Integration of functions with values in a Riesz space.

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Introduction

In this thesis one will find definitions of integrals for functions with values in a Riesz space. The idea for this subject started when I began to learn about the Bochner integral (which is an integral for functions with values in a Banach space) and about Riesz spaces. The purpose of this thesis was to find a useful definition of integration for functions with values in a Riesz space that may not be a Banach space.

The first attempt for a definition of an integral for functions with values in a Riesz space came from the Riemann integral. The idea is to approximate a function f by certain functions from above and from below. Quickly it became clear that using simple functions to approximate a function f would lead to fewer integrable functions then the classical Lebesgue integral for $\mathbb{R}$-valued functions (the classical Lebesgue integral, see Conventions and Notations). This is due to the fact that simple functions are bounded, so that a function f is bounded if there exist simple functions s, t with $s \geq f \geq t$. Therefore I wanted to try other functions for the approximation. For this, the concept of a σ-simple function is introduced in paragraph 2.2. There are more σ-simple functions than simple functions because a σ-simple function has countably many values. Approximating a function f with σ-simple functions seems to work better. In this thesis I call the integral that arises this way the R-integral (R due to Riemann (integral) and Riesz (space)). This works better since $\mathbb{R}$-valued functions are integrable in the classical sense if and only if they are R-integrable. In section 5 some examples of R-integrable functions with values in a space of continuous functions will be given.

The second idea for a definition of an integral for functions with values in a Riesz space was also inspired by the Riemann integral. But now a function will be approximated in a sort of uniform way. The integral that arises this way is called the U-integral (because of the uniform approximation and because U-integration only makes sense if the Riesz space is uniformly complete). This integral is introduced in paragraph 2.4. The definition is formulated in terms of the Bochner integral. One will see in Theorem 2.66 that a function is U-integrable if and only if it can be uniformly approximated.

For the rest I looked at the Bochner and Pettis integral. In section 2 the Bochner integral is considered for functions with values in a Banach lattice. And in paragraph 2.5 the strong and weak Pettis integrals are introduced. These are slightly adapted versions of the Pettis integral for Banach spaces (in such way that it is suitable for certain Riesz spaces).

In section 3 I compare the different definitions of integrals. There one will see that some sorts of integrability imply other ones. But on the other hand there are also examples of functions which are integrable in the one sense and not in the other. For example there is a function that is Bochner integrable but not R-integrable, but there also is a R-integrable function that is not Bochner integrable. In this case I proved that if a function is integrable in both senses, then the integrals coincide. I didn't completely manage in this thesis to compare the R-integral with the strong Pettis integral. For this case there also are functions which are integrable in the one sense and not in the other. In case a function is both strongly Pettis integrable and R-integrable, the question whether the integrals agree is still open. In case a function is positive and both strongly Pettis integrable and R-integrable, then the integrals coincide (as will be shown in Theorem 3.19). But because the strongly Pettis integrable functions -in general- do not form a Riesz space (see Example 2.73), it is not clear how this result could be used to compare the integrals for the general case.

In section 4 there will be more results for R-integrable functions. For all integrals I tried to prove statements similar to the ones that are true for the classical integral for $\mathbb{R}$-valued functions. In this section I tried to find a definition of measurability, so that the R-integrable functions would form Riesz ideal in the space of this kind of measurable functions. Here one could still do a lot of work. I will give a definition of measurability, called R-measurability. For this definition of measurability the question whether the space of R-measurable functions is a Riesz space is still open.

Finally I want to thank my supervisor A.C.M. van Rooij for his guidance, which was very useful. Also his criticism and comments on my written documents have made a lot of things in this thesis easier and better to read.

Conventions and Notations

We assume the reader to be familiar with the notions Riesz space, Riesz subspace, Riesz ideal, principal ideal, Archimedean, σ-Dedekind complete etc.. For an introduction to the theory of Riesz spaces, one could look in the books [JR77] and [LZ71].
We also assume the reader to be familiar with the notions σ-algebra, measure, measure space, measurable function, integrable function and integral (of functions $X \to \mathbb{R}$, where $(X, \mathcal{A}, \mu)$ is a measure space). For theory on integration, one could look in the books [Hal50] and [Zaa67].
We give an example of a construction of an integral for $\mathbb{R}$-valued functions, that we will refer to as the *classical integral*:
First, one defines the integral on nonnegative simple functions (Definition 1.19). Then one defines the integral of a nonnegative measurable function $f : X \to \mathbb{R}$ by

$$\int f \, \mathrm{d}\mu := \sup\{\int s \, \mathrm{d}\mu : s \leq f, \; s \text{ is a nonnegative simple function}\}.$$

A measurable function $f : X \to \mathbb{R}$ is then called integrable if $\int f^+ \, \mathrm{d}\mu, \int f^- \, \mathrm{d}\mu < \infty$ and the integral of f is then defined by

$$\int f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu.$$

We will write $\mathcal{L}(X, \mathcal{A}, \mu)$ for the space of integrable functions $X \to \mathbb{R}$.
We also assume the reader to be familiar with the following results in the theory of integration:

- $\mathcal{L}(X, \mathcal{A}, \mu)$ is a Riesz ideal in the Riesz space of measurable functions and $\int \cdot \, \mathrm{d}\mu : \mathcal{L}(X, \mathcal{A}, \mu) \to \mathbb{R}$ is linear and positive.
- $\int |f| \, \mathrm{d}\mu = 0$ implies $f = 0$ μ-a.e. for all measurable functions $f : X \to \mathbb{R}$.
- The Monotone Convergence Theorem and Lebesgue's Dominated Convergence Theorem.

Troughout the whole document, $(X, \mathcal{A}, \mu)$ is a complete measure space and $\mu : \mathcal{A} \to [0, \infty]$ is a positive measure.

We will use the following convention:

$$\infty \cdot 0 = 0 = 0 \cdot \infty.$$

This convention is made so that we can write -for example- $0 = \int 0 \, \mathrm{d}\lambda = 0 \cdot \lambda(\mathbb{R})$, where λ is the Lebesgue measure on $\mathbb{R}$.

With "Let V be a normed vector space", so without mentioning the norm, we mean "Let V be a normed vector space with norm $\| \cdot \| : V \to [0, \infty)$". Thus when the norm is not mentioned, we write $\| \cdot \|$ for the norm.

We use the following notations:

- $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. The letters i, j, n, m and k are usually used for natural numbers.
- $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_0)$ denotes the measure space $\mathbb{N}$ with the σ-algebra $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) and μ_0 is the counting measure.
- ℓ^∞ is the subspace of $\mathbb{R}^\mathbb{N}$ that consists of bounded sequences, equipped with the supremum norm.
- c is the subspace of ℓ^∞ that consists of the converging sequences.
- c_0 is the subspace of c that consists of sequences that converge to zero.
- c_{00} is the subspace of c_0 that consists of sequences with only finitely many nonzero values.
- ℓ^1, is the subspace of ℓ^∞ that consists of the absolute summable sequences.
- e_n will denote the function $\mathbf{1}_{\{n\}}$ for all $n \in \mathbb{N}$. This will be a commonly used notation in sequence spaces.
- Let E be a Riesz space. We use the following notations for a net $(a_\iota)_{\iota\in I}$ in E:

  $$a_\iota \downarrow_\iota \text{ for } \iota \geq \iota' \Rightarrow_\iota a_\iota \leq a_{\iota'}, \qquad a_\iota \downarrow_\iota a \text{ for } a_\iota \downarrow_\iota \text{ and } \inf_{\iota\in I} a_\iota = a,$$
  $$a_\iota \uparrow_\iota \text{ for } \iota \geq \iota' \Rightarrow_\iota a_\iota \geq a_{\iota'}, \qquad a_\iota \uparrow_\iota a \text{ for } a_\iota \uparrow_\iota \text{ and } \sup_{\iota\in I} a_\iota = a.$$

  Let X is a set and $f, g : X \to E$. We use the following notations

$f \leq g$	for	$f(x) \leq g(x) \quad (x \in X)$,
f^+	for the function	$x \mapsto (f(x))^+$,
f^-	for the function	$x \mapsto (f(x))^-$,
$\lvert f\rvert$	for the function	$x \mapsto \lvert f(x)\rvert$.

  For a net of functions $(f_\iota)_{\iota\in I}$ in E^X we write:

  $$f_\iota \downarrow_\iota \text{ (in } E^X) \text{ for } f_\iota(x) \downarrow_\iota \quad (x \in X), \quad f_\iota \downarrow_\iota f \text{ for } f_\iota(x) \downarrow_\iota f(x) \quad (x \in X),$$
  $$f_\iota \uparrow_\iota \text{ (in } E^X) \text{ for } f_\iota(x) \uparrow_\iota \quad (x \in X), \quad f_\iota \uparrow_\iota f \text{ for } f_\iota(x) \uparrow_\iota f(x) \quad (x \in X).$$

  We will sometimes write $\downarrow$ or $\uparrow$ instead of $\downarrow_\iota$ or $\uparrow_\iota$ (for example in case there is only one index).
- Let E be a Riesz space. We say that a statement about functions $X \to E$ (like $f \geq g$) holds *μ-almost everywhere* (abbreviated by *μ-a.e.*), if the set for which the statement doesn't hold, has measure zero. For instance, $f \leq g$ μ-a.e. means that $\mu(\{x \in X : f(x) \not\leq g(x)\}) = 0$.
- For $p \geq 1$, $\mathcal{L}^p(X, \mathcal{A}, \mu)$ is the vector space of all measurable functions $f : X \to \mathbb{R}$ for which $|f|^p$ is integrable. $\|\cdot\|_{\mathcal{L}^p}$ is the semi-norm on $\mathcal{L}^p(X, \mathcal{A}, \mu)$ given by

  $$\|f\|_{\mathcal{L}^p} = \left(\int |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \qquad (f \in \mathcal{L}^p(X, \mathcal{A}, \mu)).$$

  Details can be found in chapter 7 of [PR].

- For $p \geq 1$, $L^p(X, \mathcal{A}, \mu)$ is the Banach lattice $\mathcal{L}^p(X, \mathcal{A}, \mu)/\mathcal{N}^p$, where $\mathcal{N}^p = \{f \in \mathcal{L}^p(X, \mathcal{A}, \mu) : \|f\|_{\mathcal{L}^p} = 0\} = \{f \in \mathcal{L}^p(X, \mathcal{A}, \mu) : f = 0 \ \mu\text{-a.e.}\}$. $\|\cdot\|_{L^p}$ is the norm on $L^p(X, \mathcal{A}, \mu)$ given by $\|f + \mathcal{N}^p\|_{L^p} = \|f\|_{\mathcal{L}^p}$ for $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$. Details can be found in chapter 7 of [PR].

- We define the integral of $f + \mathcal{N}^1$ for some $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ by $\int f + \mathcal{N}^1 \, \mathrm{d}\mu := \int f \, \mathrm{d}\mu$ (this can be done since $f = g$ μ-a.e. in $\mathcal{L}^1(X, \mathcal{A}, \mu)$ implies $\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu$).

- $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$ is the vector space of all measurable functions $f : X \to \mathbb{R}$ for which there exists an $M \in [0, \infty)$ such that $|f(x)| \leq M$ for μ-almost all $x \in X$. $\|\cdot\|_{\mathcal{L}^\infty}$ is the semi-norm on $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$ given by

$$\|f\|_{\mathcal{L}^\infty} = \inf\{M > 0 : |f(x)| \leq M \text{ for } \mu\text{-almost all } x \in X\},$$

  for $f \in \mathcal{L}^\infty(X, \mathcal{A}, \mu)$. Details can be found in chapter 7 of [PR].

- $L^\infty(X, \mathcal{A}, \mu)$ is the Banach lattice $\mathcal{L}^\infty(X, \mathcal{A}, \mu)/\mathcal{N}^\infty$, where $\mathcal{N}^\infty = \{f \in \mathcal{L}^\infty(X, \mathcal{A}, \mu) : \|f\|_{\mathcal{L}^\infty} = 0\} = \{f \in \mathcal{L}^\infty : f = 0 \ \mu\text{-a.e.}\}$. $\|\cdot\|_{L^\infty}$ is the norm given by $\|f + \mathcal{N}^\infty\|_{L^\infty} = \|f\|_{\mathcal{L}^\infty}$ for $f \in \mathcal{L}^\infty(X, \mathcal{A}, \mu)$. Details can be found in chapter 7 of [PR].

- If $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$ for some $p \in [1, \infty]$, then for the element $f + \mathcal{N}^p$ of $L^p(X, \mathcal{A}, \mu)$ we will also write f.

- For Lebesgue measurable subsets X of $\mathbb{R}$ (for example $[0, 1]$ and $\mathbb{R}$) we will use the notation $(X, \mathcal{B}, \lambda)$ for the space X with σ-algebra $\mathcal{B}$ consisting of the Lebesgue measurable subsets of X and λ the Lebesgue measure.

1 Preliminaries

1.1 The Riesz dual of a Riesz space

1.1 Definition. Let E be a vector space (over $\mathbb{R}$). Then $E^{\#}$ is the vector space of all linear functions $E \to \mathbb{R}$.
For a Riesz space E, the *Riesz dual* of E, $E^{\sim}$, is the set of all functions ϕ in $E^{\#}$ with the property:

$$\text{For all } a \in E^{+} \text{ the set } \{\phi(b) : |b| \leq a\} \text{ is bounded.}$$

For $\phi, \psi \in E^{\sim}$ we define

$$\phi \leq \psi \iff \psi - \phi \text{ is increasing.}$$

It is not hard to see that $E^{\sim}$ is an ordered vector space.

1.2 Theorem. *Let E be a Riesz space. $E^{\sim}$ is a Dedekind complete Riesz space.*

Proof. See Theorem 83.4 in Chapter 12 §83 of [Zaa83] □

1.3 Theorem. *Let E be a Riesz space. Let $\phi \in E^{\sim}$. Then $|\phi|(a) = \sup\{\phi(b) : |b| \leq a\}$ for $a \in E^{+}$.*

Proof. See Theorem 83.6 in Chapter 12 §83 of [Zaa83] □

1.4 Example of an Archimedean Riesz space with trivial Riesz dual.
Consider the measure space $([0,1], \mathcal{B}, \lambda)$. For a Lebesgue measurable function $f : [0,1] \to \mathbb{R}$ let $[f]_\lambda$ be the set of all Lebesgue measurable functions that are λ-almost everywhere equal to f, let $M = \{[f]_\lambda : f \text{ is Lebesgue measurable}\}$. It is not hard to see that M is a Riesz space under the ordering given by:

$$[f]_\lambda \leq [g]_\lambda \iff f \leq g \quad \lambda\text{-a.e..}$$

M is Archimedean:
Suppose that $a, b \in M^{+}$ are such that $na \leq b$ for all $n \in \mathbb{N}$. So for all $f \in a, g \in b$ we have $nf \leq g \quad \lambda$-a.e. so $\lambda(\{x \in [0,1] : nf(x) > g(x)\}) = \lambda(\{x \in [0,1] : f(x) > \frac{1}{n}g(x)\}) = 0$. But then $\lambda(\{x \in [0,1] : f(x) > 0\}) = \lambda(\bigcup_{n\in\mathbb{N}}\{x \in [0,1] : f(x) > \frac{1}{n}g(x)\}) \leq \sum_{n\in\mathbb{N}} \lambda(\{x \in [0,1] : f(x) > \frac{1}{n}g(x)\}) = 0$, i.e. $\lambda(\{x \in [0,1] : f(x) > 0\}) = 0$ and $f = 0 \quad \lambda$-a.e. and thus $a = 0$.
$M^{\sim} = \{0\}$:
Suppose $M^{\sim} \neq \{0\}$. Then there are $\varphi \in M^{\sim}$ and a Lebesgue measurable function f with $\varphi([f]_\lambda) \neq 0$. Because $\varphi = \varphi^{+} - \varphi^{-}$ and $f = f^{+} - f^{-}$, we may assume $\varphi \geq 0$ and $f \geq 0$. Then $\varphi([f]_\lambda) = \varphi([f\mathbf{1}_{[0,\frac{1}{2}]}]_\lambda) + \varphi([f\mathbf{1}_{[\frac{1}{2},1]}]_\lambda) > 0$, so $\varphi([f\mathbf{1}_{[0,\frac{1}{2}]}]_\lambda) > 0$ or $\varphi([f\mathbf{1}_{[\frac{1}{2},1]}]_\lambda) > 0$. Let $I_1 \in \{[0,\frac{1}{2}], [\frac{1}{2},1]\}$ be such that $\varphi([f\mathbf{1}_{I_1}]_\lambda) > 0$. Inductively one proves there exists a sequence $I_1 \supset I_2 \supset \cdots$ where I_n is a closed interval of length 2^{-n} for which $\varphi([f\mathbf{1}_{I_n}]_\lambda) > 0$. Let $a \in [0,1]$ be such that $\{a\} = \bigcap_{n\in\mathbb{N}} I_n$. Let $J_n := I_n \setminus \{a\}$. Then $J_1 \supset J_2 \supset \cdots$, $\bigcap_{n\in\mathbb{N}} J_n = \emptyset$ and J_n is a Lebesgue measurable set for which $\varphi([f\mathbf{1}_{J_n}]_\lambda) > 0$ for all $n \in \mathbb{N}$. Then for each sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $[0,\infty)$, the sum $\sum_{n\in\mathbb{N}} \alpha_n f(x)\mathbf{1}_{J_n}(x)$ exists for all $x \in [0,1]$. So $\sum_{n\in\mathbb{N}} \alpha_n f\mathbf{1}_{J_n}$ is a measurable function on $[0,1]$ (because it is the pointwise limit of the sequence

$(\sum_{n=1}^{N} \alpha_n g \mathbf{1}_{J_n})_{N\in\mathbb{N}})$. Thus $\sum_{n\in\mathbb{N}} \alpha_n [f\mathbf{1}_{J_n}]_\lambda \in M$. Because $\sum_{n\in\mathbb{N}} \alpha_n f\mathbf{1}_{J_n} \geq \alpha_k f\mathbf{1}_{J_k}$ for all $k \in \mathbb{N}$ and because φ is increasing we have

$$\varphi(\sum_{n\in\mathbb{N}} \alpha_n [f\mathbf{1}_{J_n}]_\lambda) \geq \alpha_k \varphi([f\mathbf{1}_{J_k}]_\lambda) \quad (k \in \mathbb{N}).$$

By letting $\alpha_n = \frac{n}{\varphi([f\mathbf{1}_{J_n}]_\lambda)}$ we get $\varphi(\sum_{n\in\mathbb{N}} \alpha_n [f\mathbf{1}_{J_n}]_\lambda) \geq k$ for all $k \in \mathbb{N}$, which leads to a contradiction.

1.5 Definition. A Riesz space E is called a *normed Riesz space* if there is a norm $\|\cdot\|$ for which the following holds:

$$\|a\| = \||a|\|, \quad 0 \leq a \leq b \Rightarrow \|a\| \leq \|b\| \qquad (a, b \in E).$$

A norm for which this holds is called a *Riesz norm* .
We call a normed Riesz space E a *Banach lattice* if E is complete with respect to the metric defined by the norm.

1.6 Lemma. *Let E be a normed Riesz space. If $\|a_n - a\| \to 0$ and $a_n \geq b$ for all $n \in \mathbb{N}$, then $a \geq b$.*

Proof. See Lemma 83.11 in Chapter 12 §83 of [Zaa83] □

1.7 Lemma. *Let E be a normed Riesz space. Suppose $a_n \uparrow$ and $\|a_n - a\| \to 0$. Then $a_n \uparrow a$.*

Proof. By Lemma 1.6 it is clear that $a \geq a_n$ for all $n \in \mathbb{N}$, because a is the limit of the sequence $(a_i)_{i=n}^{\infty}$ for all $n \in \mathbb{N}$. Suppose $h \geq a_n$ for all $n \in \mathbb{N}$, so $h - a_n \geq 0$ for all $n \in \mathbb{N}$. Then with Lemma 1.6 we have $h - a \geq 0$, i.e. $h \geq a$. Thus $a = \sup_{n\in\mathbb{N}} a_n$. □

1.8 Theorem. *Let E be a normed Riesz space. Then E' is a Riesz ideal in $E^\sim$. E' with the operator norm is a normed Riesz space. Moreover if E is norm complete, i.e. E is a Banach lattice, then $E' = E^\sim$.*

Proof. See Theorem 85.6 in Chapter 12 §85 of [Zaa83]. □

1.9 Definition. Let E be a Riesz space. We call a function $\phi \in E^\sim$ *order continuous* if $u_\iota \downarrow 0$ implies $|\phi|(u_\iota) \downarrow 0$ for all nets $(u_\iota)_{\iota\in I}$ in E. In the same way a function $\phi \in E^\sim$ is called *σ-order continuous* if $u_n \downarrow 0$ implies $|\phi(u_n)| \downarrow 0$ for all sequences $(u_n)_{n\in\mathbb{N}}$ in E. The following notations are used:

$$E_c^\sim := \{\phi \in E^\sim : \phi \text{ is } \sigma\text{-order continuous }\},$$
$$E_n^\sim := \{\phi \in E^\sim : \phi \text{ is order continuous }\}.$$

1.10 Comment. Let E be Riesz space. Notice that $E_c^\sim$ and $E_n^\sim$ are both Riesz ideals of $E^\sim$. Let $\phi \in E_c^{\sim+}$. Then $u_n \downarrow 0$ implies $\phi(u_n) \downarrow 0$ $\big((u_n)_{n\in\mathbb{N}} \subset E\big)$. Suppose $u_n \downarrow u$; then of course also $\phi(u_n) \downarrow \phi(u)$ by linearity of ϕ. In the same way $u_n \uparrow u$ implies $\phi(u_n) \uparrow \phi(u)$.

1.11 Example: The (classical) integral is σ-order continuous.
Let E be the Riesz space $\mathcal{L}(X, \mathcal{A}, \mu)$. The map $\int \cdot \, \mathrm{d}\mu : \mathcal{L}(X, \mathcal{A}, \mu) \to \mathbb{R}$ is positive and therefore an element of $E^{\sim}$. We show that $\int \cdot \, \mathrm{d}\mu$ is order continuous. Suppose $f_n \downarrow 0$ in $\mathcal{L}(X, \mathcal{A}, \mu)$. Then for all $x \in X$ and $\lambda > 0$, there exists a $n \in \mathbb{N}$ such that $f_n \not\geq \lambda \mathbf{1}_{\{x\}}$, i.e. $f_n(x) < \lambda$. This means that $f_n(x) \downarrow 0$ for all $x \in X$. Then by Lebesgue's Dominated Convergence Theorem we have $\int f_n \, \mathrm{d}\mu \downarrow 0$. Note that even if $f_n(x) \downarrow 0$ for μ-almost all $x \in X$, then also $\int f_n \, \mathrm{d}\mu \downarrow 0$.

1.12 Definition. Let $\|\cdot\|$ be a Riesz norm for a normed Riesz space E. $\|\cdot\|$ is called an *order continuous norm* if $u_\iota \downarrow 0$ implies $\|u_\iota\| \downarrow 0$ for all nets $(u_\iota)_{\iota \in I}$ in E.
$\|\cdot\|$ is called a *σ-order continuous norm* if $u_n \downarrow 0$ implies $\|u_n\| \downarrow 0$ for all sequences $(u_n)_{n \in \mathbb{N}}$ in E.

1.13 Theorem. *The norm $\|\cdot\|_{L^p}$ for the Banach lattice $L^p(X, \mathcal{A}, \mu)$ is σ-order continuous.*

Proof. Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence in $L^p(X, \mathcal{A}, \mu)$ such that $f_n \downarrow 0$. This means (in $\mathcal{L}^p(X, \mathcal{A}, \mu)$) $f_n(x) \downarrow 0$ for μ-almost all $x \in X$. But then also $(f_n(x))^p \downarrow 0$ for μ-almost all $x \in X$. By Example 1.11 we have

$$\int |f_n|^p \, \mathrm{d}\mu = \int f_n^p \, \mathrm{d}\mu \downarrow 0.$$

Therefore $\|f_n\|_{L^p} \downarrow 0$. Thus $\|\cdot\|_{L^p}$ is σ-order continuous. ☐

1.14 Example: c_0 is a Banach lattice with σ-order continuous norm.
Suppose $(a_n)_{n \in \mathbb{N}}$ is a sequence in c_0 with $a_n \downarrow 0$. Then $a_n(m) \downarrow_n 0$ for all $n, m \in \mathbb{N}$. Let $\varepsilon > 0$ and $M \in \mathbb{N}$ be such that $|a_1(m)| < \varepsilon$ for all $m \geq M$. Then we have

$$|a_n(m)| < \varepsilon \qquad (n \in \mathbb{N}, m \geq M),$$

because $0 \leq a_n \leq a_1$ for all $n \in \mathbb{N}$.
Since $a_n(m) \downarrow_n 0$ for all $m \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n(m)| < \varepsilon$ for all $m \leq M$. Then for all $n \geq N$ we have

$$\|a_n\|_{c_0} = \sup_{m \in \mathbb{N}} |a_n(m)| \leq \varepsilon.$$

So $\|a_n\|_{c_0} \downarrow 0$ and thus $\|\cdot\|_{c_0}$ is σ-order continuous.

1.15 Theorem. *Let E be a Banach lattice with a σ-order continuous norm $\|\cdot\|$. Then $E^{\sim} = E_c^{\sim}$.*

Proof. Let $T \in E^{\sim}$. We show that T is σ-order continuous. By Theorem 1.8 we have $E' = E^{\sim}$ and thus $T \in E'$. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in E with $u_n \downarrow 0$. Because $\|\cdot\|$ is σ-order continuous, we have $|Tf_n| \leq \|T\| \|f_n\| \downarrow 0$. ☐

1.16 Corollary. $(\ell^1)^{\sim} = (\ell^1)_c^{\sim}$, $c_0^{\sim} = (c_0^{\sim})_c$ *and* $L^p(X, \mathcal{A}, \mu)^{\sim} = L^p(X, \mathcal{A}, \mu)_c^{\sim}$ *for all* $(p \in [1, \infty))$.

1.17 Example; the Riesz duals of c and c_0.
First we introduce a notation:
We will write $\ell^1(\mathbb{N} \cup \{\infty\})$ for the space of all functions $b : \mathbb{N} \cup \{\infty\} \to \mathbb{R}$ for which $\sum\{|b(n)| : n \in \mathbb{N} \cup \{\infty\}\} = \sum_{n\in\mathbb{N}} |b(n)| + |b(\infty)| < \infty$. For elements b of $\ell^1(\mathbb{N} \cup \{\infty\})$ we will write b_n for $b(n)$ for all $n \in \mathbb{N} \cup \{\infty\}$. We consider ℓ^1 to be the subspace of $\ell^1(\mathbb{N} \cup \{\infty\})$ consisting of all $b \in \ell^1(\mathbb{N} \cup \{\infty\})$ for which $b_\infty = 0$.

