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Existence result for a class of generalized standard materials with thermomechanical coupling

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Abstract

This paper deals with the study of a three-dimensional model of thermomechanical coupling for viscous solids exhibiting hysteresis effects. This model is written in accordance with the formalism of generalized standard materials. It is composed by the momentum equilibrium equation combined with the flow rule, which describes some stress-strain dependance, and the heat-transfer equation. An existence result for this thermodynamically consistent problem is obtained by using a fixed-point argument and some qualitative properties of the solutions are established.

1 Description of the problem

Motivated by the study of visco-elasto-plastic materials and Shape-Memory Alloys (SMA), we consider in this paper a thermomechanical coupling for a class of Generalized Standard Materials (GSM) exhibiting hysteresis effects. More precisely, in the framework of GSM due to Halphen and Nguyen (see [HaN75]) the mechanical behavior of the material is described by the *momentum equilibrium equation* combined with a constitutive law (*flow rule*) and the unknowns are the *displacement* u and an *internal variable* z which allows to take into account some dissipation at the microscopic level. Indeed, plasticity and phase transitions are inelastic processes which involve some loss of energy, transformed into heat. Thus it is necessary to take into account the thermal process in the description of the problem.

The model considered here is based on the Helmholtz free energy $W(\boldsymbol{\varepsilon}, z, \theta)$, depending on the *infinitesimal strain tensor* $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(u) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ for the *displacement* u, the *internal variable* z and the *temperature* θ. Here $(\cdot)^{\mathsf{T}}$ denotes the transpose of a tensor. We assume that W can be decomposed as follows

$$W(\boldsymbol{\varepsilon}, z, \theta) \stackrel{\text{def}}{=} W^{\text{mech}}(\boldsymbol{\varepsilon}, z) - W^{\theta}(\theta) + \theta W^{\text{coup}}(\boldsymbol{\varepsilon}, z), \tag{1.1}$$

which ensures that entropy separates the thermal and mechanical variables (see (1.3)). Let us emphasize that the last term in the right hand side of (1.1) allows for coupling effects between the temperature and both the displacement and the internal variable. We make the assumption of small deformations. The momentum equilibrium equation and the flow rule are given by

$$-\operatorname{div}(\boldsymbol{\sigma}^{\mathsf{el}} + \mathbf{A}\dot{\boldsymbol{\varepsilon}}) = f, \tag{1.2a}$$

$$\partial\Psi(\dot{z}) + \mathbf{B}\dot{z} + \boldsymbol{\sigma}^{\mathsf{inel}} \ni 0, \tag{1.2b}$$

where f is a given loading, $\boldsymbol{\sigma}^{\mathsf{el}} \stackrel{\text{def}}{=} \partial_{\boldsymbol{\varepsilon}} W(\boldsymbol{\varepsilon}, z, \theta)$, $\boldsymbol{\sigma}^{\mathsf{inel}} \stackrel{\text{def}}{=} \partial_z W(\boldsymbol{\varepsilon}, z, \theta)$, $\mathbf{A}$ and $\mathbf{B}$ are two viscosity tensors and Ψ is the dissipation potential. As it is common in modeling hysteresis effects in mechanics, we assume that Ψ is convex, positively homogeneous of degree 1 and $0 \in \partial\Psi(0)$ which ensures that $\boldsymbol{\sigma}^{\mathsf{inel}}.\dot{z} \le 0$.

Then the specific *entropy* is defined by the Gibb's relation

$$s \stackrel{\text{def}}{=} -\partial_\theta W(\boldsymbol{\varepsilon}, z, \theta) = \partial_\theta W^{\theta}(\theta) - W^{\text{coup}}(\boldsymbol{\varepsilon}, z), \tag{1.3}$$

and the *entropy equation*

$$\theta\dot{s} - \operatorname{div}(\kappa\nabla\theta) = \mathbf{A}\dot{\boldsymbol{\varepsilon}}{:}\dot{\boldsymbol{\varepsilon}} + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z}), \tag{1.4}$$

gives some balance between the *heat flux* $j = -\kappa\nabla\theta$, where κ is the *heat conductivity*, and the *dissipation rate* $\xi \overset{\text{def}}{=} \mathbf{A}\dot{\boldsymbol{\varepsilon}}{:}\dot{\boldsymbol{\varepsilon}} + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z}) \geq 0$. If the system is thermally isolated and $\theta > 0$, we have

$$\int_\Omega \dot{s}\,\mathrm{d}x = \int_\Omega \frac{\mathrm{div}(\kappa\nabla\theta)}{\theta}\,\mathrm{d}x + \int_\Omega \frac{\xi}{\theta}\,\mathrm{d}x = \int_\Omega \frac{\kappa\nabla\theta\cdot\nabla\theta}{\theta^2}\,\mathrm{d}x + \int_\Omega \frac{\xi}{\theta}\,\mathrm{d}x \geq 0,$$

which guarantees that the second law of thermodynamics is satisfied. Furthermore, let

$$W^{\mathrm{in}}(\boldsymbol{\varepsilon}, z, \theta) \overset{\text{def}}{=} W(\boldsymbol{\varepsilon}, z, \theta) + \theta s$$

be the *internal energy*. By using the chain rule and (1.2)–(1.4), we obtain

$$\int_\Omega \dot{W}^{\mathrm{in}}(\boldsymbol{\varepsilon}, z, \theta)\,\mathrm{d}x = \int_\Omega f\cdot\dot{u}\,\mathrm{d}x + \int_{\partial\Omega} \kappa\nabla\theta\cdot\mathbf{n}\,\mathrm{d}x,$$

which gives the total energy balance in terms of the internal energy, the power of external load and heat. Hence the model considered here is thermodynamically consistent.

We assume in the sequel that

$$W^{\mathrm{mech}}(\boldsymbol{\varepsilon}, z) \overset{\text{def}}{=} \tfrac{1}{2}\mathbf{E}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\textbf{inel}}){:}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\textbf{inel}}) + \tfrac{\alpha}{2}|\nabla z|^2 + H_1(z), \quad \boldsymbol{\varepsilon}^{\textbf{inel}} \overset{\text{def}}{=} \mathbf{Q}z, \tag{1.5a}$$

$$W^{\theta} \overset{\text{def}}{=} c(\theta\ln(\theta)-\theta), \tag{1.5b}$$

$$W^{\mathrm{coup}}(\boldsymbol{\varepsilon}, z) \overset{\text{def}}{=} \beta\mathbf{I}{:}\boldsymbol{\varepsilon} + H_2(z), \tag{1.5c}$$

where c is the *heat capacity*, $\beta \geq 0$ is the *isotropic thermal expansion coefficient*, $\mathbf{I}$ is the identity matrix, $\alpha \geq 0$ is a coefficient that measures some non local interaction effects for the internal variable z, $\mathbf{E}$ is the *elasticity tensor*, H_i, $i = 1, 2$, are two *hardening functionals* and $\mathbf{Q}$ is an affine mapping from a finite dimensional real vector space $\mathcal{Z}$ to $\mathbb{R}^{3\times3}_{\mathsf{sym}}$. More precisely, $\mathbf{Q}$ is decomposed as follows

$$\forall z \in \mathcal{Z}: \ \mathbf{Q}z \overset{\text{def}}{=} \widetilde{\mathbf{Q}}z + \mathrm{Q},$$

with $\widetilde{\mathbf{Q}} \in \mathcal{L}(\mathcal{Z}, \mathbb{R}^{3\times3}_{\mathsf{sym}})$ and $\mathrm{Q} \in \mathbb{R}^{3\times3}_{\mathsf{sym}}$. We observe that by inserting on the one hand (1.5a) and (1.5c) into (1.2) and by carrying on the other hand (1.5b), (1.5c) and (1.3) into (1.4), we obtain

$$-\,\mathrm{div}(\mathbf{E}(\boldsymbol{\varepsilon}(u)-\mathbf{Q}z)+\beta\theta\mathbf{I}+\mathbf{A}\boldsymbol{\varepsilon}(\dot{u})) = f, \tag{1.6a}$$

$$\partial\Psi(\dot{z}) + \mathbf{B}\dot{z} - \widetilde{\mathbf{Q}}^{\mathsf{T}}\mathbf{E}(\boldsymbol{\varepsilon}(u)-\mathbf{Q}z) + \partial_z H_1(z) + \theta\partial_z H_2(z) - \alpha\Delta z \ni 0, \tag{1.6b}$$

$$c\dot{\theta} - \mathrm{div}(\kappa\nabla\theta) = \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}){:}\boldsymbol{\varepsilon}(\dot{u}) + \theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u})+\partial_z H_2(z).\dot{z}) + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z}), \tag{1.6c}$$

together with boundary conditions

$$u = 0, \quad \alpha\nabla z\cdot\mathbf{n} = 0, \quad \kappa\nabla\theta\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\times[0,\tau), \tag{1.7}$$

and initial conditions

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \theta(\cdot,0) = \theta^0 \quad \text{in} \quad \Omega. \tag{1.8}$$

Here $\Omega \subset \mathbb{R}^3$ is a reference configuration and $\mathbf{n}$ denotes the outward normal to the boundary $\partial\Omega$ of Ω. As usual, $\dot{(\,)}$, ∂_z^i and ∂ denote the time derivative $\frac{\partial}{\partial t}$, the i-th derivative with respect to z and the subdifferential in the sense of convex analysis (see [Bre73]), respectively. Moreover $\boldsymbol{\varepsilon}_1{:}\boldsymbol{\varepsilon}_2$ and $z_1.z_2$ denote the inner product of $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ in the space of symmetric 3×3 tensors $\mathbb{R}^{3\times3}_{\mathsf{sym}}$ and z_1 and z_2 in the finite dimensional real vector space $\mathcal{Z}$.

The increasing interest in smart materials for industrial applications has deeply stimulated the study of such models in engineering as well as in mathematical literature during the last decade. If the coupling with heat equation (1.6c) is ignored (for instance, if the characteristic dimension of the material is small in at least one direction, the temperature can be considered as a data), the problem (1.6a)–(1.6b) together with (1.7)–(1.8) is nowadays quite well understood; existence results can be obtained either by using classical methods for maximal monotone operators (see [AlC04]) or more specific techniques for rate-independent processes when the viscosity tensors vanish (see [MiT04, Mie05, FrM06, MiR07, Mie07, MiP07, MRS08]). On the contrary, if the temperature is considered as an unknown, the coupling with the thermal process, which is not rate-independent, does not allow to use the previous techniques and the problem becomes much more difficult. Indeed, the natural functional framework for the right-hand side of (1.6c) seems at a first glance to be $\mathrm{L}^1(0,\tau;\mathrm{L}^1(\Omega))$ since we usually expect the displacements to be in $\mathrm{W}^{1,2}(0,\tau;\mathrm{W}^{1,2}(\Omega))$. This difficulty has been overcome in a serie of recent papers by using the so-called enthalpy transformation. More precisely, assuming that the heat conductivity is a continuous function of θ such that

$$\exists\gamma>1,\ \exists c^c>0,\ \forall\theta\geq 0:\ c(\theta)\geq c^c(1{+}\theta)^{\gamma-1}, \tag{1.9}$$

a new unknown, the *enthalpy*, is defined by

$$\vartheta \overset{\text{def}}{=} \int_0^\theta c(s)\,\mathrm{d}s,$$

and the heat equation is replaced by the enthalpy equation:

$$\dot\vartheta-\mathrm{div}\big(\tfrac{\kappa}{c(\zeta(\vartheta))}\nabla\vartheta\big)=\mathbf{A}\boldsymbol{\varepsilon}(\dot u){:}\boldsymbol{\varepsilon}(\dot u)+\zeta(\vartheta)(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot u){+}\partial_z H_2(z).\dot z)+\mathbf{B}\dot z.\dot z+\Psi(\dot z),$$

with $|\theta{=}\zeta(\vartheta)|\leq\big(\frac{\gamma}{c^c}\max(\vartheta,0)\big)^{\frac{1}{\gamma}}$. Roughly speaking, this change of unknown weakens the coupling effects (the greater is γ, the weaker is the coupling effects) and allows to build a solution either by using a time-discretization ([BaR08, Rou10, BaR11]) or by using a fixed-point argument ([PaP11a, PaP11b, PaP11c]). Unfortunately, assumption (1.9) on the heat conductivity is not always satisfied and we will consider in this paper the more standard case where c is a function of x. In such a case, the enthalpy is simply $\vartheta \overset{\text{def}}{=} c(x)\theta$ and does not provide any help in the mathematical analysis of the system (1.6)–(1.8). In other words, we have to manage directly with the original coupling (1.6a), (1.6b) and (1.6c). For this problem, we will prove an existence result by using a fixed-point argument. Since the right-hand side of (1.6c) behaves as a quadratic term with respect to θ, we can not expect a global existence result without some smallness assumptions on the coupling parameters β and $\partial_z H_2$.

The paper is organized as follows. In Section 2, we introduce the assumptions on the data, and we present the main result (local existence result). Then Section 3 is devoted to its proof. In Section 4, we establish some further properties of the solution, namely we prove that the temperature remains positive and thus is physically admissible and that u, z and θ satisfy some global energy estimate. Furthermore we investigate sufficient conditions to get a global solution. Finally, in Section 5, we present some examples which fit our modelization.

2 Statement of the result

We consider a reference configuration $\Omega\subset\mathbb{R}^3$, which is a bounded domain such that $\partial\Omega\in\mathrm{C}^{2+\rho}$ with $\rho>0$. Let us begin this section by introducing some assumptions on the data as well as obvious consequences following from these assumptions used later on in this work.

(A–1) The *dissipation potential* Ψ is positively homogeneous of degree 1, satisfies the triangle inequality and remains bounded on the unit ball of $\mathcal{Z}$, i.e., we have

$$\forall \gamma \geq 0,\ \forall z \in \mathcal{Z}:\ \Psi(\gamma z) = \gamma \Psi(z), \tag{2.1a}$$
$$\forall z_1, z_2 \in \mathcal{Z}:\ \Psi(z_1{+}z_2) \leq \Psi(z_1) + \Psi(z_2), \tag{2.1b}$$
$$\exists C^\Psi > 0,\ \forall z \in \mathcal{Z}:\ 0 \leq \Psi(z) \leq C^\Psi |z|. \tag{2.1c}$$

It is clear that (2.1) implies that Ψ is convex and continuous. With (2.1c), we can also check immediately that $0 \in \partial\Psi(0)$.

