1 Topologies and notions of convergence

In this document I made a small overview (for myself) about the relations and differences between topologies and notions of convergence. Example 1.9 is an example where the topology is not defined by the convergence of sequences. Example 1.14 illustrates that the topology generated by a notion of convergence of sequences may have a convergent sequences than does not converge that notion of convergence.

Let X be a set, let $\mathcal{P}(X)$ denote the power set of X, i.e., the set containing all subsets of X.

Definition 1.1. $\mathcal{T} \subset \mathcal{P}(X)$ is called a *topology* on X if

- $\emptyset, X \in \mathcal{T}$,
- $A \cap B \in \mathcal{T}$ if $A, B \in \mathcal{T}$,
- if $\mathcal{U} \subset \mathcal{T}$ then $\bigcup \mathcal{U} \in \mathcal{T}$.

1.2. If $\mathcal{F} \subset \mathcal{P}(X)$ is a set for which

- $\emptyset, X \in \mathcal{F}$,
- $A \cup B \in \mathcal{F}$ if $A, B \in \mathcal{F}$,
- if $\mathcal{V} \subset \mathcal{F}$ then $\bigcap \mathcal{V} \in \mathcal{F}$,

then $\mathcal{T} = \{A^c : A \in \mathcal{F}\}$ is a topology on X.

Definition 1.3. A map $C : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *closure operator* if

(a) $C(\emptyset) = \emptyset$,

(b) $A \subset C(A)$ for all $A \in \mathcal{P}(X)$,

(c) $C(C(A)) = C(A)$ for all $A \in \mathcal{P}(X)$,

(d) $C(A \cup B) = C(A) \cup C(B)$ for all $A, B \in \mathcal{P}(X)$.

Theorem 1.4. *[1, Theorem 1.8] Let C be a closure operator. Then $\mathcal{T} = \{C(A)^c : A \in \mathcal{P}(X)\}$ is a topology on X and $C(A)$ is the $\mathcal{T}$-closure of A.*

Proof. Let $\mathcal{F} = \{C(A) : A \in \mathcal{P}(X)\}$. Clearly $\emptyset, X \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$ if $A, B \in \mathcal{F}$. Let $\mathbb{A} \subset \mathcal{P}(X)$. It suffices to show that $\bigcup_{A \in \mathbb{A}} C(A) \in \mathcal{F}$ (see 1.2). Note that (d) of Definition 1.3 implies that $C(B) \subset C(A)$ if $B \subset A$, $A, B \in \mathcal{P}(X)$. Let $B := \bigcup_{A \in \mathbb{A}} C(A)$. Then

$$C(B) \subset C(C(A)) = C(A) \text{ for all } A \in \mathbb{A}, \tag{1}$$

i.e.,

$$C(B) \subset \bigcup_{A \in \mathbb{A}} C(A) = B. \tag{2}$$

Thus $C(B) = B$ and so $B \in \mathcal{F}$.

Let $B \subset X$. Then the closure of B with respect to $\mathcal{T}$, $\overline{B}$, equals the intersection $\bigcap\{F : F \in \mathcal{F}, B \subset F\}$. As $C(B) \in \mathcal{F}$, $\overline{B} \subset C(B)$. As $\overline{B} \in \mathcal{F}$ and $B \subset \overline{B}$, we have $C(B) \subset C(\overline{B}) = \overline{B}$ and so $\overline{B} = C(B)$. □

Definitions 1.5. (a) Let $\mathbb{A}$ be a set and $\succeq$ be a relation on $\mathbb{A}$. $\succeq$ is called a *preorder* if $a \succeq a$ for all $a \in \mathbb{A}$ and if $a \succeq b$ and $b \succeq c$ imply $a \succeq c$ for all $a, b, c \in \mathbb{A}$. $\succeq$ is said to *directed* $\mathbb{A}$ if for all $a, b \in \mathbb{A}$ there exists a $c \in \mathbb{A}$ with $c \succeq a$ and $c \succeq b$. In this case $\mathbb{A}$ equipped with $\succeq$ is said to be directed. If $\mathbb{A}_i$ are directed sets (with direction $\succeq_i$) for all $i \in I$, for some set I, then we equip the product $\prod_{i \in I} \mathbb{A}_i$ with the relation $\succeq$ defined by

$$(a_i)_{i \in I} \succeq (b_i)_{i \in I} \iff a_i \succeq_i b_i \text{ for all } i \in I. \tag{3}$$

For convenience we write $\succeq$ for any direction.

