-diffusion system with dirichlet boundary conditions. *Nonlinear Analysis: Real World Applications*, 5(4):645–665, 2004.

[13] E. Dancer, D. Hilhorst, M. Mimura, and L. Peletier. Spatial segregation limit of a competition–diffusion system. *European Journal of Applied Mathematics*, 10(02):97–115, 1999.

[14] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*, volume 22 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2011. ISBN 978-0-898719-36-9. doi: 10.1137/1.9780898719826. URL http://dx.doi.org/10.1137/1.9780898719826. Metrics, analysis, differential calculus, and optimization.

[15] G. Dziuk and C. M. Elliott. Finite element methods for surface PDEs. *Acta Numer.*, 22:289–396, 2013. ISSN 0962-4929. doi: 10.1017/S0962492913000056. URL https://doi.org/10.1017/S0962492913000056.

[16] C. M. Elliott and T. Ranner. Finite element analysis for a coupled bulk–surface partial differential equation. *IMA Journal of Numerical Analysis*, 33(2):377–402, 2013.

[17] C. M. Elliott and T. Ranner. A unified theory for continuous-in-time evolving finite element space approximations to partial differential equations in evolving domains. *IMA Journal of Numerical Analysis*, 41(3):1696–1845, 2021.

[18] C. M. Elliott and V. Styles. An ALE ESFEM for solving PDEs on evolving surfaces. *Milan J. Math.*, 80(2):469–501, 2012. ISSN 1424-9286. doi: 10.1007/s00032-012-0195-6. URL http://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1007/s00032-012-0195-6.

[19] C. M. Elliott and C. Venkataraman. Error analysis for an ALE evolving surface finite element method. *Numerical Methods for Partial Differential Equations*, 2014. ISSN 1098-2426. doi: 10.1002/num.21930. URL http://dx.doi.org/10.1002/num.21930.

[20] C. M. Elliott, T. Ranner, and C. Venkataraman. Coupled bulk-surface free boundary problems arising from a mathematical model of receptor-ligand dynamics. *SIAM J. Math. Anal.*, 49(1):360–397, 2017. ISSN 0036-1410,1095-7154. doi: 10.1137/15M1050811. URL https://doi.org/10.1137/15M1050811.

[21] R. G. Endres and N. S. Wingreen. Accuracy of direct gradient sensing by single cells. *Proceedings of the National Academy of Sciences*, 105(41):15749–15754, 2008.

[22] K. Fellner, E. Latos, and B. Q. Tang. Well-posedness and exponential equilibration of a volume-surface reaction–diffusion system with nonlinear boundary coupling. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 35, pages 643–673. Elsevier, 2018.

[23] P. García-Peñarrubia, J. J. Gálvez, and J. Gálvez. Mathematical modelling and computational study of two-dimensional and three-dimensional dynamics of receptor–ligand interactions in signalling response mechanisms. *Journal of Mathematical Biology*, pages 1–30, 2013.

[24] H. Garcke, J. Kampmann, A. Rätz, and M. Röger. A coupled surface-Cahn–Hilliard bulk-diffusion system modeling lipid raft formation in cell membranes. *Mathematical Models and Methods in Applied Sciences*, 26(06):1149–1189, 2016.

[25] A. Glitzky, M. Liero, and G. Nika. Analysis of a bulk-surface thermistor model for large-area organic LEDs. *Port. Math.*, 78(2):187–210, 2021. ISSN 0032-5155,1662-2758. doi: 10.4171/pm/2066. URL https://doi.org/10.4171/pm/2066.

[26] A. B. Goryachev and A. V. Pokhilko. Dynamics of cdc42 network embodies a turing-type mechanism of yeast cell polarity. *FEBS letters*, 582(10):1437–1443, 2008.

[27] E. E. Holmes, M. A. Lewis, J. Banks, and R. Veit. Partial differential equations in ecology: spatial interactions and population dynamics. *Ecology*, pages 17–29, 1994.

[28] H. Hutridurga and C. Venkataraman. Heterogeneity and strong competition in ecology. *European Journal of Applied Mathematics*, pages 1–25, 2018.

[29] A. Jilkine, A. F. Marée, and L. Edelstein-Keshet. Mathematical model for spatial segregation of the rho-family gtpases based on inhibitory crosstalk. *Bulletin of mathematical biology*, 69(6):1943–1978, 2007.

[30] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[31] H. Levine and W.-J. Rappel. Membrane-bound Turing patterns. *Physical Review E*, 72(6):061912, 2005.

[32] J. Linderman and D. Lauffenburger. Analysis of intracellular receptor/ligand sorting. calculation of mean surface and bulk diffusion times within a sphere. *Biophysical Journal*, 50(2):295 – 305, 1986. ISSN 0006-3495. doi: http://dx.doi.org/10.1016/S0006-3495(86)83463-4. URL http://www.sciencedirect.com/science/article/pii/S0006349586834634.