(A–2) The *hardening functionals* H_i, $i = 1, 2$, belong to $\mathrm{C}^2(\mathcal{Z}; \mathbb{R})$ and satisfy the following inequalities

$$\exists c^{H_1}, \widetilde{c}^{H_1} \geq 0,\ \forall z \in \mathcal{Z}:\ H_1(z) \geq c^{H_1}|z|^2 - \widetilde{c}^{H_1}, \tag{2.2a}$$
$$\exists C^{H_i}_{zz} > 0,\ \forall z \in \mathcal{Z}:\ |\partial^2_z H_i(z)| \leq C^{H_i}_{zz}. \tag{2.2b}$$

Note that (2.2b) leads to

$$\exists C^{H_i}_z > 0,\ \forall z \in \mathcal{Z}:\ |\partial_z H_i(z)| \leq C^{H_i}_z(1{+}|z|),\ \ |H_i(z)| \leq C^{H_i}_z(1{+}|z|^2). \tag{2.3}$$

(A–3) The *elasticity tensor* $\mathbf{E} : \Omega \to \mathcal{L}(\mathbb{R}^{3\times3}_{\mathrm{sym}}; \mathbb{R}^{3\times3}_{\mathrm{sym}})$ is a symmetric positive definite operator such that

$$\exists c^{\mathbf{E}} > 0,\ \forall \boldsymbol{\varepsilon} \in \mathrm{L}^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym}}):\ c^{\mathbf{E}}\|\boldsymbol{\varepsilon}\|^2_{\mathrm{L}^2(\Omega)} \leq \int_\Omega \mathbf{E}\boldsymbol{\varepsilon}{:}\boldsymbol{\varepsilon}\,\mathrm{d}x, \tag{2.4a}$$
$$\forall i, j, k = 1, 2, 3:\ \mathbf{E}(\cdot), \tfrac{\partial \mathrm{E}_{i,j}(\cdot)}{\partial x_k} \in \mathrm{L}^\infty(\Omega). \tag{2.4b}$$

(A–4) The *viscosity tensors* $\mathbf{A}$ and $\mathbf{B}$ are symmetric positive definite such that

$$\exists c^{\mathbf{A}}, C^{\mathbf{A}} > 0,\ \forall \boldsymbol{\varepsilon} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}:\ c^{\mathbf{A}}|\boldsymbol{\varepsilon}|^2 \leq \mathbf{A}\boldsymbol{\varepsilon}{:}\boldsymbol{\varepsilon} \leq C^{\mathbf{A}}|\boldsymbol{\varepsilon}|^2, \tag{2.5a}$$
$$\exists c^{\mathbf{B}}, C^{\mathbf{B}} > 0,\ \forall z \in \mathcal{Z}:\ c^{\mathbf{B}}|z|^2 \leq \mathbf{B}z.z \leq C^{\mathbf{B}}|z|^2. \tag{2.5b}$$

(A–5) The *inelastic strain* is given by $\boldsymbol{\varepsilon}^{\textbf{inel}} \overset{\text{def}}{=} \mathbf{Q}z = \widetilde{\mathbf{Q}}z + \mathrm{Q}$ with

$$\widetilde{\mathbf{Q}} \in \mathcal{L}(\mathcal{Z}, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \quad \text{and} \quad \mathrm{Q} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}. \tag{2.6}$$

(A–6) The *external loading* f satisfies

$$f \in \mathrm{H}^1(0, T; \mathrm{L}^2(\Omega)) \quad \text{with} \quad T > 0. \tag{2.7}$$

(A–7) The *heat capacity* $c : \Omega \to \mathbb{R}$ and the *conductivity* $\kappa^c : \Omega \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ satisfy the following inequalities

$$\exists C^c, c^c > 0:\ c^c \leq c(x) \leq C^c \ \text{ a.e. } x \in \Omega, \tag{2.8a}$$
$$\exists c^\kappa > 0,\ \forall v \in \mathbb{R}^3:\ \kappa(x)v.v \geq c^\kappa|v|^2 \ \text{ a.e. } x \in \Omega, \tag{2.8b}$$
$$\exists C^\kappa > 0:\ |\kappa(x)| \leq C^\kappa \ \text{ a.e. } x \in \Omega. \tag{2.8c}$$

Finally, we assume that $\alpha \geq 0$ and either $\alpha > 0$ and $c^{H_1} > 0$ or $\alpha = 0$ and $\partial_z H_2 \equiv 0$. Note that the boundary condition $\alpha\nabla z{\cdot}\mathbf{n} = 0$ on $\partial\Omega \times [0,\tau)$ will disappear if $\alpha = 0$. We use later the following notations; $\mathrm{W}^{m,r}_{\mathrm{Dir}}(\Omega) \overset{\text{def}}{=} \{\xi \in \mathrm{W}^{m,r}(\Omega) : \ \xi = 0 \ \text{ on } \ \partial\Omega\}$ and $\mathrm{W}^{m,r}_{\mathrm{Neu}}(\Omega) \overset{\text{def}}{=} \{\xi \in \mathrm{W}^{m,r}(\Omega) : \nabla\xi{\cdot}\mathbf{n} = 0 \ \text{ on } \ \partial\Omega\}$ with $m \geq 1$ and $r \geq 2$ are two integers.

As usual Korn's inequality will play a role in the mathematical analysis developed in the next sections. We have assumed that $\partial\Omega$ is of class $\mathrm{C}^{2+\rho}$, so we have

$$\exists C^{\mathsf{Korn}} > 0,\ \forall u \in \mathrm{W}^{1,2}_{\mathrm{Dir}}(\Omega) :\ \|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{L}^2(\Omega)} \geq C^{\mathsf{Korn}}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)}, \tag{2.9}$$

for further details on Korn's inequality, the reader is referred to [KoO88, DuL76].

As a starting point in the study of the problem (1.6)–(1.8), let us consider the heat equation

$$c\dot{\theta} - \mathrm{div}(\kappa\nabla\theta) = f^{\widetilde{\theta}}, \tag{2.10}$$

with initial and boundary conditions

$$\theta(\cdot,0) = \theta^0 \quad \text{in} \quad \Omega, \quad \kappa\nabla\theta{\cdot}\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times [0,\tau). \tag{2.11}$$

If $f^{\widetilde{\theta}} \in \mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))$ and $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega) \overset{\text{def}}{=} \{\xi \in \mathrm{W}^{1,2}(\Omega) : \ \kappa\nabla\xi{\cdot}\mathbf{n} = 0 \ \text{ on } \ \partial\Omega\}$, this problem admits a unique solution $\theta \in \mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega))$ (see [Eva10]).

Now we recall existence and uniqueness results for the system composed by the momentum equilibrium equation and the flow rule (1.6a)–(1.6b) when the temperature is a given data. More precisely, let $\widetilde{\theta}$ be given in $\mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))$. We consider the following problem: Find $u : [0,\tau] \to \mathbb{R}^3$ and $z : [0,\tau] \to \mathcal{Z}$ such that

$$-\,\mathrm{div}(\mathbf{E}(\boldsymbol{\varepsilon}(u){-}\mathbf{Q}z){+}\beta\widetilde{\theta}\mathbf{I}{+}\mathbf{A}\boldsymbol{\varepsilon}(\dot{u})) = f, \tag{2.12a}$$

$$\partial\Psi(\dot{z}) + \mathbf{B}\dot{z} - \widetilde{\mathbf{Q}}^{\mathsf{T}}\mathbf{E}(\boldsymbol{\varepsilon}(u){-}\mathbf{Q}z) + \partial_z H_1(z) + \widetilde{\theta}\partial_z H_2(z) - \alpha\Delta z \ni 0, \tag{2.12b}$$

with boundary conditions

$$u = 0, \quad \alpha\nabla z{\cdot}\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times [0,\tau), \tag{2.13}$$

and initial conditions

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0 \quad \text{in} \quad \Omega. \tag{2.14}$$

Proposition 2.1 *Let $\tau \in (0,T]$ and $\widetilde{\theta}$ be given in $\mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))$ with $p \in [4,6]$. Assume that* (2.1)*,* (2.2)*,* (2.4)*,* (2.5)*,* (2.6)*,* (2.7)*, $u^0 \in \mathrm{W}^{1,p}_{\mathrm{Dir}}(\Omega)$ and $z^0 \in \mathrm{W}^{2,p}_{\mathrm{Neu}}(\Omega)$ if $\alpha > 0$ or $z^0 \in \mathrm{L}^p(\Omega)$ if $\alpha = 0$ hold. Then the problem* (2.12)–(2.14) *admits a unique solution $u \in \mathrm{W}^{1,q}(0,\tau;\mathrm{W}^{1,p}_{\mathrm{Dir}}(\Omega))$ and $z \in \mathrm{L}^{q/2}(0,\tau;\mathrm{W}^{2,p}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{C}^0([0,\tau];\mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{W}^{1,q/2}(0,\tau;\mathrm{L}^p(\Omega)) \cap \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^{p/2}(\Omega))$ if $\alpha > 0$ and $z \in \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^p(\Omega))$ if $\alpha = 0$ for any $q > 8$. Furthermore $\widetilde{\theta} \mapsto (u,z)$ maps any bounded subset of $\mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))$ into a bounded subset of $\mathrm{W}^{1,q}(0,\tau;\mathrm{W}^{1,p}_{\mathrm{Dir}}(\Omega)){\times}(\mathrm{L}^{q/2}(0,\tau;\mathrm{W}^{2,p}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{C}^0([0,\tau];\mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{W}^{1,q/2}(0,\tau;\mathrm{L}^p(\Omega)) \cap \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^{p/2}(\Omega)))$ when $\alpha > 0$ or into a bounded subset of $\mathrm{W}^{1,q}(0,\tau;\mathrm{W}^{1,p}_{\mathrm{Dir}}(\Omega)) \times \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^p(\Omega))$ when $\alpha = 0$.*

The key tool to prove existence, uniqueness and boundedness results for (2.12)–(2.14) consists in interpreting this system of partial differential equations as an ordinary differential equation in an appropriate Banach space. For the detailed proof, the reader is referred to [PaP11a, Thm. 4.1, Prop. 4.2, Lem. 4.4–4.5] and [PaP11c, Thm. 3.1, Prop. 3.2, Lem. 3.4–3.5] when $\alpha > 0$ and to [PaP11b, Thm. 4.1] when $\alpha = 0$.

So we may prove the existence of a solution for the coupled problem (1.6)–(1.8) by combining via a fixed-point argument the results of Proposition 2.1 with the existence results for the heat equation with $f^{\widetilde{\theta}}$ given by

$$f^{\widetilde{\theta}} \stackrel{\text{def}}{=} \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}):\boldsymbol{\varepsilon}(\dot{u}) + \widetilde{\theta}(\beta\mathbf{I}:\boldsymbol{\varepsilon}(\dot{u})+\partial_z H_2(z).\dot{z}) + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z}).$$

We will obtain

Theorem 2.2 *Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), (2.8), $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega)$, $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ *and* $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ *if* $\alpha > 0$ *and* $z^0 \in \mathrm{L}^4(\Omega)$ *if* $\alpha = 0$ *hold. Then there exists* $\tau \in (0, T]$ *such that the problem* (1.6)–(1.8) *admits a solution on* $[0, \tau]$ *such that* $\theta \in \mathrm{L}^\infty(0, \tau; \mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega)) \cap \mathrm{C}^0(0, \tau; \mathrm{L}^4(\Omega))$, $\dot{\theta} \in \mathrm{L}^2(0, \tau; \mathrm{L}^2(\Omega))$, $u \in \mathrm{W}^{1,q}(0, \tau; \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega))$, $z \in \mathrm{L}^{q/2}(0, \tau; \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{C}^0([0, \tau]; \mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{W}^{1,q/2}(0, \tau; \mathrm{L}^4(\Omega)) \cap \mathrm{W}^{1,q}(0, \tau; \mathrm{L}^2(\Omega))$ *when* $\alpha > 0$, $z \in \mathrm{W}^{1,q}(0, \tau; \mathrm{L}^4(\Omega))$ *when* $\alpha = 0$*, for any* $q > 8$.

Next, reminding that the problem is thermodynamically consistent if $\theta > 0$, we establish at Proposition 4.1 that the solution obtained in the previous theorem is physically admissible, i.e. remains positive whenever $\theta^0 \geq \bar{\theta}$ almost everywhere in Ω, with $\bar{\theta} > 0$. Finally, a global energy estimate is obtained in Proposition 4.2 and sufficient conditions on β and $\partial_z H_2$ are proposed to get a global solution (u, z, θ) defined on $[0, T]$.

3 Proof of Theorem 2.2

This section is dedicated to the proof of Theorem 2.2 by using a fixed-point argument. More precisely, for any given $\widetilde{\theta} \in \mathrm{C}^0([0, \tau]; \mathrm{L}^4(\Omega))$ with $\tau \in (0, T]$, let $f^{\widetilde{\theta}} \stackrel{\text{def}}{=} \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}):\boldsymbol{\varepsilon}(\dot{u}) + \widetilde{\theta}(\beta\mathbf{I}:\boldsymbol{\varepsilon}(\dot{u})+\partial_z H_2(z).\dot{z}) + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z})$ where (u, z) is the unique solution of (2.12)–(2.14). Using Proposition 2.1, we obtain $f^{\widetilde{\theta}} \in \mathrm{L}^2(0, \tau; \mathrm{L}^2(\Omega))$ and thus the heat-transfer equation (2.10)–(2.11) possesses a unique solution $\theta \in \mathrm{L}^\infty(0, \tau; \mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega))$ such that $\dot{\theta} \in \mathrm{L}^2(0, \tau; \mathrm{L}^2(\Omega))$. This allows us to define a mapping

$$\Phi^{\widetilde{\theta},\theta}_\tau : \ \mathrm{C}^0([0, \tau]; \mathrm{L}^4(\Omega)) \to \mathrm{C}^0([0, \tau]; \mathrm{L}^4(\Omega))$$
$$\widetilde{\theta} \mapsto \theta.$$

Our aim consists in proving that this mapping satisfies the assumptions of Schauder's fixed point theorem for some positive $\tau \in (0, T]$.

Let us define the set $\mathcal{Q}_\tau \stackrel{\text{def}}{=} \Omega \times (0, \tau)$ with $\tau \in (0, T]$. In the sequel, the notations for the constants introduced in the proofs are valid only in the proof.

Proposition 3.1 *Let* τ *belongs to* $(0, T]$*. Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), (2.8), $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega)$, $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ *and* $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ *if* $\alpha > 0$ *and* $z^0 \in \mathrm{L}^4(\Omega)$ *if* $\alpha = 0$ *hold. Then* $\Phi^{\widetilde{\theta},\theta}_\tau$ *maps any bounded subset of* $\mathrm{C}^0([0, \tau]; \mathrm{L}^4(\Omega))$ *into a bounded relatively compact subset of* $\mathrm{C}^0([0, \tau]; \mathrm{L}^4(\Omega))$.

Proof. We recall first existence, uniqueness and regularity results for the heat-transfer equation. More precisely, let consider the system (2.10)–(2.11). We assume that (2.8) holds and that the initial temperature $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega)$ and $f^{\widetilde{\theta}} \in \mathrm{L}^2(0, \tau; \mathrm{L}^2(\Omega))$. By using Galerkin's method (see for instance

[Eva10]), we may prove that this problem admits a unique solution $\theta \in \mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega))$ with $\dot\theta \in \mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))$.

Moreover we have the following a priori estimates

$$\|\theta(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{2c^\kappa}{c^c}\|\nabla\theta\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} \le \tfrac{1}{c^c}\big(C^c\|\theta^0\|^2_{\mathrm{L}^2(\Omega)}+\|f^{\widetilde\theta}\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))}\big)\exp\big(\tfrac{\tau}{c^c}\big) \tag{3.1}$$

and

$$c^c\|\dot\theta\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} + c^\kappa\|\nabla\theta(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} \le C^\kappa\|\nabla\theta^0\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{1}{c^c}\|f^{\widetilde\theta}\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} \tag{3.2}$$

for almost every $t \in [0,\tau]$. Therefore we add (3.1) and (3.2), we have

$$\begin{aligned} &c^c\|\dot\theta\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} + \min(1,c^\kappa)\|\theta(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \tfrac{2c^\kappa}{c^c}\|\nabla\theta\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} \\ &\le \max\big(\tfrac{C^c}{c^c}\exp\big(\tfrac{\tau}{c^c}\big), C^\kappa\big)\|\theta^0\|^2_{\mathrm{W}^{1,2}(\Omega)} + \tfrac{1}{c^c}\big(\exp\big(\tfrac{\tau}{c^c}\big)+1\big)\|f^{\widetilde\theta}\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} \end{aligned} \tag{3.3}$$

for almost every $t \in [0,\tau]$. We introduce now the following functional space

$$\mathrm{V}((\tau_1,\tau_2)\times\Omega) \overset{\text{def}}{=} \big\{\theta \in \mathrm{L}^\infty(\tau_1,\tau_2;\mathrm{W}^{1,2}(\Omega)) : \ \dot\theta \in \mathrm{L}^2(\tau_1,\tau_2;\mathrm{L}^2(\Omega))\big\},\ 0 \le \tau_1 < \tau_2 \le T,$$

endowed with the norm

$$\forall\theta \in \mathrm{V}((\tau_1,\tau_2)\times\Omega) : \ \|\theta\|_{\mathrm{V}((\tau_1,\tau_2)\times\Omega)} \overset{\text{def}}{=} \|\theta\|_{\mathrm{L}^\infty(\tau_1,\tau_2;\mathrm{W}^{1,2}(\Omega))} + \|\dot\theta\|_{\mathrm{L}^2(\tau_1,\tau_2;\mathrm{L}^2(\Omega))}.$$

Then it follows from (3.3) that there exists a generic constant $C_\theta > 0$, independent of τ, such that the solution of problem (2.10)–(2.11) satisfies

$$\|\theta\|_{\mathrm{V}((0,\tau)\times\Omega)} \le C_\theta\exp\big(\tfrac{\tau}{c^c}\big)\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}+\|f^{\widetilde\theta}\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))}\big). \tag{3.4}$$

With Proposition 2.1, it is plain to see that for any $\widetilde\theta$ belonging to a bounded subset of $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$, $f^{\widetilde\theta} \overset{\text{def}}{=} \mathbf{A}\boldsymbol{\varepsilon}(\dot u){:}\boldsymbol{\varepsilon}(\dot u) + \widetilde\theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot u)+\partial_z H_2(z).\dot z) + \mathbf{B}\dot z.\dot z + \Psi(\dot z)$ belongs to a bounded subset of $\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))$ for any $q > 8$. Furthermore Hölder's inequality gives

$$\|f^{\widetilde\theta}\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} \le \tau^{\frac{q-8}{2q}}\|f^{\widetilde\theta}\|_{\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))}.$$

We insert (3.4) into (3.3), we find

$$\|\theta\|_{\mathrm{V}((0,\tau)\times\Omega)} \le C_\theta\exp\big(\tfrac{\tau}{c^c}\big)\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}+\tau^{\frac{q-8}{2q}}\|f^{\widetilde\theta}\|_{\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))}\big).$$

Thus it is clear that $\Phi^{\widetilde\theta,\theta}_\tau$ maps any bounded subset of $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ into a bounded subset of $\mathrm{V}((0,\tau)\times\Omega)$. However $\mathrm{V}((0,\tau)\times\Omega)$ is compactly embedded into $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ (see [Sim87]), which allows us to conclude. □

Proposition 3.2 *Let τ belongs to $\in (0,T]$. Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), (2.8), $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\mathrm{Neu}}(\Omega)$, $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ *and* $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ *if* $\alpha > 0$ *and* $z^0 \in \mathrm{L}^4(\Omega)$ *if* $\alpha = 0$ *hold. Then the mapping* $\Phi^{\widetilde\theta,\theta}_\tau$ *is continuous from* $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ *into* $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$.