By 8.16 and 8.19 of [PR] the formula

$$\phi_b(a) = \sum_{n\in\mathbb{N}} a_n b_n + \lim_{n\to\infty} a_n b_\infty \qquad (a \in c, b \in \ell^1(\mathbb{N} \cup \{\infty\}),$$

determines linear isometries both from ℓ^1 onto c_0' and from $\ell^1(\mathbb{N} \cup \{\infty\})$ onto c'. So

$$\begin{aligned} c_0^\sim &= c_0' = \{\phi_b : b \in \ell^1\} \\ c^\sim &= c' = \{\phi_b : b \in \ell^1(\mathbb{N} \cup \{\infty\})\} \end{aligned}$$

Note that $b \geq 0$ implies $\phi_b \geq 0$. If $\phi_b \geq 0$, then $b_n = \phi_b(e_n) \geq 0$ for all $n \in \mathbb{N}$ and $b_\infty = \lim_{N\to\infty} \sum_{n=N}^\infty b_n + b_\infty = \lim_{N\to\infty} \phi_b(\sum_{n=N}^\infty e_n) \geq 0$. Thus

$$\begin{aligned} c_0^{\sim+} &= \{\phi_b : b \in \ell^{1+}\} \\ c^{\sim+} &= \{\phi_b : b \in \ell^1(\mathbb{N} \cup \{\infty\})^+\} \end{aligned}$$

By Corollary 1.16, $\phi_b : c_0 \to \mathbb{R}$ is σ-order continuous for all $b \in \ell^1$. We will show

$$c_c^\sim = \{\phi_b : b \in \ell^1\}.$$

Suppose $(u_m)_{m\in\mathbb{N}}$ is a sequence in c for which we have $u_m \downarrow 0$ (u_m is written for the sequence $(u_m(n))_{n\in\mathbb{N}}$). Let $n \in \mathbb{N}$. Suppose $\lambda \in \mathbb{R}$ is such that $u_m(n) \geq \lambda$ for all $m \in \mathbb{N}$. Then $u_m \geq \lambda e_n$ for all $m \in \mathbb{N}$ and thus $\lambda \leq 0$. Thus

$$u_m \downarrow 0 \Longrightarrow u_m(n) \downarrow 0 \quad (n \in \mathbb{N}).$$

Let $b \in \ell^{1+}$. Then $bu_m = (b(n)u_m(n))_{n\in\mathbb{N}}$ is an element of ℓ^1 for all $m \in \mathbb{N}$ and $bu_m \downarrow 0$ in ℓ^1, therefore $\phi_b(u_m) = \sum_{n\in\mathbb{N}} b(n)u_m(n) \downarrow 0$. Thus ϕ_b is σ-order continuous for all $b \in \ell^1$.
Let $b \in \ell^1(\mathbb{N} \cup \{\infty\})^+$. Let

$$v_m = \sum_{i=m}^{\infty} e_i \qquad (m \in \mathbb{N}).$$

Then $v_m \downarrow 0$, but $v_m(\infty) = 1$ for all $m \in \mathbb{N}$. So $\phi_b(v_m) = \sum_{n\in\mathbb{N}} b(n)v_m(n) + b_\infty \geq b_\infty$ for all $m \in \mathbb{N}$. So ϕ_b is σ-order continuous if and only if $b_\infty = 0$, i.e. if and only if $b \in \ell^1$.

1.18 Example; the σ-order continuous functions in $C[0,1]^\sim$.
Let $\varphi \in C[0,1]^\sim_c$. Suppose $\varphi \neq 0$. So there exists an $f \in C[0,1]^+$ such that $\varphi(f) > 0$. Then also $\varphi(\mathbf{1}) > 0$ (because there exists a $\lambda \in (0,1)$ for which $\lambda f \leq \mathbf{1}$). We assume $\varphi(\mathbf{1}) = 1$.

Let $q \in [0,1]$. For $k \in \mathbb{N}$ let $f_k : [0,1] \to \mathbb{R}$ be the continuous function given by $f_k(x) = (1 - k|x-q|)^+$ (see Figure 1). Then $0 \leq f_k \leq \mathbf{1}$ and $f_k(q) = 1$ for all $k \in \mathbb{N}$. And $f_k \downarrow$ and $f_k(x) \downarrow 0$ for all $x \in [0,1] \setminus \{q\}$. Thus $f_k \downarrow 0$ in $C[0,1]$ and $\varphi(f_k) \downarrow 0$.

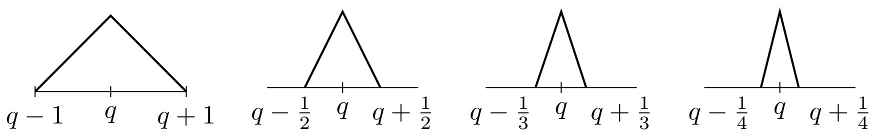

Figure 1: f_1, f_2, f_3, f_4

Let $q_1, q_2, \dots \in [0,1]$ be such that $\{q_1, q_2, \dots\} = \mathbb{Q} \cap [0,1]$. By the above, for all $n \in \mathbb{N}$ there is a $g_n \in C[0,1]$ with $0 \leq g_n \leq \mathbf{1}$, $g_n(q_n) = 1$ and $\varphi(g_n) \leq 3^{-n}$. Let $h_n := \mathbf{1} - (g_1 \vee \dots \vee g_n)$. Then $h_n \downarrow$ in $C[0,1]$ and

$$\varphi(h_n) \geq \varphi(\mathbf{1} - g_1 - \dots - g_n) \geq 1 - 3^{-1} - 3^{-2} - \cdots 3^{-n} > 1 - \sum_{n=1}^{\infty} 3^{-n} = \frac{1}{2}.$$

But $h_n(q_n) \leq (\mathbf{1} - g_n)(q_n) = 0$ for all $n \in \mathbb{N}$. So $h_n(x) \downarrow 0$ for all $x \in \mathbb{Q} \cap [0,1]$ and thus $h_n \downarrow 0$ in $C[0,1]$. This leads to a contradiction, since we assumed φ to be σ-order continuous. We conclude

$$C[0,1]^\sim_c = \{0\}.$$

1.2 Integrals for functions with values in a Banach space

1.19 Definition. Let E be a vector space. A function $s : X \to E$ is called a *step function* if it can be written as

$$s = \sum_{i=1}^{N} a_i \mathbf{1}_{A_i},$$

for some $N \in \mathbb{N}$, $A_i \in \mathcal{A}$ and $a_i \in E$ for $i \in \{1, \dots, N\}$ (note that we may assume that the A_i's are disjoint). Such a step function is called a *simple function* (with the "i" of "integrable") if $\mu(A_i) < \infty$ for all $i \in \{1, \dots, N\}$.

If $\sum_{i=1}^{N} a_i \mathbf{1}_{A_i} = \sum_{j=1}^{M} b_j \mathbf{1}_{B_j}$ for some $N, M \in \mathbb{N}$, $A_i, B_j \in \mathcal{A}$ with $\mu(A_i), \mu(B_j) < \infty$ and $a_i, b_j \in E$ for $i \in \{1, \dots, N\}, j \in \{1, \dots, M\}$, then $\sum_{i=1}^{N} \mu(A_i) a_i = \sum_{j=1}^{M} \mu(B_j) b_j$ (see also 2.13).
We define the integral of a simple function s as above by

$$\int s \, \mathrm{d}\mu = \sum_{i=1}^{N} \mu(A_i) a_i.$$

A linear combination, $\lambda s + t$, of simple functions s, t and $\lambda \in \mathbb{R}$ is a simple function and its integral is the linear combination of the integrals: $\int \lambda s + t \, \mathrm{d}\mu = \lambda \int s \, \mathrm{d}\mu + \int t \, \mathrm{d}\mu$.
If E is a Riesz space and s and t are simple functions, then also $s \wedge t$ and $s \vee t$ are simple functions.

1.2.1 The Bochner integral

Notation 1
In case V is a normed vector space and $f : X \to V$, $\|f\|$ will be written for the function $x \mapsto \|f(x)\|$.

1.20 Definition. Let V be a Banach space. A function $f : X \to V$ is called *strongly measurable* if there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of step functions such that $s_n \to f$ μ-a.e..

1.21 Definition. Let V be a Banach space. A function $f : X \to V$ is called *weakly measurable* if $\phi \circ f : X \to \mathbb{R}$ is (strongly) measurable for all $\phi \in V'$.

1.22 Definition. Let V be a Banach space. A function $f : X \to V$ is called *Borel measurable* if $f^{-1}(U) \in \mathcal{A}$ for all open sets $U \subset V$.

1.23 Definition. Let V be a Banach space. A function $f : X \to V$ is called *μ-essentially separably valued* if there exists a $Y \in \mathcal{A}$ such that $\mu(X \setminus Y) = 0$ and such that $f(Y)$ is separable.

1.24 Theorem. *Let V be a Banach space.*

I. The set of strongly measurable functions $X \to V$ is a vector space,

II. The set of weakly measurable functions $X \to V$ is a vector space,

III. The set of μ-essentially separably valued measurable functions $X \to V$ is a vector space.

Proof. For all the sets in the statement it will be clear that if $f : X \to V$ is an element of the set, then λf is also an element of the set for $\lambda \in \mathbb{R}$. Therefore we only prove that the above sets are closed under addition.
I. Suppose $f, g : X \to V$ are strongly measurable. If $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences of step functions with $s_n \to f$ μ-a.e. and $t_n \to g$ μ-a.e. then $s_n + t_n \to f + g$ μ-a.e..
II. Suppose $f, g : X \to V$ are weakly measurable. Then $\phi \circ (f + g) = \phi \circ f + \phi \circ g$ is (strongly) measurable for all $\phi \in V'$.
III. Let $f, g : X \to V$ be μ-essentially separably valued. Let $Y_1 \subset X$ and $Y_2 \subset X$ be such that $\mu(X \setminus Y_1) = 0 = \mu(X \setminus Y_2)$ and such that $f(Y_1)$ and $g(Y_2)$ are separable. Then $\mu(X \setminus (Y_1 \cap Y_2)) = \mu((X \setminus Y_1) \cup (X \setminus Y_2)) = 0$ and $(f+g)(Y_1 \cap Y_2) = f(Y_1 \cap Y_2) + g(Y_1 \cap Y_2)$ is separable. Thus $f + g$ is μ-essentially separably valued. □

1.25 Theorem. ***(Pettis' Measurability Theorem)***
Let V be a Banach space. Suppose $f : X \to V$ is supported by a set of σ-finite measure, i.e. there exists a σ-finite $Y \in \mathcal{A}$ with $\{x \in X : f(x) \neq 0\} \subset Y$. Then the following are equivalent:

I. f is strongly measurable,

II. f is weakly measurable and μ-essentially separably valued,

III. f is Borel measurable and μ-essentially separably valued.

Proof. See Proposition 2.15 in §2.3 of [Rya10]. □

1.26 Theorem. *Let V be a Banach space. Let $f : X \to V$ be a function. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of strongly measurable functions that converges pointwise to f, i.e. $f_n(x) \to f(x)$ for $x \in X$. Then f is strongly measurable.*

Proof. Let $\phi \in V'$. Then $\phi \circ f_n$ converges pointwise to $\phi \circ f$. So therefore $\phi \circ f$ is (strongly) measurable. By Theorem 1.25, $f_n(X)$ is μ-essentially separably valued for all n. So for all n, there exists a $A_n \in \mathcal{A}$ with $\mu(A_n) = 0$, such that $f_n(X \setminus A_n)$ is separable. Let $A = \bigcup A_n$. Then $\mu(A) = 0$ and $f_n(X \setminus A) \subset f_n(X \setminus A_n)$ is separable for all n. Then $\overline{\bigcup_{n\in\mathbb{N}} f_n(X \setminus A)}$ is separable. Since $f(X \setminus A)$ is a subset of $\overline{\bigcup_{n\in\mathbb{N}} f_n(X \setminus A)}$, we conclude that f is μ-essentially separably valued. With Theorem 1.25 we conclude that f is strongly measurable. □

1.27 Comment. Notice that if f is strongly measurable and s is a simple function, then $f - s$ is strongly measurable and therefore Borel measurable. Because $\|\cdot\| : V \to [0, \infty)$ is continuous, the map $\|f - s\| : X \to [0, \infty)$ is measurable.

1.28 Definition. Let V be a Banach space. A strongly measurable function $f : X \to V$ is called *Bochner integrable* if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions such that $s_n \to f$ μ-a.e. and

$$\lim_{n\to\infty} \int \|f - s_n\| \, \mathrm{d}\mu = 0.$$

1.29 Theorem. ***(Bochner's Theorem)*** *Let V be a Banach space. Then a function $f : X \to V$ is Bochner integrable if and only if f is strongly measurable and $\|f\|$ is integrable.*

Proof. See Proposition 2.16 in §2.3 of [Rya10]. □

1.30 Proposition. *Let V be a Banach space. Suppose that $f : X \to V$ is a Bochner integrable function and $(s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ are sequences of simple functions such that $s_n \to f$ μ-a.e. and $t_n \to f$ μ-a.e. and*

$$\lim_{n\to\infty} \int \|f - s_n\| \, \mathrm{d}\mu = 0, \qquad \lim_{n\to\infty} \int \|f - t_n\| \, \mathrm{d}\mu = 0.$$

Then both $(\int s_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ and $(\int t_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ converge and

$$\lim_{n\to\infty} \int s_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int t_n \, \mathrm{d}\mu.$$

Proof. Let $(r_n)_{n\in\mathbb{N}}$ be the sequence given by $r_{2n} = s_n$, $r_{2n+1} = t_n$. Then $\|f - r_n\|$ is integrable and $\lim_{n\to\infty} \int \|f - r_n\| \, \mathrm{d}\mu = 0$.
For $m, n \in \mathbb{N}$ we have

$$\left\| \int r_m \, \mathrm{d}\mu - \int r_n \, \mathrm{d}\mu \right\| \leq \int \|r_m - r_n\| \, \mathrm{d}\mu \leq \int \|f - r_m\| \, \mathrm{d}\mu + \int \|f - r_n\| \, \mathrm{d}\mu.$$

Therefore $\left(\int r_n \, \mathrm{d}\mu \right)_{n\in\mathbb{N}}$ is a Cauchy sequence in V. Hence this sequence and thus also $(\int s_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ and $(\int t_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ converge and have the same limit. □

As the name "Bochner integrable" suggests, there is an integral for Bochner integrable functions. Due to Proposition 1.30 we can define an integral:

1.31 Definition. Let V be a Banach space. Let $f : X \to V$ be Bochner integrable. We define the *Bochner integral* of f by:

$$\int f \, \mathrm{d}\mu := \lim_{n\to\infty} \int s_n \, \mathrm{d}\mu,$$

where $(s_n)_{n\in\mathbb{N}}$ is a sequence of simple functions such that $s_n \to f$ μ-a.e. and $\int \|f - s_n\| \, \mathrm{d}\mu \to 0$.

1.32 Proposition. *Let V be a Banach space. Let $f, g : X \to V$ be Bochner integrable functions.*

I. $\lambda f + g$ is Bochner integrable for all $\lambda \in \mathbb{R}$ and $\int \lambda f + g \, \mathrm{d}\mu = \lambda \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$.

II. $\|f\|$ is integrable and $\left\| \int f \, \mathrm{d}\mu \right\| \leq \int \|f\| \, \mathrm{d}\mu$.

III. Let W be a Banach space. Suppose $T : V \to W$ is continuous and linear. Then $T \circ f$ is Bochner integrable and

$$T\Big(\int f \ \mathrm{d}\mu\Big) = \int T \circ f \ \mathrm{d}\mu.$$

In particular, for $W = \mathbb{R}$ we have $\phi(\int f \ \mathrm{d}\mu) = \int \phi \circ f \ \mathrm{d}\mu$ for all $\phi \in V'$.

IV. Let $V = \mathbb{R}$. Then a function $f : X \to \mathbb{R}$ is Bochner integrable if and only if it is integrable in the classical sense and the Bochner integral f coincides with the classical integral of f.

Proof. See Appendix B §6 of [DE09] □

Notation 2
We use the following notations:

$$\begin{aligned} \mathcal{L}_B(X, \mathcal{A}, \mu, V) &= \{f : X \to V : f \text{ is Bochner integrable}\}, \\ \mathcal{N}(X, \mathcal{A}, \mu, V) &= \{f : X \to V : f = 0 \ \mu\text{-a.e.}\}, \\ L_B(X, \mathcal{A}, \mu, V) &= \mathcal{L}_B(X, \mathcal{A}, \mu, V)/\mathcal{N}(X, \mathcal{A}, \mu, V). \end{aligned}$$

1.33 Comment. The function $\|\cdot\|_{\mathcal{L}_B(X,\mathcal{A},\mu,V)} : \mathcal{L}_B(X, \mathcal{A}, \mu, V) \to [0, \infty)$ given by

$$\|f\|_{\mathcal{L}_B(X,\mathcal{A},\mu,V)} = \int \|f\| \ \mathrm{d}\mu \qquad (f \in \mathcal{L}_B(X, \mathcal{A}, \mu, V)),$$

is a seminorm on $\mathcal{L}_B(X, \mathcal{A}, \mu, V)$. Notice that $f \in \mathcal{N}(X, \mathcal{A}, \mu, V) \iff \|f\| = 0$ μ-a.e. $\iff \|f\|_{\mathcal{L}_B(X,\mathcal{A},\mu,V)} = 0$.
The space $L_B(X, \mathcal{A}, \mu, V)$ then carries a norm, given by

$$\|f + \mathcal{N}(X, \mathcal{A}, \mu, V)\|_{L_B(X,\mathcal{A},\mu,V)} := \|f\|_{\mathcal{L}_B(X,\mathcal{A},\mu,V)} \qquad (f \in \mathcal{L}_B(X, \mathcal{A}, \mu, V).$$

If $f, g \in \mathcal{L}_B(X, \mathcal{A}, \mu, V)$ are such that $f = g$ μ-a.e., then $\int \|f - s_n\| \ \mathrm{d}\mu \to 0$ implies $\int \|g - s_n\| \ \mathrm{d}\mu \to 0$ and thus $\int f \ \mathrm{d}\mu = \int g \ \mathrm{d}\mu$. Thus we can and do define the Bochner integral of an element $f + \mathcal{N}(X, \mathcal{A}, \mu, V)$ in $L_B(X, \mathcal{A}, \mu, V)$ by

$$\int f + \mathcal{N}(X, \mathcal{A}, \mu, V) \ \mathrm{d}\mu := \int f \ \mathrm{d}\mu.$$

Notation 3
As is done for the $\mathbb{R}$-valued function mentioned in Conventions and Notations: If $f \in \mathcal{L}_B(X, \mathcal{A}, \mu, V)$, then for the element $f + \mathcal{N}(X, \mathcal{A}, \mu, V)$ of $L_B(X, \mathcal{A}, \mu, V)$ we will also write f.

1.34 Theorem. *Let V be a Banach space. Then*

$$(L_B(X, \mathcal{A}, \mu, V), \ \ \|\cdot\|_{L_B(X,\mathcal{A},\mu,V)})$$

is a Banach space.

Proof. See Theorem 3 in Chapter 6 §31 of [Zaa67] □

1.35 Theorem. *Let V be a Banach space. Suppose $f_n : X \to V$ is Bochner integrable for all $n \in \mathbb{N}$ and $\|f_n(x)\| \le g(x)$, $(x \in X, n \in \mathbb{N})$ for some $g \in \mathcal{L}(X, \mathcal{A}, \mu)$. And suppose that $f_n(x) \to f(x)$ μ-a.e.. Then f is Bochner integrable and $\lim_{n\to\infty} \int \|f_n - f\| \ \mathrm{d}\mu = 0$, and thus $\int f \ \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \ \mathrm{d}\mu$.*

Proof. See Theorem 4 in Chapter 6 §31 of [Zaa67] □

1.2.2 The Pettis integral

1.36 Definition. Let A and B be sets. We say that a subset F of B^A *separates the points* of A, if for $a_1, a_2 \in A$ with $a_1 \neq a_2$ there is an $f \in F$ such that $f(a_1) \neq f(a_2)$.

1.37 Lemma. *Let V be a Banach space. V' separates the points of V, i.e. for $u, v \in V$ with $u \neq v$ there exists a $\phi \in V'$ with $\phi(u) \neq \phi(v)$.*

Proof. It is sufficient to prove that for $v \neq 0$ there is an $\phi \in V'$ such that $\phi(v) \neq 0$. Let $v \in V \setminus \{0\}$. Define $g : \mathbb{R}v \to \mathbb{R}$ by $g(\lambda v) = \lambda \|v\|$. Then we have $|g(\lambda v)| \leq \|\lambda v\|$. By the Hahn-Banach theorem (see III.6.4 of [Con07]) there exists a $\phi \in V'$ such that $\phi|_{\mathbb{R}v} = g$. And thus $\phi(v) = \|v\| \neq 0$. □

1.38 Definition. Let V be a Banach space. A function $f : X \to V$ is called *Pettis integrable* if there is a $v \in V$ such that for all $\phi \in V'$:

$$\phi \circ f \in \mathcal{L}(X, \mathcal{A}, \mu) \qquad \text{and} \qquad \phi(v) = \int \phi \circ f \, \mathrm{d}\mu.$$

There is only one v with this property, because V' separates the points of V (by Lemma 1.37). This v we call the *Pettis integral* of f, denoted by (P)-$\int f \, \mathrm{d}\mu$ (or simply $\int f \, \mathrm{d}\mu$).

1.39 Comment. Note that all simple functions are Pettis integrable and that the Pettis integral of a simple function coincides with the integral of a simple function defined as in 1.19.

1.40 Comment. Comparing the Bochner integral with the Pettis integral. Let V be a Banach space. By Theorem 1.32 it is clear that a Bochner integrable function is Pettis integrable and the Bochner integral coincides with the Pettis integral. The converse doesn't hold:
Consider the measure space $([0,1], \mathcal{B}, \lambda)$. Consider the Banach lattice c_0. Let $f : [0,1] \to c_0$ be the function

$$f = \sum_{n \in \mathbb{N}} \frac{2^n}{n} e_n \mathbf{1}_{[2^{-n}, 2^{-n+1})}.$$

Let $\phi \in c_0' = c_0^\sim$ (by Theorem 1.8). Because $\phi = \phi^+ - \phi^-$, we assume $\phi \geq 0$. Then by the monotone convergence theorem and by continuity of ϕ, we have

$$\begin{aligned}\int \phi \circ f \, \mathrm{d}\mu &= \int \sum_{n \in \mathbb{N}} \frac{2^n}{n} \phi(e_n) \mathbf{1}_{[2^{-n}, 2^{-n+1})} \, \mathrm{d}\mu = \sum_{n \in \mathbb{N}} \frac{2^n}{n} \phi(e_n) \int \mathbf{1}_{[2^{-n}, 2^{-n+1})} \, \mathrm{d}\mu \\ &= \sum_{n \in \mathbb{N}} \frac{2^n}{n} \phi(e_n)(2^{-n+1} - 2^{-n}) = \sum_{n \in \mathbb{N}} \frac{1}{n} \phi(e_n) = \phi(\sum_{n \in \mathbb{N}} \frac{1}{n} e_n).\end{aligned}$$

So we see that f is Pettis integrable with $\int f \, \mathrm{d}\mu = \sum_{n \in \mathbb{N}} \frac{1}{n} e_n$. Because $\|e_n\| = 1$ for all $n \in \mathbb{N}$, we have

$$\|f\| = \sum_{n \in \mathbb{N}} \frac{2^n}{n} \mathbf{1}_{[2^{-n}, 2^{-n+1})}.$$

$\|f\|$ is not integrable since $\sum_{n \in \mathbb{N}} \frac{2^n}{n} \lambda([2^{-n}, 2^{-n+1})) = \sum_{n \in \mathbb{N}} \frac{1}{n} = \infty$. Therefore f is a Pettis integrable function which is not Bochner integrable.

2 Integrals for functions with values in a Riesz space

Before defining integrals for functions with values in a Riesz space, there will already be shown that in general a Monotone Convergence Theorem does not hold. This is done for an integral that is defined on simple functions as in Definition 1.19 (for all integrals in this thesis the integrals of simple functions agree):

2.1 Example. Let X be the direct product group $\{0,1\}^{\mathbb{N}}$. Then there exists a Haar measure μ on X (see Theorem 1.3.4 of [DE09]). Let $\mathcal{A}$ denote the associated σ-algebra. Let E be the Riesz space $C(X)$. We will show that there is a sequence of simple functions $(s_n)_{n\in\mathbb{N}}$ in E^X such that

$$s_n \downarrow 0, \qquad \int s_n \, \mathrm{d}\mu \geq \frac{1}{2}\mathbf{1}_X \qquad (n \in \mathbb{N}).$$

The following notations will be used in this example:
For $a_1, \dots, a_n \in \{0,1\}$:

$$\langle a_1, \dots, a_n \rangle = \{x \in X : x_1 = a_1, \dots, x_n = a_n\},$$
$$\langle a_1, \dots, a_n \rangle^* = \{x \in X : x_{n+1} = a_1, \dots, x_{2n} = a_n\}.$$

A set of the form $\langle a_1, \dots, a_n \rangle$ is called a cylinder. For this cylinder we have $\mu(\langle a_1, \dots, a_n \rangle) = 2^{-n}$. Notice that the set of cylinders forms a basis for the topology of X.
Notice also that $\{\langle a_1, \dots, a_n \rangle^* : a_1, \dots, a_n \in \{0,1\}\}$ is a set of disjoint sets that cover X for all $n \in \mathbb{N}$.
$\star$ For $n \in \mathbb{N}$ define $t_n : X \to E$, by

$$(t_n(x))(y) = \begin{cases} 1 & \text{if } (y_{n+1}, \dots, y_{2n}) = (x_1, \dots, x_n), \\ 0 & \text{otherwise.} \end{cases}$$

Let $a_1, \dots, a_n \in \{0,1\}$. Then $t_n(x) = \mathbf{1}_{\langle a_1, \dots, a_n \rangle^*}$ for all $x \in \langle a_1, \dots, a_n \rangle$. Thus t_n is a simple function with

$$\begin{aligned} \int t_n \, \mathrm{d}\mu &= \sum_{a_1, \dots, a_n \in \{0,1\}} \mu(\langle a_1, \dots, a_n \rangle) \mathbf{1}_{\langle a_1, \dots, a_n \rangle^*} \\ &= 2^{-n} \sum_{a_1, \dots, a_n \in \{0,1\}} \mathbf{1}_{\langle a_1, \dots, a_n \rangle^*} = 2^{-n}\mathbf{1}_X. \end{aligned}$$

$\star$ For $n \geq 2$ let s_n be the simple function given by $s_n = \mathbf{1} - (t_2 \vee \dots \vee t_n)$, where $\mathbf{1}$ represents the function $x \mapsto \mathbf{1}_X \quad (x \in X)$. We have $s_n \downarrow$ and $s_n \geq 0$ for all $n \in \mathbb{N}$. Then $\int s_n \, \mathrm{d}\mu \geq \int \mathbf{1} - (t_2 + \dots + t_n) \, \mathrm{d}\mu = \mathbf{1}_X - (2^{-2}\mathbf{1}_X + \cdots 2^{-n}\mathbf{1}_X) \geq \frac{1}{2}\mathbf{1}_X$.
$\star$ Finally we show $s_n \downarrow 0$, i.e. $s_n(x) \downarrow 0$ (in E) for all $x \in X$. For this it is sufficient to prove that $\{y \in X : (s_n(x))(y) \downarrow 0\}$ is dense in X for all $x \in X$. Let $x \in X$. We will even show that the set $S = \{y \in X : \exists n \ [(s_n(x))(y) = 0]\}$ is dense in X. Let $a_1, \dots, a_n \in \{0,1\}$ for some $n \in \mathbb{N}$. Then $y = (a_1, \dots, a_n, x_1, \dots, x_n, 0, 0, \dots)$ is an element of $\langle a_1, \dots, a_n \rangle$ and also an element of S, because $(t_n(x))(y) = 1$.

2.1 The Bochner integral on Riesz space valued functions

2.2 Comment. Let E be a Banach lattice. Note that the Bochner integral for Bochner integrable functions $f : X \to E$ is an integral for functions with values in a Riesz space.