[33] A. Logioti, B. Niethammer, M. Röger, and J. J. Velázquez. A parabolic free boundary problem arising in a model of cell polarization. *SIAM Journal on Mathematical Analysis*, 53(1):1214–1238, 2021.

[34] A. Logioti, B. Niethammer, M. Röger, and J. J. Velázquez. Qualitative properties of solutions to a mass-conserving free boundary problem modeling cell polarization. *Communications in Partial Differential Equations*, 48(7-8):1065–1101, 2023.

[35] A. Logioti, B. Niethammer, M. Röger, and J. J. Velázquez. Interface behavior for the solutions of a mass conserving free boundary problem modeling cell polarization. *arXiv preprint arXiv:2402.03034*, 2024.

[36] G. MacDonald, J. A. Mackenzie, M. Nolan, and R. Insall. A computational method for the coupled solution of reaction–diffusion equations on evolving domains and manifolds: Application to a model of cell migration and chemotaxis. *Journal of computational physics*, 309:207–226, 2016.

[37] J. Mackenzie, M. Nolan, and R. Insall. Local modulation of chemoattractant concentrations by single cells: dissection using a bulk-surface computational model. *Interface Focus*, 6(5):20160036, 2016.

[38] A. Madzvamuse, A. H. W. Chung, and C. Venkataraman. Stability analysis and simulations of coupled bulk-surface reaction-diffusion systems. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 471(2175), 2015. ISSN 1364-5021. doi: 10.1098/rspa.2014.0546.

[39] A. Marciniak-Czochra and M. Ptashnyk. Derivation of a macroscopic receptor-based model using homogenization techniques. *SIAM Journal on Mathematical Analysis*, 40(1):215–237, jan 2008. doi: 10.1137/050645269. URL http://dx.doi.org/10.1137/050645269.

[40] R. McLennan, L. Dyson, K. W. Prather, J. A. Morrison, R. E. Baker, P. K. Maini, and P. M. Kulesa. Multiscale mechanisms of cell migration during development: theory and experiment. *Development*, 139(16):2935–2944, 2012.

[41] R. McLennan, L. J. Schumacher, J. A. Morrison, J. M. Teddy, D. A. Ridenour, A. C. Box, C. L. Semerad, H. Li, W. McDowell, D. Kay, et al. Neural crest migration is driven by a few trailblazer cells with a unique molecular signature narrowly confined to the invasive front. *Development*, 142(11):2014–2025, 2015.

[42] R. McLennan, L. J. Schumacher, J. A. Morrison, J. M. Teddy, D. A. Ridenour, A. C. Box, C. L. Semerad, H. Li, W. McDowell, D. Kay, et al. Vegf signals induce trailblazer cell identity that drives neural crest migration. *Developmental biology*, 407(1):12–25, 2015.

[43] A. Mielke. Thermomechanical modeling of energy-reaction-diffusion systems, including bulk-interface interactions. *Discrete Contin. Dyn. Syst. Ser. S*, 6(2):479–499, 2013. ISSN 1937-1632,1937-1179. doi: 10.3934/dcdss.2013.6.479. URL https://doi.org/10.3934/dcdss.2013.6.479.

[44] Y. Mori, A. Jilkine, and L. Edelstein-Keshet. Wave-pinning and cell polarity from a bistable reaction-diffusion system. *Biophysical journal*, 94(9):3684–3697, 2008.

[45] R. Nittka. Inhomogeneous parabolic Neumann problems. *Czechoslovak Math. J.*, 64(139)(3): 703–742, 2014. ISSN 0011-4642. doi: 10.1007/s10587-014-0127-4. URL http://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1007/s10587-014-0127-4.

[46] B. Perthame. *Parabolic Equations in Biology: Growth, reaction, movement and diffusion*. Springer, 2015.

[47] B. Perthame, F. Quirós, and J. L. Vázquez. The hele–shaw asymptotics for mechanical models of tumor growth. *Archive for Rational Mechanics and Analysis*, 212:93–127, 2014.

[48] M. Ptashnyk and C. Venkataraman. Multiscale analysis and simulation of a signaling process with surface diffusion. *Multiscale Modeling & Simulation*, 18(2):851–886, 2020.

[49] A. Rätz and M. Röger. Symmetry breaking in a bulk-surface reaction-diffusion model for signaling networks. *Nonlinearity*, 27:1805–1827, 2014.

[50] V. Sharma and J. Morgan. Global existence of solutions to reaction-diffusion systems with mass transport type boundary conditions. *SIAM Journal on Mathematical Analysis*, 48(6):4202–4240, 2016.

[51] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987. ISSN 0003-4622. doi: 10.1007/BF01762360. URL https://doi.org/10.1007/BF01762360.