Proof. Let us consider a converging sequence $(\widetilde{\theta}_n)_{n\in\mathbb{N}} \in (\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega)))^{\mathbb{N}}$ and let $\widetilde{\theta}_*$ be its limit. We denote by (u_n, z_n) the solution of (2.12)–(2.14) with $\widetilde{\theta} = \widetilde{\theta}_n$, and $\theta_n \overset{\text{def}}{=} \Phi_\tau^{\widetilde{\theta},\theta}(\widetilde{\theta}_n)$ for all $n \geq 0$. Similarly, let (u_*, z_*) be the solution of (2.12)–(2.14) with $\widetilde{\theta} = \widetilde{\theta}_*$, and $\theta_* \overset{\text{def}}{=} \Phi_\tau^{\widetilde{\theta},\theta}(\widetilde{\theta}_*)$. Since $(\widetilde{\theta}_n)_{n\in\mathbb{N}}$ is a bounded family of $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$, we infer that $(\theta_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{V}((0,\tau)\times\Omega)$. It follows that $(\theta_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ (see [Sim87]). Hence, there exists a subsequence, still denoted by $(\theta_n)_{n\in\mathbb{N}}$, such that

$$\begin{aligned}
&\theta_n \to \theta \ \text{ in } \ \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega)),\\
&\theta_n \rightharpoonup \theta \ \text{ in } \ \mathrm{L}^2(0,\tau;\mathrm{W}^{1,2}(\Omega)) \ \text{ weak},\\
&\dot{\theta}_n \rightharpoonup \dot{\theta} \ \text{ in } \ \mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega)) \ \text{ weak},
\end{aligned}$$

and for all $n \geq 0$, we have $\theta_n(\cdot,0) = \theta^0$ and

$$\begin{aligned}
&\int_{Q_\tau} c(x)\dot{\theta}_n(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} \kappa(x)\nabla\theta_n(x,t)\nabla\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{Q_\tau} f^{\widetilde{\theta}_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.5}$$

for all $\xi \in \mathrm{W}^{1,2}(\Omega)$ and $w \in \mathcal{D}(0,\tau)$. We observe that since $(\theta_n)_{n\in\mathbb{N}}$ converges strongly to θ in $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$, we have immediately $\theta(\cdot,0) = \theta^0$. In order to pass to the limit in (3.5), it remains to study the convergence of $(f^{\widetilde{\theta}_n})_{n\in\mathbb{N}}$. We begin with the study of the convergence of $(u_n, z_n)_{n\in\mathbb{N}}$.

It is convenient to introduce the following functional space $\mathrm{X}^\alpha(\Omega) = \mathrm{W}^{1,2}_{\text{Neu}}(\Omega)$ if $\alpha > 0$ and $\mathrm{X}^\alpha(\Omega) = \mathrm{L}^2(\Omega)$ if $\alpha = 0$.

Lemma 3.3 *Let* $\tau \in (0,T]$. *Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), $u^0 \in \mathrm{W}^{1,4}_{\text{Dir}}(\Omega)$ *and* $z^0 \in \mathrm{W}^{2,4}_{\text{Neu}}(\Omega)$ *if* $\alpha > 0$ *and* $z^0 \in \mathrm{L}^4(\Omega)$ *if* $\alpha = 0$ *hold. Then the mapping* $\widetilde{\theta} \mapsto (u,z)$, *where* (u,z) *is the unique solution of* (2.12)–(2.14), *is continuous from* $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ *into* $\mathrm{W}^{1,2}(0,\tau;\mathrm{W}^{1,2}_{\text{Dir}}(\Omega)\times\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}_{\text{Dir}}(\Omega)\times\mathrm{X}^\alpha(\Omega))$.

Proof. We consider $\widetilde{\theta}_i \in \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ and for $i = 1,2$, we denote by (u_i, z_i) the solution of the following system:

$$-\operatorname{div}(\mathbf{E}(\boldsymbol{\varepsilon}(u_i) - \mathbf{Q}z_i) + \beta\widetilde{\theta}_i\mathbf{I} + \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}_i)) = f, \tag{3.6a}$$

$$\partial\Psi(\dot{z}_i) + \mathbf{B}\dot{z}_i - \widetilde{\mathbf{Q}}^{\mathsf{T}}\mathbf{E}(\boldsymbol{\varepsilon}(u_i) - \mathbf{Q}z_i) + \partial_z H_1(z_i) + \widetilde{\theta}_i\partial_z H_2(z_i) - \alpha\Delta z_i \ni 0, \tag{3.6b}$$

with boundary conditions

$$u_i = 0, \quad \alpha\nabla z_i\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\times[0,\tau), \tag{3.7}$$

and initial conditions

$$u_i(\cdot,0) = u^0, \quad z_i(\cdot,0) = z^0 \quad \text{in} \quad \Omega. \tag{3.8}$$

On the one hand, with the definition of the subdifferential $\partial\Psi(\dot{z}_i)$ (see [Bre73]), we have

$$\begin{aligned}
&\int_\Omega -\mathbf{E}(\boldsymbol{\varepsilon}(u_i) - \mathbf{Q}z_i):(\widetilde{\mathbf{Q}}\dot{z}_{3-i} - \widetilde{\mathbf{Q}}\dot{z}_i)\,\mathrm{d}x + \int_\Omega \mathbf{B}\dot{z}_i.(\dot{z}_{3-i} - \dot{z}_i)\,\mathrm{d}x\\
&- \alpha\int_\Omega \Delta z_i.(\dot{z}_{3-i} - \dot{z}_i)\,\mathrm{d}x + \int_\Omega \partial_z H_1(z_i).(\dot{z}_{3-i} - \dot{z}_i)\,\mathrm{d}x\\
&+ \int_\Omega \widetilde{\theta}_i\partial_z H_2(z_i).(\dot{z}_{3-i} - \dot{z}_i)\,\mathrm{d}x + \int_\Omega \Psi(\dot{z}_{3-i})\,\mathrm{d}x - \int_\Omega \Psi(\dot{z}_i)\,\mathrm{d}x \geq 0
\end{aligned} \tag{3.9}$$

for almost every $t \in [0,\tau]$. On the other hand, we multiply (3.6a) by $\dot{u}_{3-i} - \dot{u}_i$, we integrate this expression over Ω and we add it to to (3.9). We obtain

$$\begin{aligned}
&\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u_i) - \mathbf{Q} z_i) : ((\boldsymbol{\varepsilon}(\dot{u}_{3-i}) - \widetilde{\mathbf{Q}} \dot{z}_{3-i}) - (\boldsymbol{\varepsilon}(\dot{u}_i) - \widetilde{\mathbf{Q}} \dot{z}_i)) \,\mathrm{d}x \\
&+ \beta \int_\Omega \widetilde{\theta}_i \mathbf{I} : (\boldsymbol{\varepsilon}(\dot{u}_{3-i}) - \boldsymbol{\varepsilon}(\dot{u}_i)) \,\mathrm{d}x + \int_\Omega \mathbf{A} \boldsymbol{\varepsilon}(\dot{u}_i) : (\boldsymbol{\varepsilon}(\dot{u}_{3-i}) - \boldsymbol{\varepsilon}(\dot{u}_i)) \,\mathrm{d}x \\
&+ \int_\Omega \mathbf{B} \dot{z}_i . (\dot{z}_{3-i} - \dot{z}_i) \,\mathrm{d}x - \alpha \int_\Omega \Delta z_i . (\dot{z}_{3-i} - \dot{z}_i) \,\mathrm{d}x + \int_\Omega \partial_z H_1(z_i) . (\dot{z}_{3-i} - \dot{z}_i) \,\mathrm{d}x \\
&+ \int_\Omega \widetilde{\theta}_i \partial_z H_2(z_i) . (\dot{z}_{3-i} - \dot{z}_i) \,\mathrm{d}x - \int_\Omega f \cdot (\dot{u}_{3-i} - \dot{u}_i) \,\mathrm{d}x \\
&+ \int_\Omega \Psi(\dot{z}_{3-i}) \,\mathrm{d}x - \int_\Omega \Psi(\dot{z}_i) \,\mathrm{d}x \geq 0
\end{aligned} \tag{3.10}$$

for almost every $t \in [0,\tau]$. Therefore, we take $i = 1,2$ in (3.10), and thus we add these two inequalities, we obtain

$$\begin{aligned}
&\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(\bar{u}) - \widetilde{\mathbf{Q}} \bar{z}) : (\boldsymbol{\varepsilon}(\dot{\bar{u}}) - \widetilde{\mathbf{Q}} \dot{\bar{z}}) \,\mathrm{d}x + \int_\Omega \mathbf{A} \boldsymbol{\varepsilon}(\dot{\bar{u}}) : \boldsymbol{\varepsilon}(\dot{\bar{u}}) \,\mathrm{d}x + \int_\Omega \mathbf{B} \dot{\bar{z}} . \dot{\bar{z}} \,\mathrm{d}x \\
&- \alpha \int_\Omega \Delta \bar{z} . \dot{\bar{z}} \,\mathrm{d}x + \int_\Omega (\partial_z H_1(z_1) - \partial_z H_1(z_2)) . \dot{\bar{z}} \,\mathrm{d}x \\
&\leq -\beta \int_\Omega \bar{\theta} \mathbf{I} : \boldsymbol{\varepsilon}(\dot{\bar{u}}) \,\mathrm{d}x - \int_\Omega (\widetilde{\theta}_1 \partial_z H_2(z_1) - \widetilde{\theta}_2 \partial_z H_2(z_2)) . \dot{\bar{z}} \,\mathrm{d}x
\end{aligned}$$

with $\bar{u} \stackrel{\text{def}}{=} u_1 - u_2$, $\bar{z} \stackrel{\text{def}}{=} z_1 - z_2$ and $\bar{\theta} \stackrel{\text{def}}{=} \widetilde{\theta}_1 - \widetilde{\theta}_2$. Let $C^{H_1} > 0$ and define

$$\delta_\alpha(t) \stackrel{\text{def}}{=} \tfrac{1}{2} \int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(\bar{u}) - \widetilde{\mathbf{Q}} \bar{z}) : (\boldsymbol{\varepsilon}(\bar{u}) - \widetilde{\mathbf{Q}} \bar{z}) \,\mathrm{d}x - \tfrac{\alpha}{2} \int_\Omega \Delta \bar{z} . \bar{z} \,\mathrm{d}x + \tfrac{C^{H_1}}{2} \int_\Omega |\bar{z}|^2 \,\mathrm{d}x \tag{3.11}$$

for all $t \in [0,\tau]$. By using assumptions (2.4) and (2.6) combined with Korn's inequality, we find that there exists $c^\delta > 0$ such that

$$\forall t \in [0,\tau] : \; \delta_\alpha(t) \geq c^\delta \big(\|\bar{u}(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|\bar{z}(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} \big) + \tfrac{\alpha}{2} \|\nabla \bar{z}(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)}. \tag{3.12}$$

Furthermore Proposition 2.1 implies that the mapping $\delta_\alpha(\cdot)$ is continuous on $[0,\tau]$ and its derivative in the sense of distributions belongs to $\mathrm{L}^1(0,\tau)$. Then $\delta_\alpha(\cdot)$ is absolutely continuous on $[0,\tau]$ and with (2.2b), (2.5) and (3.11), we get

$$\begin{aligned}
&\dot{\delta}_\alpha(t) + c^{\mathbf{A}} \|\boldsymbol{\varepsilon}(\dot{\bar{u}})\|^2_{\mathrm{L}^2(\Omega)} + c^{\mathbf{B}} \|\dot{\bar{z}}\|^2_{\mathrm{L}^2(\Omega)} \leq (C^{H_1}_{zz} + C^{H_1}) \int_\Omega |\bar{z}| |\dot{\bar{z}}| \,\mathrm{d}x \\
&- \beta \int_\Omega \bar{\theta} \mathbf{I} : \boldsymbol{\varepsilon}(\dot{\bar{u}}) \,\mathrm{d}x - \int_\Omega (\widetilde{\theta}_1 \partial_z H_2(z_1) - \widetilde{\theta}_2 \partial_z H_2(z_2)) . \dot{\bar{z}} \,\mathrm{d}x
\end{aligned} \tag{3.13}$$

for almost every $t \in [0,\tau]$.

Let us distinguish now the cases $\alpha = 0$ and $\alpha > 0$.

If $\alpha = 0$, then $\partial_z H_2 \equiv 0$ and (3.13) reduces to

$$\dot{\delta}_0(t) + c^{\mathbf{A}} \|\boldsymbol{\varepsilon}(\dot{\bar{u}})\|^2_{\mathrm{L}^2(\Omega)} + c^{\mathbf{B}} \|\dot{\bar{z}}\|^2_{\mathrm{L}^2(\Omega)} \leq (C^{H_1}_{zz} + C^{H_1}) \int_\Omega |\bar{z}| |\dot{\bar{z}}| \,\mathrm{d}x - \beta \int_\Omega \bar{\theta} \mathbf{I} : \boldsymbol{\varepsilon}(\dot{\bar{u}}) \,\mathrm{d}x$$

for almost every $t \in [0,\tau]$. The two terms of the right hand side can be estimated by using Cauchy-Schwarz's inequality, it comes that

$$\begin{aligned}\dot{\delta}_0(t) + \tfrac{c^{\mathbf{A}}}{2}\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{c^{\mathbf{B}}}{2}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)} &\le \tfrac{9\beta^2}{2c^{\mathbf{A}}}\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{(C^{H_1}_{zz}+C^{H_1})^2}{2c^{\mathbf{B}}}\|\bar{z}\|^2_{\mathrm{L}^2(\Omega)}\\ &\le \tfrac{9\beta^2}{2c^{\mathbf{A}}}\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{(C^{H_1}_{zz}+C^{H_1})^2}{2c^{\mathbf{B}}c^{\delta}}\delta_\alpha(t).\end{aligned}$$

Therefore we integrate over $(0,t)$ and we use Grönwall's lemma, we find

$$\begin{aligned}&\delta_0(t) + \tfrac{c^{\mathbf{A}}}{2}\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2(\Omega))} + \tfrac{c^{\mathbf{B}}}{2}\|\dot{z}\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2(\Omega))}\\ &\le \tfrac{9\beta^2}{2c^{\mathbf{A}}}t\|\bar{\theta}\|^2_{\mathrm{C}^0([0,\tau];\mathrm{L}^2(\Omega))}\exp\big(\tfrac{(C^{H_1}_{zz}+C^{H_1})^2}{2c^{\mathbf{B}}c^{\delta}}\tau\big)\end{aligned}$$

for all $t \in [0,\tau]$.