2.3 Theorem. *Let E be a Banach lattice.* $\int \cdot \, \mathrm{d}\mu : \mathcal{L}_B(X, \mathcal{A}, \mu, E) \to E, f \mapsto \int f \, \mathrm{d}\mu$ *is a positive linear map.*

Proof. Linearity is shown in 1.34. Suppose $f \in \mathcal{L}_B(X, \mathcal{A}, \mu, E)^+$. Because $|f(x) - s_n(x) \vee 0| = |f(x) \vee 0 - s_n(x) \vee 0| \le |f(x) - s_n(x)|$ for all $x \in X$ we have $\int \|f - s_n^+\| \, \mathrm{d}\mu \to 0$ and thus $\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int s_n^+ \, \mathrm{d}\mu \ge 0$. □

2.4 Theorem. *Let E be a Banach lattice.*

I. The set of strongly measurable functions $X \to E$ is a Riesz space,

II. The set of μ-essentially separably valued measurable functions $X \to E$ is a Riesz space.

Proof. By 1.24 we already know that the sets in the statement are (ordered) vector spaces. We prove that $|f|$ is an element of one of the above sets as soon as f is an element of that set.
I. Suppose $f : X \to E$ is strongly measurable. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $s_n \to f$ μ-a.e.. Then $|s_n|$ is a step function for all $n \in \mathbb{N}$ and $|s_n| \to |f|$ μ-a.e..
II. Suppose $f : X \to E$ is μ-essentially separably valued. Let $Y \in \mathcal{A}$ be such that $\mu(X \setminus Y) = 0$ and $f(Y)$ is separable. Let D be a countable dense subset of $f(Y)$. Then $|f|(X) = \{|a| : a \in f(Y)\}$. Let $|D| = \{|d| : d \in D\}$. Let $b \in |f|(Y)$ and $\varepsilon > 0$. Then $b = |a|$ for some $a \in f(Y)$. Let $d \in D$ be such that $\|a - d\| < \varepsilon$. Then $|d| \in |D|$ and $\big\||a| - |d|\big\| \le \|a - d\| < \varepsilon$. Therefore $|D|$ is a countable dense subset of $|f|(Y)$. Thus f is μ-essentially separably valued. □

2.5 Theorem. *Let E be a Banach lattice. Then $(L_B(X, \mathcal{A}, \mu, E), \|\cdot\|_{L_B(X,\mathcal{A},\mu,E)})$ is a Banach lattice which is a Riesz ideal in the Riesz space of strongly measurable functions.*

Proof. Suppose f is Bochner integrable and $(s_n)_{n\in\mathbb{N}}$ is a sequence of simple functions such that $\lim_{n\to\infty} \int \|f - s_n\| \, \mathrm{d}\mu = 0$. $\big\||f| - |s_n|\big\|$ is measurable because $|f| - |s_n|$ is strongly measurable. Because $\big||f(x)| - |s_n(x)|\big| \le |f(x) - s_n(x)| \quad (x \in X, n \in \mathbb{N})$, $\||f| - |s_n|\|$ is integrable for all $n \in \mathbb{N}$ and

$$\int \||f| - |s_n|\| \, \mathrm{d}\mu \le \int \|f - s_n\| \, \mathrm{d}\mu \to 0.$$

Because $|s_n|$ is a simple function for all $n \in \mathbb{N}$, $|f|$ is Bochner integrable. Therefore $\mathcal{L}_B(X, \mathcal{A}, \mu, E)$ is a Riesz space. $L_B(X, \mathcal{A}, \mu, E)$ is a normed Riesz space since

$$\||f|\|_{L_B(X,\mathcal{A},\mu,E)} = \int \||f|\| \, \mathrm{d}\mu = \int \|f\| \, \mathrm{d}\mu = \|f\|_{L_B(X,\mathcal{A},\mu,E)}$$

and $0 \le f \le g$ implies $\|f(x)\| \le \|g(x)\|$ for all $x \in X$, and therefore

$$\|f\|_{L_B(X,\mathcal{A},\mu,E)} = \int \|f\| \, \mathrm{d}\mu \le \int \|g\| \, \mathrm{d}\mu = \|g\|_{L_B(X,\mathcal{A},\mu,E)}$$

With Theorem 1.34 we conclude that $L_B(X, \mathcal{A}, \mu, E)$ is a Banach lattice.
By Theorem 1.29 it follows that $L_B(X, \mathcal{A}, \mu, E)$ is a Riesz ideal in the Riesz space of strongly measurable functions. □

2.6 Proposition. *Let E be a Banach lattice. Suppose that the norm is σ-order continuous. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Bochner integrable functions and $f : X \to E$, with $f_n \uparrow f$ μ-a.e. and $\sup_{n\in\mathbb{N}} \int \|f_n\| \, \mathrm{d}\mu < \infty$. Then f is Bochner integrable and*

$$\sup_{n\in\mathbb{N}} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

Proof. Because $f - f_n \downarrow 0$ μ-a.e., we have $\|f(x) - f_n(x)\| \downarrow 0$ for μ-almost all $x \in X$ (since $\|\cdot\|$ is σ-order continuous). So f_n converges μ-a.e. to f. By 1.26 f is strongly measurable. Because $\|f_n\| \uparrow \|f\|$ and $\sup_{n\in\mathbb{N}} \int \|f_n\| \, \mathrm{d}\mu < \infty$, $\|f\|$ is integrable by the Monotone Convergence Theorem. By Theorem 1.29 we conclude that f is Bochner integrable. Then $\|f - f_n\|$ is integrable for all $n \in \mathbb{N}$, and $\|f(x) - f_n(x)\| \downarrow 0$ for all $x \in X$. Therefore we have $\int \|f - f_n\| \, \mathrm{d}\mu \to 0$. So $\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$. Because $(\int f_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ is an increasing sequence, we thus have $\int f_n \, \mathrm{d}\mu \uparrow \int f \, \mathrm{d}\mu$ (by Lemma 1.7). □

2.2 σ-simple functions

2.7 Definition. Let E be a Riesz space. A function $\rho : X \to E$ for which $\rho(X)$ is countable and $\rho^{-1}(a) \in \mathcal{A}$ for all $a \in E$, is called a *σ-step function*.
Note that the set of σ-step functions is a Riesz space.

Notation 4
If ρ is a σ-step function, $(a_n)_{n\in\mathbb{N}}$ is a sequence in E such that $\rho(X) = \{a_n : n \in \mathbb{N}\}$ or such that $\rho(X) \setminus \{0\} = \{a_n : n \in \mathbb{N}\}$ and $(A_n)_{n\in\mathbb{N}}$ a sequence of disjoint sets in $\mathcal{A}$ such that $\rho(x) = a_n$ for all $x \in A_n$ and all $n \in \mathbb{N}$, then we write

$$\rho = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}.$$

2.8 Definition. Let E be a Riesz space. A (σ-step) function $\rho : X \to E^+$ is called a *positive σ-simple function* if it can be written as

$$\rho = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n},$$

for a sequence $(a_n)_{n\in\mathbb{N}}$ in E^+ and a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$, such that the set $\{\sum_{n=1}^N \mu(A_n)a_n : N \in \mathbb{N}\}$ has a supremum in E (so in particular we have $\mu(A_n) < \infty$ for all n for which $a_n \neq 0$). Note that we may assume that $\bigcup_{n\in\mathbb{N}} A_n = X$.
A function $\sigma : X \to E$ is called a *σ-simple function* if σ^+ and σ^- are positive σ-simple functions.

2.9 Comment. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_0)$. Let E be a Riesz space. Then a function $h : \mathbb{N} \to E$ (so h is a sequence in E) for which $h = (h_n)_{n\in\mathbb{N}}$, is a σ-simple function if and only if the sets $\{\sum_{n=1}^N h_n^+ : N \in \mathbb{N}\}$ and $\{\sum_{n=1}^N h_n^- : N \in \mathbb{N}\}$ have a supremum in E.

2.10 Comment. We will use Lemma 2.12 and Theorem 2.13 to conclude Corollary 2.14. And by Corollary 2.14 we can define an integral for σ-simple functions, as will be done in Definition 2.15.

2.11 Lemma. *Let E be a Riesz space. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be sequences in E and $a, b \in E$. Then we have*

$$a_n \uparrow a, \quad b_n \uparrow b \quad \Longrightarrow \quad a_n + b_n \uparrow a + b.$$

Proof. It is clear that $a_n + b_n \leq a + b$. Suppose $h \in E$ is such that $h \geq a_n + b_n$ for all $n \in \mathbb{N}$, then $h \geq a_m + b_n$ for all $n, m \in \mathbb{N}$. For this reason one has $h \geq a + b$. $\square$

2.12 Lemma. *Let E be a Riesz space. Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be disjoint sequences in $\mathcal{A}$ with $\bigcup_{n\in\mathbb{N}} A_n = X = \bigcup_{n\in\mathbb{N}} B_n$ and let $(c_n)_{n\in\mathbb{N}}$ be a sequence in E^+. Then the sets*

$$\begin{aligned} S :=& \{\sum_{n=1}^{N} \mu(A_n)c_n : N \in \mathbb{N}\}, \\ T :=& \{\sum_{n=1}^{N}\sum_{k=1}^{K} \mu(A_n \cap B_k)c_n : N, K \in \mathbb{N}\}, \\ U :=& \{\sum_{n=1}^{N}\sum_{k=1}^{N} \mu(A_n \cap B_k)c_n : N \in \mathbb{N}\}, \end{aligned}$$

have the same upper bounds in E.

Proof. For all $n, K \in \mathbb{N}$ we have $\mu(A_n) \geq \sum_{k=1}^{K} \mu(A_n \cap B_k)$. Therefore

$$\sum_{n=1}^{N} \mu(A_n)c_n \geq \sum_{n=1}^{N}\sum_{k=1}^{K} \mu(A_n \cap B_k)c_n \qquad (N, K \in \mathbb{N}).$$

So every upper bound of S is an upper bound of T.
For all $n \in \mathbb{N}$ we have $\bigcup_{k\in\mathbb{N}} A_n \cap B_k = A_n$ and thus $\sum_{k\in\mathbb{N}} \mu(A_n \cap B_k) = \mu(A_n)$. Therefore for all $N \in \mathbb{N}$ we have (by Lemma 2.11, for $K \to \infty$):

$$\sum_{n=1}^{N}\sum_{k=1}^{K} \mu(A_n \cap B_k)c_n \uparrow_K \sum_{n=1}^{N} \mu(A_n)c_n.$$

Thus every upper bound of T is an upper bound of S. It will be clear that T and U have the same upper bounds. □

2.13 Theorem. *Let E be a Riesz space. Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be disjoint sequences in $\mathcal{A}$ with $\bigcup_{n\in\mathbb{N}} A_n = X = \bigcup_{n\in\mathbb{N}} B_n$ and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in E^+.*

- *If $\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} \geq \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{B_n}$, then an upper bound of $\{\sum_{n=1}^{N} \mu(A_n)a_n : N \in \mathbb{N}\}$ is an upper bound of $\{\sum_{k=1}^{K} \mu(B_k)b_k : K \in \mathbb{N}\}$.*
- *If $\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} = \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{B_n}$, then $\{\sum_{n=1}^{N} \mu(A_n)a_n : N \in \mathbb{N}\}$ and $\{\sum_{k=1}^{K} \mu(B_k)b_k : K \in \mathbb{N}\}$ have the same upper bounds.*

Proof. In case $\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} \geq \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{B_n}$, then $\mu(A_n \cap B_k)a_n \geq \mu(A_n \cap B_k)b_k$ for all $n, k \in \mathbb{N}$. By Lemma 2.12 an upper bound for $\{\sum_{n=1}^{N} \mu(A_n)a_n : N \in \mathbb{N}\}$ is an upper bound for $\{\sum_{n=1}^{N}\sum_{k=1}^{K} \mu(A_n \cap B_k)a_n : N, K \in \mathbb{N}\}$ and thus also for $\{\sum_{n=1}^{N}\sum_{k=1}^{K} \mu(A_n \cap B_k)b_k : N, K \in \mathbb{N}\}$ and thus for $\{\sum_{k=1}^{K} \mu(B_k)b_k : K \in \mathbb{N}\}$.
The second statement is a consequence of the first one. □

2.14 Corollary. *Let E be a Riesz space. If σ is a positive σ-simple function, then there exists an $a \in E$ such that the following holds:*
If $\sigma = \sum_{k\in\mathbb{N}} b_k \mathbf{1}_{B_k}$ for some sequence $(b_k)_{k\in\mathbb{N}}$ in E^+ and some sequence $(B_k)_{k\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$, then $\{\sum_{k=1}^{K} \mu(B_k)b_k : K \in \mathbb{N}\}$ has a supremum and

$$\sup_{K\in\mathbb{N}} \sum_{k=1}^{K} \mu(B_k)b_k = a.$$

2.15 Definition. Let E be a Riesz space. Let $\sigma : X \to E^+$ be a positive σ-simple function with

$$\sigma = \sum_{n \in \mathbb{N}} a_n \mathbf{1}_{A_n},$$

for a sequence $(a_n)_{n\in\mathbb{N}}$ in E^+ and a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$. We define the integral of σ by

$$\int \sigma \, \mathrm{d}\mu = \sup_{N \in \mathbb{N}} \sum_{n=1}^{N} \mu(A_n) a_n.$$

Let $\sigma : X \to E$ be a σ-simple function. We define the integral of σ by

$$\int \sigma \, \mathrm{d}\mu = \int \sigma^+ \, \mathrm{d}\mu - \int \sigma^- \, \mathrm{d}\mu.$$

2.16 Comment. In general it is not true that the supremum (or a linear combination) of two σ-simple functions is again σ-simple function, as will be shown in the following example.

2.17 Example. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_0)$. Let E be the Riesz space c. Let $f : \mathbb{N} \to c$ and $g : \mathbb{N} \to c$ be given by

$$\begin{aligned} f &= (e_1, e_2, e_3, \dots), \\ g &= (0, e_1 + e_2, 0, e_3 + e_4, \dots). \end{aligned}$$

Then f and g are σ-simple functions (by Comment 2.9), both with integrals $\mathbf{1}$. But then

$$\begin{aligned} f \vee g &= (e_1, e_1 + e_2, e_3, e_3 + e_4, \dots), \\ f - g &= (e_1, -e_1, e_3, -e_3, \dots), \\ (f - g)^+ &= (e_1, 0, e_3, 0, \dots). \end{aligned}$$

So $f \vee g$ and $f - g$ are not σ-simple functions.

2.18 Comment. So the supremum of two σ-simple functions and the difference of two σ-simple functions don't have to be σ-simple (but for some Riesz spaces they do, as we will see in Theorem 2.31). But the sum of two positive σ-simple functions is a positive σ-simple function (as will be shown in Theorem 2.21).

2.19 Lemma. *Let E be a Riesz space. Let $\sigma, \tau : X \to E$ be σ-step functions. Then there exist sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$ such that*

$$\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} \qquad \tau = \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{A_n}.$$

Proof. Suppose $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ and $\tau = \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{B_n}$ for some sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ in E^+ and sequences $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$ with $\bigcup_{n\in\mathbb{N}} A_n = X = \bigcup_{m\in\mathbb{N}} B_n$. Define $a_{nm} := a_n$ and $b_{nm} := b_m$ and $C_{nm} = A_n \cap B_m$ for all $n, m \in \mathbb{N}$. Then $\sigma = \sum_{n,m\in\mathbb{N}} a_{nm} \mathbf{1}_{C_{nm}}$ and $\tau = \sum_{n,m\in\mathbb{N}} b_{nm} \mathbf{1}_{C_{nm}}$. Let $q : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. Then

$$\sigma = \sum_{n\in\mathbb{N}} a_{q(n)} \mathbf{1}_{C_{q(n)}} \qquad \tau = \sum_{n\in\mathbb{N}} b_{q(n)} \mathbf{1}_{C_{q(n)}}.$$

□

2.20 Comment. Let E be a Riesz space. Suppose $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ with $0 < \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Let $\Sigma((A_n)_{n\in\mathbb{N}})$ be the Riesz space of σ-step functions ρ for which there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in E such that

$$\rho = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}.$$

Then there exists a bijection $\Sigma((A_n)_{n\in\mathbb{N}}) \to E^{\mathbb{N}}$, given by

$$\psi : \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} \mapsto \sum_{n\in\mathbb{N}} \mu(A_n) a_n \mathbf{1}_{\{n\}}.$$

A σ-step function $\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ is σ-simple if and only if $\sum_{n\in\mathbb{N}} \mu(A_n) a_n \mathbf{1}_{\{n\}}$ is σ-simple. If $\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ is σ-simple then

$$\int \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} \,\mathrm{d}\mu = \int \sum_{n\in\mathbb{N}} \mu(A_n) a_n \mathbf{1}_{\{n\}} \,\mathrm{d}\mu.$$

Notice that ψ is a Riesz isomorphism. This (in combination with Lemma 2.19) will be used to simplify the proofs of theorems that will follow.

2.21 Theorem. *Let E be a Riesz space. Let σ and τ be positive σ-simple functions $X \to E$. Let $\lambda \geq 0$. Then $\lambda\sigma + \tau$ is a positive σ-simple function and*

$$\int \lambda\sigma + \tau \,\mathrm{d}\mu = \lambda \int \sigma \,\mathrm{d}\mu + \int \tau \,\mathrm{d}\mu.$$

Proof. By Comment 2.20 it is sufficient to prove this for positive σ-simple functions $\sigma, \tau : \mathbb{N} \to E$. Let σ and τ be given by

$$\sigma = (a_1, a_2, a_3, \dots), \qquad \tau = (b_1, b_2, b_3, \dots),$$

for some sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ in E^+. Then

$$\sigma + \tau = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots)$$

By Lemma 2.11 $\sum_{n=1}^N a_n + b_n = \sum_{n=1}^N a_n + \sum_{n=1}^N b_n \uparrow \sup_{N\in\mathbb{N}} \sum_{n=1}^N a_n + \sup_{N\in\mathbb{N}} \sum_{n=1}^N b_n = \int \sigma \,\mathrm{d}\mu + \int \tau \,\mathrm{d}\mu$. So $\sigma + \tau$ is σ-simple with

$$\int \sigma + \tau \,\mathrm{d}\mu = \int \sigma \,\mathrm{d}\mu + \int \tau \,\mathrm{d}\mu.$$

By definition of a positive σ-simple function $\lambda\sigma$ is a positive σ-simple function and $\int \lambda\sigma \,\mathrm{d}\mu = \lambda \int \sigma \,\mathrm{d}\mu$. □

2.22 Theorem. *Let E be a Riesz space. Then the following holds for σ-simple functions $\sigma, \tau : X \to E$ and a σ-step function $\rho : X \to E$:*

- *If $\rho = 0$ μ-a.e., then ρ is σ-simple and $\int \rho \,\mathrm{d}\mu = 0$,*
- *If $\rho = \sigma$ μ-a.e., then ρ is σ-simple and $\int \rho \,\mathrm{d}\mu = \int \sigma \,\mathrm{d}\mu$,*
- *If $\sigma \leq \tau$ μ-a.e., then $\int \sigma \,\mathrm{d}\mu \leq \int \tau \,\mathrm{d}\mu$.*

Proof. If $\pi = \tau$ μ-a.e. for σ-step functions π, τ, then $\pi^+ = \tau^+$ μ-a.e. and $\pi^- = \tau^-$ μ-a.e.. Therefore we assume $\rho \geq 0$. Suppose $\rho = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ for some sequence $(a_n)_{n\in\mathbb{N}}$ in E and a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$. We assume $a_n > 0$ for all $n \in \mathbb{N}$.
• If $\rho = 0$ μ-a.e., then $\mu(A_n) = 0$ for all $n \in \mathbb{N}$ and thus $\sum_{n=1}^N \mu(A_n)a_n = 0$ for all $N \in \mathbb{N}$. Therefore ρ is σ-simple and $\int \rho \, \mathrm{d}\mu = \sup_{N\in\mathbb{N}} \sum_{n=1}^N \mu(A_n)a_n = 0$.
• Suppose $\rho = \sigma$ μ-a.e.. And suppose $\sigma = \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{A_n}$ for some sequence $(b_n)_{n\in\mathbb{N}}$ in E^+ (see Lemma 2.19). Let $Y = \{x \in X : \rho(x) \neq \sigma(x)\}$. We have

$$\rho = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n\setminus Y} + \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n\cap Y},$$
$$\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n\setminus Y} + \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{A_n\cap Y}.$$

Notice that both $\sum_{n\in\mathbb{N}} b_n \mathbf{1}_{A_n\cap Y}$ and $\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n\cap Y}$ are 0 μ-a.e.. Therefore these are σ-simple functions with integral equal to 0. Because

$$\sum_{n=1}^N \mu(A_n \setminus Y)a_n = \sum_{n=1}^N \mu(A_n)a_n \qquad (N \in \mathbb{N}),$$

$\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n\setminus Y}$ is a σ-simple function with integral equal to $\int \sigma \, \mathrm{d}\mu$. By Theorem 2.21 we conclude that ρ is a σ-simple function with integral equal to $\int \sigma \, \mathrm{d}\mu$.
• Suppose that $\sigma \leq \tau$ μ-a.e.. Let $N = \{x \in X : \sigma(x) \not\leq \tau(x)\}$. Let $\tilde{\sigma} = \sigma \mathbf{1}_{X\setminus N} + \tau \mathbf{1}_N$. Then $\tilde{\sigma} = \sigma$ μ-a.e. and $\tilde{\sigma} \leq \tau$. Theorem 2.13 implies $\int \tilde{\sigma} \, \mathrm{d}\mu \leq \int \tau \, \mathrm{d}\mu$. And because $\int \sigma \, \mathrm{d}\mu = \int \tilde{\sigma} \, \mathrm{d}\mu$, we conclude $\int \sigma \, \mathrm{d}\mu \leq \int \tau \, \mathrm{d}\mu$. □

2.23 Theorem. *Let E be a Riesz space. Let σ and τ be σ-simple functions.*

- *If $0 \leq \tau \leq \sigma$ μ-a.e. then $\sigma - \tau$ is σ-simple, with $\int \sigma - \tau \, \mathrm{d}\mu = \int \sigma \, \mathrm{d}\mu - \int \tau \, \mathrm{d}\mu$.*
- *If $\sigma = \sigma \mathbf{1}_A$ μ-a.e. and $\tau = \tau \mathbf{1}_{X\setminus A}$ μ-a.e. for some $A \in \mathcal{A}$, then $\sigma + \tau$ is σ-simple. And $\int \sigma + \tau \, \mathrm{d}\mu = \int \sigma \, \mathrm{d}\mu + \int \tau \, \mathrm{d}\mu$.*

Proof. Suppose $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ and $\tau = \sum_{n\in\mathbb{N}} b_n \mathbf{1}_{A_n}$ for some sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in E and a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$.
• Suppose $0 \leq \tau \leq \sigma$ μ-a.e.. By Theorem 2.22 we may assume $0 \leq \tau \leq \sigma$. Then

$$\sum_{n=1}^N \mu(A_n)(a_n - b_n) = \sum_{n=1}^N \mu(A_n)a_n - \sum_{n=1}^N \mu(A_n)b_n.$$

With Lemma 2.11 (notice that $\mu(A_n)(a_n - b_n), \mu(A_n)a_n, \mu(A_n)b_n \geq 0$ for all $n \in \mathbb{N}$) we conclude that $\{\sum_{n=1}^N \mu(A_n)(a_n - b_n) : N \in \mathbb{N}\}$ has a supremum in E that is equal to $\int \sigma \, \mathrm{d}\mu - \int \tau \, \mathrm{d}\mu$.
• Suppose $\sigma = \sigma \mathbf{1}_A$ μ-a.e. and $\tau = \tau \mathbf{1}_{X\setminus A}$ μ-a.e. for some $A \in \mathcal{A}$. By Theorem 2.22 we may assume $\sigma = \sigma \mathbf{1}_A$ and $\tau = \tau \mathbf{1}_{X\setminus A}$. Then $a_n \neq 0$ implies $b_n = 0$ for all $n \in \mathbb{N}$. Then $(\sigma + \tau)^+ = \sum_{n\in\mathbb{N}} (a_n + b_n)^+ \mathbf{1}_{A_n} = \sum_{n\in\mathbb{N}} a_n^+ \mathbf{1}_{A_n} + \sum_{n\in\mathbb{N}} b_n^+ \mathbf{1}_{A_n} = \sigma^+ + \tau^+$. By Theorem 2.21 $(\sigma + \tau)^+$ is a positive σ-simple function. In the same way $(\sigma + \tau)^- = \sigma^- + \tau^-$ is a positive σ-simple function. Thus $\sigma + \tau = \sigma^+ + \tau^+ - (\sigma^- + \tau^-)$ is a σ-simple function, with $\int \sigma + \tau \, \mathrm{d}\mu = \int \sigma^+ + \tau^+ \, \mathrm{d}\mu - \int \sigma^- + \tau^- \, \mathrm{d}\mu = \int \sigma \, \mathrm{d}\mu + \int \tau \, \mathrm{d}\mu$. □

2.24 Definition. Let E be a Riesz space. E is called *R-complete* if the following holds for sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ in E: If $\{a_n + b_m : n, m \in \mathbb{N}\}$ has a supremum in E than also $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ have a supremum in E.

2.25 Theorem. *Let E be a Riesz space. Then the following are equivalent:*

I. *E is R-complete.*

II. *For all sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$: If $\{a_n - b_m : n, m \in \mathbb{N}\}$ has infimum 0, then $\{a_n : n \in \mathbb{N}\}$ has an infimum in E.*

III. *For all sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ with $a_n \uparrow, b_n \uparrow$: If $\{a_n + b_n : n \in \mathbb{N}\}$ has a supremum in E, then so does $\{a_n : n \in \mathbb{N}\}$.*

IV. *For all sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ with $a_n \downarrow, b_n \uparrow$ and $a_n \geq b_n$ for all $m \in \mathbb{N}$: If $\{a_n - b_m : n, m \in \mathbb{N}\}$ has infimum 0, then $\{a_n : n \in \mathbb{N}\}$ has an infimum in E.*

V. *For all sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$: If $\{\sum_{n=1}^N \sum_{m=1}^M a_n + b_m : N, M \in \mathbb{N}\}$ has a supremum in E, then so does $\{\sum_{n=1}^N a_n : N \in \mathbb{N}\}$.*

VI. *For all sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ with $a_n \geq 0$, $b_n \geq 0$: If $\{\sum_{n=1}^N a_n + b_n : N \in \mathbb{N}\}$ has a supremum in E, then so does $\{\sum_{n=1}^N a_n : N \in \mathbb{N}\}$.*

VII. *For all sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ with $b_n \geq a_n \geq 0$: If $\{\sum_{n=1}^N b_n : N \in \mathbb{N}\}$ has a supremum in E, then so does $\{\sum_{n=1}^N a_n : N \in \mathbb{N}\}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be sequences in E.
I $\Longrightarrow$ II. Suppose I and suppose $a_n \geq b_m$ for $n, m \in \mathbb{N}$. If $\{a_n - b_m : n, m \in \mathbb{N}\}$ has infimum equal to 0, then $\{-a_n + b_m : n, m \in \mathbb{N}\}$ has a supremum in E. Thus $\{-a_n : n \in \mathbb{N}\}$ has a supremum in E, i.e. $\{a_n : n \in \mathbb{N}\}$ has an infimum in E.
II $\Longrightarrow$ I. Suppose II and suppose $\{a_n + b_m : n, m \in \mathbb{N}\}$ has supremum z in E. Then $z - a_n \geq b_m$ for all $n, m \in \mathbb{N}$ and $\{(z - a_n) - b_m : n, m \in \mathbb{N}\}$ has infimum 0. Therefore $\{(z - a_n) : n \in \mathbb{N}\}$ has an infimum in E and thus $\{a_n : n \in \mathbb{N}\}$ has a supremum in E.
I $\iff$ III & II $\iff$ IV. Note that III and IV are special cases of I and II, respectively. Then III implies I (and similarly IV implies II) because of the fact that $\{a_n : n \in \mathbb{N}\}$ has a supremum in E if and only if $\{\sup_{n\leq N} a_n : N \in \mathbb{N}\}$ has a supremum in E.
I $\Longrightarrow$ V. Note that V follows from I, by writing $c_N = \sum_{n=1}^N a_n$ and $d_N = \sum_{n=1}^N b_n$ and then applying I to the sequences $(c_N)_{N\in\mathbb{N}}$ and $(d_N)_{N\in\mathbb{N}}$.
V $\Longrightarrow$ VI. Trivial.
VI $\Longrightarrow$ III. Suppose VI and $a_n \uparrow, b_n \uparrow$. By switching to sequences $(a_n - a_1)_{n\in\mathbb{N}}$ and $(b_n - b_1)_{n\in\mathbb{N}}$ we may assume $a_1, b_1 = 0$. Write $c_n = a_{n+1} - a_n$ and $d_n = b_{n+1} - b_n$ for $n \in \mathbb{N}$. Then $c_n \geq 0$, $d_n \geq 0$ and $\sum_{n=1}^N c_n + d_n = a_{N+1} + b_{N+1}$. Then apply VI to the sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$.
VI $\iff$ VII. Trivial. □

2.26 Definition. Let E be a Riesz space. Then E is said to have the *filling property* if for all sequences $(a_n)_{n\in\mathbb{N}}$ in E and $z \in E$ for which $a_n \uparrow$ and $a_n \leq z$ $(n \in \mathbb{N})$ there exists a sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n \uparrow$ such that $a_n + b_n \uparrow z$.

2.27 Theorem. *Let E be a Riesz space. Then E is σ-Dedekind complete if and only if E is R-complete and has the filling property.*

Proof. *"only if"*. Suppose E is σ-Dedekind complete. It will be clear that E is R-complete. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in E and $z \in E$ be such that $a_n \uparrow$ and $a_n \leq z$ $(n \in \mathbb{N})$. Then $\{a_n : n \in \mathbb{N}\}$ has a supremum in E, say a. Let $b_n = z - a$ for all $n \in \mathbb{N}$, then $b_n \uparrow$ and $a_n + b_n \uparrow z$.
"if". Suppose E is R-complete and has the filling property. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in E and $z \in E$ be such that $a_n \uparrow$ and $a_n \leq z$ $(n \in \mathbb{N})$. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in E with $b_n \uparrow$ such that $a_n + b_n \uparrow z$. Because E is R-complete, $\{a_n : n \in \mathbb{N}\}$ has a supremum in E. Therefore E is σ-Dedekind complete. $\square$

2.28 Example of a Riesz space that has the filling property, which is not R-complete.
We will prove (I) that c has the filling property. And we will show (II) that c is not σ-Dedekind complete and (thus by Theorem 2.27) not R-complete.