(b) If $\mathbb{A}$ is a directed set and $\mathbb{B} \subset \mathbb{A}$ is such that for all $\alpha \in \mathbb{A}$ there exists a $\beta \in \mathbb{B}$ with $\beta \succeq \alpha$, then $\mathbb{B}$ is called a *cofinal* subset of $\mathbb{A}$.

(c) A *net* in X is a function on a directed set $\mathbb{A}$ into X, $f : \mathbb{A} \to X$, also written as $(f_\alpha)_{\alpha \in \mathbb{A}}$.

(d) A net $(g_\beta)_{\beta \in \mathbb{B}}$ is called *subnet* of a net $(f_\alpha)_{\alpha \in \mathbb{A}}$ if there exists a function $\phi : \mathbb{B} \to \mathbb{A}$ such that

- $g = f \circ \phi$, i.e., $g_\beta = f_{\phi(\beta)}$ for all $\beta \in \mathbb{B}$,
- for all $\alpha \in \mathbb{A}$ there is a $\beta \in \mathbb{B}$ such that $\gamma \succeq \beta$ implies $\phi(\gamma) \succeq \alpha$.

If $(f_\alpha)_{\alpha \in \mathbb{A}}$ is a net and $\mathbb{B} \subset \mathbb{A}$ cofinal, then $(f_\beta)_{\beta \in \mathbb{B}}$ is a subset of $(f_\alpha)_{\alpha \in \mathbb{A}}$.

Definition 1.6. Let $\mathfrak{C}$ be a set consisting of pairs (f, x) with f a net in X and $x \in X$. We say that f $\mathfrak{C}$-converges to x, and write $\mathfrak{C} - \lim_\alpha f_\alpha = x$, to denote that $(f, x) \in \mathfrak{C}$. $\mathfrak{C}$ is called a *convergence class* for X if it satisfies the following conditions

(a) If $\mathbb{A}$ is a directed set then the constant net $(x)_{\alpha \in \mathbb{A}}$ $\mathfrak{C}$-converges to x.

(b) If f is a net that $\mathfrak{C}$-converges to x, then so does every subnet of f.

(c) If f does not $\mathfrak{C}$-converge to x, then there exists a subnet of f such that no subset of it $\mathfrak{C}$-converges to x.

(d) Let $\mathbb{A}$ be a directed set and $\mathbb{B}_\alpha$ be directed for all $\alpha \in \mathbb{A}$. Let $\mathbb{F} = \mathbb{A} \times \prod_{\alpha \in A} B_\alpha$ and for (α, q) in $\mathbb{F}$ let $R(\alpha, q) = (\alpha, q(\alpha))$. If

$$\mathfrak{C} - \lim_{\alpha \in \mathbb{A}} \lim_{\beta \in \mathbb{B}_\alpha} f(\alpha, \beta) = x,$$

then $f \circ R$ $\mathfrak{C}$-converges to x.

Theorem 1.7. *[1, Theorem 2.9] Let $\mathfrak{C}$ be a convergence class for X. For $A \in \mathcal{P}(X)$ let $C(A)$ be the set of $x \in X$ such that there exists a net f in X such that f $\mathfrak{C}$-converges to x. Then C is a closure operator, and $(f, x) \in \mathfrak{C}$ if and only if f converges to x in the topology associated with C (see Theorem 1.4).*

Remark 1.8. Actually, as we will see in the proof, (a), (b) and (d) of Definition 1.6 imply that C is a closure operator. (c) is crucial for the fact that $\mathfrak{C}$-convergence of a net is the same as convergence in the topology associated with C.