If $\alpha \neq 0$, the following decomposition is used to estimate the last term in (3.13), namely

$$(\widetilde{\theta}_1\partial_z H_2(z_1) - \widetilde{\theta}_2\partial_z H_2(z_2)).\dot{\bar{z}} = (\bar{\theta}\partial_z H_2(z_1) + \widetilde{\theta}_2(\partial_z H_2(z_1) - \partial_z H_2(z_2))).\dot{\bar{z}}.$$

Then it follows that

$$\begin{aligned}&\dot{\delta}_\alpha(t) + \tfrac{c^{\mathbf{A}}}{2}\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{3c^{\mathbf{B}}}{4}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)} \le \tfrac{9\beta^2}{2c^{\mathbf{A}}}\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{(C^{H_1}_{zz}+C^{H_1})^2}{c^{\mathbf{B}}}\|\bar{z}\|^2_{\mathrm{L}^2(\Omega)}\\ &+\int_\Omega \big(|\bar{\theta}||\partial_z H_2(z_1)||\dot{\bar{z}}| + |\widetilde{\theta}_2||\partial_z H_2(z_1) - \partial_z H_2(z_2)||\dot{\bar{z}}|\big)\,\mathrm{d}x.\end{aligned} \tag{3.14}$$

Observe that (2.2b), (2.3) and Young's inequality give

$$\begin{aligned}&\int_\Omega \big(|\bar{\theta}||\partial_z H_2(z_1)||\dot{\bar{z}}| + |\widetilde{\theta}_2||\partial_z H_2(z_1) - \partial_z H_2(z_2)||\dot{\bar{z}}|\big)\,\mathrm{d}x\\ &\le C^{H_2}_z\int_\Omega \big(1+|z_1|\big)|\bar{\theta}||\dot{\bar{z}}|\,\mathrm{d}x + C^{H_2}_{zz}\int_\Omega |\widetilde{\theta}_2||\bar{z}||\dot{\bar{z}}|\,\mathrm{d}x \le \tfrac{C^{H_2}_z}{2\gamma_1}\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)}\\ &+ \tfrac{C^{H_2}_z}{2\gamma_2}\int_\Omega |\bar{\theta}|^2|z_1|^2\,\mathrm{d}x + \tfrac{C^{H_2}_{zz}}{2\gamma_3}\int_\Omega |\widetilde{\theta}_2|^2|\bar{z}|^2\,\mathrm{d}x + \tfrac{C^{H_2}_z(\gamma_1+\gamma_2)+C^{H_2}_{zz}\gamma_3}{2}\|\dot{\bar{z}}\|^2_{\mathrm{L}^2(\Omega)},\end{aligned}$$

with $\gamma_i > 0$, $i = 1,2,3$. We notice that $z_1 \in \mathrm{L}^{q/2}(0,\tau;\mathrm{W}^{2,4}_{\text{Neu}}(\Omega))$ and $\mathrm{W}^{2,4}_{\text{Neu}}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$ with continuous embedding, thus we have

$$\begin{aligned}&\int_\Omega \big(|\bar{\theta}||\partial_z H_2(z_1)||\dot{\bar{z}}| + |\theta_2||\partial_z H_2(z_1) - \partial_z H_2(z_2)||\dot{\bar{z}}|\big)\,\mathrm{d}x \le \tfrac{C^{H_2}_z}{2\gamma_1}\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)}\\ &+ \tfrac{C^{H_2}_z}{2\gamma_2}\|z_1\|^2_{\mathrm{L}^\infty(\Omega)}\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{C^{H_2}_{zz}}{2\gamma_3}\|\widetilde{\theta}_2\|^2_{\mathrm{L}^4(\Omega)}\|\bar{z}\|^2_{\mathrm{L}^4(\Omega)} + \tfrac{C^{H_2}_z(\gamma_1+\gamma_2)+C^{H_2}_{zz}\gamma_3}{2}\|\dot{\bar{z}}\|^2_{\mathrm{L}^2(\Omega)}.\end{aligned} \tag{3.15}$$

We insert (3.15) in (3.14) and we choose $\gamma_1 = \gamma_2 = \frac{c^{\mathbf{B}}}{4C^{H_2}_z}$ and $\gamma_3 = \frac{c^{\mathbf{B}}}{2C^{H_2}_{zz}}$. Using the continuous embedding $\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$ and (3.12), we obtain

$$\begin{aligned}&\dot{\delta}_\alpha(t) + \tfrac{c^{\mathbf{A}}}{2}\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{c^{\mathbf{B}}}{4}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)} \le \big(\tfrac{9\beta^2}{2c^{\mathbf{A}}} + \tfrac{2(C^{H_2}_z)^2}{c^{\mathbf{B}}} + \tfrac{2(C^{H_2}_z)^2}{c^{\mathbf{B}}}\|z_1\|^2_{\mathrm{L}^\infty(\Omega)}\big)\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)}\\ &+ \tfrac{1}{c^\delta_\alpha c^{\mathbf{B}}}\big((C^{H_1}_{zz}+C^{H_1})^2 + C_1(C^{H_2}_{zz})^2\|\widetilde{\theta}_2\|^2_{\mathrm{L}^4(\Omega)}\big)\delta_\alpha(t)\end{aligned} \tag{3.16}$$

for almost every $t \in [0,\tau]$, where C_1 is the generic constant involved in the continuous embedding of $\mathrm{W}^{1,2}(\Omega)$ into $\mathrm{L}^4(\Omega)$ and $c^\delta_\alpha \overset{\text{def}}{=} \min\big(c^\delta, \frac{\alpha}{2}\big)$. Let us define

$$c(\widetilde{\theta}_2) \overset{\text{def}}{=} \tfrac{1}{c^\delta_\alpha c^{\mathbf{B}}}\big((C^{H_1}_{zz}+C^{H_1})^2 + C_1(C^{H_2}_{zz})^2\|\widetilde{\theta}_2\|^2_{\mathrm{C}^0([0,\tau],\mathrm{L}^4(\Omega))}\big).$$

By using Grönwall's lemma, we get

$$\delta_\alpha(t) + \tfrac{c^{\mathbf{A}}}{2}\|\boldsymbol{\varepsilon}(\dot{\bar{u}})\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2(\Omega))} + \tfrac{c^{\mathbf{B}}}{4}\|\dot{\bar{z}}\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2(\Omega))}$$
$$\leq \big(\tfrac{9\beta^2}{2c^{\mathbf{A}}}\tau + \tfrac{2(C_z^{H_2})^2}{c^{\mathbf{B}}}\tau + \tfrac{2(C_z^{H_2})^2}{c^{\mathbf{B}}}\|z_1\|^2_{\mathrm{L}^2(0,\tau;\mathrm{L}^\infty(\Omega))}\big)\|\bar{\theta}\|^2_{\mathrm{C}^0([0,\tau];\mathrm{L}^2(\Omega))}(1+\tau c(\widetilde{\theta}_2)\exp(c(\widetilde{\theta}_2)\tau))$$

for all $t \in [0,\tau]$. □

As a corollary, it is possible to prove that

Lemma 3.4 *Let* $\tau \in (0,T]$. *Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ *and* $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ *if* $\alpha > 0$ *and* $z^0 \in \mathrm{L}^4(\Omega)$ *if* $\alpha = 0$ *hold. Then the mapping* $\widetilde{\theta} \mapsto f^{\widetilde{\theta}}$ *with* $f^{\widetilde{\theta}} = \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}){:}\boldsymbol{\varepsilon}(\dot{u}) + \widetilde{\theta}(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u}) + \partial_z H_2(z).\dot{z}) + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z})$, *where* (u,z) *is the unique solution of* (2.12)–(2.14), *is continuous from* $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ *into* $\mathrm{L}^r(0,\tau;\mathrm{L}^{4/3}(\Omega))$, *with* $\frac{1}{r} = \frac{2}{q} + \frac{1}{2}$.

Proof. We consider once again $\widetilde{\theta}_i \in \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ and for $i = 1,2$, we denote by (u_i,z_i) the solution of the system (3.6)–(3.8). With the definition of $f^{\widetilde{\theta}}$ we have

$$f^{\widetilde{\theta}_1} - f^{\widetilde{\theta}_2} = \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}_1+\dot{u}_2){:}\boldsymbol{\varepsilon}(\dot{u}_1-\dot{u}_2) + (\widetilde{\theta}_1-\widetilde{\theta}_2)(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u}_1)+\partial_z H_2(z_1).\dot{z}_1)$$
$$+ \widetilde{\theta}_2(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u}_1-\dot{u}_2)+\partial_z H_2(z_1).\dot{z}_1-\partial_z H_2(z_2).\dot{z}_2)+\mathbf{B}(\dot{z}_1+\dot{z}_2).(\dot{z}_1-\dot{z}_2)+\Psi(\dot{z}_1)-\Psi(\dot{z}_2).$$

Thus it comes that

$$|f^{\widetilde{\theta}_1}-f^{\widetilde{\theta}_2}| \leq \|\mathbf{A}\||\boldsymbol{\varepsilon}(\dot{u}_1+\dot{u}_2)||\boldsymbol{\varepsilon}(\dot{\bar{u}})| + |\bar{\theta}|(3\beta|\boldsymbol{\varepsilon}(\dot{u}_1)|+C_z^{H_2}(1+|z_1|)|\dot{z}_1|)$$
$$+ |\widetilde{\theta}_2|(3\beta|\boldsymbol{\varepsilon}(\dot{\bar{u}})|+|\partial_z H_2(z_1).\dot{z}_1-\partial_z H_2(z_2).\dot{z}_2|) + \|\mathbf{B}\||\dot{z}_1+\dot{z}_2||\dot{\bar{z}}| + |\Psi(\dot{z}_1)-\Psi(\dot{z}_2)|.$$

But (2.1c) and (2.1b) lead to

$$|\Psi(\dot{z}_1)-\Psi(\dot{z}_2)| \leq C^\Psi|\dot{z}_1-\dot{z}_2| = C^\Psi|\dot{\bar{z}}|$$

and (2.3) and (2.2b) give

$$|\partial_z H_2(z_1).\dot{z}_1-\partial_z H_2(z_2).\dot{z}_2| \leq |\partial_z H_2(z_1)||\dot{\bar{z}}| + |\partial_z H_2(z_1)-\partial_z H_2(z_2)||\dot{z}_2|$$
$$\leq C_z^{H_2}(1+|z_1|)|\dot{\bar{z}}| + C_{zz}^{H_2}|\bar{z}||\dot{z}_2|.$$

Then, reminding that $\partial_z H_2 \equiv 0$ whenever $\alpha = 0$, the boundedness properties of (u,z) stated at Proposition 2.1 and the continuity property of $\widetilde{\theta} \mapsto (u,z)$ proved in Lemma 3.3 allow us to conclude by using Young's inequality. □

Now we may conclude the proof of Proposition 3.2. Indeed, since $(\widetilde{\theta}_n)_{n\in\mathbb{N}}$ converges strongly to $\widetilde{\theta}_*$ in $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$, we infer from Lemma 3.4 that

$$\lim_{n\to+\infty}\int_{\mathcal{Q}_\tau} f^{\widetilde{\theta}_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_\tau} f^{\widetilde{\theta}_*}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t$$

for all $\xi \in \mathrm{W}^{1,2}(\Omega)$ and $w \in \mathcal{D}(0,\tau)$. Therefore we may pass to the limit in all the terms of (3.5) to get

$$\int_{\mathcal{Q}_\tau} c(x)\dot{\theta}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_\tau}\kappa(x)\nabla\theta(x,t)\nabla\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\mathcal{Q}_\tau} f^{\widetilde{\theta}_*}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t \tag{3.17}$$

for all $\xi \in \mathrm{W}^{1,2}(\Omega)$ and $w \in \mathcal{D}(0,\tau)$. It follows that θ is solution of problem (2.10)–(2.11) with the data $f^{\widetilde{\theta}_*}$. Besides by uniqueness of the solution, it comes that $\theta = \theta_*$ and the whole sequence $(\theta_n)_{n\in\mathbb{N}}$ converges to θ_* in $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$. $\square$

Corollary 3.5 *Let $\tau \in (0,T]$. Assume that (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), (2.8), $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\mathrm{Neu}}(\Omega)$, $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ and $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ if $\alpha > 0$ and $z^0 \in \mathrm{L}^4(\Omega)$ if $\alpha = 0$ hold. Then there exists $\tau \in (0,T]$ such that $\Phi_\tau^{\widetilde{\theta},\theta}$ admits a fixed point in $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$.*

Proof. We have already proved in the previous propositions that $\Phi_\tau^{\widetilde{\theta},\theta}$ is a continuous mapping from $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ into $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ and maps any bounded subset $\mathcal{C} \subset \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ into a bounded relatively compact subset. Hence we will be able to conclude by using Schauder's fixed point theorem (see [Eva10]) if we can find a closed convex bounded subset $\mathcal{C}$ of $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ such that $\Phi_\tau^{\widetilde{\theta},\theta}(\mathcal{C}) \subset \mathcal{C}$.

Let $C_1 > 0$ be the generic constant involved in the continuous embedding of $\mathrm{W}^{1,2}(\Omega)$ into $\mathrm{L}^4(\Omega)$ and let $R^\theta > C_1 C_\theta \exp(\frac{T}{c^c})\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}$, where C_θ is the constant defined in Proposition 3.1. For any $\tau \in (0,T]$ and $\widetilde{\theta} \in \mathcal{C} \overset{\text{def}}{=} \bar{B}_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}(0,R^\theta)$, we denote $\theta = \Phi_\tau^{\widetilde{\theta},\theta}(\widetilde{\theta})$ and we have (see (3.4))

$$\|\theta\|_{\mathrm{V}((0,\tau)\times\Omega)} \leq C_\theta \exp\big(\tfrac{\tau}{c^c}\big)\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)} + \|f^{\widetilde{\theta}}\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))}\big)$$

and $\theta \in \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$. Thus we have

$$\begin{aligned} &\|\theta\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))} = \|\theta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^4(\Omega))} \\ &\leq C_1 C_\theta \exp\big(\tfrac{\tau}{c^c}\big)\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)} + \tau^{\frac{q-8}{2q}}\|f^{\widetilde{\theta}}\|_{\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))}\big) \end{aligned} \tag{3.18}$$

for any $q > 8$. Since $\lim_{\tau\to 0} \tau^{\frac{q-8}{2q}} = 0$, we only need to prove that $\|f^{\widetilde{\theta}}\|_{\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))}$ remains bounded independently of τ. Let us emphasize that Proposition 2.1 implies that $f^{\widetilde{\theta}}$ remains in a bounded subset of $\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))$ but this does not allow us to conclude since we don't know if the diameter of this bounded subset depends on τ or not. In order to cope with this difficulty, we consider the extension of $\widetilde{\theta}$ to $[0,T]$ by zero on $(\tau,T]$. We denote by $\widetilde{\theta}_{\text{ext}}$ this extension. Of course, for any $\widetilde{\theta} \in \mathcal{C}$, we have $\widetilde{\theta}_{\text{ext}} \in \mathrm{L}^q(0,T;\mathrm{L}^4(\Omega))$ for any $q > 8$ and

$$\|\widetilde{\theta}_{\text{ext}}\|_{\mathrm{L}^q(0,T;\mathrm{L}^4(\Omega))} = \|\widetilde{\theta}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} = \tau^{\frac{1}{q}}\|\widetilde{\theta}\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))} \leq T^{\frac{1}{q}} R^\theta.$$

Then we define $(u_{\text{ext}}, z_{\text{ext}})$ as the unique solution of problem (2.12)–(2.14) with τ replaced by T and $\widetilde{\theta}$ replaced by $\widetilde{\theta}_{\text{ext}}$. Since $\widetilde{\theta}_{\text{ext}}$ remains in the closed ball $\bar{B}_{\mathrm{L}^q(0,T;\mathrm{L}^4(\Omega))}(0,T^{1/q}R^\theta)$, Proposition 2.1 implies that $(u_{\text{ext}}, z_{\text{ext}})$ remains in a bounded subset of $\mathrm{W}^{1,q}(0,\tau;\mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)) \times (\mathrm{L}^{q/2}(0,\tau;\mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{C}^0([0,\tau];\mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{W}^{1,q/2}(0,\tau;\mathrm{L}^4(\Omega)) \cap \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^2(\Omega)))$ if $\alpha > 0$ or $\mathrm{W}^{1,q}(0,\tau;\mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)) \times \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^4(\Omega))$ if $\alpha = 0$. It follows that

$$f^{\widetilde{\theta}_{\text{ext}}} \overset{\text{def}}{=} \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}_{\text{ext}}){:}\boldsymbol{\varepsilon}(\dot{u}_{\text{ext}}) + \widetilde{\theta}_{\text{ext}}(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u}_{\text{ext}}) + \partial_z H_2(z_{\text{ext}}).\dot{z}_{\text{ext}}) + \mathbf{B}\dot{z}_{\text{ext}}.\dot{z}_{\text{ext}} + \Psi(\dot{z}_{\text{ext}})$$

remains in a bounded subset of $\mathrm{L}^{q/4}(0,T;\mathrm{L}^2(\Omega))$, i.e. there exists a constant $C(R^\theta)$, depending only on R^θ and the data, such that $\|f^{\widetilde{\theta}_{\text{ext}}}\|_{\mathrm{L}^{q/4}(0,T;\mathrm{L}^2(\Omega))} \leq C(R^\theta)$. But $f^{\widetilde{\theta}}$ coincide with $f^{\widetilde{\theta}_{\text{ext}}}$ on $[0,\tau]$ and

$$\|f^{\widetilde{\theta}}\|_{\mathrm{L}^{q/4}(0,\tau;\mathrm{L}^2(\Omega))} = \|f^{\widetilde{\theta}_{\text{ext}}}\mathbf{1}_{[0,\tau]}\|_{\mathrm{L}^{q/4}(0,T;\mathrm{L}^2(\Omega))} \leq \|f^{\widetilde{\theta}_{\text{ext}}}\|_{\mathrm{L}^{q/4}(0,T;\mathrm{L}^2(\Omega))} \leq C(R^\theta). \tag{3.19}$$