(I) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in E and $z \in E$ be such that $a_n \uparrow$ and $a_n \leq z$ $(n \in \mathbb{N})$. Notice that

$$\beta_i := z(i) - \sup_{n\in\mathbb{N}} a_n(i) \geq 0 \qquad (i \in \mathbb{N}).$$

Define $b_n := (\beta_1, \beta_2, \dots, \beta_n, 0, 0 \dots) = \sum_{i=1}^n \beta_i e_i$ for all $n \in \mathbb{N}$. Then $b_n \uparrow$ and

$$(a_n + b_n)(m) \uparrow z(m) \qquad (m \in \mathbb{N}).$$

Therefore $a_n + b_n \uparrow z$.

(II) Let $a_n := \sum_{i=1}^n e_{2i}$, thus $a_1 = (0,1,0,0,\dots)$, $a_2 = (0,1,0,1,0,0,\dots)$, $a_3 = (0,1,0,1,0,1,0,0,\dots)$ etc. Then $a_n \uparrow$ and $a_n \leq \mathbf{1}$ for all $n \in \mathbb{N}$. But $\{a_n : n \in \mathbb{N}\}$ has no supremum in c. Therefore c is not σ-Dedekind complete.

2.29 Example of a Riesz space that is R-complete, which is not σ-Dedekind complete and thus does not have the filling property.
We will prove (I) that the lexicographic plane $\mathbb{R}^2$ is R-complete. And we will show (II) that the lexicographic plane is not Archimedean (thus not σ-Dedekind complete) and does not have the filling property (by Theorem 2.27).

(I) First we examine which sequences in the lexicographic plane have a supremum. Recall that the lexicographic plane is equipped with the ordering $\leq_{lex}$ that is defined by

$$\begin{aligned}(x_1, x_2) \leq_{lex} (y_1, y_2) \iff & \text{ either } x_1 < y_1 \\ & \text{ or } x_1 = y_1, x_2 \leq y_2,\end{aligned}$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Suppose $((a_n, b_n))_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^2$. Suppose that $\{a_n : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. Let $a = \sup_{n\in\mathbb{N}} a_n$. If $a_n < a$ for all $n \in \mathbb{N}$, then (a, λ) is an upper bound for $\{(a_n, b_n) : n \in \mathbb{N}\}$ for all $\lambda \in \mathbb{R}$. So $\{(a_n, b_n) : n \in \mathbb{N}\}$ has no supremum if $a_n < a$ for all $n \in \mathbb{N}$. Therefore $\{(a_n, b_n) : n \in \mathbb{N}\}$ can only have a supremum if there exists $n \in \mathbb{N}$ for which $a = a_n$.
Suppose $((a_n, b_n))_{n\in\mathbb{N}}$ and $((c_n, d_n))_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}^2$ with $(a_n, b_n) \uparrow$, $(c_n, d_n) \uparrow$. Suppose $\{(a_n, b_n) + (c_n, d_n) : n \in \mathbb{N}\} = \{(a_n + c_n, b_n + d_n) : n \in \mathbb{N}\}$ has a supremum in E. Then there exists a $N \in \mathbb{N}$ such that $a_n + c_n =$

$\sup_{m\in\mathbb{N}} a_m + c_m$ for all $n \geq N$. Because $a_n \uparrow$ and $c_n \uparrow$, there exists $a, c \in \mathbb{R}$ such that $a_n = a$ and $c_n = c$ for $n \geq N$. Then

$$\begin{aligned}\sup\{(a_n + c_n, b_n + d_n) : n \in \mathbb{N}\} &= \sup\{(a + c, b_n + d_n) : n \geq N\} \\ &= (a + c, \sup_{n\geq N} b_n + d_n).\end{aligned}$$

Let $b = \sup_{n\geq N} b_n$ and $d = \sup_{n\geq N} d_n$. Then (a, b) is a supremum for $\{(a_n, b_n) : n \in \mathbb{N}\}$ and (c, d) is a supremum for $\{(c_n, d_n) : n \in \mathbb{N}\}$. We conclude that the lexicographic plane is R-complete.

(II) The element $(0, 1)$ is infinitesimal, because $(0, n) \leq_{lex} (1, 0)$ for all $n \in \mathbb{N}$. Therefore the lexicographic plane is not Archimedean. Because σ-Dedekind complete spaces are Archimedean, the lexicographic plane is not σ-Dedekind complete.

2.30 Theorem. *Let E be an R-complete Riesz space. Let $\sigma : X \to E$ be a positive σ-simple function and $\rho : X \to E$ be σ-step function. Suppose that $|\rho| \leq \sigma$ μ-a.e.. Then ρ is a σ-simple function.*

Proof. By Theorem 2.22 we may assume $|\rho| \leq \sigma$. Then by Comment 2.20 it is sufficient to consider $\sigma, \rho : \mathbb{N} \to E$. We prove that ρ^+ is a positive σ-simple function. Suppose ρ^+ and σ are given by

$$\rho^+ = (a_1, a_2, a_3, \dots), \qquad \sigma = (b_1, b_2, b_3, \dots),$$

for some sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ in E^+. Because $\{\sum_{n=1}^N b_n : N \in \mathbb{N}\}$ has a supremum in E, so does $\{\sum_{n=1}^N a_n : N \in \mathbb{N}\}$ by Theorem 2.25 (I $\iff$ VII). $\square$

2.31 Theorem. *Let E be an R-complete Riesz space. Then the set of σ-simple functions is a Riesz space. And for σ-simple functions σ, τ and $\lambda \in \mathbb{R}$ we have:*

$$\int \lambda\sigma + \tau \, d\mu = \lambda \int \sigma \, d\mu + \int \tau \, d\mu.$$

Proof. By definition σ^+ and σ^- are positive σ-simple functions for a σ-simple function σ.
We will show that $\sigma - \tau$ is a σ-simple function with $\int \sigma - \tau \, d\mu = \int \sigma \, d\mu - \int \tau \, d\mu$ for all positive σ-simple functions σ and τ. By Comment 2.20 it is sufficient to prove this for positive σ-simple functions $\sigma, \tau : \mathbb{N} \to E$.
So suppose $\sigma, \tau : \mathbb{N} \to E$ are positive σ-simple functions. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in E^+ such that

$$\sigma = (a_1, a_2, a_3, \dots) \qquad \tau = (b_1, b_2, b_3, \dots).$$

Then

$$\sigma - \tau = (a_1 - b_1, a_2 - b_2, a_3 - b_3, \dots).$$

Because $|\sigma - \tau| \leq \sigma + \tau$, by Theorem 2.30 $\sigma - \tau$ is σ-simple ($\sigma + \tau$ is σ-simple by Theorem 2.21). Because $(\sigma - \tau)^+ + \tau = (\sigma - \tau)^- + \sigma$ and $(\sigma - \tau)^+, \tau, (\sigma - \tau)^-, \sigma$ are positive σ-simple functions, by Theorem 2.21 we have

$$\int (\sigma - \tau)^+ \, d\mu + \int \tau \, d\mu = \int (\sigma - \tau)^- \, d\mu + \int \sigma \, d\mu.$$

Therefore

$$\int \sigma - \tau \, \mathrm{d}\mu = \int (\sigma - \tau)^+ \, \mathrm{d}\mu - \int (\sigma - \tau)^- \, \mathrm{d}\mu = \int \sigma \, \mathrm{d}\mu - \int \tau \, \mathrm{d}\mu.$$

Let σ and τ be σ-simple functions and $\lambda \in \mathbb{R}$. Observing that $\lambda\sigma + \tau = (\lambda\sigma)^+ + \tau^+ - ((\lambda\sigma)^- + \tau^-)$, by Theorem 2.21 we then conclude that $\lambda\sigma + \tau$ is a σ-simple function with $\int \lambda\sigma + \tau \, \mathrm{d}\mu = \lambda \int \sigma \, \mathrm{d}\mu + \int \tau \, \mathrm{d}\mu$. $\square$

2.32 Theorem. *Let E be a Riesz space. Suppose there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$ with $0 < \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Then the space of σ-simple functions $X \to E$ is a Riesz space if and only if E is R-complete.*

Proof. By Theorem 2.31 we only have to show the "only if" part. So suppose the space of σ-simple functions $X \to E$ is a Riesz space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{A}$ with $0 < \mu(A_n) < \infty$ for all $n \in \mathbb{N}$.
The set of σ-simple functions σ for which there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in E such that $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ then is a Riesz subspace of the space of σ-simple functions. Then by Comment 2.20 the σ-simple functions $\mathbb{N} \to E$ also form a Riesz space.
Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in E^+ such that $\{\sum_{n=1}^N a_n + b_n : N \in \mathbb{N}\}$ has a supremum in E. Then

$$\sigma := (a_1, b_1, 0, a_2, b_2, 0, a_3, b_3, 0, \dots), \quad \tau := (a_1, 0, b_1, a_2, 0, b_2, a_3, 0, b_3, \dots),$$

are positive σ-simple functions. But then

$$\sigma \wedge \tau = (a_1, 0, 0, a_2, 0, 0, a_3, 0, 0, \dots),$$

is also a postive σ-simple function. So then $\{\sum_{n=1}^N a_n : N \in \mathbb{N}\}$ has a supremum in E. With Theorem 2.25 we conclude E is R-complete. $\square$

2.33 Corollary. *Let E be an R-complete Riesz space. Then the space of σ-simple functions is a Riesz ideal in the Riesz space of σ-step functions.*

Proof. This is a consequence of Theorem 2.30 and Theorem 2.31. $\square$

2.3 The R-integral on Riesz space valued functions.

2.34 Definition. Let E be a Riesz space. A function $f : X \to E$ is called *R-integrable* if there are sequences $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}$ of σ-simple functions such that

$$\sigma_n \geq f \geq \tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}),$$
$$\text{the sequence } (\textstyle\int \sigma_n \, \mathrm{d}\mu)_{n\in\mathbb{N}} \text{ has an infimum in } E,$$
$$\text{the sequence } (\textstyle\int \tau_n \, \mathrm{d}\mu)_{n\in\mathbb{N}} \text{ has a supremum in } E \text{ and}$$

$$\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Notation 5
In the sequel we will sometimes avoid writing that a sequence has a supremum or an infimum if there follows a statement about the supremum or infimum. So for example, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in a Riesz space E and $b \in E$. Then we write "$\inf_{n\in\mathbb{N}} a_n = b$ " instead of "the sequence $(a_n)_{n\in\mathbb{N}}$ has an infimum in E and $\inf_{n\in\mathbb{N}} a_n = b$".

2.35 Proposition. *Let E be a Riesz space. Suppose $f : X \to E$ is R-integrable and $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}, (\sigma'_n)_{n\in\mathbb{N}}, (\tau'_n)_{n\in\mathbb{N}}$ are sequences of σ-simple functions such that*

$$\sigma_n, \sigma'_n \geq f \geq \tau_n, \tau'_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}),$$

$$\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu,$$

$$\inf_{n\in\mathbb{N}} \int \sigma'_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau'_n \, \mathrm{d}\mu.$$

Then $\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int \sigma'_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau'_n \, \mathrm{d}\mu.$

Proof. Because $\sigma_n \geq \tau'_m$ μ-a.e. and $\sigma'_n \geq \tau_m$ μ-a.e., we have $\int \sigma_n \, \mathrm{d}\mu \geq \int \tau'_m \, \mathrm{d}\mu$ and $\int \sigma'_n \, \mathrm{d}\mu \geq \int \tau_m \, \mathrm{d}\mu$ for all $n, m \in \mathbb{N}$ (by Theorem 2.22). Thus

$$\inf_{n\in\mathbb{N}} \int \sigma'_n \, \mathrm{d}\mu \geq \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu \geq \sup_{n\in\mathbb{N}} \int \tau'_n \, \mathrm{d}\mu.$$

□

2.36 Comment. We are now ready to give a definition of an integral of an R-integrable function. For σ-simple functions we already have an integral (see Definition 2.15).
Let E be a Riesz space. Let $\sigma : X \to E$ be a σ-simple function. Note that σ is an R-integrable function. Let $(\sigma_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ be sequences of σ-simple functions such that

$$\sigma_n \geq \sigma \geq \tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}),$$

$$\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Then by Proposition 2.35 $\int \sigma \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu$. As one will read in the next definition, this proves that the R-integral of a σ-simple function agrees with the integral that we already had for a σ-simple function.

2.37 Definition. Let E be a Riesz space. Let $f : X \to E$ be R-integrable. We define the *R-integral* of f, denoted by (R)-$\int f \, \mathrm{d}\mu$ (or simply $\int f \, \mathrm{d}\mu$) by:

$$\text{(R)-}\int f \, \mathrm{d}\mu = \inf_{n \in \mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu,$$

where $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}$ are sequences of σ-simple functions with

$$\sigma_n \geq f \geq \tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}),$$

$$\inf_{n \in \mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Notice that if $g : X \to E$ is a function for which $g = f$ μ-a.e., then g is R-integrable with $\int g \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$.

2.38 Comment. As will be proved in Theorem 2.51: A function $f : X \to \mathbb{R}$ is R-integrable if and only if f is integrable in the classical sense. In case f is R-integrable then the R-integral coincides with the classical integral.

2.39 Comment. Let $f : X \to E$ and $(\tau_n)_{n\in\mathbb{N}}, (\sigma_n)_{n\in\mathbb{N}}$ be sequences of σ-simple functions with $\sigma_n \geq f \geq \tau_n$ μ-a.e. for all $n \in \mathbb{N}$. Suppose the sets $\{\int \sigma_n \, \mathrm{d}\mu : n \in \mathbb{N}\}$ and $\{\int \tau_n \, \mathrm{d}\mu : n \in \mathbb{N}\}$ have a supremum and infimum in E, respectively. Then we have

$$\inf_{n,m \in \mathbb{N}} \int \sigma_n - \tau_m \, \mathrm{d}\mu = 0 \qquad \iff \qquad \inf_{n \in \mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Note that if E is R-complete and $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}$ are sequences of σ-simple functions with $\sigma_n \geq f \geq \tau_n$ μ-a.e. for all $n \in \mathbb{N}$, then $\inf_{n,m\in\mathbb{N}} \int \sigma_n - \tau_m \, \mathrm{d}\mu = 0$ implies that f is R-integrable. Indeed if $\inf_{n,m\in\mathbb{N}} \int \sigma_n - \tau_m \, \mathrm{d}\mu = 0$ then by Theorem 2.25 the sets $\{\int \sigma_n \, \mathrm{d}\mu : n \in \mathbb{N}\}$ and $\{\int \tau_n \, \mathrm{d}\mu : n \in \mathbb{N}\}$ have a supremum and infimum in E, respectively.

2.40 Theorem. *Let E be a Riesz space. Suppose $f : X \to E^+$ is R-integrable. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of simple functions with $0 \leq t_n \leq f$ μ-a.e. such that the supremum of $\{\int t_n \, \mathrm{d}\mu : n \in \mathbb{N}\}$ exists and is equal to $\int f \, \mathrm{d}\mu$.*

Proof. Suppose $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of positive σ-simple functions with $\tau_n \leq f$ μ-a.e. for $n \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$. Suppose $\tau_n = \sum_{m\in\mathbb{N}} b_m^n \mathbf{1}_{B_m^n}$ for a sequence $(b_m^n)_{m\in\mathbb{N}}$ in E^+ and a sequence $(B_m^n)_{m\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu &= \sup_{n\in\mathbb{N}} \sup_{M\in\mathbb{N}} \sum_{m=1}^{M} \mu(B_m^n) b_m^n \\
&= \sup_{N\in\mathbb{N}} \sup \Big\{ \sum_{m=1}^{N} \mu(B_m^n) b_m^n : n \in \{1, \dots, N\} \Big\}.
\end{aligned}$$

Let $t_N = \sup \Big\{ \sum_{m=1}^{N} \mu(B_m^n) b_m^n : n \in \{1, \dots, N\} \Big\}$. Then t_N is a simple function for all $N \in \mathbb{N}$ and $\sup_{N\in\mathbb{N}} \int t_N \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$. $\square$

2.41 Comment. We have shown in Comment 2.36 that if a σ-step function is σ-simple, then it is R-integrable and the R-integral coincides with the integral that we defined for a σ-simple function in Definition 2.15. The converse holds for positive σ-step functions, as we will see in Theorem 2.42. However, as we will see in Example 2.43, there exists a σ-step function ($\mathbb{N} \to c$) that is R-integrable and not σ-simple.

2.42 Theorem. *Let E be a Riesz space. Let $\rho : X \to E^+$ be a positive σ-step function. If ρ is R-integrable, then ρ is σ-simple.*

Proof. Suppose ρ is R-integrable and suppose $\rho = \sum_{k\in\mathbb{N}} a_k \mathbf{1}_{A_k}$ for some sequence $(a_k)_{k\in\mathbb{N}}$ in E^+ and a sequence of disjoint sets $(A_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ with $X = \bigcup_{k\in\mathbb{N}} A_k$.
Let $(\sigma_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ be sequences of σ-simple functions with

$$\begin{aligned} &\sigma_n \geq \rho \geq \tau_n \quad \mu\text{-a.e.}, \\ &\inf_{n\in\mathbb{N}} \int \sigma_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \,\mathrm{d}\mu = (\mathrm{R})\text{-}\!\int \rho \,\mathrm{d}\mu. \end{aligned}$$

By Theorem 2.40 we assume τ_n is simple for all $n \in \mathbb{N}$. Then for all $K \in \mathbb{N}$ the function $\tau_n \mathbf{1}_{\bigcup_{k=1}^K A_k}$ is simple. Because $X = \bigcup_{k\in\mathbb{N}} A_k$ we have

$$\sup_{K\in\mathbb{N}} \int \tau_n \mathbf{1}_{\bigcup_{k=1}^K A_k} \,\mathrm{d}\mu = \int \tau_n \,\mathrm{d}\mu.$$

Because $\rho \leq \sigma_n$ μ-a.e. for all $n \in \mathbb{N}$, we have

$$\sum_{k=1}^K \mu(A_k) a_k \leq \int \sigma_n \,\mathrm{d}\mu \qquad (K, n \in \mathbb{N}).$$

And thus $(\mathrm{R})\text{-}\!\int \rho \,\mathrm{d}\mu \geq \sum_{k=1}^K \mu(A_k) a_k$ for all $K \in \mathbb{N}$.
Suppose $h \in E$ is such that $h \geq \sum_{k=1}^K \mu(A_k) a_k$ for all $K \in \mathbb{N}$. Because $\tau_n \leq \rho$ μ-a.e. for all $n \in \mathbb{N}$, we have $\int \tau_n \mathbf{1}_{\bigcup_{k=1}^K A_k} \,\mathrm{d}\mu \leq \sum_{k=1}^K \mu(A_k) a_k \leq h$ for all $K, n \in \mathbb{N}$. Then

$$(\mathrm{R})\text{-}\!\int \rho \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \sup_{K\in\mathbb{N}} \int \tau_n \mathbf{1}_{\bigcup_{k=1}^K A_k} \,\mathrm{d}\mu \leq h.$$

So $\{\sum_{k=1}^K \mu(A_k) a_k : K \in \mathbb{N}\}$ has supremum $(\mathrm{R})\text{-}\!\int \rho \,\mathrm{d}\mu$ in E, i.e. ρ is σ-simple. $\square$

2.43 Example. Recall Example 2.17. f and g are σ-simple functions for which

$$\begin{aligned} f - g &= (e_1, -e_1, e_3, -e_3, \dots), \\ (f-g)^+ &= (e_1, 0, e_3, 0, \dots), \end{aligned}$$

are not σ-simple. Because $(f-g)^+$ is a positive σ-step function that is not σ-simple, by Theorem 2.42, it is not R-integrable. By this we see that the set of R-integrable functions $\mathbb{N} \to c$ is not a Riesz space.
We will show that $f - g$ is R-integrable as a function $X \to c$, see however

Example 3.13.
Let $(\sigma_n)_{n\in\mathbb{N}}$ be the sequence of σ-simple functions given by

$$\begin{aligned}
\sigma_1 &= (e_1, -e_1,\ e_3+e_4,\ e_3+e_4,\ e_5+e_6,\ e_5+e_6,\ e_7+e_8,\ e_7+e_8, \dots),\\
\sigma_2 &= (e_1, -e_1,\ e_3,\ -e_3,\ e_5+e_6,\ e_5+e_6,\ e_7+e_8,\ e_7+e_8, \dots),\\
\sigma_3 &= (e_1, -e_1,\ e_3,\ -e_3,\ e_5,\ -e_5,\ e_7+e_8,\ e_7+e_8, \dots),\\
&\ \vdots
\end{aligned}$$

Then $\sigma_n \geq f-g$ for all $n\in\mathbb{N}$. In the same way let $(\tau_n)_{n\in\mathbb{N}}$ be the sequence of σ-simple functions given by

$$\begin{aligned}
\tau_1 &= (e_1, -e_1, -e_3-e_4,\ -e_3-e_4,\ -e_5-e_6,\ -e_5-e_6, -e_7-e_8,\ -e_7-e_8, \dots),\\
\tau_2 &= (e_1, -e_1,\ e_3,\ -e_3,\ -e_5-e_6,\ -e_5-e_6, -e_7-e_8,\ -e_7-e_8, \dots),\\
\tau_3 &= (e_1, -e_1,\ e_3,\ -e_3,\ e_5,\ -e_5,\ -e_7-e_8,\ -e_7-e_8, \dots),\\
&\ \vdots
\end{aligned}$$

Then $\tau_n \leq f-g$ for all $n\in\mathbb{N}$. We have

$$\begin{aligned}
&\int \sigma_1 \,\mathrm{d}\mu = (0,0,2,2,2,2,2,2,\dots), && \int \tau_1 \,\mathrm{d}\mu = (0,0,-2,-2,-2,-2,\dots),\\
&\int \sigma_2 \,\mathrm{d}\mu = (0,0,0,0,2,2,2,2,\dots), && \int \tau_2 \,\mathrm{d}\mu = (0,0,0,0,-2,-2,-2,\dots),\\
&\int \sigma_3 \,\mathrm{d}\mu = (0,0,0,0,0,0,2,2,\dots), && \int \tau_3 \,\mathrm{d}\mu = (0,0,0,0,0,0,-2,-2,\dots),\\
&\quad\vdots
\end{aligned}$$

So $\inf_{n\in\mathbb{N}} \int \sigma_n \,\mathrm{d}\mu = 0 = \sup_{n\in\mathbb{N}} \int \tau_n \,\mathrm{d}\mu$. Therefore $f-g$ is R-integrable.

2.44 Theorem. *Let E be an R-complete Riesz space. Let $f : X \to E$ be R-integrable. Then f^+ and f^- are R-integrable.*

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}$ be sequences of σ-simple functions with

$$\sigma_n \geq f \geq \tau_n \quad \mu\text{-a.e.} \quad (n\in\mathbb{N}), \quad \inf_{n\in\mathbb{N}} \int \sigma_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \,\mathrm{d}\mu.$$

Then

$$\begin{aligned}
&\sigma_n^+ \geq f^+ \geq \tau_n^+ \quad \mu\text{-a.e.} \quad (n\in\mathbb{N}),\\
&\tau_n^- \geq f^- \geq \sigma_n^- \quad \mu\text{-a.e.} \quad (n\in\mathbb{N}),\\
&0 \leq \sigma_n^+ - \tau_m^+, \tau_m^- - \sigma_n^- \leq \sigma_n - \tau_m \quad \mu\text{-a.e.} \quad (n,m\in\mathbb{N}).
\end{aligned}$$

Then (see Theorem 2.23) $0 \leq \int \sigma_n^+ - \tau_m^+ \,\mathrm{d}\mu, \int \tau_m^- - \sigma_n^- \,\mathrm{d}\mu \leq \int \sigma_n - \tau_m \,\mathrm{d}\mu$ for all $n,m\in\mathbb{N}$. Therefore $\inf_{n,m\in\mathbb{N}} \int \sigma_n^+ - \tau_m^+ \,\mathrm{d}\mu = 0$ and $\inf_{n,m\in\mathbb{N}} \int \tau_m^- - \sigma_n^- \,\mathrm{d}\mu = 0$. As is noted in Comment 2.39 this implies that f^+ and f^- are R-integrable. $\square$

2.45 Corollary of Theorem 2.42. *Let E be an R-complete Riesz space. Then a σ-step function $\rho : X \to E$ is R-integrable if and only if it is σ-simple.*

Proof. If ρ is R-integrable then so ρ^+ and ρ^- are R-integrable by Theorem 2.44. Then by Theorem 2.42 ρ^+ and ρ^- are σ-simple and thus ρ is σ-simple. $\square$

2.46 Theorem. *Let E be an R-complete Riesz space. A function $f : X \to E$ is R-integrable if and only if there are sequences $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}$ of σ-simple functions such that $\sigma_n \downarrow$, $\tau_n \uparrow$ and*

$$\sigma_n \geq f \geq \tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}),$$
$$\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Proof. The "if" part will be clear.
Let $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}$ be sequences of σ-simple functions such that $\sigma_n \geq f \geq \tau_n \quad \mu$-a.e. $\quad (n \in \mathbb{N})$, and $\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu$.
Let $\rho_n := \inf_{m\leq n} \sigma_m$ and $\pi_n := \sup_{m\leq n} \tau_m$. Then $(\rho_n)_{n\in\mathbb{N}}, (\pi_n)_{n\in\mathbb{N}}$ are sequences of σ-simple functions (by Theorem 2.30) and $\rho_n \geq f \geq \pi_n$ μ-a.e. for all $n \in \mathbb{N}$. Then by Theorem 2.22

$$\int \sigma_n \, \mathrm{d}\mu \geq \int \rho_n \, \mathrm{d}\mu \geq \sup_{m\in\mathbb{N}} \int \tau_m \, \mathrm{d}\mu \qquad (n \in \mathbb{N}),$$
$$\int \tau_n \, \mathrm{d}\mu \leq \int \pi_n \, \mathrm{d}\mu \leq \inf_{m\in\mathbb{N}} \int \sigma_m \, \mathrm{d}\mu \qquad (n \in \mathbb{N}).$$

So we conclude $\inf_{n\in\mathbb{N}} \rho_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \pi_n \, \mathrm{d}\mu$. $\square$

2.47 Theorem. *Let E be an R-complete Riesz space. The R-integrable functions $X \to E$ form a Riesz subspace of E^X. We call this space*

$$\mathcal{L}_R(X, \mathcal{A}, \mu, E).$$

Then $\int \cdot \, \mathrm{d}\mu : \mathcal{L}_R(X, \mathcal{A}, \mu, E) \to E$, $f \mapsto \int f \, \mathrm{d}\mu$ is a positive linear map.