Proof of Theorem 1.7. It will be clear that $C(\emptyset) = \emptyset$. By (a) of Definition 1.6 it follows that $A \subset C(A)$ for all $A \in \mathcal{P}(X)$. If $x \in C(A)$, then by definition of C, $x \in C(A \cup B)$ for all $B \in \mathcal{P}(X)$. Therefore $C(A) \cup C(B) \subset C(A \cup B)$. If $(f_\alpha)_{\alpha \in \mathbb{A}}$ is a net in $A \cup B$ that $\mathfrak{C}$-converges to x, then either $\mathbb{A}_A = \{\alpha \in \mathbb{A} : f_\alpha \in A\}$, or $\mathbb{A}_B = \{\alpha \in \mathbb{A} : f_\alpha \in B\}$ is cofinal in $\mathbb{A}$, providing a subset in either A or B that $\mathfrak{C}$-converges by (b) of Definition 1.6. Whence $C(A) \cup C(B) = C(A \cup B)$. It rests us to show that $C(C(A)) = C(A)$ for $A \in \mathcal{P}(X)$. Let $(f_\alpha)_{\alpha \in \mathbb{A}}$ be a net in $C(A)$ that $\mathfrak{C}$-converges to an x in $C(C(A))$. For all $\alpha \in \mathbb{A}$ there is a directed set B_α and a net $(g_{\alpha,\beta})_{\beta \in \mathbb{B}_\alpha}$ that $\mathfrak{C}$-converges to f_α. By condition (d) of Definition 1.6 there exists a net in A that $\mathfrak{C}$-converges to x, whence $x \in C(A)$ and $C(C(A)) = C(A)$.

If $(f_\alpha)_{\alpha \in \mathbb{A}}$ does not converge to x with respect to the topology, then there exists an open neighbourhood U of x and a cofinal $\mathbb{B} \subset \mathbb{A}$ such that $f_\beta \in U^c$ for all $\beta \in \mathbb{B}$. As $C(U^c) = U^c$, $(f_\beta)_{\beta \in \mathbb{B}}$ does not $\mathfrak{C}$-converge to x. By (b) of Definition 1.6 $(f_\alpha)_{\alpha \in \mathbb{A}}$ does not converge to x.

Suppose that a net g converges to x with respect to the topology but which does not $\mathfrak{C}$-converge to x. By (c) of Definition 1.6 there is a subnet of g, $(f_\alpha)_{\alpha \in \mathbb{A}}$ such that no subnet of f $\mathfrak{C}$-converges to x. We construct such a subnet to obtain a contradiction. For $\alpha \in \mathbb{A}$ let

$$M_\alpha = \{\beta \in \mathbb{A} : \beta \succeq \alpha\}, \tag{4}$$

$$A_\alpha = \{f_\beta : \beta \in M_\alpha\}. \tag{5}$$

As $f_\alpha \to x$, x is an element of the closure of A_α, $\overline{A_\alpha}$ for all $\alpha \in \mathbb{A}$. Therefore, for all $\alpha \in \mathbb{A}$, there exists a directed set $\mathbb{B}_\alpha$ and a net $(z_{\alpha,\beta})_{\beta \in \mathbb{B}_\alpha}$ in M_α such that $\mathfrak{C} - \lim_{\beta \in \mathbb{B}_\alpha} f \circ z(\alpha, \beta) = x$. By (d) of Definition 1.6 (in combination with (a)), $f \circ z \circ R$ $\mathfrak{C}$-converges to x. If $\alpha \in \mathbb{A}$ and $\alpha_1 \succeq \alpha$ then $z \circ R(\alpha_1, q) = z(\alpha_1, q(\alpha_1)) \in M_{\alpha_1}$ and is therefore $\succeq \alpha$. Therefore $f \circ z \circ R$ is a subnet of f. $\square$

There exist several notions of convergence of sequences. As one would have a class $\mathfrak{C}$ of pairs (f, x) with f sequences in X and $x \in X$, then (d) of Definition 1.6 can not be satisfied. As (d) is used in the proof of Theorem 1.7 to prove that $C(C(A)) = C(A)$, it may be that $C(C(A))$ is strictly larger than $C(A)$ if (d) fails to hold. This is illustrated in, for example, Rudin [3, Chapter 3 Exercise 9]. This example, but so the following examples, illustrate that topologies might not be determined by the notion of convergence of sequences. Moreover, as we illustrate in Example 1.14 there might be notion of convergence of sequences and a sequence that does not converge in the sense of this notion of convergence, but which does converge in the sense of the topology generated by this notion of convergence.

Example 1.9. Let X be an uncountable set equipped with the topology determined by

$$A \text{ is closed} \iff A \text{ is countable or } A = X. \tag{6}$$

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence that converges to an x in X. Then

$$U := (X \setminus \{x_n : n \in \mathbb{N}\}) \cup \{x\} \tag{7}$$

is an open neighbourhood of y. As there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$, this implies that $x_n = y$ for $n \geq N$. This implies that

$$\{x \in X : x \text{ is a limit of a sequence in } A\} = A \tag{8}$$

for all $A \subset X$. Therefore the topology is not determined by the notion of convergence of sequences.