Then by introducing (3.19) into (3.18) and by choosing $\tau \in (0,T]$ such that

$$C_1 C_\theta \exp\left(\tfrac{\tau}{c^c}\right)\left(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)} + \tau^{\frac{q-8}{2q}} C(R^\theta)\right) \leq R^\theta,$$

we may conclude. □

We can consider $\tau \in (0,T]$ such that $\Phi_\tau^{\widetilde{\theta},\theta}$ admits a fixed point θ in $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ and we define (u,z) as the unique solution of problem (2.12)–(2.14) with $\widetilde{\theta} = \theta$. By definition of $\Phi_\tau^{\widetilde{\theta},\theta}$, (u,z,θ) is a solution of the coupled problem (1.6)–(1.8) and by combining the regularity results for (u,z) given at Proposition 2.1 with the regularity results for the heat-transfer equation recalled in the proof of Proposition 3.1, we get $\theta \in \mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega)) \cap \mathrm{C}^0(0,\tau;\mathrm{L}^4(\Omega))$, $\dot{\theta} \in \mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))$, $u \in \mathrm{W}^{1,q}(0,\tau;\mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega))$, $z \in \mathrm{L}^{q/2}(0,\tau;\mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{C}^0([0,\tau];\mathrm{W}^{1,2}_{\mathrm{Neu}}(\Omega)) \cap \mathrm{W}^{1,q/2}(0,\tau;\mathrm{L}^4(\Omega)) \cap \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^2(\Omega))$ when $\alpha > 0$, $z \in \mathrm{W}^{1,q}(0,\tau;\mathrm{L}^4(\Omega))$ when $\alpha = 0$, for any $q > 8$. Hence the proof of Theorem 2.2 is complete.

4 Further properties of the solution

Let us recall that system (1.6)–(1.8) is thermodynamically consistent if the temperature remains positive (see Section 1). So we begin this section by proving that the solutions (u,z,θ) of (1.6)–(1.8) are physically admissible, i.e. $\theta(x,t) > 0$ almost everywhere in $\mathcal{Q}_\tau$. To this aim we introduce the following assumption for the initial temperature:

(A–8) There exists $\bar{\theta} > 0$ such that

$$\theta^0(x) \geq \bar{\theta} > 0 \text{ a.e. } x \in \Omega. \tag{4.1}$$

Proposition 4.1 *Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), (2.8), $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\mathrm{Neu}}(\Omega)$, $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ *and* $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ *if* $\alpha > 0$ *and* $z^0 \in \mathrm{L}^4(\Omega)$ *if* $\alpha = 0$ *hold. Assume also that condition* (4.1) *is satisfied and* $\kappa \in \mathrm{C}^1(\bar{\Omega})$. *Then, any solution* (u,z,θ) *of problem* (1.6)–(1.8) *defined on* $[0,\tau]$, $\tau \in (0,T]$, *is thermodynamically admissible, i.e.* $\theta(x,t) > 0$ *for almost every* $(x,t) \in \mathcal{Q}_\tau$.

Proof. The key tool of the proof is the classical Stampacchia's truncation method (see [Bre83]). So we consider a function $\mathcal{G} \in \mathrm{C}^1(\mathbb{R};\mathbb{R})$ such that

(i) $\exists C^{\mathcal{G}'} > 0,\ \forall \sigma \in \mathbb{R} :\ |\mathcal{G}'(\sigma)| \leq C^{\mathcal{G}'}$,

(ii) $\mathcal{G}$ is strictly increasing on $(0,\infty)$,

(iii) $\forall \sigma \leq 0 :\ \mathcal{G}(\sigma) = 0$,

and we define $\Gamma(\sigma) \overset{\text{def}}{=} \int_0^\sigma \mathcal{G}(s)\,\mathrm{d}s$ for all $\sigma \in \mathbb{R}$. Now let (u,z,θ) be a solution of (1.6)–(1.8) on $[0,\tau]$. We will prove that θ is positive almost everywhere in $\mathcal{Q}_\tau$ in two steps: first we will establish that θ is non negative, then that θ remains bounded from below by a positive quantity.

Since we have assumed that $\kappa \in \mathrm{C}^1(\bar{\Omega})$, we can infer that $\theta \in \mathrm{L}^2(0,\tau,\mathrm{W}^{2,2}(\Omega))$. Indeed, θ is a fixed point of $\Phi_\tau^{\widetilde{\theta},\theta}$, thus

$$-\operatorname{div}(\kappa(x)\nabla\theta) = f^\theta - c(x)\dot{\theta}$$

with $f^\theta = \mathbf{A}\boldsymbol{\varepsilon}(\dot u){:}\boldsymbol{\varepsilon}(\dot u) + \theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot u){+}\partial_z H_2(z).\dot z) + \mathbf{B}\dot z.\dot z + \Psi(\dot z) \in \mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))$ and $c(x)\dot\theta \in \mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))$. It follows that $-\operatorname{div}(\kappa(x)\nabla\theta) \in \mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))$. We can consider the time variable as a parameter and the linearity of the operator $-\operatorname{div}(\kappa(x)\nabla\cdot)$ combined with classical regularity properties (see [Bre83]) yield the announced result.

Then we introduce the mapping $\varphi : [0,\tau] \to \mathbb{R}$ given by

$$\varphi(t) \overset{\text{def}}{=} \exp\Big(-\tfrac{1}{c^c}\int_0^t \tfrac{9\beta^2}{2c^{\mathbf{A}}}\|\theta(\cdot,s)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}s\Big) \tag{4.2}$$

for all $t \in [0,\tau]$ if $\alpha = 0$ and by

$$\varphi(t) \overset{\text{def}}{=} \exp\Big(-\tfrac{1}{c^c}\int_0^t \big(\tfrac{9\beta^2}{2c^{\mathbf{A}}}{+}\tfrac{(C_z^{H_2})^2}{c^{\mathbf{B}}}\big(1{+}\|z(\cdot,s)\|^2_{\mathrm{L}^\infty(\Omega)}\big)\big)\|\theta(\cdot,s)\|_{\mathrm{L}^\infty(\Omega)}\,\mathrm{d}s\Big) \tag{4.3}$$

for all $t \in [0,\tau]$ if $\alpha > 0$. Reminding that $z \in \mathrm{L}^{q/2}(0,\tau;\mathrm{W}^{2,4}_{\text{Neu}}(\Omega))$ for any $q > 8$ if $\alpha > 0$ and $\mathrm{W}^{2,2}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$, we can deduce that $\varphi \in \mathrm{W}^{1,1}(0,\tau)$ and $0 \le \varphi(t) \le 1$ for almost every $t \in [0,\tau]$. Next we define $\Theta_{\theta_\varphi}(t) \overset{\text{def}}{=} \int_\Omega c(x)\Gamma(\theta_\varphi(x,t))\,\mathrm{d}x$ with $\theta_\varphi(x,t) \overset{\text{def}}{=} -\theta(x,t)\varphi(t)$ for almost every $(x,t) \in \mathcal{Q}_\tau$. Since $\theta \in \mathrm{V}((0,\tau){\times}\Omega) \cap \mathrm{L}^2(0,\tau,\mathrm{W}^{2,2}(\Omega))$, we get $\theta_\varphi \in \mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}(\Omega)) \cap \mathrm{W}^{1,1}(0,\tau;\mathrm{L}^2(\Omega))$ and

$$\dot\theta_\varphi(x,t){=}\big({-}\dot\theta(x,t){+}\tfrac{\theta(x,t)}{c^c}\tfrac{9\beta^2}{2c^{\mathbf{A}}}\|\theta(\cdot,t)\|_{\mathrm{L}^\infty(\Omega)}\big)\varphi(t) \text{ if } \alpha = 0,$$
$$\dot\theta_\varphi(x,t){=}\big({-}\dot\theta(x,t){+}\tfrac{\theta(x,t)}{c^c}\big(\tfrac{9\beta^2}{2c^{\mathbf{A}}}{+}\tfrac{(C_z^{H_2})^2}{c^{\mathbf{B}}}\big(1{+}\|z(\cdot,t)\|^2_{\mathrm{L}^\infty(\Omega)}\big)\big)\|\theta(\cdot,t)\|_{\mathrm{L}^\infty(\Omega)}\big)\varphi(t) \text{ if } \alpha > 0$$

for almost every $(x,t) \in \mathcal{Q}_\tau$. Thus Θ_{θ_φ} is absolutely continuous on $[0,\tau]$ and we have

$$\begin{aligned}\dot\Theta_{\theta_\varphi}(t) &= \int_\Omega c(x)\mathcal{G}(\theta_\varphi)\dot\theta_\varphi\,\mathrm{d}x = -\int_\Omega \mathcal{G}(\theta_\varphi)(\operatorname{div}(\kappa\nabla\theta){+}\mathbf{A}\boldsymbol{\varepsilon}(\dot u){:}\boldsymbol{\varepsilon}(\dot u)\\ &+\theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot u){+}\partial_z H_2(z).\dot z){+}\mathbf{B}\dot z.\dot z{+}\Psi(\dot z))\varphi\,\mathrm{d}x - \int_\Omega c(x)\mathcal{G}(\theta_\varphi)\theta\dot\varphi\,\mathrm{d}x\\ &= -\int_\Omega \mathcal{G}'(\theta_\varphi)\kappa\nabla\theta_\varphi{:}\nabla\theta_\varphi\,\mathrm{d}x - \int_\Omega \mathcal{G}(\theta_\varphi)(\mathbf{A}\boldsymbol{\varepsilon}(\dot u){:}\boldsymbol{\varepsilon}(\dot u)\\ &+\theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot u){+}\partial_z H_2(z).\dot z){+}\mathbf{B}\dot z.\dot z{+}\Psi(\dot z))\varphi\,\mathrm{d}x - \int_\Omega c(x)\mathcal{G}(\theta_\varphi)\theta\dot\varphi\,\mathrm{d}x\end{aligned} \tag{4.4}$$

for almost every $t \in [0,\tau]$. We evaluate now the second term of the right hand side of (4.4). By using (2.3), (2.5) and Cauchy-Schwarz's inequality, we get

$$\mathbf{A}\boldsymbol{\varepsilon}(\dot u){:}\boldsymbol{\varepsilon}(\dot u) + \beta\theta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot u) \ge c^{\mathbf{A}}|\boldsymbol{\varepsilon}(\dot u)|^2 - 3\beta|\theta||\boldsymbol{\varepsilon}(\dot u)| \ge \tfrac{c^{\mathbf{A}}}{2}|\boldsymbol{\varepsilon}(\dot u)|^2 - \tfrac{9\beta^2}{2c^{\mathbf{A}}}|\theta|^2, \tag{4.5}$$

and if $\alpha > 0$

$$\begin{aligned}&\mathbf{B}\dot z.\dot z + \theta\partial_z H_2(z).\dot z \ge c^{\mathbf{B}}|\dot z|^2 - |\theta||\partial_z H_2(z)||\dot z|\\ &\ge c^{\mathbf{B}}|\dot z|^2 - C_z^{H_2}|\theta|(1{+}|z|)|\dot z| \ge \tfrac{c^{\mathbf{B}}}{2}|\dot z|^2 - \tfrac{(C_z^{H_2})^2}{c^{\mathbf{B}}}(1{+}|z|^2)|\theta|^2.\end{aligned} \tag{4.6}$$

We insert (4.5) and (4.6) into (4.4), then reminding that $\mathcal{G}'(\theta_\varphi) \ge 0$ almost everywhere, we obtain

$$\dot\Theta_{\theta_\varphi}(t) \le \int_\Omega \mathcal{G}(\theta_\varphi)\tfrac{9\beta^2}{2c^{\mathbf{A}}}\big(|\theta|^2{+}\tfrac{c(x)}{c^c}\|\theta\|_{\mathrm{L}^\infty(\Omega)}\theta\big)\varphi\,\mathrm{d}x$$

if $\alpha = 0$ and

$$\dot\Theta_{\theta_\varphi}(t) \le \int_\Omega \mathcal{G}(\theta_\varphi)\big(\tfrac{9\beta^2}{2c^{\mathbf{A}}}{+}\tfrac{(C_z^{H_2})^2}{c^{\mathbf{B}}}\big(1{+}\|z\|^2_{\mathrm{L}^\infty(\Omega)}\big)\big)\big(|\theta|^2{+}\tfrac{c(x)}{c^c}\|\theta\|_{\mathrm{L}^\infty(\Omega)}\theta\big)\varphi\,\mathrm{d}x$$

if $\alpha > 0$, for almost every $t \in [0, \tau]$. Now we observe that $\mathcal{G}(\Theta_{\theta_\varphi})$ vanishes whenever θ is non negative and

$$|\theta|^2 + \tfrac{c(x)}{c^c}\|\theta\|_{\mathrm{L}^\infty(\Omega)}\theta = |\theta|\big(|\theta| - \tfrac{c(x)}{c^c}\|\theta\|_{\mathrm{L}^\infty(\Omega)}\big) \le |\theta|\big(|\theta| - \|\theta\|_{\mathrm{L}^\infty(\Omega)}\big) \le 0$$

whenever θ is non positive. Hence $\dot{\Theta}_{\theta_\varphi}(t) \le 0$ for almost every $t \in [0, \tau]$. Since we have $\Theta_{\theta_\varphi}(0) = \int_\Omega c(x)\Gamma(-\theta^0(x))\,\mathrm{d}x = 0$, we infer that $\Theta_{\theta_\varphi}(t) \le 0$ for all $t \in [0, \tau]$. It follows that $\Gamma(\theta_\varphi(x, t)) = 0$ for almost every $(x, t) \in \mathcal{Q}_\tau$ implying that $\theta_\varphi(x, t) = -\theta(x, t)\varphi(t) \le 0$ i.e. $\theta(x, t) \ge 0$ for almost every $(x, t) \in \mathcal{Q}_\tau$.