Proof. Suppose f and g are R-integrable and $(\sigma_n)_{n\in\mathbb{N}}, (\sigma_n')_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}, (\tau_n')_{n\in\mathbb{N}}$ are sequences of σ-simple functions such that

$$\sigma_n \geq f \geq \tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \quad \sigma_n \downarrow, \tau_n \uparrow, \quad \inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu,$$
$$\sigma_n' \geq g \geq \tau_n' \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \quad \sigma_n' \downarrow, \tau_n' \uparrow, \quad \inf_{n\in\mathbb{N}} \int \sigma_n' \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n' \, \mathrm{d}\mu.$$

Then $(\sigma_n + \sigma_n')_{n\in\mathbb{N}}$ and $(\tau_n + \tau_n')_{n\in\mathbb{N}}$ are sequences of σ-simple functions (by Theorem 2.31) with

$$\sigma_n + \sigma_n' \geq f + g \geq \tau_n + \tau_n' \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \quad \sigma_n + \sigma_n' \downarrow, \tau_n + \tau_n' \uparrow.$$

The sequences $(\int \tau_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ and $(\int \tau_n' \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ are increasing by 2.22. And the sequences $(\int \sigma_n \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ and $(\int \sigma_n' \, \mathrm{d}\mu)_{n\in\mathbb{N}}$ are decreasing. Therefore with Lemma 2.11 we have

$$\sup_{n\in\mathbb{N}} \int \tau_n + \tau_n' \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu + \sup_{n\in\mathbb{N}} \int \tau_n' \, \mathrm{d}\mu,$$
$$\inf_{n\in\mathbb{N}} \int \sigma_n + \sigma_n' \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu + \inf_{n\in\mathbb{N}} \int \sigma_n' \, \mathrm{d}\mu.$$

And because

$$\inf_{n\in\mathbb{N}}\int \sigma_n \,\mathrm{d}\mu + \inf_{n\in\mathbb{N}}\int \sigma_n' \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}}\int \tau_n \,\mathrm{d}\mu + \sup_{n\in\mathbb{N}}\int \tau_n' \,\mathrm{d}\mu,$$

we then have

$$\inf_{n\in\mathbb{N}}\int \sigma_n + \sigma_n' \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}}\int \tau_n + \tau_n' \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu.$$

Thus $f+g$ is R-integrable with $\int f+g \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$. It is easy to see that $f \in \mathcal{L}_R(X,\mathcal{A},\mu,E)$ implies $\lambda f \in \mathcal{L}_R(X,\mathcal{A},\mu,E)$, with $\int \lambda f \,\mathrm{d}\mu = \lambda \int f \,\mathrm{d}\mu$ for $\lambda \in \mathbb{R}$. With 2.44 we conclude that $\mathcal{L}_R(X,\mathcal{A},\mu,E)$ is a Riesz subspace of E^X. If $f \in \mathcal{L}_R(X,\mathcal{A},\mu,E)^+$ then $\tau_n \geq 0$ for all $n \in \mathbb{N}$ and thus $\sup_{n\in\mathbb{N}} \int \tau_n \,\mathrm{d}\mu \geq 0$ since $\int \tau_n \,\mathrm{d}\mu \geq 0$ for all $n \in \mathbb{N}$. Therefore $\int \cdot \,\mathrm{d}\mu$ is a positive linear map. □

2.48 Corollary of Theorem 2.32. *Let E be a Riesz space. Suppose there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$ with $0 < \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Then the space of R-integrable functions $X \to E$ is a Riesz space if and only if E is R-complete.*

Proof. By Theorem 2.47 we only have to show the "only if" part. But the "only if" part follows from Theorem 2.42 combined with the proof of Theorem 2.32.□

2.49 Theorem. *Let E be a Riesz space. Let $f : X \to E$. Suppose that $(g_n)_{n\in\mathbb{N}}$, $(h_n)_{n\in\mathbb{N}}$ are sequences in $\mathcal{L}_R(X,\mathcal{A},\mu,E)$ and*

$$g_n \geq f \geq h_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \quad \inf_{n\in\mathbb{N}}\int g_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}}\int h_n \,\mathrm{d}\mu.$$

Then $f \in \mathcal{L}_R(X,\mathcal{A},\mu,E)$ and $\int f \,\mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int g_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int h_n \,\mathrm{d}\mu$.

Proof. For $n \in \mathbb{N}$, let $(\sigma_m^n)_{m\in\mathbb{N}}, (\tau_m^n)_{m\in\mathbb{N}}$ be sequences of σ-simple functions such that

$$\begin{aligned} &\sigma_m^n \geq g_n \quad \mu\text{-a.e.} \quad (m \in \mathbb{N}), && \inf_{m\in\mathbb{N}}\int \sigma_m^n \,\mathrm{d}\mu = \int g_n \,\mathrm{d}\mu, \\ &h_n \geq \tau_m^n \quad \mu\text{-a.e.} \quad (m \in \mathbb{N}), && \sup_{m\in\mathbb{N}}\int \tau_m^n \,\mathrm{d}\mu = \int h_n \,\mathrm{d}\mu. \end{aligned}$$

Then the sets $\{\sigma_m^n : n,m \in \mathbb{N}\}, \{\tau_m^n : n,m \in \mathbb{N}\}$ consist of countably many σ-simple functions (and therefore we are able to "turn them in to sequences"), for which

$$\begin{aligned} \inf_{(n,m)\in\mathbb{N}^2}\int \sigma_m^n \,\mathrm{d}\mu &= \inf_{n\in\mathbb{N}}\inf_{m\in\mathbb{N}}\int \sigma_m^n \,\mathrm{d}\mu = \inf_{n\in\mathbb{N}}\int g_n \,\mathrm{d}\mu \\ &= \sup_{n\in\mathbb{N}}\int h_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}\int \tau_m^n \,\mathrm{d}\mu = \sup_{(n,m)\in\mathbb{N}^2}\int \tau_m^n \,\mathrm{d}\mu. \end{aligned}$$

We conclude f is R-integrable and $\int f \,\mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int g_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int h_n \,\mathrm{d}\mu$.□

We will now look at R-integration for functions with values in the Riesz space $E = \mathbb{R}$.

2.50 Lemma. *A positive σ-simple function $\sigma : X \to \mathbb{R}$ is integrable (in the classical sense) and its integral coincides with the R-integral, i.e.* $(\mathrm{R})\text{-}\!\int \sigma \,\mathrm{d}\mu = \int \sigma \,\mathrm{d}\mu$.

Proof. Suppose $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$, for a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$, such that the sequence $(\sum_{n=1}^{N} \mu(A_n)a_n)_{N\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Then $s_N := \sum_{n=1}^{N} a_n \mathbf{1}_{A_n}$ are simple functions for $N \in \mathbb{N}$ and therefore integrable. Because the sequence $(s_N)_{N\in\mathbb{N}}$ is increasing and $\sigma(x) = \sup_{N\in\mathbb{N}} s_N(x)$ for all $x \in X$, we conclude with the Monotone Convergence Theorem that σ is integrable and $\int \sigma \,\mathrm{d}\mu = \sup_{N\in\mathbb{N}} \int s_N \,\mathrm{d}\mu = (\mathrm{R})\text{-}\!\int \sigma \,\mathrm{d}\mu$. $\square$

In the proof of the next theorem, the following is used to make certain σ-simple functions:

$$(0,1] = \Big(0, \frac{1}{2^n}\Big] \cup \Big(\frac{1}{2^n}, \frac{2}{2^n}\Big] \cup \Big(\frac{2}{2^n}, \frac{3}{2^n}\Big] \cup \cdots \cup \Big(\frac{2^n-1}{2^n}, 1\Big] = \bigcup_{i=1}^{2^n} \Big(\frac{i-1}{2^n}, \frac{i}{2^n}\Big],$$

A picture of this partition for $n = 1, 2, 3$ is given in Figure 2:

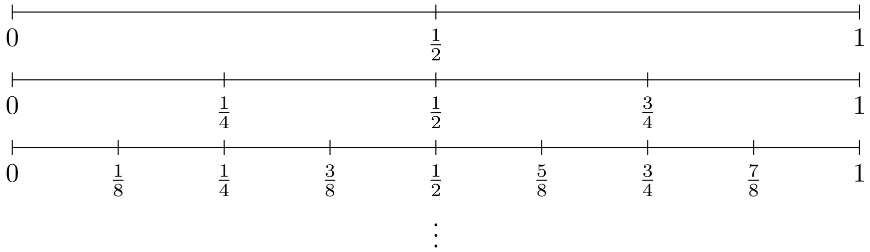

Figure 2: $(0,1] = \bigcup_{i=1}^{2^n} \big(\frac{i-1}{2^n}, \frac{i}{2^n}\big]$.

$$(0,1] = \bigcup_{k=0}^{\infty} (2^{-(k+1)}, 2^{-k}], \qquad (0, \frac{1}{2^n}] = \bigcup_{k=0}^{\infty} \Big(\frac{2^{-(k+1)}}{2^n}, \frac{2^{-k}}{2^n}\Big],$$

$$(\frac{i-1}{2^n}, \frac{i}{2^n}] = \bigcup_{k=0}^{\infty} \Big(\frac{i-1+2^{-(k+1)}}{2^n}, \frac{i-1+2^{-k}}{2^n}\Big],$$

$$(0,1] = \bigcup_{i=1}^{2^n} \bigcup_{k=0}^{\infty} \Big(\frac{i-1+2^{-(k+1)}}{2^n}, \frac{i-1+2^{-k}}{2^n}\Big].$$

A picture of this partition for $n = 1, 2, 3$ is given in Figure 3:

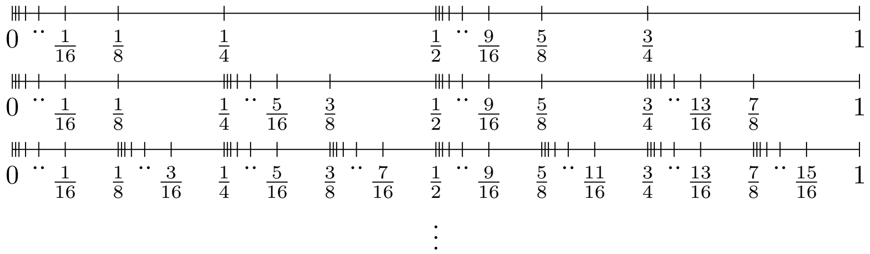

Figure 3: $(0,1] = \bigcup_{i=1}^{2^n} \bigcup_{k=0}^{\infty} \big(\frac{i-1+2^{-(k+1)}}{2^n}, \frac{i-1+2^{-k}}{2^n}\big]$.

2.51 Theorem. *A function $f : X \to \mathbb{R}$ is R-integrable if and only if f is integrable in the classical sense. The R-integral of such a function f coincides with the integral of f, i.e.* (R)-$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$.

Proof. Because $\mathcal{L}(X, \mathcal{A}, \mu)$ and $\mathcal{L}_R(X, \mathcal{A}, \mu, \mathbb{R})$ are Riesz spaces and both $\int \cdot \, \mathrm{d}\mu$ and (R)-$\int \cdot \, \mathrm{d}\mu$ are linear, we may assume f to be positive.

Suppose $f : X \to [0, \infty)$ is integrable. So f is measurable and $\int f \, \mathrm{d}\mu < \infty$. For $n \in \mathbb{N}$ and $i \in \{1, \dots, 2^n\}, k \in \mathbb{N}$, let

$$I_{i,k}^n = \{x \in X : \frac{i - 1 + 2^{-(k+1)}}{2^n} < f(x) \le \frac{i - 1 + 2^{-k}}{2^n}\}.$$

And for $n \in \mathbb{N}$ and $i \ge 2^n + 1$, let

$$I_i^n = \{x \in X : \frac{i-1}{2^n} < f(x) \le \frac{i}{2^n}\}.$$

Note that $\{I_{i,k}^n : 1 \le i \le 2^n, k \in \mathbb{N}\} \cup \{I_i^n : i \ge 2^n + 1\}$ is a set of disjoint sets in $\mathcal{A}$ (by measurability of f) for all $n \in \mathbb{N}$. Let

$$\sigma_n = \sum_{i=1}^{2^n} \sum_{k=0}^{\infty} \frac{i - 1 + 2^{-k}}{2^n} \mathbf{1}_{I_{i,k}^n} + \sum_{i=2^n+1}^{\infty} \frac{i}{2^n} \mathbf{1}_{I_i^n},$$

$$\tau_n = \sum_{i=1}^{2^n} \sum_{k=0}^{\infty} \frac{i - 1 + 2^{-(k+1)}}{2^n} \mathbf{1}_{I_{i,k}^n} + \sum_{i=2^n+1}^{\infty} \frac{i-1}{2^n} \mathbf{1}_{I_i^n}.$$

Then σ_n and τ_n are measurable functions and $0 \le \tau_n \le f \le \sigma_n$. Therefore τ_n is integrable, which implies (by Theorem 2.42) that τ_n is a σ-simple function. Since $0 \le \sigma_n \le 2\tau_n$, also σ_n is integrable and thus a σ-simple function. And $\sigma_n \downarrow f$, $\tau_n \uparrow f$, because $0 \le \sigma_n(x) - f(x) \le 2^{-n}$, $0 \le f(x) - \tau_n(x) \le 2^{-n} \quad (n \in \mathbb{N})$. Therefore by Lebesgue's Dominated Convergence Theorem we have

$$\inf_{n \in \mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \int \inf_{n \in \mathbb{N}} \sigma_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu = \int \sup_{n \in \mathbb{N}} \tau_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

So f is R-integrable and $\int f \, \mathrm{d}\mu =$ (R)-$\int f \, \mathrm{d}\mu$.

Conversely, suppose $f : X \to [0, \infty)$ is R-integrable. Let $(\sigma_n)_{n \in \mathbb{N}}, (\tau_n)_{n \in \mathbb{N}}$ be sequences of σ-simple functions with

$$\tau_n \le f \le \sigma_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \quad \inf_{n \in \mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Because σ_n and τ_n are integrable for all $n \in \mathbb{N}$, the functions $\inf_{n \in \mathbb{N}} \sigma_n : X \to \mathbb{R}$; $x \mapsto \inf_{n \in \mathbb{N}} \sigma_n(x)$ and $\sup_{n \in \mathbb{N}} \tau_n : X \to \mathbb{R}$; $x \mapsto \sup_{n \in \mathbb{N}} \tau_n(x)$ are integrable by Lebesgue's Dominated Convergence Theorem and:

$$\int \inf_{n \in \mathbb{N}} \sigma_n \, \mathrm{d}\mu = \inf_{n \in \mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu = \int \sup_{n \in \mathbb{N}} \tau_n \, \mathrm{d}\mu.$$

Because $\inf_{n \in \mathbb{N}} \sigma_n \ge \sup_{n \in \mathbb{N}} \tau_n$ we have $\int |\inf_{n \in \mathbb{N}} \sigma_n - \sup_{n \in \mathbb{N}} \tau_n| \, \mathrm{d}\mu = 0$. Therefore $\inf_{n \in \mathbb{N}} \sigma_n = \sup_{n \in \mathbb{N}} \tau_n$ μ-a.e.. Because $\inf_{n \in \mathbb{N}} \sigma_n \ge f \ge \sup_{n \in \mathbb{N}} \tau_n$, $f = \inf_{n \in \mathbb{N}} \sigma_n = \sup_{n \in \mathbb{N}} \tau_n$ μ-a.e.. Therefore f is integrable and

$$\int f \, \mathrm{d}\mu = \int \inf_{n \in \mathbb{N}} \sigma_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu = \text{(R)-}\int f \, \mathrm{d}\mu.$$

$\square$

2.4 The U-integral on Riesz space valued functions

2.52 Definition. A sequence $(a_n)_{n\in\mathbb{N}}$ in a Riesz space E is said to *converge relative uniformly* to an element $a \in E$ if there exists a $u \in E^+$ such that for all $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| \leq \varepsilon u$. The notation

$$a_n \xrightarrow{ru} a,$$

is used to denote that the sequence $(a_n)_{n\in\mathbb{N}}$ converges relative uniformly to a. We call a sequence $(a_n)_{n\in\mathbb{N}}$ *relative uniformly convergent* if there exists an $a \in E$ such that $(a_n)_{n\in\mathbb{N}}$ converges relative uniformly to a.

2.53 Theorem. *Let E be a Riesz space. Let $a_n, b_n, a, b \in E$ for all $n \in \mathbb{N}$.*

(i) Suppose E is Archimedean. If $a_n \xrightarrow{ru} a$, $a_n \xrightarrow{ru} b$, then $a = b$.

(ii) Let $\alpha, \beta \in \mathbb{R}$. If $a_n \xrightarrow{ru} a$, $b_n \xrightarrow{ru} b$ then

$$\alpha a_n + \beta b_n \xrightarrow{ru} \alpha a + \beta b, \qquad a_n \vee b_n \xrightarrow{ru} a \vee b, \qquad a_n \wedge b_n \xrightarrow{ru} a \wedge b.$$

(iii) Suppose E is Archimedean. If $a_n \downarrow$ and $a_n \xrightarrow{ru} a$ then $a_n \downarrow a$ and if $b_n \uparrow$, $b_n \xrightarrow{ru} b$ then $b_n \uparrow b$.

Proof. (i) Let $\varepsilon > 0$. Let $u, e \in E^+$ and $N, M \in \mathbb{N}$ be such that

$$n \geq N \Rightarrow |a - a_n| \leq \varepsilon u, \quad m \geq M \Rightarrow |b - a_m| \leq \varepsilon e.$$

Then $|a - b| \leq |a - a_n| + |a_n - b| \leq \varepsilon(e + u)$ for all $n \geq N \vee M$. Thus for all $\varepsilon > 0$ we have $|a - b| \leq \varepsilon(e + u)$. Because E is Archimedean, we have $a = b$.
(ii) Let $\varepsilon > 0$. Let $u, e \in E^+$ and $N, M \in \mathbb{N}$ be such that

$$n \geq N \Rightarrow |a - a_n| \leq \varepsilon u, \quad m \geq M \Rightarrow |b - b_m| \leq \varepsilon e.$$

Then $|\alpha a+\beta b-(\alpha a_n+\beta b_n)| \leq |\alpha||a-a_n|+|\beta||b-b_n| \leq \varepsilon(|\alpha|u+|\beta|e)$. Therefore $\alpha a_n + \beta b_n \xrightarrow{ru} \alpha a + \beta b$. Because $a \vee b = \frac{a+b}{2} + \frac{|a-b|}{2}$ and $a \wedge b = \frac{a+b}{2} - \frac{|a-b|}{2}$, we only have to show $|a_n| \xrightarrow{ru} |a|$, but this follows from $n \geq N \Rightarrow ||a| - |a_n|| \leq |a - a_n| \leq \varepsilon u$.
(iii) Because E is Archimedean and $a_n \xrightarrow{ru} a$, we have $|a_n - a| \downarrow 0$. $a_n \downarrow$ implies $(a_n - a)^- \uparrow$. Then $(a_n - a)^- = 0$ for all $n \in \mathbb{N}$ because $0 \leq (a_n - a)^- \leq |a_n - a| \downarrow 0$. Therefore $a_n - a = |a_n - a| \downarrow 0$, and thus $a_n \downarrow a$. $\square$

Notation 6
For E a Riesz space and $u \in E^+$, we will write E_u for the principal ideal generated by u. Thus $E_u = \bigcup_{n\in\mathbb{N}}[-nu, nu]$, where $[a, b] = \{d \in E : a \leq d \leq b\}$.

2.54 Definition. Let E be an Archimedean Riesz space. Let $u \in E^+$. For $a \in E_u$, define

$$\|a\|_u := \inf\{\lambda \in [0, \infty) : |a| \leq \lambda u\}.$$

Because E is Archimedean $|a| \leq \|a\|_u u$ for all $a \in E$ and $\|\cdot\|_u$ is a Riesz norm on E_u. Notice that if E is a normed Riesz space, with norm $\|\cdot\|$, then because $|a| \leq \|a\|_u u$ we have

$$\|a\| \leq \|u\|\|a\|_u \qquad (a \in E_u). \qquad (*)$$

2.55 Definition. An Archimedean Riesz space E is called *uniformly complete* if for all $u \in E^+$, the principal ideal generated by u, E_u, is complete with respect to the metric defined by the norm $\|\cdot\|_u$. Because $\|\cdot\|_u$ is a Riesz norm for all $u \in E^+$, E is uniformly complete if and only if $(E_u, \|\cdot\|_u)$ is a Banach lattice for all $u \in E^+$. If the norm is not mentioned, then E_u is equipped with the norm $\|\cdot\|_u$.

2.56 Theorem. *Let E be a Banach lattice. Then E is uniformly complete.*

Proof. Let $u \in E^+$ and $(a_n)_{n\in\mathbb{N}} \subset E_u$ be a $\|\cdot\|_u$-Cauchy sequence. We have to show that this sequence has a limit with respect to $\|\cdot\|_u$ in E_u. By inequality $(*)$, mentioned in Definition 2.54, $(a_n)_{n\in\mathbb{N}}$ is also a $\|\cdot\|$-Cauchy sequence. So there exists an $a \in E$, such that $\|a_n - a\| \to 0$. Let $\varepsilon > 0$. By assumption there exists a $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|a_n - a_m| \leq \varepsilon u$. By Lemma 1.6 we therefore have $|a_n - a| \leq \varepsilon u$ for $n \geq N$ (notice that thus $a \in E_u$). And thus $\|a_n - a\|_u \to 0$. □

2.57 Theorem. *Let E be an Archimedean R-complete Riesz space. Then E is uniformly complete.*

Proof. Let $u \in E^+$. Then E_u is Archimedean. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in E_u such that $\{a_n + b_m : n, m \in \mathbb{N}\}$ has a supremum in E_u. Then $\{a_n : n \in \mathbb{N}\}$ has a supremum in E. Because $\sup_{n\in\mathbb{N}} a_n \leq \sup\{a_n + b_m : n, m \in \mathbb{N}\}$, even $\sup_{n\in\mathbb{N}} a_n \in E_u$. So E_u is Archimedean and R-complete.
We prove that E_u is complete with respect to the norm $\|\cdot\|_u$. This is the case if the following holds for all sequences $(a_n)_{n\in\mathbb{N}}$ in E_u (see 10.2 of [PR]):

If $\sum_{n\in\mathbb{N}} \|a_n\|_u < \infty$, then $(a_n)_{n\in\mathbb{N}}$ is summable in E_u.

So suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence in E_u with $\sum_{n\in\mathbb{N}} \|a_n\|_u < \infty$. Because $\||a_n|\| = \|a_n\|$ for all $n \in \mathbb{N}$ and $a_n^+, a_n^- \leq |a_n|$, we have

$$\sum_{n\in\mathbb{N}} \|a_n^+\|_u, \sum_{n\in\mathbb{N}} \|a_n^-\|_u < \infty.$$

So if both the sequences $(a_n^+)_{n\in\mathbb{N}}$ and $(a_n^-)_{n\in\mathbb{N}}$ are summable in E_u, then $(a_n)_{n\in\mathbb{N}}$ is summable in E_u. Therefore we may assume $a_n \geq 0$ for all $n \in \mathbb{N}$. Because $a_n = |a_n| \leq \|a_n\|_u u$ for all $n \in \mathbb{N}$, we have

$$0 \leq \sum_{n=1}^{N} a_n \leq \sum_{n\in\mathbb{N}} \|a_n\|_u u \qquad (N \in \mathbb{N}).$$

Let $b_n = \|a_n\|_u u - a_n$ for $n \in \mathbb{N}$ (notice that $b_n \geq 0$ for all $n \in \mathbb{N}$). Then $\sum_{n\in\mathbb{N}} \|a_n\|_u u$ is the supremum of $\{\sum_{n=1}^{N} a_n + b_n : N \in \mathbb{N}\} = \{\sum_{n=1}^{N} \|a_n\|_u u : N \in \mathbb{N}\}$ in E_u. Because E_u is R-complete, $\{\sum_{n=1}^{N} a_n : N \in \mathbb{N}\}$ has a supremum in E_u. For all $M \in \mathbb{N}$ and $N \geq M+1$ we have, $\sum_{n=M}^{N} a_n \leq \sum_{n\geq M+1} \|a_n\|_u u$ and thus $\sup_{N\geq M+1} \sum_{n=M}^{N} a_n \leq \sum_{n\geq M+1} \|a_n\|_u u$. Therefore

$$\left\| \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} a_n - \sum_{n=1}^{M} a_n \right\|_u = \left\| \sup_{N\geq M+1} \sum_{n=1}^{N} a_n \right\|_u \leq \sum_{n\geq M+1} \|a_n\|_u \xrightarrow{M\to\infty} 0,$$

i.e. $(a_n)_{n\in\mathbb{N}}$ is summable in E_u. □

2.58 Definition. Let E be a uniformly complete Riesz space. A function $f : X \to E$ is called *U-integrable* if there is a $u \in E^+$ such that $f(X) \subset E_u$ and the function $f : X \to E_u$ is Bochner integrable. Thus f is U-integrable if there are $u \in E^+$ and a sequence of simple functions $(s_n)_{n\in\mathbb{N}}$ with values in E_u such that $\|f - s_n\|_u \to 0$ μ-a.e. and $\int \|f - s_n\|_u \; \mathrm{d}\mu \to 0$.

2.59 Lemma. *Let E be an Archimedean Riesz space. Let $u, e \in E^+$ and $\kappa > 0$. If $u \leq \kappa e$ then $\|\cdot\|_e \leq \kappa\|\cdot\|_u$ and $E_u \subset E_e$. Moreover if $u = \kappa e$ then $\|\cdot\|_e = \kappa\|\cdot\|_u$ and $E_u = E_e$.*

Proof. Suppose $u, e \in E^+$ and $\kappa > 0$. By definition of the principal ideal we have $E_u \subset E_e$. For $a \in E_u$ we have $|a| \leq \|a\|_u u \leq \|a\|_u \kappa e$, so $\|a\|_e \leq \|a\|_u \kappa$ by definition of $\|\cdot\|_e$. □

2.60 Theorem. *Let E be a uniformly complete Riesz space. Let $u, e \in E^+$ and $\kappa > 0$. Suppose $u \leq \kappa e$. Let $f : X \to E$. Suppose $f(X) \subset E_u$ and $f : X \to E_u$ is Bochner integrable. Then $f : X \to E_e$ is Bochner integrable and the Bochner integrals agree.*
Furthermore, if $f, g : X \to E$ are U-integrable, then there exists a $u \in E^+$ such that $f(X), g(X) \subset E_u$ and $f, g : X \to E_u$ are Bochner integrable.

Proof. Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence of simple functions with values in E_u such that $\|s_n - f\|_u \to 0$ μ-a.e. and $\int \|s_n - f\|_u \; \mathrm{d}\mu \to 0$. Then by Lemma 2.59 we also have $\|s_n - f\|_e \to 0$ μ-a.e. and $\int \|s_n - f\|_e \; \mathrm{d}\mu \to 0$. Let $a \in E_u$ be the Bochner integral of $f : X \to E_u$. Then $a \in E_e$ and $\|a - \int s_n \; \mathrm{d}\mu\|_e \leq \kappa\|a - \int s_n \; \mathrm{d}\mu\|_u$, so the Bochner integrals agree.
Suppose $f, g : X \to E$ are U-integrable. Let $u \in E^+$ and $v \in E^+$ be such that $f(X) \subset E_u$ and $g(X) \subset E_v$ and such that $f : X \to E_u$, $g : X \to E_v$ are Bochner integrable. Then $e = u \vee v \in E^+$ and $u \leq e$, $v \leq e$ and thus $f, g : X \to E_e$ are Bochner integrable. □

2.61 Proposition. *Let E be a uniformly complete Riesz space. Let $f : X \to E$, $u, v \in E^+$. Suppose $f(X) \subset E_u$ and $f(X) \subset E_v$ and suppose $f : X \to E_u$ is Bochner integrable with integral a and suppose $f : X \to E_v$ is Bochner integrable with integral b. Then $a = b$.*

Proof. Let $e = u \vee v$. Then by Theorem 2.60 $f : X \to E_e$ is Bochner integrable and if c is the Bochner integral of this f, then $a = c$ and $b = c$. □

2.62 Definition. Let E be a uniformly complete Riesz space. Let $f : X \to E$ be U-integrable. We define the *U-integral* of f by:

$$\int f \; \mathrm{d}\mu := \lim_{n\to\infty} \int s_n \; \mathrm{d}\mu,$$

where $(s_n)_{n\in\mathbb{N}}$ is a sequence of simple functions for which there is a $u \in E^+$ such that $f(X) \subset E_u$, $\|s_n - f\|_u \to 0$ μ-a.e. and $\int \|f - s_n\|_u \; \mathrm{d}\mu \to 0$ (the limit $\lim_{n\to\infty} \int s_n \; \mathrm{d}\mu$ is taken in the Banach lattice $(E_u, \|\cdot\|_u)$). Thus the U-integral of f is the Bochner integral of $f : X \to E_u$.
If E is a Banach lattice, a U-integrable function $f : X \to E$ is Bochner integrable (this is a consequence of the inequality $(*)$, mentioned in Definition 2.54, see also Theorem 3.9).