For the example we want to provide, we introduce the notion of a Banach lattice. ℓ^1 is an example of a Banach lattice, and is the example we consider.

Definition 1.10. An *ordered vector space* is a vector space E over $\mathbb{R}$ equipped with an ordering $\leq$ such that

$$\begin{aligned} &x \leq y \Longrightarrow \lambda x \leq \lambda y \qquad (x, y \in E, \lambda \geq 0), \\ &x \leq y \Longrightarrow x + z \leq y + z \qquad (x, y, z \in E). \end{aligned}$$

A Banach space E equipped with an ordering $\leq$ is called a *Banach lattice*, if it is an ordered vector space, if the supremum of x and y, $x \vee y$ exists for all $x, y \in E$ (the supremum of x and y is the element h such that $h \geq x$ and $h \geq y$, such that for all j with $j \geq x$ and $j \geq y$ it holds that $j \geq h$), and if

$$\|x\| = \| \, |x| \, \| \qquad (x \in E).$$

where $|x| = x \vee (-x)$.

Definition 1.11. Let E be a Banach lattice. A sequence $(x_n)_{n \in \mathbb{N}}$ in E is said to *converge relatively uniformly* to an x in E, written $x_n \xrightarrow{u} x$, if there exists a $a \in E$ and a sequence of skalars $(\varepsilon_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ such that

$$\begin{aligned} &|x_n - x| \leq \varepsilon_n a \qquad (n \in \mathbb{N}), \\ &\varepsilon_n \to 0. \end{aligned}$$

Theorem 1.12. *[4, Theorem 105.15] Let E be a Banach lattice. Let $\mathcal{T}$ be the topology generated by the (relatively) uniform convergence. Then $\mathcal{T}$ is the norm topology.*

Proof. Let $A \subset E$. If A is norm closed and $x_1, x_2, \dots \in A$ and $x_n \xrightarrow{u} x$, then $x_n \to x$ in norm and so $x \in A$, i.e., A is $\mathcal{T}$-closed.

Suppose A is closed with respect to $\mathcal{T}$. Let $x_1, x_2, \dots \in A$ be such that $x_n \to x$ in norm. We prove that $x \in A$. We may as well assume that $x_1, x_2, \dots$ are such that $\sum_{n \in \mathbb{N}} \|x_n - x\| < \infty$.

Let us prove that $x_n \xrightarrow{u} x$ (which is [2, Theorem 3.9]), so that $x \in A$.

Let $(\varepsilon_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ be such that $\varepsilon_n \to 0$ and

$$\sum_{n \in \mathbb{N}} \varepsilon_n^{-1} \|x_n - x\| < \infty.$$

As $\sum_{n\in\mathbb{N}} \|\varepsilon_n^{-1}\ |x_n - x|\ \| = \sum_{n\in\mathbb{N}} \varepsilon_n^{-1}\|x_n - x\| < \infty$, $a = \sum_{n\in\mathbb{N}} \varepsilon_n^{-1}|x_n - x|$ exist in E and

$$|x_n - x| \leq \varepsilon_n a,$$

i.e., $x_n \xrightarrow{u} x$. □

Theorem 1.13. *[4, Theorem 105.15] Let E be a Banach lattice. Let $\mathcal{T}$ be the topology generated by the (relatively) uniform convergence. Then $\mathcal{T}$ is the norm topology.*

Proof. Let $A \subset E$. If A is norm closed and $x_1, x_2, \dots \in A$ and $x_n \xrightarrow{u} x$, then $x_n \to x$ in norm and so $x \in A$, i.e., A is $\mathcal{T}$-closed.

Suppose A is closed with respect to $\mathcal{T}$. Let $x_1, x_2, \dots \in A$ be such that $x_n \to x$ in norm. We prove that $x \in A$. We may as well assume that $x_1, x_2, \dots$ are such that $\sum_{n\in\mathbb{N}} \|x_n - x\| < \infty$. Then $x_n \xrightarrow{u} x$ by [2, Theorem 3.9], whence $x \in A$. □

Example 1.14. Consider the Banach lattice $E = \ell^1$ and the sequence $(x_n)_{n\in\mathbb{N}}$ in E given by

$$x_n = \tfrac{1}{n} e_n, \tag{9}$$

where $e_n = \mathbb{1}_{\{n\}}$. Then $x_n \to 0$ in norm, but not uniformly.

References

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