Let us establish now that the temperature $\theta(x, t)$ remains positive for almost every $(x, t) \in \mathcal{Q}_\tau$. To this aim, we define $\widetilde{\Theta}_{\widetilde{\theta}_\varphi}(t) \overset{\text{def}}{=} \int_\Omega c(x)\Gamma(\widetilde{\theta}_\varphi(x, t))\,\mathrm{d}x$ with $\widetilde{\theta}_\varphi(x, t) \overset{\text{def}}{=} -\theta(x, t) + \bar{\theta}\varphi(t)$ for almost every $(x, t) \in \mathcal{Q}_\tau$. Since $\theta \in \mathrm{W}^{1,2}(0, \tau; \mathrm{L}^2(\Omega))$, we infer that $\widetilde{\Theta}_{\theta_\varphi}$ is absolutely continuous on $[0, \tau]$ and we have

$$\begin{aligned}\dot{\widetilde{\Theta}}_{\widetilde{\theta}_\varphi}(t) &= \int_\Omega c(x)\mathcal{G}(\widetilde{\theta}_\varphi)\dot{\widetilde{\theta}}_\varphi\,\mathrm{d}x = -\int_\Omega \mathcal{G}(\widetilde{\theta}_\varphi)(\mathrm{div}(\kappa\nabla\theta) + \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}){:}\boldsymbol{\varepsilon}(\dot{u}) \\ &+\theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u}) + \partial_z H_2(z).\dot{z}) + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z}) - c(x)\bar{\theta}\dot{\varphi})\,\mathrm{d}x = -\int_\Omega \mathcal{G}'(\widetilde{\theta}_\varphi)\kappa\nabla\widetilde{\theta}_\varphi{:}\nabla\widetilde{\theta}_\varphi\,\mathrm{d}x \\ &-\int_\Omega \mathcal{G}(\widetilde{\theta}_\varphi)(\mathbf{A}\boldsymbol{\varepsilon}(\dot{u}){:}\boldsymbol{\varepsilon}(\dot{u}) + \theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u}) + \partial_z H_2(z).\dot{z}) + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z}) - c(x)\bar{\theta}\dot{\varphi})\,\mathrm{d}x\end{aligned} \tag{4.7}$$

for almost every $t \in [0, \tau]$. We estimate the right hand side of (4.7) by using the same tricks as previously, we obtain

$$\dot{\widetilde{\Theta}}_{\widetilde{\theta}_\varphi}(t) \le \int_\Omega \mathcal{G}(\widetilde{\theta}_\varphi)\big(\tfrac{9\beta^2}{2c^{\mathbf{A}}}|\theta|^2 + c(x)\bar{\theta}\dot{\varphi}\big)\,\mathrm{d}x$$

if $\alpha = 0$ and

$$\dot{\widetilde{\Theta}}_{\widetilde{\theta}_\varphi}(t) \le \int_\Omega \mathcal{G}(\widetilde{\theta}_\varphi)\big(\big(\tfrac{9\beta^2}{2c^{\mathbf{A}}} + \tfrac{(C_z^{H_2})^2}{c^{\mathbf{B}}}\big(1 + |z|^2\big)\big)|\theta|^2 + c(x)\bar{\theta}\dot{\varphi}\big)\,\mathrm{d}x$$

if $\alpha > 0$, for almost every $t \in [0, \tau]$. It follows from (4.2) and (4.3) that

$$\dot{\widetilde{\Theta}}_{\widetilde{\theta}_\varphi}(t) \le \int_\Omega \mathcal{G}(\widetilde{\theta}_\varphi)\tfrac{9\beta^2}{2c^{\mathbf{A}}}\big(|\theta|^2 - \tfrac{c(x)}{c^c}\|\theta\|_{\mathrm{L}^\infty(\Omega)}\bar{\theta}\varphi\big)\,\mathrm{d}x$$

if $\alpha = 0$ and

$$\dot{\widetilde{\Theta}}_{\widetilde{\theta}_\varphi}(t) \le \int_\Omega \mathcal{G}(\widetilde{\theta}_\varphi)\big(\tfrac{9\beta^2}{2c^{\mathbf{A}}} + \tfrac{(C_z^{H_2})^2}{c^{\mathbf{B}}}\big(1 + \|z\|^2_{\mathrm{L}^\infty(\Omega)}\big)\big)\big(|\theta|^2 - \tfrac{c(x)}{c^c}\|\theta\|_{\mathrm{L}^\infty(\Omega)}\bar{\theta}\varphi\big)\,\mathrm{d}x$$

if $\alpha > 0$, for almost every $t \in [0, \tau]$. Then we observe that $\mathcal{G}(\widetilde{\theta}_\varphi)$ vanishes whenever $\theta \ge \bar{\theta}\varphi$, and

$$|\theta|^2 - \tfrac{c(x)}{c^c}\|\theta\|_{\mathrm{L}^\infty(\Omega)}\bar{\theta}\varphi \le \|\theta\|_{\mathrm{L}^\infty(\Omega)}\big(|\theta| - \tfrac{c(x)}{c^c}\bar{\theta}\varphi\big) \le 0$$

whenever $0 \le \theta \le \bar{\theta}\varphi$. Since we have already proved that θ is non negative almost everywhere in $\mathcal{Q}_\tau$, we may infer that $\dot{\widetilde{\Theta}}_{\widetilde{\theta}_\varphi}(t) \le 0$ for almost every $t \in [0, \tau]$. Therefore $\widetilde{\Theta}_{\widetilde{\theta}_\varphi}(t) \le \widetilde{\Theta}_{\widetilde{\theta}_\varphi}(0) = \int_\Omega c(x)\Gamma(-\theta^0 + \bar{\theta})\,\mathrm{d}x = 0$ for all $t \in [0, \tau]$. It follows that $\Gamma(\widetilde{\theta}_\varphi(x, t)) = 0$ for almost every $(x, t) \in \mathcal{Q}_\tau$, which implies that

$$\widetilde{\theta}_\varphi(x, t) = -\theta(x, t) + \bar{\theta}\varphi \le 0$$

for almost every $(x, t) \in \mathcal{Q}_\tau$. ☐

Furthermore the solutions of problem (1.6)–(1.8) satisfy the following global estimate:

Proposition 4.2 *Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), (2.8), $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega)$, $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ *and* $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ *if* $\alpha > 0$ *and* $z^0 \in \mathrm{L}^4(\Omega)$ *if* $\alpha = 0$ *hold. Assume also that condition* (4.1) *is satisfied,* $\kappa \in \mathrm{C}^1(\bar{\Omega})$ *and* $c^{H_1} > 0$*. Then, there exists a constant* $C_0 > 0$*, depending only the data such that for any solution* (u, z, θ) *of problem* (1.6)–(1.8) *defined on* $[0, \tau]$, $\tau \in (0, T]$*, we have*

$$\forall t \in [0,\tau] : \ \|u(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + \|z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} + \alpha\|\nabla z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} + \|\theta(\cdot,t)\|_{\mathrm{L}^1(\Omega)} \le C_0.$$

Proof. First we choose $\dot{u}$ as a test-function in (1.6a) and the constant function equal to 1 in (1.6c). We get

$$\int_{\mathcal{Q}_t} (\mathbf{E}(\boldsymbol{\varepsilon}(u) - \mathbf{Q}z) + \beta\theta\mathbf{I} + \mathbf{A}\boldsymbol{\varepsilon}(\dot{u})){:}\boldsymbol{\varepsilon}(\dot{u})\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathcal{Q}_t} f\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}s \tag{4.8}$$

and

$$\begin{aligned}\int_\Omega c(\cdot)\theta(\cdot,t)\,\mathrm{d}x = \int_\Omega c(\cdot)\theta^0\,\mathrm{d}x + \int_{\mathcal{Q}_t} \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}){:}\boldsymbol{\varepsilon}(\dot{u})\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathcal{Q}_t} \mathbf{B}\dot{z}.\dot{z}\,\mathrm{d}x\,\mathrm{d}s \\ + \int_{\mathcal{Q}_t} \theta(\beta\mathbf{I}{:}\boldsymbol{\varepsilon}(\dot{u}) + \partial_z H_2(z).\dot{z})\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathcal{Q}_t} \Psi(\dot{z})\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.9}$$

Then we use the definition of the subdifferential $\partial\Psi$; for almost every $s \in [0, \tau]$ and all $\widetilde{z} \in \mathrm{L}^2(\Omega, \mathcal{Z})$, we have

$$\begin{aligned}&\int_\Omega (\mathbf{B}\dot{z}(\cdot,s) - \widetilde{\mathbf{Q}}^{\mathsf{T}}\mathbf{E}(\boldsymbol{\varepsilon}(u(\cdot,s)) - \mathbf{Q}z(\cdot,s)).(\widetilde{z} - \dot{z}(\cdot,s))\,\mathrm{d}x - \int_\Omega \alpha\Delta z(\cdot,s).(\widetilde{z} - \dot{z}(\cdot,s))\,\mathrm{d}x \\ &+ \int_\Omega \partial_z H_1(z(\cdot,s)) + \theta(\cdot,s)\partial_z H_2(z(\cdot,s)).(\widetilde{z} - \dot{z}(\cdot,s))\,\mathrm{d}x \\ &+ \int_\Omega \Psi(\widetilde{z})\,\mathrm{d}x - \int_\Omega \Psi(\dot{z}(\cdot,s))\,\mathrm{d}x \ge 0.\end{aligned}$$

But Ψ is positively homogeneous of degree 1, so by choosing successively $\widetilde{z} \equiv 0$ and $\widetilde{z} = 2\dot{z}(\cdot,s)$ and integrating over $[0,t] \subset [0,\tau]$, we obtain

$$\begin{aligned}\int_{\mathcal{Q}_t} (\mathbf{B}\dot{z} - \widetilde{\mathbf{Q}}^{\mathsf{T}}\mathbf{E}(\boldsymbol{\varepsilon}(u) - \mathbf{Q}z) + \partial_z H_1(z) + \theta(\cdot,s)\partial_z H_2(z) - \alpha\Delta z).\dot{z}\,\mathrm{d}x\,\mathrm{d}s \\ + \int_{\mathcal{Q}_t} \Psi(\dot{z})\,\mathrm{d}x\,\mathrm{d}s = 0.\end{aligned} \tag{4.10}$$

Now we add (4.8), (4.9) and (4.10), we get

$$\begin{aligned}&\tfrac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u(\cdot,t)) - \mathbf{Q}z(\cdot,t)){:}(\boldsymbol{\varepsilon}(u(\cdot,t)) - \mathbf{Q}z(\cdot,t))\,\mathrm{d}x + \int_\Omega H_1(z(\cdot,t))\,\mathrm{d}x \\ &+ \tfrac{\alpha}{2}\|\nabla z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} + \int_\Omega c(\cdot)\theta(\cdot,t)\,\mathrm{d}x = \tfrac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u^0) - \mathbf{Q}z^0){:}(\boldsymbol{\varepsilon}(u^0) - \mathbf{Q}z^0)\,\mathrm{d}x \\ &+ \tfrac{\alpha}{2}\|\nabla z^0\|^2_{\mathrm{L}^2(\Omega)} + \int_\Omega H_1(z^0)\,\mathrm{d}x + \int_\Omega c(\cdot)\theta^0\,\mathrm{d}x + \int_{\mathcal{Q}_t} f\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.11}$$

We estimate from below the two first terms of the left hand side by using (2.2a), (2.4a) and (2.6). Indeed, for any $\lambda \in (0,1)$, we find

$$\begin{aligned}&\tfrac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u(\cdot,t)) - \mathbf{Q}z(\cdot,t)){:}(\boldsymbol{\varepsilon}(u(\cdot,t)) - \mathbf{Q}z(\cdot,t))\,\mathrm{d}x + \int_\Omega H_1(z(\cdot,t))\,\mathrm{d}x \\ &\ge \tfrac{1}{2}(1-\lambda)c^{\mathbf{E}}\|\boldsymbol{\varepsilon}(u(\cdot,t))\|^2_{\mathrm{L}^2(\Omega)} + \big(1 - \tfrac{1}{\lambda}\big)\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\big(\|\widetilde{\mathbf{Q}}\|^2\|z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} + |\mathrm{Q}|^2|\Omega|\big) \\ &+ c^{H_1}\|z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} - \widetilde{c}^{H_1}|\Omega|.\end{aligned}$$

We may choose $\lambda \in (0,1)$ such that

$$\big(1-\tfrac{1}{\lambda}\big)\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\|\widetilde{\mathbf{Q}}\|^2 + c^{H_1} > 0,$$

i.e.

$$1 > \lambda > \frac{\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\|\widetilde{\mathbf{Q}}\|^2}{\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\|\widetilde{\mathbf{Q}}\|^2+c^{H_1}}.$$

Then

$$\begin{aligned}&\tfrac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u(\cdot,t))-\mathbf{Q}z(\cdot,t)):(\boldsymbol{\varepsilon}(u(\cdot,t))-\mathbf{Q}z(\cdot,t))\,\mathrm{d}x + \int_\Omega H_1(z(\cdot,t))\,\mathrm{d}x\\ &\geq C\big(\|u(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)}+\|z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)}\big) - \widetilde{C},\end{aligned}$$

with

$$\begin{aligned}C &\stackrel{\text{def}}{=} \min\big(\tfrac{1}{2}(1-\lambda)c^{\mathbf{E}}C^{\text{Korn}}, \big(1-\tfrac{1}{\lambda}\big)\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\|\widetilde{\mathbf{Q}}\|^2+c^{H_1}\big),\\ \widetilde{C} &\stackrel{\text{def}}{=} \big(\tfrac{1}{\lambda}-1\big)\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}|\mathrm{Q}|^2|\Omega| + \widetilde{c}^{H_1}|\Omega|.\end{aligned}$$

Now we integrate by parts the last term of the right hand side of (4.11) to get

$$\begin{aligned}&\tfrac{C}{2}\|u(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)} + C\|z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{\alpha}{2}\|\nabla z(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)} + \int_\Omega c(\cdot)\theta(\cdot,t)\,\mathrm{d}x\\ &\leq \tfrac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u^0)-\mathbf{Q}z^0):(\boldsymbol{\varepsilon}(u^0)-\mathbf{Q}z^0)\,\mathrm{d}x + \tfrac{\alpha}{2}\|\nabla z^0\|^2_{\mathrm{L}^2(\Omega)}\\ &+ \int_\Omega H_1(z^0)\,\mathrm{d}x + \int_\Omega c(\cdot)\theta^0\,\mathrm{d}x + \widetilde{C} + \|f\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}\|u^0\|_{\mathrm{L}^2(\Omega)}\\ &+ \tfrac{1}{2C}\|f\|^2_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))} + \tfrac{1}{2}\|\dot{f}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} + \tfrac{1}{2}\int_0^t \|u\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s.\end{aligned}$$

Then, reminding that θ remains non negative, Grönwall's lemma allows us to conclude. ☐

Let us find now some sufficient conditions on the data which will lead to a global existence result, i.e. existence of a solution of problem (1.6)–(1.8) defined on the whole interval $[0,T]$. First we observe that the heat-transfer equation (1.6c) and the system composed of the momentum equilibrium equation and the flow rule (1.6a)–(1.6b) are totally decoupled if $\beta = 0$ and $\partial_z H_2 \equiv 0$. In such a case, we may obtain a solution of (1.6)–(1.8) by applying Proposition 2.1 to solve (2.12)–(2.14) with $\widetilde{\theta} = 0$ and $\tau = T$, then by finding the solution θ of (2.10)–(2.11) with

$$f^{\widetilde{\theta}} = \mathbf{A}\boldsymbol{\varepsilon}(\dot{u}):\boldsymbol{\varepsilon}(\dot{u}) + \mathbf{B}\dot{z}.\dot{z} + \Psi(\dot{z}).$$

Hence we will consider only the case of non vanishing coupling parameters $\beta \neq 0$ or $\partial_z H_2 \not\equiv 0$. By using more detailed estimates for the mapping $\widetilde{\theta} \mapsto (u,z)$, we can obtain more precise estimates for $f^{\widetilde{\theta}}$ which will allow us to prove that the mapping $\Phi_T^{\widetilde{\theta},\theta}$ possesses a fixed point in $\mathrm{C}^0([0,T];\mathrm{L}^4(\Omega))$.