2.63 Theorem. *Let E be a uniformly complete unitary Riesz space, with unit $e \in E^+$. Let $f : X \to E$. Then f is U-integrable if and only if $f : X \to E_e$ is Bochner integrable (here E_e is the space E equipped with the norm $\|\cdot\|_e$). For such integrable f we have*

$$\text{(B)-}\int f \, \mathrm{d}\mu = \text{(U)-}\int f \, \mathrm{d}\mu.$$

Proof. The only if part will be clear by definition of U-integrability.
If f is U-integrable, there exists a $u \in E^+$ such that $f : X \to E_u$ is Bochner integrable. Because e is a unit, there exists a $\kappa > 0$ such that $u \leq \kappa e$. By Theorem 2.60 $f : X \to E_e$ is Bochner integrable and the integrals agree. □

2.64 Corollary. *A function $f : X \to \mathbb{R}$ is U-integrable if and only if f is integrable in the classical sense. For such integrable function f, the U-integral and the classical integral coincide.*

Proof. This is a consequence of Theorem 2.63 and Theorem 1.32, because $\mathbb{R}$ is uniformly complete with unit 1. □

2.65 Theorem. *Let E be a uniformly complete Riesz space. The set of U-integrable functions $X \to E$ forms a Riesz space. We call this space*

$$\mathcal{L}_U(X, \mathcal{A}, \mu, E).$$

Then $\int \cdot \, \mathrm{d}\mu : \mathcal{L}_U(X, \mathcal{A}, \mu, E) \to E,\ f \mapsto \int f \, \mathrm{d}\mu$ is a positive linear map.

Proof. Suppose $f, g : X \to E$ are U-integrable and let $u \in E^+$ be such that $f : X \to E_u$ and $g : X \to E_u$ are Bochner integrable (see Theorem 2.60). Then also $\lambda f + g : X \to E_u$ is Bochner integrable for $\lambda \in \mathbb{R}$. Therefore $\lambda f + g$ is U-integrable and $\int \lambda f + g \, \mathrm{d}\mu = \lambda \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$ for $\lambda \in \mathbb{R}$.
If $f : X \to E$ is U-integrable, then $|f|$ is U-integrable by Theorem 2.5. Therefore $\mathcal{L}_U(X, \mathcal{A}, \mu, E)$ is a Riesz space.
Positivity of the map $f \mapsto \int f \, \mathrm{d}\mu$ follows by 2.3. □

2.66 Theorem. *Let E be a uniformly complete Riesz space. Let $f : X \to E$ be a function. Then the following are equivalent:*

(I) f is U-integrable.

(II) There are $u \in E^+$, a sequence of simple functions $X \to E$, $(s_n)_{n\in\mathbb{N}}$ and a sequence of positive σ-simple functions $X \to [0, \infty)$, $(\rho_n)_{n\in\mathbb{N}}$, such that

$$s_n - u\rho_n \leq f \leq s_n + u\rho_n \quad \mu\text{-a.e.}, \qquad \int \rho_n \, \mathrm{d}\mu \to 0,$$
$$\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int s_n - u\rho_n \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int s_n + u\rho_n \, \mathrm{d}\mu.$$

(III) There are $u \in E^+$, a σ-simple function $\pi : X \to [0, \infty)$ and a sequence of simple functions $X \to E$, $(s_n)_{n\in\mathbb{N}}$ such that

$$s_n - \frac{1}{n} u\pi \leq f \leq s_n + \frac{1}{n} u\pi \quad \mu\text{-a.e.},$$
$$\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int s_n - \frac{1}{n} u\pi \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} s_n + \frac{1}{n} u\pi \, \mathrm{d}\mu.$$

Proof. (I) $\Rightarrow$ (II). Assume f is U-integrable. Let $u \in E$ and $(s_n)_{n\in\mathbb{N}}$ be a sequence of simple functions such that $\|s_n - f\|_u \to 0$ μ-a.e. and $\int \|f - s_n\|_u \, \mathrm{d}\mu \to 0$. By Theorem 2.51 for all $n \in \mathbb{N}$ there is a σ-simple function $\rho_n : X \to \mathbb{R}$ with

$$\|f - s_n\|_u \le \rho_n \quad \mu\text{-a.e.} \quad \text{and} \quad \int \rho_n \, \mathrm{d}\mu - \int \|f - s_n\|_u \, \mathrm{d}\mu < \frac{1}{n} \quad (n \in \mathbb{N}).$$

So $\int \rho_n \, \mathrm{d}\mu \to 0$. Then we have $|f(x) - s_n(x)| \le u\rho_n(x) \quad (x \in X, n \in \mathbb{N})$ and thus

$$s_n - u\rho_n \le f \le s_n + u\rho_n \qquad (n \in \mathbb{N}).$$

And because s_n is simple and $u\rho_n$ is σ-simple, $s_n - u\rho_n$ and $s_n + u\rho_n$ are σ-simple functions. We have

$$\begin{aligned}\Big\| \int f \, \mathrm{d}\mu - \int s_n \pm u\rho_n \, \mathrm{d}\mu \Big\|_u &\le \Big\| \int f - s_n \, \mathrm{d}\mu \Big\|_u + \int \rho_n \, \mathrm{d}\mu \\ &\le \int \|f - s_n\|_u \, \mathrm{d}\mu + \int \rho_n \, \mathrm{d}\mu \to 0.\end{aligned}$$

By Lemma 1.7 we conclude

$$\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int s_n - u\rho_n \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int s_n + u\rho_n \, \mathrm{d}\mu.$$

(II) $\Rightarrow$ (III). Assume (II). By switching to subsequences we may assume that $\int \rho_n \, \mathrm{d}\mu \le \frac{1}{n2^n}$ for all $n \in \mathbb{N}$. Then the sequence $(\sum_{i=1}^N i\rho_i)_{N\in\mathbb{N}}$ is an increasing sequence of integrable functions $X \to \mathbb{R}$ (see Theorem 2.31) for which

$$\sum_{i=1}^N \int i\rho_i \, \mathrm{d}\mu \le 1 \qquad (N \in \mathbb{N}).$$

By the monotone convergence theorem there is an integrable function a such that $a = \sum_{i=1}^\infty i\rho_i$ μ-a.e.. Therefore $\rho_n \le \frac{1}{n} a$ μ-a.e.. By Theorem 2.51 there is a σ-simple function π with $\pi \ge a$. We have $\|f - s_n\|_u \le \frac{1}{n}\pi \quad \mu$-a.e. and thus $s_n - \frac{1}{n}u\pi \le f \le s_n + \frac{1}{n}u\pi \quad \mu$-a.e.. In a similar way as in (I) $\Rightarrow$ (II):

$$\Big\| \int f \, \mathrm{d}\mu - \int s_n \pm \frac{1}{n}u\pi \, \mathrm{d}\mu \Big\|_u \le \Big\| \int f - s_n \, \mathrm{d}\mu \Big\|_u + \frac{1}{n}u \int \pi \, \mathrm{d}\mu \to 0,$$

and thus (by Lemma 1.7)

$$\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int s_n - \frac{1}{n}u\pi \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int s_n + \frac{1}{n}u\pi \, \mathrm{d}\mu.$$

(III) $\Rightarrow$ (I). Assume (III). Because $\|f - s_n\|_u \le \frac{1}{n}\pi$ μ-a.e. we have $\|f - s_n\|_u \to 0$ μ-a.e. and $\int \|f - s_n\|_u \, \mathrm{d}\mu \le \frac{1}{n} \int \pi \, \mathrm{d}\mu \to 0$. So f is U-integrable. $\square$

2.5 The Pettis integral on Riesz space valued functions

Recall the definition of the Pettis integral of §1.2.2. In Lemma 1.37 we have seen that V' separates the points of a Banach space V. For a Riesz space E, $E^{\sim}$ doesn't have to separate the points of E as we have seen in Example 1.4. But by assuming that $E^{\sim}$ or a subset of it separates te points, we can give a similar definition of the Pettis integral for functions with values in E:

2.67 Definition. Let E be a Riesz space, with the property that $E^{\sim}$ ($E_c^{\sim}$) separates the points of E. A function $f : X \to E$ is called *strongly Pettis integrable* (*weakly Pettis integrable*) if there is a $v \in E$ such that for all $\phi \in E^{\sim}$ (for all $\phi \in E_c^{\sim}$):

$$\phi \circ f \in \mathcal{L}(X, \mathcal{A}, \mu) \qquad \text{and} \qquad \phi(v) = \int \phi \circ f \, \mathrm{d}\mu.$$

There is only one v with this property, because $E^{\sim}$ ($E_c^{\sim}$) separates the points of E. This v we call the *strong Pettis integral* (*weak Pettis integral*) of f, denoted by (sP)-$\int f \, \mathrm{d}\mu$ $\Big($(wP)-$\int f \, \mathrm{d}\mu\Big)$ or simply $\int f \, \mathrm{d}\mu$.

Notation 7
We use the following notations:

$$\mathcal{L}_{wP}(X, \mathcal{A}, \mu, E) = \{f : X \to E : f \text{ is weakly Pettis integrable}\},$$
$$\mathcal{L}_{sP}(X, \mathcal{A}, \mu, E) = \{f : X \to E : f \text{ is strongly Pettis integrable}\}.$$

2.68 Comment. In the above definition we could replace "for all $\phi \in E^{\sim}$" respectively "for all $\phi \in E_c^{\sim}$" by "for all $\phi \in E^{\sim +}$" respectively "for all $\phi \in E_c^{\sim +}$", because $E^{\sim}$ and $E_c^{\sim}$ are Riesz spaces.

2.69 Comment. Note that all simple functions are strongly/weakly Pettis integrable and that the strong/weak Pettis integral of a simple function coincides with the integral of a simple function defined as in 1.19.

2.70 Comment. (Comparing the Pettis integrals)
It will be clear that if $E_c^{\sim}$ separates the points of E, then a strongly Pettis integrable function is weakly Pettis integrable and the weak Pettis integral coincides with the strong Pettis integral. There are spaces for which $E^{\sim}$ separates the points of E and for which $E_c^{\sim} = \{0\}$ (Example 1.18). If $E_c^{\sim}$ (and thus $E^{\sim}$) separates the points of E, then a weakly Pettis integrable function doesn't have to be strongly Pettis integrable as we will see in Example 3.20.
Also notice that if E is a Banach lattice, then $E' = E^{\sim}$ (by Theorem 1.8). Therefore a function $f : X \to E$ is Pettis integrable if and only if it is strongly Pettis integrable and the strong Pettis integral is equal to the Pettis integral.

2.71 Lemma. *Let E be a Riesz space. Suppose that $E^{\sim}$ ($E_c^{\sim}$) separates the points of E. Let $a \in E$. Suppose $\phi(a) \geq 0$ for all $\varphi \in E^{\sim +}$ (for all $\varphi \in E_c^{\sim}$). Then $a \geq 0$.*

Proof. Suppose $E^{\sim}$ separates the points of E. Suppose $a \ngeq 0$. Then $a^- > 0$ and therefore there exists a $\phi \in E^{\sim +}$ with $\phi(a^-) > 0$. Let $p : E \to [0, \infty)$ be defined by

$$p(x) := \phi(x^+) \qquad (x \in E).$$

Then $p(\lambda x) = \phi((\lambda x)^+) = \lambda\phi(x^+) = \lambda p(x)$ for all $x \in E$ and $\lambda \geq 0$. And $p(x+y) = \phi((x+y)^+) \leq \phi(x^+ + y^+) = \phi(x^+) + \phi(y^+) = p(x) + p(y)$ for all $x, y \in E$. Thus p is sublinear. Let $D := \mathbb{R}a^+ + \mathbb{R}a^-$. Then D is a linear subspace of E. And for $\lambda, \kappa \in \mathbb{R}$ we have (using that $a^+ \perp a^-$)

$$\begin{aligned}(\lambda a^+ + \kappa a^-)^+ &= (\lambda^+ a^+ - \lambda^- a^+ + \kappa^+ a^- - \kappa^- a^-)^+ \\ &= \lambda^+ a^+ + \kappa^+ a^-.\end{aligned}$$

Therefore

$$\begin{aligned}p(\lambda a^+ + \kappa a^-) &= \phi((\lambda a^+ + \kappa a^-)^+) \\ &= \phi(\lambda^+ a^+ + \kappa^+ a^-) \geq \kappa\phi(a^-).\end{aligned}$$

By Hahn-Banach (see III.6.2 of [Con07]) there exists a linear function $\psi : E \to \mathbb{R}$ with

$$\begin{aligned}&\psi(\lambda a^+ + \kappa a^-) = \kappa\phi(a^-) && (\lambda, \kappa \in \mathbb{R}), \\ &\psi(x) \leq p(x) && (x \in E).\end{aligned}$$

So $x \leq 0$ implies $\psi(x) \leq p(x) = 0$, i.e. $\psi \in E^{\sim+}$. But $\psi(a) = -\phi(a^-) < 0$. So for $a \ngeq 0$ there exists a $\psi \in E^{\sim+}$ such that $\psi(a) < 0$. Thus if $a \in E$ is such that $\phi(a) \geq 0$ for all $\phi \in E^{\sim+}$, then $a \geq 0$.
Suppose $E_c^\sim$ separates the points of E. Suppose $a \ngeq 0$, i.e. $a^- > 0$. Then there exists a $\phi \in E_c^{\sim+}$ with $\phi(a^-) > 0$. We are done when we can prove that the ψ obtained as above is σ-order continuous. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in E for which $u_n \downarrow 0$. Then $0 \leq \psi(u_n) \leq p(u_n) = \phi(u_n^+) = \phi(u_n) \downarrow 0$. So ψ is indeed an element of $E_c^{\sim+}$. □

2.72 Corollary. *Let E be a Riesz space for which $E^\sim$ respectively $E_c^\sim$ separates the points of E. The maps*

$$(\mathrm{sP})\text{-}\int \cdot \, \mathrm{d}\mu : \mathcal{L}_{sP}(X, \mathcal{A}, \mu, E) \to E, \qquad f \mapsto (\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu,$$

and

$$(\mathrm{wP})\text{-}\int \cdot \, \mathrm{d}\mu : \mathcal{L}_{wP}(X, \mathcal{A}, \mu, E) \to E, \qquad f \mapsto (\mathrm{wP})\text{-}\int f \, \mathrm{d}\mu,$$

respectively, are positive linear maps.

2.73 Example of a Riesz space, even a Banach lattice for which the strongly and the weakly Pettis integrable functions do not form a Riesz space.
Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_0)$.
Recall Example 1.17 (we will use notation introduced in this example). Let $f : \mathbb{N} \to c_0$ be given

$$f = (f_n)_{n\in\mathbb{N}} = (e_1, -e_1, e_2, -e_2, e_3, -e_3, \dots).$$

Then for $n \in \mathbb{N}$ and $i \in \mathbb{N}$:

$$f_n(i) = \begin{cases} 1 & \text{if } 2i - 1 = n, \\ -1 & \text{if } 2i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $b \in \ell^1$. Then

$$\int |\phi_b \circ f| \, \mathrm{d}\mu_0 = \sum_{n\in\mathbb{N}} \left|\phi_b \circ f(n)\right| = \sum_{n\in\mathbb{N}} \Big| \sum_{i\in\mathbb{N}} b_i f_n(i) \Big| \leq \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} \left| b_i f_n(i) \right|$$
$$= \sum_{i\in\mathbb{N}} \sum_{n\in\mathbb{N}} \left| b_i f_n(i) \right| = \sum_{i\in\mathbb{N}} 2|b_i| < \infty \qquad (*)$$

Thus $\phi \circ f$ is integrable for all $\phi \in c_0^\sim$. Of course also $\int |\phi_b \circ |f|| \, \mathrm{d}\mu_0 \leq \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} |b_i f_n(i)|$ for all $b \in \ell^1$, thus also $\phi \circ |f|$ is integrable for all $\phi \in c_0^\sim$. Let $b \in \ell^1$. By $(*)$ we can apply Fubini's Theorem:

$$\int \phi_b \circ f \, \mathrm{d}\mu_0 = \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} b_i f_n(i) = \sum_{i\in\mathbb{N}} \sum_{n\in\mathbb{N}} b_i f_n(i) = \sum_{i\in\mathbb{N}} b_i - b_i = 0.$$

Thus $\int \phi \circ f \, \mathrm{d}\mu_0 = 0$ for all $\phi \in c_0^\sim$. Thus f is (strongly and weakly) Pettis integrable with $\int f \, \mathrm{d}\mu_0 = 0$.
Suppose $v \in c_0$ is such that $\phi_b(v) = \int \phi_b \circ |f| \, \mathrm{d}\mu_0 = \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} b_i |f_n(i)| = \sum_{i\in\mathbb{N}} \sum_{n\in\mathbb{N}} b_i |f_n(i)| = \sum_{i\in\mathbb{N}} b_i + b_i = 2 \sum_{i\in\mathbb{N}} b_i$ for all $b \in \ell^1$, then $v_n = \phi_{e_n}(v) = 2$ for all $n \in \mathbb{N}$, which leads to a contradiction with $v \in c_0$. So $|f|$ is not (strongly or weakly) Pettis integrable.

2.74 Comment. A strongly Pettis integrable function is weakly measurable. We have seen that the strongly Pettis integrable functions do not have to form a Riesz space. As one might have noticed, in Theorem 2.4, we did not prove that the space of weakly measurable functions is a Riesz space. This is because this does not have to be true, as will be seen in the next example.

2.75 Example of a weakly measurable function f, for which $|f|$ is not weakly measurable.
This is a more detailed version of Example 1.1 of Chapter II in [Jeu82].
Consider the measure space $([0,1], \mathcal{B}, \lambda)$. Let D be a subset of $[0,1]$ that is not Lebesgue measurable (see [Hal50] Chapter III, 16D). Let X be the direct product group $\{-1,1\}^{[0,1]}$. Then there exists a Haar measure μ on X (see Theorem 1.3.4 of [DE09]). Because X is compact, μ is finite. Let E be the Banach lattice and Hilbert space $L^2(X, \mathcal{B}(X), \mu)$. Let $\{f_\omega : \omega \in [0,1]\}$ be the set of coordinate functions on X, i.e. $f_\omega(x) = x(\omega)$ for $x \in \{-1,1\}^{[0,1]}$. Because f_ω is continuous and bounded for all $\omega \in [0,1]$ and μ is finite, $f_\omega \in L^2(X, \mathcal{B}(X), \mu)$.
Suppose $\omega_1, \omega_2 \in [0,1]$ and $\omega_1 \neq \omega_2$. Let $y \in \{-1,1\}^{[0,1]}$ be given by $y(\omega) = 1$ for all $\omega \neq \omega_1$ and $y(\omega_1) = -1$. Let $A = \{x \in X : x(\omega_1) = x(\omega_2)\}$ and $B = \{x \in X : x(\omega_1) = -x(\omega_2)\}$. Then $B = yA$ and thus $\mu(A) = \mu(B)$. Because $f_{\omega_1}(x) f_{\omega_2}(x) = \mathbf{1}_{\{x\in X : x(\omega_1) = x(\omega_2)\}}(x) - \mathbf{1}_{\{x \in X : x(\omega_1) = -x(\omega_2)\}}(x)$ for all $x \in X$ we have

$$\langle f_{\omega_1}, f_{\omega_2} \rangle = \int f_{\omega_1}(x) f_{\omega_2}(x) \, \mathrm{d}\mu(x) = \mu(A) - \mu(B) = 0.$$

Let $\omega \in [0,1]$. Then $f_\omega(x)^2 = 1$ for all $x \in X$, therefore $\langle f_\omega, f_\omega \rangle = 1$. We conclude $\{f_\omega : \omega \in [0,1]\}$ is an orthonormal set in $L^2(X, \mathcal{B}(X), \mu)$.
Define $f : [0,1] \to E$ by

$$f(\omega) = \mathbf{1}_D(\omega) f_\omega \qquad (\omega \in [0,1]).$$

Let $\phi \in E'$. If $\omega \in D$ and $\langle \phi, f_\omega \rangle \neq 0$, then $\phi \circ f(\omega) = \langle \phi, f(\omega) \rangle \neq 0$. Because $\{f_\omega : \omega \in [0,1]\}$ is an orthonormal set, by Corollary I.4.9 of [Con07], we have

$\langle \phi, f_\omega \rangle \neq 0$ for at most countably many $\omega \in [0,1]$. Therefore $\phi \circ f = 0$ μ-a.e. and thus $\phi \circ f$ is measurable for all $\phi \in E'$.
Note that for all $\omega \in [0,1]$ and $x \in X$, $|f_\omega|(x) = |f_\omega(x)| = 1$. Therefore $|f| = \mathbf{1}_D$. So for $\phi = \mathbf{1}$ in E', then $\phi \circ |f|(\omega) = \langle \mathbf{1}, \mathbf{1} \rangle = 1$ for $\omega \in D$ and $\phi \circ |f|(\omega) = 0$ for $\omega \in [0,1] \setminus D$. Thus $\phi \circ |f| = \mathbf{1}_D$. Therefore $|f|$ is not weakly measurable.

2.76 Comment. We have $L^2(X, \mathcal{A}, \mu)_c^\sim = L^2(X, \mathcal{A}, \mu)^\sim = L^2(X, \mathcal{A}, \mu)'$ by Theorem 1.16. Therefore by Example 2.75 we also see that the set of functions f for which $\phi \circ f$ is measurable for all $\phi \in E_c^\sim$ does not have to be a Riesz space.

2.77 Theorem. *Let E be a Riesz space. Suppose that $E_c^\sim$ separates the points of E. Let f_n be a sequence of weakly Pettis integrable functions with $0 \leq f_1 \leq f_2 \leq \cdots$. Suppose for all $x \in X$, $\sup_{n\in\mathbb{N}} f_n(x)$ exists in E and also that $\sup_{n\in\mathbb{N}} \int f_n \, \mathrm{d}\mu$ exists in E. Then the function $f : X \to E$ given by $f(x) = \sup_{n\in\mathbb{N}} f_n(x)$ is weakly Pettis integrable and*

$$\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int f_n \, \mathrm{d}\mu.$$

Proof. For all $\phi \in E_c^{\sim+}$, $(\phi \circ f_n)_{n\in\mathbb{N}}$ is an increasing sequence of μ-integrable functions. By Comment 1.10 we have

$$\sup_{n\in\mathbb{N}} \phi \circ f_n = \phi \circ \sup_{n\in\mathbb{N}} f_n = \phi \circ f \qquad (\phi \in E_c^{\sim+}).$$

Let $\phi \in E_c^{\sim+}$. By the monotone convergence theorem of Levi, we have

$$\phi \circ f = \sup_{n\in\mathbb{N}} \phi \circ f_n \in \mathcal{L}(X, \mathcal{A}, \mu) \quad \text{and}$$
$$\int \phi \circ f \, \mathrm{d}\mu = \int \sup_{n\in\mathbb{N}} \phi \circ f_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \phi \circ f_n \, \mathrm{d}\mu.$$

Writing $v_{f_n} = \int f_n \, \mathrm{d}\mu$ we have by σ-order continuity of ϕ

$$\phi(\sup_{n\in\mathbb{N}} v_{f_n}) = \sup_{n\in\mathbb{N}} \phi(v_{f_n}) = \sup_{n\in\mathbb{N}} \int \phi \circ f_n \, \mathrm{d}\mu.$$

So for all $\phi \in E_c^{\sim+}$ we have

$$\phi \circ f \in \mathcal{L}(X, \mathcal{A}, \mu) \qquad \text{and} \qquad \phi(\sup_{n\in\mathbb{N}} v_{f_n}) = \int \phi \circ f \, \mathrm{d}\mu.$$

So we see that f is weakly Pettis integrable and $\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int f_n \, \mathrm{d}\mu$. $\square$

3 Comparing the integrals

3.1 Comparing the R-integral with the Bochner integral

3.1 Lemma. *Let E be a Banach lattice. All σ-step functions $\rho : X \to E$ are strongly measurable.*

Proof. Suppose $\rho = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ for some sequence $(a_n)_{n\in\mathbb{N}}$ in E and a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$. Define $s_n := \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}$ for all $n \in \mathbb{N}$. Then s_n is a step function for all $n \in \mathbb{N}$. And for all $x \in X$ we have $\|\rho(x) - s_n(x)\| \to 0$. Therefore ρ is strongly measurable. $\square$

3.2 Example of an R-integrable function, even a σ-simple function, that is not Bochner integrable.
Consider the measure space $([0,1], \mathcal{B}, \lambda)$. Let E be the Banach lattice $C_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$. For $n \in \mathbb{N}$ let $a_n \in C_b(\mathbb{R})$ be the function given by

$$a_n(x) = (1 - 2|x-n|) \vee 0.$$

And let $A_n = [n, n+1)$ for $n \in \mathbb{N}$. Then $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ is a σ-step function $\mathbb{R} \to E$.

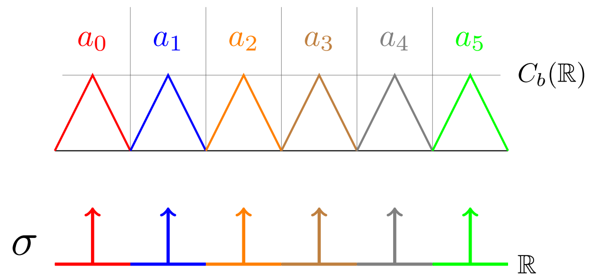

Figure 4: A picture of σ

We have $\|a_n\|_\infty = 1$ for all $n \in \mathbb{N}$, therefore $\|\sigma\|_\infty = \sum_{n\in\mathbb{N}} \mathbf{1}_{A_n} = \mathbf{1}_{[1,\infty)}$. So $\|\sigma\|_\infty$ is not integrable. But the set $\{\sum_{n=1}^{N} \mu(A_n) a_n : N \in \mathbb{N}\}$ has a supremum in $C_b(\mathbb{R})$, because $\sum_{n\in\mathbb{N}} a_n \in C_b(\mathbb{R})$. So σ is a σ-simple function and thus R-integrable, but $\|\sigma\|$ is not integrable and therefore σ is not Bochner integrable.

3.3 Theorem. *Let E be a Banach lattice. Let $\sigma : X \to E$ be a σ-step function. If $\|\sigma\| : X \to [0,\infty)$ is integrable (and thus σ is Bochner integrable by Theorem 1.29 and Lemma 3.1), then σ is σ-simple and thus R-integrable.*

Proof. Because $\sigma = \sigma^+ - \sigma^-$, with σ^+ and σ^- positive, we assume σ to be positive. Suppose that $\sigma : X \to E$ can be written as $\sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$, for a sequence $(a_n)_{n\in\mathbb{N}}$ in E^+ and a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$. If $\|\sigma\|$ is integrable then we have $\int \|\sigma\| \, \mathrm{d}\mu = \sum_{n\in\mathbb{N}} \mu(A_n)\|a_n\| < \infty$. By completeness of E, $\sum_{n\in\mathbb{N}} \mu(A_n) a_n$ exists. Therefore the sequence $(\sum_{n=1}^{N} \mu(A_n) a_n)_{N\in\mathbb{N}}$ is increasing and has limit $\sum_{n\in\mathbb{N}} \mu(A_n) a_n$. By Lemma 1.7 the sequence has a supremum in E, which is $\sum_{n\in\mathbb{N}} \mu(A_n) a_n$. $\square$

3.4 Example of a Bochner integrable function with values in a Banach lattice that is not R-integrable.

Consider the group $(0,1]$ with addition modulo 1, which is isomorphic to the circle group $\mathbb{T}$ (i.e. the multiplicative group of complex numbers of absolute value 1). Consider the measure space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$, where $\mathbb{T}$ is the circle group, μ is the Haar measure on $\mathbb{T}$ and $\mathcal{B}(\mathbb{T})$ the associated σ-algebra ($\mu(xA) = \mu(A)$ for all $x \in \mathbb{T}$ and $A \in \mathcal{B}(\mathbb{T})$ and $\mu(\mathbb{T}) = 1$). The Haar measure on $\mathbb{T}$ coincides with the Lebesgue measure on $(0,1]$, in the following way:

Let $\phi : (0,1] \to \mathbb{T}$ be given by $\phi(t) = e^{2\pi i t}$. Then $\mu(A) = \lambda(\phi^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{T})$.

In this example we write $\mathbb{T}$ for the space $(0,1]$ with addition modulo 1, equipped with the Lebesgue measure on the Lebesgue measurable subsets of $(0,1]$.

Let E be the Banach lattice $L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$. Then E is isomorphic in a natural way to the Banach lattice $L^1([0,1], \mathcal{B}[0,1], \lambda)$. This E is separable (see 9.6 of [PR])

• The function $h : (0,1] \to \mathbb{R}$ given by $h(x) = \frac{1}{\sqrt{x}}$ is integrable. So there is a sequence of simple functions $(h_n)_{n\in\mathbb{N}}$ such that $h_n \uparrow h$ and thus (by the Monotone Convergence Theorem) $\int h_n \, \mathrm{d}\lambda \uparrow \int h \, \mathrm{d}\lambda$.

• Let $f : \mathbb{T} \to L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ be the map given by $f(x) = L_x h$, where $L_x h : y \mapsto h(y-x)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{T}$ that converges to some $x \in \mathbb{T}$. Then $\|L_{x_n} h - L_x h\|_{L^1(\mathbb{T})} \to 0$ by Lemma 1.4.2 of [DE09]. Thus f is continuous and hence Borel measurable. Because $L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ is separable we conclude with Theorem 1.25 that f is strongly measurable. And because $\|f\| : x \mapsto \|L_x h\|_{L^1(\mathbb{T})} = \|h\|_{L^1(\mathbb{T})}$, we see that $\|f\|$ is integrable. Then with Theorem 1.29 we conclude that f is Bochner integrable.

• We show that f is not R-integrable. This we do by showing that there is no σ-simple function σ with $f \leq \sigma$ λ-a.e.. And this is a consequence of the following (which will be proved):

If $a \in L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ and $A \in \mathcal{B}(\mathbb{T})$ are such that $f(x) \leq a$ for all $x \in A$, then $\lambda(A) = 0$.

Suppose $a \in L^1(\mathbb{T})$ and $A \in \mathcal{B}(\mathbb{T})$ are such that $f(x) \leq a$ for all $x \in A$. Then for all $x \in A$ and μ-almost all $y \in \mathbb{T}$:

$$a(y) \geq (f(x))(y) = h(y-x) = \frac{1}{\sqrt{y-x}}.$$

Let $K \in \mathbb{N}$. Then for μ-almost all $x \in A$ we have $a(\frac{1}{K^2} + x) \geq K$. Therefore $\lambda(\{y \in \mathbb{T} : a(y) \geq K\}) \geq \lambda(\frac{1}{K^2} + A) = \lambda(A)$. Because $a \in L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$, we have $\lim_{K\to\infty} \lambda(\{y \in \mathbb{T} : a(y) \geq K\}) = 0$ and thus $\lambda(A) = 0$.

3.5 Corollary of Theorem 3.19. *Let E be an R-complete Banach lattice. Suppose $f : X \to E$ is R-integrable and Bochner integrable. Then*

$$\text{(R)-}\int f \, \mathrm{d}\mu = \text{(B)-}\int f \, \mathrm{d}\mu.$$

Proof. By Theorem 2.5 and Theorem 2.47 f^+ and f^- are R-integrable and Bochner integrable. Therefore we may assume $f \geq 0$.

Then the proposed identity is a corollary of Theorem 3.19 and Theorem 1.32 (see Comment 1.40). $\square$

3.2 Comparing the U-integral with the R-integral

3.6 Theorem. *Let E be a uniformly complete Riesz space. Suppose that a function $f : X \to E$ is U-integrable. Then f is also R-integrable and*

$$\text{(R)-}\int f \, \mathrm{d}\mu = \text{(U)-}\int f \, \mathrm{d}\mu.$$

Proof. By Theorem 2.66 there are $u \in E^+$, a σ-simple function $\pi : X \to [0, \infty)$ and a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions $X \to E$ such that

$$s_n - \frac{1}{n} u\pi \leq f \leq s_n + \frac{1}{n} u\pi \quad \mu\text{-a.e.},$$
$$\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int s_n - \frac{1}{n} u\pi \, \mathrm{d}\mu = \inf_{n\in\mathbb{N}} s_n + \frac{1}{n} u\pi \, \mathrm{d}\mu.$$

Because s_n and π are simple and σ-simple respectively, $s_n - \frac{1}{n}u\pi$ and $s_n + \frac{1}{n}u\pi$ are σ-simple for all $n \in \mathbb{N}$. Thus f is R-integrable and the R-integral of f coincides with the U-integral. $\square$

3.7 Example of an R-integrable function, even a σ-simple function, that is not U-integrable.
Consider the σ-simple function $\sigma : \mathbb{R} \to C_b(\mathbb{R})$ of Example 3.2. We have already seen that $\sigma : X \to C_b(\mathbb{R})$ is not Bochner integrable ($C_b(\mathbb{R})$ is equipped with $\| \cdot \|_\infty = \| \cdot \|_{\mathbf{1}}$, where $\mathbf{1} \in C_b(\mathbb{R})$ is the function $x \mapsto 1$). Because $C_b(\mathbb{R})$ is unitary and uniformly complete, we conclude by Theorem 2.63 that σ is not U-integrable.

3.3 Comparing the U-integral with the Bochner integral

3.8 Example of a Bochner integrable function that is not U-integrable. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_0)$. Let E be the Banach lattice ℓ^1. Let $f : \mathbb{N} \to E$ be given by

$$f = (f_n)_{n\in\mathbb{N}} = (e_1, \frac{1}{4}e_2, \frac{1}{9}e_3, \frac{1}{16}e_4, \frac{1}{25}e_5, \dots).$$

$\|f\|$ is the function given by $n \mapsto n^{-2}$. So because $\sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$, f is Bochner integrable. f is U-integrable if and only if there exists a $u \in E$ such that $f(X) \subset E_u$ and $\|f\|_u$ is integrable by Theorem 1.29 (because for all $u \in E^+$ for which $f(X) \subset E_u$, $f : X \to E_u$ is strongly measurable by Lemma 3.1). And $\|f\|_u = \sum_{n\in\mathbb{N}} \|f_n\|_u \mathbf{1}_{\{n\}}$, where

$$\begin{aligned}\|f_n\|_u &= \inf\{\lambda > 0 : f_n(k) \leq \lambda u(k) \text{ for all } k \in \mathbb{N}\}\\ &= \inf\{\lambda > 0 : \frac{1}{n^2} \leq \lambda u(n)\} = \frac{1}{n^2 u(n)}.\end{aligned}$$

So f is U-integrable if and only if there exists a $u \in E^+$ such that

$$\int \|f\| \; \mathrm{d}\mu_0 = \sum_{n\in\mathbb{N}} \|f_n\|_u = \sum_{n\in\mathbb{N}} \frac{1}{n^2 u(n)} < \infty.$$

Suppose $\|f\|_u$ is integrable for some $u \in \ell^1$. Then for all $n \in \mathbb{N}$, $u(n) \leq \frac{1}{n}$ implies $\frac{1}{n^2 u(n)} \geq \frac{1}{n}$. Hence for all $n \in \mathbb{N}$ we have $\frac{1}{n^2 u(n)} + u(n) \geq \frac{1}{n}$. But because both the sequence $(u(n))_{n\in\mathbb{N}}$ and the sequence $(\frac{1}{n^2 u(n)})_{n\in\mathbb{N}}$ are summable, the sequence $\big(u(n) + \frac{1}{n^2 u(n)}\big)_{n\in\mathbb{N}}$ should be summable. But this leads to a contradiction with $\frac{1}{n^2 u(n)} + u(n) \geq \frac{1}{n}$ for all $n \in \mathbb{N}$. So we see that f is a Bochner integrable function which is not U-integrable.

3.9 Theorem. *Let E be a Banach lattice (which is uniformly complete by Theorem 2.56). Suppose that a function $f : X \to E$ is U-integrable. Then f is Bochner integrable and*

$$\text{(B)-}\int f \; \mathrm{d}\mu = \text{(U)-}\int f \; \mathrm{d}\mu.$$

Proof. Let $\|\cdot\|$ denote the norm of the Banach lattice E. Then this theorem is a consequence of the following inequality, metioned in Definition 2.54, for $u \in E^+$:

$$\|a\| \leq \|u\| \|a\|_u \qquad (a \in E_u).$$

This theorem is also a consequence of Theorem 1.32 (III), by considering the inclusion map $T : E_u \to E$. $\square$

3.4 Comparing the U-integral with the strong and weak Pettis integral

As we have seen in Comment 1.40, a Bochner integrable function is Pettis integrable and the integrals coincide. And the converse doesn't have to hold. In this paragraph we will see that -for suitable Riesz spaces- a U-integrable function is strongly (weakly) Pettis integrable. And the converse doesn't have to hold.

3.10 Theorem. *Let E be a uniformly complete Riesz space. Suppose that $E^{\sim}$ ($E_c^{\sim}$) separates the points. Let $f : X \to E$ be U-integrable. Then f is strongly (weakly) Pettis integrable and the strong (weak) Pettis integral coincides with the U-integral.*

Proof. Let $u \in E^+$ be such that $f : X \to E_u$ is Bochner integrable. Let $\phi \in E^{\sim}$ ($\phi \in E_c^{\sim}$). Then $\phi|_{E_u} \in (E_u)^{\sim}$ and thus $\phi \in E_u'$ (by Theorem 1.8).
As we have seen in Proposition 1.32, we have

$$\phi(\int f \, \mathrm{d}\mu) = \phi|_{E_u}(\int f \, \mathrm{d}\mu) = \int \phi|_{E_u} \circ f \, \mathrm{d}\mu = \int \phi \circ f \, \mathrm{d}\mu.$$

So f is strongly (weakly) Pettis integrable and the strong (weak) Pettis integral coincides with the U-integral. $\square$

3.11 Example of a strongly and weakly Pettis integrable function with values in the Banach lattice c_0 that is not U-integrable.
Consider the measure space $([0,1], \mathcal{B}, \lambda)$. Let E be the Banach lattice c_0 (for which $c_0' = (c_0)^{\sim} = (c_0)_c^{\sim}$) and the function f as in Comment 1.40:

$$f = \sum_{n \in \mathbb{N}} \frac{2^n}{n} e_n \mathbf{1}_{[2^{-n}, 2^{-n+1})}.$$

Then f is U-integrable if and only if there exists a $u \in c_0^+$ such that $f(X) \subset E_u$ and $\|f\|_u = \sum_{n \in \mathbb{N}} \frac{2^n}{n} \|e_n\|_u \mathbf{1}_{[2^{-n}, 2^{-n+1})}$ is integrable, i.e. when $\sum_{n \in \mathbb{N}} \frac{1}{n} \|e_n\|_u < \infty$ (because for all $u \in E^+$ for which $f(X) \subset E_u$, $f : X \to E_u$ is strongly measurable by Lemma 3.1). We have $u(n) > 0$ for all $n \in \mathbb{N}$, because $f(X) \subset E_u$. We also have

$$\|e_n\|_u = \inf\{\lambda \in [0, \infty) : e_n \leq \lambda u\} = \frac{1}{u(n)}.$$

So f is U-integrable if and only if there exists a $u \in c_0^+$ such that

$$\sum_{n \in \mathbb{N}} \frac{1}{n u(n)} = \sum_{n \in \mathbb{N}} \frac{1}{n} \|e_n\|_u < \infty.$$

But for all $u \in c_0^+$, u is bounded, say $u(n) \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$. Thus $\frac{1}{u(n)} \geq \frac{1}{M}$ and

$$\sum_{n \in \mathbb{N}} \frac{1}{n u(n)} \geq \frac{1}{M} \sum_{n \in \mathbb{N}} \frac{1}{n}.$$

So f is a (strongly and weakly) Pettis integrable function (see Comment 2.70) which is not U-integrable.

3.5 Comparing the R-integral with the weak Pettis integral

3.12 Theorem. *Let E be a Riesz space. Suppose that $E_c^\sim$ separates the points. Let $f : X \to E$ be an R-integrable function. Then $\phi \circ f$ is integrable for all $\phi \in E_c^\sim$. In particular f is weakly Pettis integrable and the weak Pettis integral coincides with the U-integral.*

Proof. Suppose $\phi \in E_c^{\sim +}$. Let $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ be a positive σ-simple function. Then because ϕ is σ-order continuous and linear, we have

$$\phi(\int \sigma \, \mathrm{d}\mu) = \phi(\sup_{N\in\mathbb{N}} \sum_{i=1}^{N} \mu(A_n)a_n) = \sup_{N\in\mathbb{N}} \sum_{i=1}^{N} \mu(A_n)\phi(a_n).$$

Thus $\phi \circ \sigma$ is integrable (using Theorem 2.51) and

$$\phi(\int \sigma \, \mathrm{d}\mu) = \int \phi \circ \sigma \, \mathrm{d}\mu.$$

Let $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}$ be sequences of σ-simple functions with

$$\tau_n \le f \le \sigma_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \qquad \inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Then we have $\phi \circ \tau_n \le \phi \circ f \le \phi \circ \sigma_n \quad \mu$-a.e. $\quad (n \in \mathbb{N})$, because ϕ is positive. And because ϕ is σ-order continuous we have

$$\phi(\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu) = \inf_{n\in\mathbb{N}} \phi(\int \sigma_n \, \mathrm{d}\mu) = \inf_{n\in\mathbb{N}} \int \phi \circ \sigma_n \, \mathrm{d}\mu,$$
$$\phi(\sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu) = \sup_{n\in\mathbb{N}} \phi(\int \tau_n \, \mathrm{d}\mu) = \sup_{n\in\mathbb{N}} \int \phi \circ \tau_n \, \mathrm{d}\mu.$$

Therefore $\phi \circ f$ is integrable and $\phi(\int f \, \mathrm{d}\mu) = \int \phi \circ f \, \mathrm{d}\mu$. □

3.13 Example of a weakly and strongly Pettis integrable function with values in the Banach lattice c_0 that is not R-integrable.
Recall Example 2.73, where $f : \mathbb{N} \to c_0$ is given by

$$f = (e_1, -e_1, e_2, -e_2, e_3, -e_3, \dots).$$

We have seen that f is (weakly and strongly) Pettis integrable. Because f is a σ-step function and c_0 is R-complete, by Corollary 2.45 it is R-integrable/σ-simple if and only if f^+ and f^- are R-integrable/σ-simple. We have

$$f^+ = (e_1, 0, e_2, 0, e_3, 0, \dots),$$
$$f^- = (0, e_1, 0, e_2, 0, e_3, \dots).$$

Because $\{\sum_{n=1}^N e_n : N \in \mathbb{N}\}$ has no supremum in c_0, both f^+ and f^- are not σ-simple and thus f is not R-integrable.

So there exists a σ-step function that is weakly and strongly Pettis integrable, but not R-integrable. The σ-step function of Example 3.13 is not positive (or negative), this is essential as we will see in the next theorem.

3.14 Theorem. *Let E be a Riesz space. Let ρ be a positive σ-step function. Suppose $E^\sim$ separates the points of E. If ρ is strongly Pettis integrable, then ρ is R-integrable.*
Suppose that even $E_c^\sim$ separates the points of E. If ρ is weakly Pettis integrable, then ρ is R-integrable.

Proof. Suppose $\rho = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ for some sequence $(a_n)_{n\in\mathbb{N}}$ in E^+ and a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$.
Suppose $E^\sim$ separates the points of E and suppose ρ is strongly Pettis integrable. By definition (and with Theorem 2.51) we have for all $\phi \in E^{\sim+}$:

$$\begin{aligned}\phi((\mathrm{sP})\text{-}\int \rho \,\mathrm{d}\mu) = \int \phi\circ\rho \,\mathrm{d}\mu = \int \sum_{n\in\mathbb{N}} \phi(a_n)\mathbf{1}_{A_n} \,\mathrm{d}\mu \\ = \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} \mu(A_n)\phi(a_n) = \sup_{N\in\mathbb{N}} \phi(\sum_{n=1}^{N} \mu(A_n)a_n).\end{aligned}$$

Therefore (Lemma 2.71) (sP)-$\int \rho \,\mathrm{d}\mu \geq \sum_{n=1}^N \mu(A_n)a_n$ for all $N \in \mathbb{N}$. Suppose $h \in E$ is such that $h \geq \sum_{n=1}^N \mu(A_n)a_n$ for all $N \in \mathbb{N}$. Then for all $\phi \in E^{\sim+}$:

$$\phi(h) \geq \sup_{N\in\mathbb{N}} \phi(\sum_{n=1}^{N} \mu(A_n)a_n) = \phi((\mathrm{sP})\text{-}\int \rho \,\mathrm{d}\mu).$$

And thus $h \geq$ (sP)-$\int \rho \,\mathrm{d}\mu$, i.e. (sP)-$\int \rho \,\mathrm{d}\mu$ is the supremum of $\{\sum_{n=1}^N \mu(A_n)a_n : N \in \mathbb{N}\}$. Therefore ρ is a σ-simple function and thus R-integrable.
If $E_c^\sim$ separates the points of E and ρ is weakly Pettis integrable, the proof as above can be copied except $E^{\sim+}$ has to be replaced with $E_c^{\sim+}$. □

3.15 Corollary. *Let E be a Riesz space for which $E_c^\sim$ separates the points of E. Let $\rho : X \to E$ be a positive σ-step function. Then ρ is R-integrable (σ-simple) if and only if ρ is weakly Pettis integrable.*

3.16 Example. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_0)$. Let E be the Banach lattice $L^1(X, \mathcal{A}, \mu)$. Then $E_c^\sim = E^\sim = E'$ (see Theorem 1.16). Let $f : \mathbb{N} \to E^+$. Then (by Corollary 3.15) f is R-integrable (σ-simple) if and only if f is Pettis integrable (which is the same as weakly and strongly Pettis integrable here).

3.6 Comparing the R-integral with the strong Pettis integral

3.17 Example of a strongly and weakly Pettis integrable function that is not R-integrable.
By Corollary 1.16, Example 3.4 also yields us a strongly and weakly Pettis integrable function that is not R-integrable as we have seen in Comment 1.40 and Comment 2.70.

3.18 Lemma. *Let E be a Riesz space. Let σ be a positive σ-simple function. Then $\phi \circ \sigma$ is integrable for all $\phi \in E^\sim$ and*

$$\int \phi \circ \sigma \, \mathrm{d}\mu \leq \phi(\int \sigma \, \mathrm{d}\mu) \qquad (\phi \in E^{\sim +}),$$
$$\int \phi \circ \sigma \, \mathrm{d}\mu = \phi(\int \sigma \, \mathrm{d}\mu) \qquad (\phi \in E_c^{\sim +}).$$

Proof. Let $\phi \in E^{\sim +}$. Suppose $\sigma = \sum_{n \in \mathbb{N}} a_n \mathbf{1}_{A_n}$ for a sequence $(a_n)_{n \in \mathbb{N}}$ in E^+ and a sequence $(A_n)_{n \in \mathbb{N}}$ of disjoint sets in $\mathcal{A}$. Then for all $N \in \mathbb{N}$:

$$\sum_{n=1}^{N} \mu(A_n)\phi(a_n) = \phi(\sum_{n=1}^{N} \mu(A_n)a_n) \leq \phi(\sup_{N \in \mathbb{N}} \sum_{n=1}^{N} \mu(A_n)a_n) = \phi(\int \sigma \, \mathrm{d}\mu).$$

Therefore $\{\sum_{n=1}^{N} \mu(A_n)\phi(a_n) : N \in \mathbb{N}\}$ has a supremum in $\mathbb{R}$ and thus $\phi \circ \sigma$ is integrable (by Theorem 2.51) with $\int \phi \circ \sigma \, \mathrm{d}\mu = \sup_{N \in \mathbb{N}} \sum_{n=1}^{N} \mu(A_n)\phi(a_n) \leq \phi(\int \sigma \, \mathrm{d}\mu)$ (this is an equality if $\phi \in E_c^{\sim +}$). ☐

3.19 Theorem. *Let E be a Riesz space for which $E^\sim$ separates the points of E. Suppose $f : X \to E^+$ is R-integrable and strongly Pettis integrable. Then*

$$(\mathrm{R})\text{-}\int f \, \mathrm{d}\mu = (\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu.$$

Proof. Suppose $(\tau_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ are sequences of positive σ-simple functions with $0 \leq \tau_n \leq f \leq \sigma_n$ μ-a.e. for all $n \in \mathbb{N}$ and with $\inf_{n \in \mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu$. By 3.18 $\phi \circ \sigma_n$ is integrable for all $n \in \mathbb{N}$ and $\int \phi \circ \sigma_n \, \mathrm{d}\mu \leq \phi((\mathrm{R})\text{-}\int \sigma_n \, \mathrm{d}\mu)$ for all $\phi \in E^{\sim +}$. Then for all $\phi \in E^{\sim +}$ and $n \in \mathbb{N}$ we have

$$\phi((\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu) = \int \phi \circ f \, \mathrm{d}\mu \leq \int \phi \circ \sigma_n \, \mathrm{d}\mu \leq \phi((\mathrm{R})\text{-}\int \sigma_n \, \mathrm{d}\mu).$$

Therefore by Lemma 2.71 we have $(\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu \leq (\mathrm{R})\text{-}\int \sigma_n \, \mathrm{d}\mu$ for all $n \in \mathbb{N}$. We thus have $(\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu \leq \inf_{n \in \mathbb{N}} (\mathrm{R})\text{-}\int \sigma_n \, \mathrm{d}\mu = (\mathrm{R})\text{-}\int f \, \mathrm{d}\mu$.
We assume τ_n is a simple function for all $n \in \mathbb{N}$ (see Theorem 2.40). Because all simple functions are strongly Pettis integrable, τ_n is strongly Pettis integrable. Because $0 \leq \tau_n \leq f$, we have $(\mathrm{sP})\text{-}\int \tau_n \, \mathrm{d}\mu \leq (\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu$ by Corollary 2.72 for all $n \in \mathbb{N}$. Therefore $(\mathrm{R})\text{-}\int f \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int \tau_n \, \mathrm{d}\mu \leq (\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu$. We conclude $(\mathrm{R})\text{-}\int f \, \mathrm{d}\mu = (\mathrm{sP})\text{-}\int f \, \mathrm{d}\mu$. ☐

3.20 Example of an R-integrable (even a σ-simple) and weakly Pettis integrable function that is not strongly Pettis integrable.

Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_0)$.

Recall Example 1.17. Let $f : \mathbb{N} \to c$ be given by

$$f = (f_n)_{n\in\mathbb{N}} = (e_1, e_2, e_3, \dots).$$

Then $\sum_{n=1}^N f(n) = \sum_{n=1}^N e_n$. Thus $\{\sum_{n=1}^N f(n) : N \in \mathbb{N}\}$ has a supremum in E, namely the sequence $\mathbf{1} = (1, 1, 1, \dots)$. So f is a σ-simple function and thus R-integrable (thus also weakly Pettis integrable because $c_c^\sim$ separates the points of c) with

$$\text{(R)-}\int f \,\mathrm{d}\mu_0 = \text{(wP)-}\int f \,\mathrm{d}\mu_0 = \mathbf{1}.$$

Let $b \in \ell^1(\mathbb{N} \cup \{\infty\})$ be given by $b_n = 0$ for all $n \in \mathbb{N}$ and $b_\infty = 1$. Then $\phi_b(f_n) = \phi_b(e_n) = 0$ for all $n \in \mathbb{N}$. So

$$\int \phi_b \circ f \,\mathrm{d}\mu_0 = \sum_{n\in\mathbb{N}} \phi_b(f_n) = 0.$$

And of course $\phi_b\Big(\text{(R)-}\int f \,\mathrm{d}\mu_0\Big) = \phi_b\Big(\text{(wP)-}\int f \,\mathrm{d}\mu_0\Big) = 1$. As is mentioned in Comment 2.70, if f were strongly Pettis integrable, then (because $c_c^\sim$ separates the points of c) f would also be weakly Pettis integrable and the integrals would agree. So we see that f is not strongly Pettis integrable.

4 Further properties of the R-integral

4.1 Theorem. *Let E be an R-complete Riesz space. Let $g : X \to \mathbb{R}$ be a bounded measurable function. Let $f : X \to E$ be an R-integrable function. Then $gf : X \to E$, $gf : x \mapsto g(x)f(x)$ is an R-integrable function.*

Proof. By Theorem 2.44 and 2.47 we may assume $f \geq 0$.
• Let $B \in \mathcal{A}$. Suppose $(\sigma_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ are sequences of σ-simple functions with (see Comment 2.39 and Theorem 2.46):

$$\sigma_n \geq f \geq \tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \sigma_n \downarrow, \tau_n \uparrow,$$
$$\inf_{n\in\mathbb{N}} \int \sigma_n - \tau_n \, \mathrm{d}\mu = 0.$$

Since E is R-complete, $(\sigma_n \mathbf{1}_B)_{n\in\mathbb{N}}$ and $(\tau_n \mathbf{1}_B)_{n\in\mathbb{N}}$ are sequences of σ-simple functions by Theorem 2.30. And

$$\mathbf{1}_B\sigma_n \geq \mathbf{1}_B f \geq \mathbf{1}_B\tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \mathbf{1}_B\sigma_n \downarrow, \mathbf{1}_B\tau_n \uparrow,$$
$$0 \leq \inf_{n\in\mathbb{N}} \int \mathbf{1}_B\sigma_n - \mathbf{1}_B\tau_n \, \mathrm{d}\mu \leq \inf_{n\in\mathbb{N}} \int \sigma_n - \tau_n \, \mathrm{d}\mu = 0.$$

So $\mathbf{1}_B f$ is R-integrable for all $B \in \mathcal{A}$. Then for a step function $s : X \to \mathbb{R}$, sf is R-integrable by Theorem 2.47.
• Let g be a bounded measurable function. Then there are sequences of step functions $(G_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ with

$$G_n \geq g \geq g_n \quad \mu\text{-a.e.}, \quad \|g - g_n\|_\infty < \tfrac{1}{n}, \|g - G_n\|_\infty < \tfrac{1}{n} \quad (n \in \mathbb{N}), \ G_n \downarrow, g_n \uparrow.$$

Then
$$\int (G_n - g_n)f \, \mathrm{d}\mu \leq \frac{2}{n} \int f \, \mathrm{d}\mu \downarrow 0.$$

So $(G_n f)_{n\in\mathbb{N}}$ and $(g_n f)_{n\in\mathbb{N}}$ are sequences of R-integrable functions with

$$G_n f \geq gf \geq g_n f \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), G_n f \downarrow, g_n f \uparrow,$$
$$\inf_{n\in\mathbb{N}} \int G_n f - g_n f \, \mathrm{d}\mu = 0.$$

Then by Theorem 2.49 we conclude that gf is R-integrable. □

4.2 Lemma. *Let E be an R-complete Riesz space. Let $f : X \to E^+$ be an R-integrable function. Let $(B_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ with $B_1 \supset B_2 \supset \cdots$. Suppose $\mu(B_m) \downarrow 0$. Then*

$$\int \mathbf{1}_{B_m} f \, \mathrm{d}\mu \downarrow 0, \qquad \int \mathbf{1}_{X\setminus B_m} f \, \mathrm{d}\mu \uparrow \int f \, \mathrm{d}\mu.$$

Proof. Let $\sigma : X \to E^+$ be a σ-simple function with $\sigma \geq f$. Suppose $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$. For all $n, m \in \mathbb{N}$ we have $\mu(A_n) - \mu(B_m) \leq \mu(A_n \setminus B_m) \leq \mu(A_n)$. Therefore

$$\mu(A_n \setminus B_m) \uparrow_m \mu(A_n) \qquad (n \in \mathbb{N}).$$

We have (using Theorem 4.1)

$$\int \sigma \, \mathrm{d}\mu = \int \mathbf{1}_{B_m}\sigma \, \mathrm{d}\mu + \int \mathbf{1}_{X\setminus B_m}\sigma \, \mathrm{d}\mu \qquad (m \in \mathbb{N}).$$

By Lemma 2.11 we have

$$\begin{aligned}
\sup_{m\in\mathbb{N}} \int \mathbf{1}_{X\setminus B_m}\sigma \, \mathrm{d}\mu &= \sup_{m\in\mathbb{N}} \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} \mu(A_n \setminus B_m) a_n \\
&= \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} \sup_{m\in\mathbb{N}} \mu(A_n \setminus B_m) a_n \\
&= \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} \mu(A_n) a_n = \int \sigma \, \mathrm{d}\mu.
\end{aligned}$$

Therefore $-\int \mathbf{1}_{X\setminus B_m}\sigma \, \mathrm{d}\mu \downarrow -\int \sigma \, \mathrm{d}\mu$ and thus

$$\int \mathbf{1}_{B_m}\sigma \, \mathrm{d}\mu = \int \sigma \, \mathrm{d}\mu - \int \mathbf{1}_{X\setminus B_m}\sigma \, \mathrm{d}\mu \downarrow 0.$$

Because $0 \le f \le \sigma$ we have

$$\begin{aligned}
0 \le \int \mathbf{1}_{B_m} f \, \mathrm{d}\mu &\le \int \mathbf{1}_{B_m}\sigma \, \mathrm{d}\mu \downarrow 0, \\
\int \mathbf{1}_{X\setminus B_m} f \, \mathrm{d}\mu &= \int f \, \mathrm{d}\mu - \int \mathbf{1}_{B_m} f \, \mathrm{d}\mu \uparrow \int f \, \mathrm{d}\mu.
\end{aligned}$$