Let us begin with the case $\alpha = 0$. Then we have

Lemma 4.3 *([PaP11b, Thm. 4.1]). Let $\tau \in (0,T]$. Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7) *hold. Let $\widetilde{\theta} \in \mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))$, with $q > 8$ and $p \in [4,6]$, $u^0 \in \mathrm{W}^{1,p}_{\mathrm{Dir}}(\Omega)$ and $z^0 \in \mathrm{L}^p(\Omega)$ be given and denote by (u,z) the unique solution of* (2.12)–(2.14)*. Then, there exists a non decreasing positive mapping $\tau \mapsto C_{u,z}(\tau)$, independent of the initial data, such that*

$$\begin{aligned}&\|u\|_{\mathrm{C}^0([0,\tau];\mathrm{W}^{1,p}(\Omega))} + \|z\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^p(\Omega))} + \|\dot{u}\|_{\mathrm{L}^q(0,\tau;\mathrm{W}^{1,p}(\Omega))} + \|\dot{z}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))}\\ &\leq C^q_{u,z}(\tau)\big(\|u^0\|_{\mathrm{W}^{1,p}(\Omega)}+\|z^0\|_{\mathrm{L}^p(\Omega)}+\beta\|\widetilde{\theta}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))}+1\big).\end{aligned}$$

Let us assume from now on that $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$, $z^0 \in \mathrm{L}^4(\Omega)$, $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\mathrm{Neu}}(\Omega)$ and let $\widetilde{\theta} \in \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ with $\tau \in (0,T]$. From Lemma 4.3 we can estimate $f^{\widetilde{\theta}}$ as follows

$$
\begin{aligned}
&\|f^{\widetilde{\theta}}\|_{\mathrm{L}^{q/2}(0,\tau;\mathrm{L}^2(\Omega))} \leq \|\mathbf{A}\|\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} + 3\beta\|\widetilde{\theta}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\|\boldsymbol{\varepsilon}(\dot{u})\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\\
&+ \|\mathbf{B}\|\|\dot{z}\|^2_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} + C^\Psi\|\dot{z}\|_{\mathrm{L}^{q/2}(0,\tau;\mathrm{L}^2(\Omega))}\\
&\leq \big(\|\mathbf{A}\|+\|\mathbf{B}\|\big)(C^q_{u,z}(\tau))^2\big(\|u^0\|_{\mathrm{W}^{1,4}(\Omega)}+\|z^0\|_{\mathrm{L}^4(\Omega)}+\beta\|\widetilde{\theta}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}+1\big)^2\\
&+\big(3\beta\|\widetilde{\theta}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}+C^\Psi|\Omega|^{\frac14}\tau^{\frac1q}\big)C^q_{u,z}(\tau)\big(\|u^0\|_{\mathrm{W}^{1,4}(\Omega)}+\|z^0\|_{\mathrm{L}^4(\Omega)}+\beta\|\widetilde{\theta}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}+1\big)\\
&\leq C^q_{f^{\widetilde{\theta}}}\big(\tau,\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)\big(1+\beta^2\|\widetilde{\theta}\|^2_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\big),
\end{aligned}
$$

where $C^q_{f^{\widetilde{\theta}}}\big(\tau,\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)$ is given by

$$
\begin{aligned}
&C^q_{f^{\widetilde{\theta}}}\big(\tau,\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big) \stackrel{\text{def}}{=} \max\big(2\big(\|\mathbf{A}\|+\|\mathbf{B}\|\big)(C^q_{u,z}(\tau))^2+3C^q_{u,z}(\tau)+1,\\
&2\big(\|\mathbf{A}\|+\|\mathbf{B}\|+\tfrac94\big)(C^q_{u,z}(\tau))^2\big(\|u^0\|_{\mathrm{W}^{1,4}(\Omega)}+\|z^0\|_{\mathrm{L}^4(\Omega)}+1\big)^2+\tfrac12\big(C^\Psi|\Omega|^{\frac14}\tau^{\frac1q}C^q_{u,z}(\tau)\big)^2\\
&+C^\Psi|\Omega|^{\frac14}\tau^{\frac1q}C^q_{u,z}(\tau)\big(\|u^0\|_{\mathrm{W}^{1,4}(\Omega)}+\|z^0\|_{\mathrm{L}^4(\Omega)}+1\big)\big)
\end{aligned}
$$

for any $q > 8$. It follows that $\theta = \Phi^{\widetilde{\theta},\theta}_\tau(\widetilde{\theta})$ can be estimated as

$$
\begin{aligned}
\|\theta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}(\Omega))} &\leq C_\theta\exp(\tfrac{\tau}{c^c})\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}+\|f^{\widetilde{\theta}}\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))}\big)\\
&\leq C_\theta\exp(\tfrac{\tau}{c^c})\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}+\tau^{\frac{q-4}{2q}}\|f^{\widetilde{\theta}}\|_{\mathrm{L}^{q/2}(0,\tau;\mathrm{L}^2(\Omega))}\big)
\end{aligned}
$$

where C_θ is the constant, independent of τ and of the initial data, introduced in Proposition 3.1 (see (3.4)). Since $\theta \in \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ and $\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$, we obtain

$$
\begin{aligned}
&\|\theta\|_{C^0([0,\tau];\mathrm{L}^4(\Omega))} \leq C_1C_\theta\exp(\tfrac{\tau}{c^c})\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}\\
&+\tau^{\frac{q-4}{2q}}C^q_{f^{\widetilde{\theta}}}\big(\tau,\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)\big(1+\beta^2\|\widetilde{\theta}\|^2_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\big)\\
&\leq C^q\big(\tau,\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)},\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)\big(1+\beta^2\tau^{\frac2q}\|\widetilde{\theta}\|^2_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}\big)
\end{aligned}
$$

where C_1 is the generic constant involved in the continuous embedding of $\mathrm{W}^{1,2}(\Omega)$ into $\mathrm{L}^4(\Omega)$ and

$$
\begin{aligned}
&C^q\big(\tau,\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)},\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)\\
&\stackrel{\text{def}}{=} C_1C_\theta\exp(\tfrac{\tau}{c^c})\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}+\tau^{\frac{q-4}{2q}}C^q_{f^{\widetilde{\theta}}}\big(\tau,\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)\big).
\end{aligned}
$$

We can not expect to get a global existence result without further assumptions on β. This is not very surprising since $f^{\widetilde{\theta}}$ behaves as a quadratic coupling term if $\beta > 0$. But the mapping

$$
\gamma^q : R^\theta \mapsto C^q\big(T,\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)},\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)\big(1+\beta^2T^{\frac2q}(R^\theta)^2\big)-R^\theta
$$

admits a minimum for $R^\theta = R^\theta_{q,\min} \stackrel{\text{def}}{=} \frac{1}{2C^q(T,\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)},\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)})\beta^2T^{\frac2q}}$ and

$$
\gamma^q(R^\theta_{q,\min}) = C^q\big(T,\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)},\|u^0\|_{\mathrm{W}^{1,4}(\Omega)},\|z^0\|_{\mathrm{L}^4(\Omega)}\big)-\frac{R^\theta_{q,\min}}{2}.
$$

Hence $\gamma^q(R^\theta_{q,\min}) < 0$ if $R^\theta_{q,\min} > 2C^q\big(T, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{L}^4(\Omega)}\big)$, i.e.

$$0 < \beta < \frac{1}{2C^q(T, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{L}^4(\Omega)})T^{\frac{1}{q}}}. \tag{4.12}$$

Let us fix now $q > 8$ and assume that this condition on β holds. We choose $R^\theta = R^\theta_{q,\min}$. We may observe that, since β satisfies condition (4.12), we have

$$\begin{aligned} R^\theta_{q,\min} &= \frac{1}{2C^q(T, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{L}^4(\Omega)})\beta^2 T^{\frac{2}{q}}} \\ &> 2C^q\big(T, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{L}^4(\Omega)}\big) \\ &> C_1 C_\theta \exp\big(\tfrac{T}{c^c}\big)\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}. \end{aligned}$$

Thus we can apply the results of Corollary 3.5: there exists $\tau \in (0,T]$ such that $\Phi^{\widetilde{\theta},\theta}_\tau$ possesses a fixed point in $\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$. But the previous estimate implies also that

$$\begin{aligned} &\|\Phi^{\widetilde{\theta},\theta}_\tau(\widetilde{\theta})\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))} = \|\theta\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))} \\ &\le C^q\big(\tau, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{L}^4(\Omega)}\big)\big(1{+}\beta^2\tau^{\frac{2}{q}}\|\widetilde{\theta}\|^2_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}\big) \\ &\le C^q\big(T, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{L}^4(\Omega)}\big)\big(1{+}\beta^2 T^{\frac{2}{q}}(R^\theta_{q,\min})^2\big) \\ &= \gamma^q(R^\theta_{q,\min}) + R^\theta_{q,\min} < R^\theta_{q,\min} \end{aligned}$$

for any $\tau \in (0,T]$ and any $\widetilde{\theta} \in \bar{B}_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}(0, R^\theta_{q,\min})$. Hence we can consider $\tau = T$ and the closed convex bounded set $\mathcal{C} \stackrel{\text{def}}{=} \bar{B}_{\mathrm{C}^0([0,T];\mathrm{L}^4(\Omega))}(0, R^\theta_{q,\min})$. We have $\Phi^{\widetilde{\theta},\theta}_T(\mathcal{C}) \subset \mathcal{C}$, and using Schauder's fixed point theorem, we infer that $\Phi^{\widetilde{\theta},\theta}_T$ admits a fixed point θ in $\mathrm{C}^0([0,T];\mathrm{L}^4(\Omega))$. Then we define (u,z) as the unique solution of (2.12)–(2.14) with $\widetilde{\theta} = \theta$ and $\tau = T$. By definition of $\Phi^{\widetilde{\theta},\theta}_T$, (u,z,θ) is a global solution of the coupled problem (1.6)–(1.8) on $[0,T]$.

Now let us consider the case $\alpha > 0$.

Lemma 4.4 *([PaP11a, Lemma 4.4] and [PaP11c, Lemma 3.4]). Let $\tau \in (0,T]$. Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7) *hold. Let $\widetilde{\theta} \in \mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))$, with $q > 8$ and $p \in [4,6]$, $u^0 \in \mathrm{W}^{1,p}_{\mathrm{Dir}}(\Omega)$ and $z^0 \in \mathrm{W}^{2,p}_{\mathrm{Neu}}(\Omega)$ be given and denote by (u,z) the unique solution of* (2.12)–(2.14)*. Then, there exists a non-decreasing positive mapping $\tau \mapsto C^q_u(\tau)$, independent of the initial data, such that*

$$\begin{aligned} &\|u\|_{\mathrm{C}^0([0,\tau];\mathrm{W}^{1,p}(\Omega))} + \|\dot{u}\|_{\mathrm{L}^q(0,\tau;\mathrm{W}^{1,p}(\Omega))} \\ &\le C^q_u(\tau)\big(\|z\|_{\mathrm{L}^q(0,\tau;\mathrm{W}^{1,2}(\Omega))} {+} \beta\|\widetilde{\theta}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^p(\Omega))} {+} \|u^0\|_{\mathrm{W}^{1,p}(\Omega)} {+} 1\big). \end{aligned}$$

Let $u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$, $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ and $\theta^0 \in \mathrm{W}^{1,2}_{\kappa,\,\mathrm{Neu}}(\Omega)$ and let $\widetilde{\theta} \in \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ with $\tau \in (0,T]$. With similar computations as in Lemma 3.3 and Proposition 4.2, we can obtain

Lemma 4.5 *Let $\tau \in (0,T]$. Assume that* (2.1), (2.2), (2.4), (2.5), (2.6), (2.7), *$u^0 \in \mathrm{W}^{1,4}_{\mathrm{Dir}}(\Omega)$ and $z^0 \in \mathrm{W}^{2,4}_{\mathrm{Neu}}(\Omega)$ hold. Let $\widetilde{\theta} \in \mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))$ be given and denote by (u,z) the unique solution of* (2.12)–(2.14)*. Then*

$$\begin{aligned} &\|u\|^2_{\mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}(\Omega))} + \|z\|^2_{\mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}(\Omega))} \\ &\le C\big(\|u^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|z^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)(X{+}1)\exp(c_0(X{+}1)\tau), \end{aligned}$$

where $X \overset{\text{def}}{=} \big(\beta^2+(C_z^{H_2})^2\big)\|\widetilde{\theta}\|^2_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}$, $c_0 > 0$ *is a constant independent of the initial data and* τ, *and* $C\big(\|u^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|z^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)$ *is a non decreasing positive function of each of its arguments.*

Proof. Let $C^{H_1} > 0$ and define

$$\delta(t) \overset{\text{def}}{=} \tfrac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u)-\mathbf{Q}z):(\boldsymbol{\varepsilon}(u)-\mathbf{Q}z)\,\mathrm{d}x - \tfrac{\alpha}{2}\int_\Omega \Delta z.z\,\mathrm{d}x + \tfrac{C^{H_1}}{2}\int_\Omega |z|^2\,\mathrm{d}x$$

for all $t \in [0,\tau]$. As in Proposition 4.2 we can check that, for any $\lambda \in (0,1)$, we have

$$\tfrac{1}{2}\int_\Omega \mathbf{E}(\boldsymbol{\varepsilon}(u)-\mathbf{Q}z):(\boldsymbol{\varepsilon}(u)-\mathbf{Q}z)\,\mathrm{d}x + \tfrac{C^{H_1}}{2}\int_\Omega |z|^2\,\mathrm{d}x \geq \tfrac{1}{2}(1-\lambda)c^{\mathbf{E}}\|\boldsymbol{\varepsilon}(u)\|^2_{\mathrm{L}^2(\Omega)}$$
$$+ \big(1-\tfrac{1}{\lambda}\big)\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\big(\|\widetilde{\mathbf{Q}}\|^2\|z\|^2_{\mathrm{L}^2(\Omega)}+|\mathrm{Q}|^2|\Omega|\big) + \tfrac{C^{H_1}}{2}\|z\|^2_{\mathrm{L}^2(\Omega)}.$$

Thus we may choose $\lambda \in (0,1)$ such that

$$1 > \lambda > \frac{\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\|\widetilde{\mathbf{Q}}\|^2}{\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\|\widetilde{\mathbf{Q}}\|^2+\frac{C^{H_1}}{2}},$$

and we obtain

$$\delta(t) \geq C_\delta\big(\|u(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)}+\|z(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)}\big) - \widetilde{C}_\delta$$

for all $t \in [0,\tau]$, with

$$C_\delta \overset{\text{def}}{=} \min\big(\tfrac{1}{2}(1-\lambda)c^{\mathbf{E}}C^{\text{Korn}}, \big(1-\tfrac{1}{\lambda}\big)\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\|\widetilde{\mathbf{Q}}\|^2+\tfrac{C^{H_1}}{2}, \tfrac{\alpha}{2}\big) \text{ and } \widetilde{C}_\delta \overset{\text{def}}{=} \big(\tfrac{1}{\lambda}-1\big)|\mathrm{Q}|^2|\Omega|.$$

Moreover δ is absolutely continuous on $[0,\tau]$ and, by similar computations as in Lemma 3.3, we get

$$\dot{\delta}(t)+c^{\mathbf{A}}\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)}+c^{\mathbf{B}}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)} \leq C^{H_1}\int_\Omega z.\dot{z}\,\mathrm{d}x-\beta\int_\Omega \widetilde{\theta}\mathbf{I}:\boldsymbol{\varepsilon}(\dot{u})\,\mathrm{d}x$$
$$-\int_\Omega \partial_z H_1(z).\dot{z}\,\mathrm{d}x-\int_\Omega \widetilde{\theta}\partial_z H_2(z).\dot{z}\,\mathrm{d}x+\int_\Omega f.\dot{u}\,\mathrm{d}x$$

for almost every $t \in [0,\tau]$. We estimate the right hand side of this last inequality by using (2.3), we obtain

$$\dot{\delta}(t) + c^{\mathbf{A}}\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)} + c^{\mathbf{B}}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)} \leq (C^{H_1}+C_z^{H_1})\int_\Omega |z||\dot{z}|\,\mathrm{d}x + C_z^{H_1}\int_\Omega |\dot{z}|\,\mathrm{d}x$$
$$+ 3\beta\int_\Omega |\widetilde{\theta}||\boldsymbol{\varepsilon}(\dot{u})|\,\mathrm{d}x + C_z^{H_2}\int_\Omega |\widetilde{\theta}|(1+|z|)|\dot{z}|\,\mathrm{d}x + \int_\Omega |f||\dot{u}|\,\mathrm{d}x.$$

Then, with Cauchy-Schwarz's inequality

$$\dot{\delta}(t) + \tfrac{c^{\mathbf{A}}}{2}\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{c^{\mathbf{B}}}{4}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)}$$
$$\leq \big(\tfrac{(C^{H_1}+C_z^{H_1})^2}{c^{\mathbf{B}}}+\tfrac{2C_1^2}{c^{\mathbf{B}}}(C_z^{H_2})^2\|\widetilde{\theta}\|^2_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}\big)\|z(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)}$$
$$+ \tfrac{(C_z^{H_1})^2}{c^{\mathbf{B}}}|\Omega| + \big(\tfrac{9\beta^2}{c^{\mathbf{A}}}+\tfrac{2(C_z^{H_2})^2}{c^{\mathbf{B}}}\big)|\Omega|^{\frac{1}{2}}\|\widetilde{\theta}\|^2_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))} + \tfrac{1}{c^{\mathbf{A}}C^{\text{Korn}}}\|f\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))}$$

for almost every $t \in [0,\tau]$, where we recall that C_1 is the generic constant involved in the continuous embedding of $\mathrm{W}^{1,2}(\Omega)$ into $\mathrm{L}^4(\Omega)$. Since

$$\|z(\cdot,t)\|^2_{\mathrm{W}^{1,2}(\Omega)} \leq \tfrac{\delta(t)+\widetilde{C}_\delta}{C_\delta}$$

for all $t \in [0,\tau]$, we may define c_0 and $C\big(\|u^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|z^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)$ by

$$
\begin{aligned}
c_0 &\stackrel{\text{def}}{=} \tfrac{1}{C_\delta} \max\big(\tfrac{(C^{H_1}+C_z^{H_1})^2}{c^{\mathbf{B}}}, \tfrac{2C_1^2}{c^{\mathbf{B}}}\big),\\
C\big(\|u^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|z^0\|_{\mathrm{W}^{1,2}(\Omega)}\big) &\stackrel{\text{def}}{=} \tfrac{\delta(0)+\widetilde{C}_\delta}{C_\delta} + \tfrac{(C_z^{H_1})^2}{c^{\mathbf{B}}C_\delta}|\Omega|T + \big(\tfrac{9}{c^{\mathbf{A}}}+\tfrac{2}{c^{\mathbf{B}}}\big)\tfrac{|\Omega|^{\frac{1}{2}}}{C_\delta}T\\
&\quad + \tfrac{1}{c^{\mathbf{A}}C^{\text{Korn}}C_\delta}T\|f\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))},
\end{aligned}
$$

and the conclusion follows with Grönwall's lemma. □

Now we rewrite (2.12b) as follows

$$
\dot{z} - \alpha\mathbf{B}^{-1}\Delta z = \mathbf{B}^{-1}f^z,
$$

with $f^z \stackrel{\text{def}}{=} \widetilde{\mathbf{Q}}^{\mathsf{T}}\mathbf{E}(\varepsilon(u){-}\mathbf{Q}z) - \partial_z H_1(z) - \widetilde{\theta}\partial_z H_2(z) - \psi$ and $\psi \in \partial\Psi(\dot{z})$. With assumption (2.1c) we infer that $\psi \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^\infty(\Omega))$ with $\|\psi(\cdot,t)\|_{\mathrm{L}^\infty(\Omega)} \le C^\Psi$ almost every $t \in (0,\tau)$. Furthermore, we can estimate f^z as

$$
|f^z| \le \|\widetilde{\mathbf{Q}}\|\|\mathbf{E}\|\big(|\varepsilon(u)|{+}\|\widetilde{\mathbf{Q}}\||z|{+}|\mathrm{Q}|\big){+}(C_z^{H_1}{+}C_z^{H_2}|\widetilde{\theta}|)(1{+}|z|) + C^\Psi.
$$

Thus, using Lemma 4.5, we infer first an estimate of f^z in $\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))$ given by

$$
\|f^z\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))} \le C\Big(C\big(\|u^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|z^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)(X{+}1)\exp(c_0(X{+}1)\tau){+}X{+}1\Big), \tag{4.13}
$$

where C is a constant independent of the initial data and τ. Hence, for any $q > 8$, we have

$$
\|z\|_{\mathrm{L}^q(0,\tau;\mathrm{W}^{2,2}(\Omega))} \le C_z^q(\tau)\big(\|f^z\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))}{+}\|z^0\|_{\mathrm{W}^{2,2}(\Omega)}\big), \tag{4.14}
$$

with a non decreasing positive mapping $\tau \mapsto C_z^q(\tau)$ (see [HiR08, PrS01]). It follows that

$$
\begin{aligned}
&\|f^z\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} \le \|\widetilde{\mathbf{Q}}\|\|\mathbf{E}\|_{\mathrm{L}^\infty(\Omega)}\big(\|\varepsilon(u)\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\\
&{+}C_1\tau^{\frac{1}{q}}\|\widetilde{\mathbf{Q}}\|\|z\|_{\mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}(\Omega))}{+}\tau^{\frac{1}{q}}|\Omega|^{\frac{1}{4}}|\mathrm{Q}|\big)\\
&+ C_z^{H_1}\tau^{\frac{1}{q}}\big(C_1\|z\|_{\mathrm{L}^\infty(0,\tau;\mathrm{W}^{1,2}(\Omega))}{+}|\Omega|^{\frac{1}{4}}\big)\\
&+ C_z^{H_2}\|\widetilde{\theta}\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}\big(\tau^{\frac{1}{q}}{+}C_2\|z\|_{\mathrm{L}^q(0,\tau;\mathrm{W}^{2,2}(\Omega))}\big) + C^\Psi\tau^{\frac{1}{q}}|\Omega|^{\frac{1}{4}},
\end{aligned} \tag{4.15}
$$

where C_1 and C_2 are the two generic constants involved in the continuous embeddings of $\mathrm{W}^{1,2}(\Omega)$ into $\mathrm{L}^4(\Omega)$ and $\mathrm{W}^{2,2}(\Omega)$ into $\mathrm{L}^\infty(\Omega)$, respectively. By combining Lemma 4.4 and Lemma 4.5, we have

$$
\begin{aligned}
&\|u\|_{\mathrm{C}^0([0,\tau];\mathrm{W}^{1,4}(\Omega))} + \|\dot{u}\|_{\mathrm{L}^q(0,\tau;\mathrm{W}^{1,4}(\Omega))}\\
&\le C_u^q(\tau)\Big(\tfrac{\tau^{\frac{2}{q}}}{2}C\big(\|u^0\|_{\mathrm{W}^{1,2}(\Omega)}, \|z^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)(X{+}1)\exp(c_0(X{+}1)\tau){+}\tfrac{\tau^{\frac{2}{q}}}{2}X{+}\|u^0\|_{\mathrm{W}^{1,4}(\Omega)}{+}2\Big),
\end{aligned}
$$

and gathering (4.13), (4.14) and (4.15), we infer that

$$
\|f^z\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} \le C_{f^z}^q\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,2}(\Omega)}\big)(X{+}1)^2\exp(c_0(X{+}1)\tau),
$$

where $C_{f^z}^q\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,2}(\Omega)}\big)$ is a non decreasing positive function of each of its arguments.

Using classical maximal regularity results for parabolic equations ([HiR08, PrS01]), we obtain an analogous estimate for $\|\dot{z}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}$. More precisely, there exists $C_{u,z}^q\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}\big)$, which is a non decreasing positive function of each of its arguments, such that

$$
\|\dot{z}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} \le C_{u,z}^q\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}\big)(X{+}1)^2\exp(c_0(X{+}1)\tau),
$$

and

$$\|\boldsymbol{\varepsilon}(\dot{u})\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} \leq C^q_{u,z}\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}\big)(X{+}1)\exp(c_0(X{+}1)\tau).$$

Finally, we have

$$\begin{aligned}
&\|f^{\widetilde{\theta}}\|_{\mathrm{L}^{q/2}(0,\tau;\mathrm{L}^2(\Omega))} \leq \|\mathbf{A}\|\,\|\boldsymbol{\varepsilon}(\dot{u})\|^2_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} + \|\mathbf{B}\|\,\|\dot{z}\|^2_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\\
&+ \|\widetilde{\theta}\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}\big(3\beta\tau^{\frac{1}{q}}\|\boldsymbol{\varepsilon}(\dot{u})\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\\
&+C^{H_2}_z\big(\tau^{\frac{1}{q}}{+}C_2\|z\|_{\mathrm{L}^q(0,\tau;\mathrm{W}^{2,2}(\Omega))}\big)\|\dot{z}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\big) + C^{\Psi}|\Omega|^{\frac{1}{4}}\tau^{\frac{1}{q}}\|\dot{z}\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))}\\
&\leq C^q_{f^{\widetilde{\theta}}}(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)})(X{+}1)^4\exp(4c_0(X{+}1)\tau),
\end{aligned}$$

where once again $C^q_{f^{\widetilde{\theta}}}\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}\big)$ is a non decreasing positive function of each of its arguments. It follows that

$$\begin{aligned}
&\|\theta\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))} = \|\Phi^{\widetilde{\theta},\theta}_{\tau}(\widetilde{\theta})\|_{\mathrm{C}^0([0,\tau];\mathrm{L}^4(\Omega))}\\
&\leq C_1C_\theta\exp\big(\tfrac{\tau}{2}\big)\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}{+}\|f^{\widetilde{\theta}}\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))}\big) \leq C_1C_\theta\exp\big(\tfrac{\tau}{c^c}\big)\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}\\
&+C^q_{f^{\widetilde{\theta}}}\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)\tau^{\frac{q-4}{2q}}(X{+}1)^4\exp(4c_0(X{+}1)\tau)\big)\\
&\leq C^q\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)(X{+}1)^4\exp(4c_0(X{+}1)\tau),
\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
&C^q\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)\\
&\stackrel{\text{def}}{=} C_1C_\theta\exp\big(\tfrac{\tau}{c^c}\big)\big(\|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)} + \tau^{\frac{q-4}{2q}}C^q_{f^{\widetilde{\theta}}}\big(\tau, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}\big)\big).
\end{aligned}$$

Let us fix now $q > 8$ and define the mapping γ^q by

$$\gamma^q : R^\theta \mapsto g^q((\beta^2{+}(C^{H_2}_z)^2)(R^\theta)^2) - R^\theta,$$

with

$$g^q(X) \stackrel{\text{def}}{=} C^q\big(T, \|u^0\|_{\mathrm{W}^{1,4}(\Omega)}, \|z^0\|_{\mathrm{W}^{2,4}(\Omega)}, \|\theta^0\|_{\mathrm{W}^{1,2}(\Omega)}\big)(X{+}1)^4\exp(4c_0(X{+}1)T),$$

for all $X \geq 0$. Observing that $X \mapsto g^q(X)$ is a continuous function, we can check that for any $R^\theta > g^q(0)$, there exists $\varepsilon_q > 0$ such that $\gamma^q(R^\theta) < 0$ if

$$0 < \beta^2 + (C^{H_2}_z)^2 < \tfrac{\varepsilon_q}{(R^\theta)^2}.$$

Then, assuming that this condition holds, (4.16) shows that $\mathcal{C} \stackrel{\text{def}}{=} \bar{B}_{\mathrm{C}^0([0,T];\mathrm{L}^4(\Omega))}(0, R^\theta)$ is a closed convex bounded subset of $\mathrm{C}^0([0,T];\mathrm{L}^4(\Omega))$ such that $\Phi^{\widetilde{\theta},\theta}_T(\mathcal{C}) \subset \mathcal{C}$. By using once again Schauder's fixed point theorem we may conclude that problem (1.6)–(1.8) admits a global solution (u, z, θ) on $[0, T]$.

5 Examples

In this concluding section, we present two classes of materials which fit our modelization, namely visco-elasto-plastic materials and SMA undergoing thermal expansion.

Indeed, in the both cases, an internal variable z belonging to a finite dimensional real vector space is introduced to describe the inelastic strain due to plasticity or to phase transitions via the relation

$$\varepsilon^{\textsf{inel}} = \mathbf{Q}z$$

where $z \mapsto \mathbf{Q}z$ is an affine mapping. The Helmholtz free energy is given by

$$W(\boldsymbol{\varepsilon}(u), z, \theta) \stackrel{\text{def}}{=} \tfrac{1}{2}\mathbf{E}(\boldsymbol{\varepsilon}(u) - \mathbf{Q}z):(\boldsymbol{\varepsilon}(u) - \mathbf{Q}z) + \tfrac{\alpha}{2}|\nabla z|^2 + H(z,\theta) - c(\theta\ln(\theta) - \theta) + \beta\mathbf{I}:\boldsymbol{\varepsilon}(u),$$

where $H(z,\theta)$ is a hardening functional that may depend on the temperature, $\beta\mathbf{I}$, with $\beta \geq 0$, is the isotropic thermal expansion tensor and $\alpha \geq 0$ is a coefficient that measures non local interaction effects for the internal variable. As usual $\mathbf{E}$ denotes the elasticity tensor, $\boldsymbol{\varepsilon}(u) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ is the infinitesimal strain tensor, and c and κ are the heat capacity and conductivity.

For visco-elasto-plastic models $\mathbf{Q}$ is linear, H does not depend on θ and $\alpha = 0$ while $\mathbf{Q}$ may be linear or affine as well, $\alpha > 0$ and H depends on θ for SMA. Thus, by replacing $H(z,\theta)$ by an affine approximation $H_1(z) + \theta H_2(z)$, we may split $W(\boldsymbol{\varepsilon}(u), z, \theta)$ as

$$W^{\text{mech}}(\boldsymbol{\varepsilon}(u), z) - W^{\theta}(\theta) + \theta W^{\text{coup}}(\boldsymbol{\varepsilon}(u), z)$$

with

$$\begin{aligned} &W^{\text{mech}}(\boldsymbol{\varepsilon}(u), z) \stackrel{\text{def}}{=} \tfrac{1}{2}\mathbf{E}(\boldsymbol{\varepsilon}(u) - \mathbf{Q}z):(\boldsymbol{\varepsilon}(u) - \mathbf{Q}z) + H_1(z) + \tfrac{\alpha}{2}|\nabla z|^2, \\ &W^{\theta}(\theta) \stackrel{\text{def}}{=} c(\theta\ln(\theta) - \theta), \\ &W^{\text{coup}}(\boldsymbol{\varepsilon}(u), z) \stackrel{\text{def}}{=} \beta\mathbf{I}:\boldsymbol{\varepsilon}(u) + H_2(z). \end{aligned}$$

Let us illustrate this general setting with more precise modelizations. In the case of thermo-visco-elasto-plasticity, we can consider the Melan-Prager model corresponding to a linear kinematic hardening, i.e. we have

$$H(z,\theta) \stackrel{\text{def}}{=} H_1(z) = \tfrac{1}{2}\mathbf{L}z.z \quad \text{and} \quad H_2(z) \equiv 0,$$

with a symmetric positive definite tensor $\mathbf{L} \in \mathcal{L}(\mathcal{Z},\mathcal{Z})$, or the Prandtl-Reuss model for which $H(z,\theta) \equiv 0 = H_1(z) = H_2(z)$ (see [Mau92]).

In the case of SMA, we can consider the 3D macroscopic phenomenological model introduced by Souza, Auricchio et al. ([SMZ98, AuP02, AuP04], or so-called mixture models (see [MiT99, Mie00, HaG02, GMH02, MTL02, GHH07]). In the former case, $z \in \mathcal{Z} \stackrel{\text{def}}{=} \mathbb{R}^{3\times 3}_{\mathsf{dev}} = \{z \in \mathbb{R}^{3\times 3}_{\mathsf{sym}} : \mathbf{I}:z = 0\}$ and $\varepsilon^{\textsf{inel}} = \mathbf{Q}z = z$. Moreover the hardening functional is given by

$$H_{\text{SA}}(z,\theta) \stackrel{\text{def}}{=} c_1(\theta)|z| + c_2(\theta)|z|^2 + \chi(z),$$

where χ is the indicator function of the ball $\{z \in \mathbb{R}^{3\times 3}_{\mathsf{dev}} : |z| \leq c_3(\theta)\}$. This coefficient $c_3(\theta)$ corresponds to the maximum modulus of transformation strain that can be obtained by alignment of martensitic variants while $c_1(\theta) > 0$ is an activation threshold for initiation of martensitic phase transformations and $c_2(\theta)$ measures the occurrence of hardening with respect to the internal variable z.

In order to fit our regularity assumptions for the hardening functionals, which were assumed to be of class C^2, we consider the regularization of H_{SA} given by

$$H^{\delta}_{\text{SA}}(z,\theta) \stackrel{\text{def}}{=} c_1(\theta)\sqrt{\delta^2 + |z|^2} + c_2(\theta)|z|^2 + \tfrac{((|z| - c_3(\theta))_+)^4}{\delta(1+|z|^2)},$$

with $0 < \delta \ll 1$, (see also [MiP07] for another regularization of H_{SA}).

In the latter case, i.e. in so called mixture models, $z \in \mathcal{Z} \overset{\text{def}}{=} \mathbb{R}^{N-1}$ where $N \geq 2$ is the total number of phases and $\varepsilon^{\text{inel}} = \mathbf{Q}z$ is the effective transformation strain of the mixture, given by

$$\mathbf{Q}z \overset{\text{def}}{=} \sum_{k=1}^{N-1} z_k \varepsilon_k + \Big(1 - \sum_{k=1}^{N-1} z_k\Big)\varepsilon_N,$$

where ε_k is the transformation strain of the phase k. Then $z_1, \ldots, z_{N-1}$ and $z_N \overset{\text{def}}{=} 1 - \sum_{k=1}^{N-1} z_k$ can be interpreted as phase fractions and

$$H_{\text{mixt}}(z,\theta) = w(z,\theta) + \chi(z)$$

where χ is the indicator function of the set $[0,1]^{N-1}$. Once again we may consider a regularization of H_{mixt} given by

$$H^{\delta}_{\text{mixt}}(z,\theta) = w(z,\theta) + \sum_{k=1}^{N-1} \tfrac{((-z_k)_+)^4 + ((z_k-1)_+)^4}{\delta(1+|z_k|^2)},$$

with $0 < \delta \ll 1$.

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