□

4.3 Theorem. *Let E be an R-complete Archimedean Riesz space. Let $f : X \to E^+$ be an R-integrable function. Let $g : X \to \mathbb{R}$ be a bounded measurable function. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of bounded measurable functions with $g_n \uparrow g$ (respectively $g_n \downarrow g$) μ-a.e.. Then*

$$\int g_n f \, \mathrm{d}\mu \uparrow \int gf \, \mathrm{d}\mu \qquad (\textit{respectively } \int g_n f \, \mathrm{d}\mu \downarrow \int gf \, \mathrm{d}\mu).$$

Proof. We may assume g and g_n are positive for all $n \in \mathbb{N}$. Suppose $g_n \uparrow g$ (the proof for the case that $g_n \downarrow g$ is done in a similar way). Let $M > 0$ be such that

$$g(x) \le M, \qquad g_n(x) \le M \qquad\qquad (x \in X, n \in \mathbb{N}).$$

Let $\varepsilon > 0$. Let $B_n := \{x \in X : g(x) - g_n(x) \ge \varepsilon\}$. Because $g_n \downarrow g$ μ-a.e., we have $\mu(\bigcap_{n\in\mathbb{N}} B_n) = 0$ and thus $\mu(B_n) \downarrow 0$. Then for all $n \in \mathbb{N}$:

$$\begin{aligned}
\int gf \, \mathrm{d}\mu - \int g_n f \, \mathrm{d}\mu = \int (g - g_n) f \, \mathrm{d}\mu &\le M \int \mathbf{1}_{B_n} f \, \mathrm{d}\mu + \varepsilon \int \mathbf{1}_{X\setminus B_n} f \, \mathrm{d}\mu \\
&\le M \int \mathbf{1}_{B_n} f \, \mathrm{d}\mu + \varepsilon \int f \, \mathrm{d}\mu.
\end{aligned}$$

We show $\int g_n f \, \mathrm{d}\mu \uparrow \int gf \, \mathrm{d}\mu$ by showing that the set $\{\int gf \, \mathrm{d}\mu - \int g_n f \, \mathrm{d}\mu : n \in \mathbb{N}\}$ has infimum 0 in E. We already know $\int gf \, \mathrm{d}\mu - \int g_n f \, \mathrm{d}\mu \ge 0$ for all $n \in \mathbb{N}$. So suppose $h \in E$ is such that $h \le \int gf \, \mathrm{d}\mu - \int g_n f \, \mathrm{d}\mu$. We have to show $h \le 0$. And by the above inequality we must have

$$h \le M \int \mathbf{1}_{B_n} f \, \mathrm{d}\mu + \varepsilon \int f \, \mathrm{d}\mu \qquad (n \in \mathbb{N}).$$

Then with Lemma 4.2 we have

$$h \leq M \inf_{n \in \mathbb{N}} \int \mathbf{1}_{B_n} f \, \mathrm{d}\mu + \varepsilon \int f \, \mathrm{d}\mu = \varepsilon \int f \, \mathrm{d}\mu.$$

So for all $\varepsilon > 0$ we have $h \leq \varepsilon \int f \, \mathrm{d}\mu$. Because E is Archimedean, we conclude $h \leq 0$. And thus $\sup_{n \in \mathbb{N}} \int g_n f \, \mathrm{d}\mu = \int g f \, \mathrm{d}\mu$. □

4.4 Theorem. *Let E be an R-complete Archimedean Riesz space. Let $(A_m)_{m \in \mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{A}$ with $\bigcup_{m \in \mathbb{N}} A_m = X$. Let $f : X \to E^+$. Suppose there exists a σ-simple function σ such that $0 \leq f \leq \sigma$ μ-a.e.. Suppose $f\mathbf{1}_{A_m}$ is R-integrable for all $m \in \mathbb{N}$. Then f is R-integrable and*

$$\int f \, \mathrm{d}\mu = \sup_{M \in \mathbb{N}} \sum_{m=1}^{M} \int f\mathbf{1}_{A_m} \, \mathrm{d}\mu.$$

Proof. We may and do assume $0 \leq f \leq \sigma$. Suppose $(\tau_n^m)_{n \in \mathbb{N}}$ and $(\sigma_n^m)_{n \in \mathbb{N}}$ are sequences of σ-simple functions for all $m \in \mathbb{N}$ with:

$$\tau_n^m \leq f\mathbf{1}_{A_m} \leq \sigma_n^m \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \qquad \tau_n^m \uparrow_n, \ \sigma_n^m \downarrow_n,$$
$$\sup_{n \in \mathbb{N}} \int \tau_n^m \, \mathrm{d}\mu = \int f\mathbf{1}_{A_m} \, \mathrm{d}\mu = \inf_{n \in \mathbb{N}} \int \sigma_n^m \, \mathrm{d}\mu \qquad (m \in \mathbb{N}).$$

We may and do assume that $0 \leq \tau_n^m, \ \sigma_n^m \leq \sigma$ for all $n, m \in \mathbb{N}$. We also may and do assume $\tau_n^m = \tau_n^m \mathbf{1}_{A_m}$, $\sigma_n^m = \sigma_n^m \mathbf{1}_{A_m}$ for all $n, m \in \mathbb{N}$.
Define

$$\tau_n := \sum_{m \in \mathbb{N}} \tau_n^m, \qquad \sigma_n = \sum_{m \in \mathbb{N}} \sigma_n^m.$$

Thus $\tau_n(x) = \tau_n^m(x)$ and $\sigma_n(x) = \sigma_n^m(x)$ for all $x \in A_m$ and $n, m \in \mathbb{N}$. Then we have

$$\tau_n \leq f \leq \sigma_n \qquad \mu\text{-a.e.} \quad (n \in \mathbb{N}), \qquad \tau_n \uparrow, \ \sigma_n \downarrow.$$

Suppose that $(a_i^{nm})_{i \in \mathbb{N}}, (b_i^{nm})_{i \in \mathbb{N}}$ are sequences in E and $(A_i^{nm})_{i \in \mathbb{N}}$, $(B_i^{nm})_{i \in \mathbb{N}}$ are sequences of disjoint sets in $\mathcal{A}$ for all $n, m \in \mathbb{N}$ such that

$$\tau_n^m = \sum_{i \in \mathbb{N}} a_i^{nm} \mathbf{1}_{A_i^{nm}}, \qquad \sigma_n^m = \sum_{i \in \mathbb{N}} b_i^{nm} \mathbf{1}_{B_i^{nm}} \qquad (n, m \in \mathbb{N}).$$

Because $\tau_n^m \mathbf{1}_{A_m} = \tau_n^m$ and $\sigma_n^m \mathbf{1}_{A_m} = \sigma_n^m$ for all $n, m \in \mathbb{N}$ we assume $A_i^{nm}, B_i^{nm} \subset A_m$ for all $i, n, m \in \mathbb{N}$. Then $\{A_i^{nm} : i \in \mathbb{N}, m \in \mathbb{N}\}$ and $\{B_i^{nm} : i \in \mathbb{N}, m \in \mathbb{N}\}$ are sets of disjoint sets in $\mathcal{A}$ for all $n \in \mathbb{N}$ and

$$\tau_n = \sum_{m \in \mathbb{N}} \sum_{i \in \mathbb{N}} a_i^{nm} \mathbf{1}_{A_i^{nm}}, \qquad \sigma_n = \sum_{m \in \mathbb{N}} \sum_{i \in \mathbb{N}} b_i^{nm} \mathbf{1}_{B_i^{nm}} \qquad (n \in \mathbb{N}).$$

Because E is R-complete and $0 \leq \tau_n \leq \sigma_n \leq \sigma$ for all $n \in \mathbb{N}$, τ_n and σ_n are σ-simple functions for all $n \in \mathbb{N}$ (by Theorem 2.30).
We will show that $\inf_{n \in \mathbb{N}} \int \sigma_n - \tau_n \, \mathrm{d}\mu = 0$. We already know $0 \leq \int \sigma_n - \tau_n \, \mathrm{d}\mu$ for all $n \in \mathbb{N}$. So let $h \leq \int \sigma_n - \tau_n \, \mathrm{d}\mu$ for all $n \in \mathbb{N}$. We proof $h \leq 0$.

For all $n, M \in \mathbb{N}$ we have:

$$h \leq \int \sigma_n - \tau_n \, \mathrm{d}\mu = \int (\sigma_n - \tau_n)\mathbf{1}_{\bigcup_{m=1}^{M} A_m} \, \mathrm{d}\mu + \int (\sigma_n - \tau_n)\mathbf{1}_{\bigcup_{m=M+1}^{\infty} A_m} \, \mathrm{d}\mu$$
$$\leq \sum_{m=1}^{M} \int \sigma_n^m - \tau_n^m \, \mathrm{d}\mu + \int \sigma \mathbf{1}_{\bigcup_{m=M+1}^{\infty} A_m} \, \mathrm{d}\mu,$$

because $\sigma_n - \tau_n \leq \sigma$ for all $n \in \mathbb{N}$. Then $h \leq \inf_{n\in\mathbb{N}} \sum_{m=1}^{M} \int \sigma_n^m - \tau_n^m \, \mathrm{d}\mu + \int \sigma \mathbf{1}_{\bigcup_{m=M+1}^{\infty} A_m} \, \mathrm{d}\mu = \int \sigma \mathbf{1}_{\bigcup_{m=M+1}^{\infty} A_m} \, \mathrm{d}\mu$. And because $\mathbf{1}_{\bigcup_{m=M}^{\infty} A_m} \downarrow 0$ by Theorem 4.3 we have

$$h \leq \inf_{M\in\mathbb{N}} \int \sigma \mathbf{1}_{\bigcup_{m=M}^{\infty} A_m} \, \mathrm{d}\mu = 0.$$

Thus

$$\inf_{n\in\mathbb{N}} \int \sigma_n - \tau_n \, \mathrm{d}\mu = 0.$$

So (see Comment 2.39) f is R-integrable. By Lemma 2.11

$$\int \tau_n \, \mathrm{d}\mu = \sup_{M\in\mathbb{N}} \sup_{I\in\mathbb{N}} \sum_{m=1}^{M} \sum_{i=1}^{I} \mu(A_i^{nm}) a_i^{nm} = \sup_{M\in\mathbb{N}} \sum_{m=1}^{M} \int \tau_n^m \, \mathrm{d}\mu.$$

Then

$$\int f \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \sup_{M\in\mathbb{N}} \sum_{m=1}^{M} \int \tau_n^m \, \mathrm{d}\mu = \sup_{M\in\mathbb{N}} \sum_{m=1}^{M} \sup_{n\in\mathbb{N}} \int \tau_n^m \, \mathrm{d}\mu$$
$$= \sup_{M\in\mathbb{N}} \sum_{m=1}^{M} \int f \mathbf{1}_{A_m} \, \mathrm{d}\mu.$$

□

4.5 Lemma. *Let E be an R-complete Riesz space. Let $s : X \to E$ be a simple function. Let $\sigma : X \to \mathbb{R}$ be σ-simple. Then $\sigma s : x \mapsto \sigma(x)s(x)$ is a σ-simple function. Suppose $s = \sum_{i=1}^{k} a_i \mathbf{1}_{A_i}$ for some $a_i \in E$ and $A_i \in \mathcal{A}$ with $\mu(A_i) < \infty$ for all $i \in \{1, \ldots, k\}$. Then*

$$\int \sigma s \, \mathrm{d}\mu = \sum_{i=1}^{k} (\int \sigma \mathbf{1}_{A_i} \, \mathrm{d}\mu) a_i.$$

Proof. It is sufficient to prove that $\sigma a \mathbf{1}_A$ is integrable for all $a \in E$ and $A \in \mathcal{A}$ with $\mu(A) < \infty$. By Theorem 4.1 $\mathbf{1}_A$ is measurable and bounded so $\sigma \mathbf{1}_A$ is a σ-simple function $X \to \mathbb{R}$. But then $\sigma f = a\sigma \mathbf{1}_A$ is σ-simple with $\int a\sigma \mathbf{1}_A \, \mathrm{d}\mu = (\int \sigma \mathbf{1}_A \, \mathrm{d}\mu) a$. □

4.6 Corollary. *Let E be an R-complete Riesz space. Let $f : X \to E$ be R-integrable. Suppose there exist $a, b \in E$ and an $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that*

$$a\mathbf{1}_A \leq f \leq b\mathbf{1}_A \quad \mu\text{-a.e.}$$

Then there exist sequences $(t_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ of simple functions with $t_n \leq f \leq s_n$ such that

$$\inf_{n\in\mathbb{N}} \int t_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int s_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

Consequently the function $x \mapsto g(x)f(x)$ is R-integrable for all integrable functions $g : X \to \mathbb{R}$.

Proof. It will be clear that the first statement follows from the Theorem 2.40, by looking at the functions $b\mathbf{1}_A - f$ and $f - a\mathbf{1}_A$.
Suppose E is R-complete. We may assume f and g to be positive (by Theorem 2.47).
Let $g : X \to \mathbb{R}^+$ be integrable and let σ be a σ-simple function with $\sigma \geq g$ μ-a.e.. Suppose $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ and $(C_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\sigma = \sum_{n\in\mathbb{N}} \lambda_n \mathbf{1}_{C_n}$. Then $g\mathbf{1}_{C_n}$ is a bounded measurable function for all $n \in \mathbb{N}$. By Theorem 4.1 $fg\mathbf{1}_{C_n}$ is R-integrable for all $n \in \mathbb{N}$. Let s be a simple function with $s \geq f$. Then σs is a σ-simple function $X \to E^+$ (Lemma 4.5) with $\sigma s \geq gf$. By Theorem 4.4 we conclude that gf is R-integrable. □

4.7 Definition. Let E be a Riesz space. A subset D of E is called *order dense* in E if for all $a \in E$ with $a > 0$ there exists a $d \in D^+$ such that $0 < d \leq a$.

4.8 Definition. Let E be a vector space over $\mathbb{Q}$. We call E a $\mathbb{Q}$*-vector space*. If E is endowed with an ordering, then we call E an *ordered* $\mathbb{Q}$*-vector space* if

$$x, y \in E,\ x \leq y \qquad \Longrightarrow \qquad \begin{cases} x + a \leq y + a & (a \in E), \\ \lambda x \leq \lambda y & (\lambda \in \mathbb{Q}^+). \end{cases}$$

We call E a $\mathbb{Q}$*-Riesz space* if E is an ordered $\mathbb{Q}$-vector space that is a lattice.
Let E be a Riesz space (or a $\mathbb{Q}$-Riesz space). Let D be a $\mathbb{Q}$-vector subspace of E. Then D is called a $\mathbb{Q}$*-Riesz subspace* if

$$x, y \in D \qquad \Longrightarrow \qquad x \vee y,\ x \wedge y \in D.$$

4.9 Theorem. *Let E be a Riesz space. Suppose there exists a countable order dense subset of E. Then there exists a countable order dense $\mathbb{Q}$-Riesz subspace of E.*

Proof. Suppose D is a countable order dense subset of E. Then the following sets are countable:

$$D_1 := \{\lambda d : \lambda \in \mathbb{Q}, d \in D\},$$
$$D_m := \{d_1 + d_2, d_1 - d_2, d_1 \wedge d_2, d_1 \vee d_2 : d_1 \in D_{m-1}, d_2 \in D_{m-1}\} \quad (m \geq 2).$$

Therefore the set $C := \bigcup_{m\in\mathbb{N}} D_m$ is countable. Notice that $D_m \subset D_{m+1}$. And for $a, b \in D_m$, we have $\lambda a + \mu b,\ a \vee b,\ a \wedge b \in D_{m+1}$ for all $\lambda, \mu \in \mathbb{Q}$. It will be clear that this implies that C is a $\mathbb{Q}$-Riesz subspace of E. Since D is contained in C, C is order dense in E. □

4.10 Theorem. *Let E be an Archimedean Riesz space. A $\mathbb{Q}$-Riesz subspace D of E is order dense if and only if for all $a \in E^+$:*

$$a = \sup\{d \in D : d \leq a\}.$$

Proof. The "if" part follows by $\sup\{d \in D : d \leq a\} = \sup\{d \in D^+ : d \leq a\}$ for all $a \geq 0$.
Suppose D is order dense in E. Suppose $a \in E^+$ is such that there exists an $h \in E$ with $h < a$ and with $h \geq d$ for all $d \in D$ with $d \leq a$. Then there exists a $e \in D$ with $0 < e \leq a - h$. For all $d \in D$ with $d \leq a$ we have

$$d + e \leq h + a - h = a.$$

By induction we have for all $d \in D$ with $d \leq a$ and for all $n \in \mathbb{N}$:

$$d + ne \leq a.$$

Since E is Archimedean, this leads to a contradiction. We conclude that a is the supremum of $\{d \in D : d \leq a\}$. ☐

4.11 Definition. Let E be a Riesz space. E is called *countably generated* if E possesses a countable order dense subset (or, equivalently, a countable order dense $\mathbb{Q}$-Riesz space).

4.12 Example The Riesz spaces $\mathbb{R}^{\mathbb{N}}, \ell^\infty, c, c_0, \ell^1$ and c_{00} are countably generated.
Let E be one of the Riesz spaces $\mathbb{R}^{\mathbb{N}}, \ell^\infty, c, c_0, \ell^1, c_{00}$. Let $D = \{\lambda e_n : \lambda \in \mathbb{Q}, n \in \mathbb{N}\}$. Notice that $D \subset E$. Let $a = (a_n)_{n\in\mathbb{N}} \in E$. Suppose $a > 0$. Then there exists an $n \in \mathbb{N}$ such that $a_n > 0$. Let $\lambda \in \mathbb{Q}$ be such that $0 < \lambda \leq a_n$. Then $0 < \lambda e_n \leq a$. Thus D a countable set that is order dense in E.

4.13 Theorem. *Let Y be a separable metrisable space. Then $C(Y)$ is a countably generated Riesz space.*

Proof. Let $d : Y \times Y \to [0, \infty)$ be a metric. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y such that $\{y_n : n \in \mathbb{N}\}$ is dense in Y. Define $h_{q,s,n}$ for $q, s \in \mathbb{Q}^+ \setminus \{0\}$ and $n \in \mathbb{N}$ by

$$h_{q,s,n} : Y \to \mathbb{R}, \qquad h_{q,s,n}(y) = q\Big(\big(1 - \frac{1}{s} d(y_n, y)\big) \vee 0\Big).$$

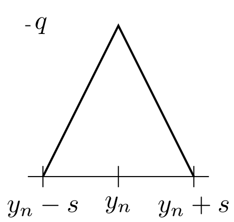

Figure 5: $h_{q,s,n}$ for $Y = \mathbb{R}$.

Note that $h_{q,s,n}(Y) \subset [0, q]$ and $h_{q,s,n}(y) = 0$ for $y \in Y$ with $d(y_n, y) > s$. We will show that the countable set $D = \{h_{q,s,n} : q, s \in \mathbb{Q}^+ \setminus \{0\}, n \in \mathbb{N}\}$ is order dense in $C(Y)$.

Let $f \in C(Y)$. Suppose $f > 0$. Then there exists an $\varepsilon > 0$, an $x \in Y$ and an $r > 0$ such that the ball with centre x and radius r, denoted by $B(x,r)$, is contained in $f^{-1}(\varepsilon,\infty)$. Let $q \in \mathbb{Q}$ be such that $0 < q < \varepsilon$. Let $n \in \mathbb{N}$ be such that $y_n \in B(x,r)$ and let $s \in \mathbb{Q}$, $s > 0$ be such that $B(y_n,s) \subset B(x,r)$. Then $h_{q,s,n}(y) \le q < \varepsilon \le f(y)$ for all $y \in B(y_n,s)$ and $h_{q,s,n}(y) = 0 \le f(y)$ for $y \in Y \setminus B(y_n,s)$. Therefore $0 < h_{q,s,n} \le f$ in $C(Y)$. Thus the set D is order dense in $C(Y)$. □

4.14 Definition. Let E be a Riesz space. We call a function $f : X \to E$ *left-order measurable* if $\{x \in X : f(x) \le a\} \in \mathcal{A}$ for all $a \in E$. We call f *right-order measurable* if $\{x \in X : f(x) \ge a\} \in \mathcal{A}$ for all $a \in E$. We call f *order measurable* if f is both left-order measurable and right-order measurable.

4.15 Lemma. *Let E be a Riesz space. Suppose $\sigma : X \to E$ is a σ-step function. Then σ is an order measurable function.*

Proof. Suppose $\sigma = \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n}$ for a sequence $(a_n)_{n\in\mathbb{N}}$ in E and a sequence of disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$. Let $a \in E$. Let $K = \{n \in \mathbb{N} : a_n \le a\}$. Then $\{x \in X : \sigma(x) \le a\} = \bigcup_{n\in K} A_n \in \mathcal{A}$. The same can be done with $-\sigma$ and $-a$ instead of σ and a. Then $\{x \in X : \sigma(x) \ge a\} = \{x \in X : -\sigma(x) \le -a\} \in \mathcal{A}$. □

Notation 8

For subsets A, B of X we write $A \overset{\mu-\text{a.e.}}{=} B$ if $\mu((A \setminus B) \cup (B \setminus A)) = 0$.

4.16 Theorem. *Let E be a Riesz space. Let $f : X \to E$ be a function.*

- *Suppose there are sequences of σ-step functions $(\rho_n)_{n\in\mathbb{N}}$ such that $\rho_n \downarrow f$ μ-a.e.. Then f is right-order measurable.*
- *Suppose there are sequences of σ-step functions $(\pi_n)_{n\in\mathbb{N}}$ such that $\pi_n \uparrow f$ μ-a.e.. Then f is left-order measurable.*

Let E be a countably generated Archimedean Riesz space. Suppose $f \ge 0$.

- *Suppose f is left-order measurable. Then there are sequences of step functions $(\pi_n)_{n\in\mathbb{N}}$ such that $\pi_n \uparrow f$ μ-a.e..*

Proof. • Let $a \in E$. Let $Y \subset X$ be the set for which

$$x \in Y \qquad \iff \qquad \rho_n(x) \downarrow f(x).$$

Then $Y \overset{\mu-\text{a.e.}}{=} X$ and thus

$$\begin{aligned}
\{x \in X : f(x) \ge a\} &\overset{\mu-\text{a.e.}}{=} \{x \in Y : f(x) \ge a\} \\
&= \{x \in Y : \inf_{n\in\mathbb{N}} \rho_n(x) \ge a\} \\
&= \bigcap_{n\in\mathbb{N}} \{x \in Y : \rho_n(x) \ge a\} \\
&\overset{\mu-\text{a.e.}}{=} \bigcap_{n\in\mathbb{N}} \{x \in X : \rho_n(x) \ge a\}.
\end{aligned}$$

By Lemma 4.15 we have $\{x \in X : \rho_n(x) \ge a\} \in \mathcal{A}$. Because the measure space $(X, \mathcal{A}, \mu)$ is complete, $\{x \in X : f(x) \ge a\} \in \mathcal{A}$.

In a similar way we have $\{x \in X : f(x) \leq a\} \overset{\mu-\text{a.e.}}{=} \bigcap_{n\in\mathbb{N}}\{x \in X : \pi_n(x) \leq a\}$.
• Let E be a countably generated Riesz space. Suppose $\{x \in X : f(x) \leq a\} \in \mathcal{A}$ for all $a \in E$. Let $D = \{d_n : n \in \mathbb{N}\}$ be a countable order dense $\mathbb{Q}$-Riesz space. Define

$$\tau_n(x) = \sup\{d_j : 1 \leq j \leq n, d_j \leq f(x)\}.$$

Because D is order dense, by Theorem 4.10 we conclude $\tau_n \uparrow f$. $\square$

4.17 Lemma. *Let E be a countably generated Riesz space. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of σ-simple functions with $\sigma_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that $\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = 0$. Then for almost all $x \in X$, the infimum of $\{\sigma_n(x) : n \in \mathbb{N}\}$ exists and is equal to 0 (however, see Example 4.20).*

Proof. Let D be a countable order dense $\mathbb{Q}$-Riesz space.
Let A be the set for which

$$x \in A \iff \forall a \in E^+ \Big[\forall n \in \mathbb{N}\big[a \leq \sigma_n(x)\big] \Rightarrow a = 0\Big].$$

So A consists of all $x \in X$ for which the infimum of $\{\sigma_n(x) : n \in \mathbb{N}\}$ exists and is equal to 0. We have to prove that this set is almost equal to X, i.e. $\mu(X \setminus A) = 0$. We have

$$x \in X \setminus A \iff \exists a \in E^+ \ \forall n \in \mathbb{N}\big[0 < a \leq \sigma_n(x)\big].$$

Let $x \in X$. Suppose there exists an $a > 0$ is such that $a \leq \sigma_n(x)$ for all $n \in \mathbb{N}$. Then there exists a $d \in D$ such that $0 < d \leq a$, so we have

$$x \in X \setminus A \iff \exists d \in D \ \forall n \in \mathbb{N}\big[0 < d \leq \sigma_n(x)\big].$$

Let $E_d = \bigcap_{n\in\mathbb{N}}\{x \in X : d \leq \sigma_n(x)\}$. Then

$$X \setminus A = \bigcup_{d\in D^+\setminus\{0\}} E_d.$$

Because D is countable, we have $X \setminus A \in \mathcal{A}$ by Lemma 4.15. For all $d \in D^+ \setminus \{0\}$ we have:

$$d\mu(E_d) = \int d\mathbf{1}_{E_d} \, \mathrm{d}\mu \leq \int \sigma_n \, \mathrm{d}\mu \qquad (n \in \mathbb{N}).$$

So therefore $d\mu(E_d) = 0$, i.e. $\mu(E_d) = 0$. But then

$$\mu(X \setminus A) \leq \sum_{d\in D^+\setminus\{0\}} \mu(E_d) = 0.$$

$\square$

4.18 Theorem. *Let E be a countably generated Riesz space. Let f be an R-integrable function. Suppose $(\sigma_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ are sequences of σ-simple functions with:*

$$\sigma_n \geq f \geq \tau_n \quad \mu\text{-a.e.} \quad (n \in \mathbb{N}),$$

$$\inf_{n\in\mathbb{N}} \int \sigma_n \, \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \, \mathrm{d}\mu.$$

Then for almost all $x \in X$

$$\inf_{n\in\mathbb{N}} \sigma_n(x) = f(x) = \sup_{n\in\mathbb{N}} \tau_n(x).$$

Consequently (by Theorem 4.16), f is order measurable.

Proof. $\sigma_n - \tau_m$ is a σ-simple function for all $n \in \mathbb{N}$ (see Theorem 2.23). And $\inf_{n,m\in\mathbb{N}} \int \sigma_n - \tau_m \, \mathrm{d}\mu = 0$ (see Comment 2.39). So by Lemma 4.17 (we may assume $\sigma_n - \tau_m \geq 0$ for all $n, m \in \mathbb{N}$) follows that for almost all $x \in X$ the infimum of $\{\sigma_n(x) - \tau_m(x) : n \in \mathbb{N}\}$ exists and is equal to 0. For such $x \in X$:

$$\forall n, m \in \mathbb{N}[k \leq \sigma_n(x) - \tau_m(x)] \text{ implies } k \leq 0 \qquad (k \in E).$$

So because $0 \leq \sigma_n(x) - f(x) \leq \sigma_n(x) - \tau_m(x)$ for all $n, m \in \mathbb{N}$ and almost all $x \in X$,

$$\forall n \in \mathbb{N}[k \leq \sigma_n(x) - f(x)] \text{ implies } k \leq 0 \qquad (k \in E).$$

I.e. the set $\{\sigma_n(x) - f(x) : n \in \mathbb{N}\}$ has an infimum for almost all $x \in X$, which is equal to 0. So for almost all $x \in X$ we have $\inf_{n\in\mathbb{N}} \sigma_n(x) = f(x)$. Because $0 \leq f(x) - \tau_m(x) \leq \sigma_n(x) - \tau_m(x)$ for all $n, m \in \mathbb{N}$ and almost all $x \in X$, we also have $\sup_{n\in\mathbb{N}} \tau_n(x) = f(x)$ for almost all $x \in X$. □

4.19 Corollary of Lemma 4.17. *Let E be a countably generated Riesz space. Let $f : X \to E^+$ be an R-integrable function. If $\int f \, \mathrm{d}\mu = 0$, then $f = 0$ μ-a.e..*

4.20 Example. We will give an example of a Riesz space E and an R-integrable function $f : X \to E^+$ for which $\int f \, \mathrm{d}\mu = 0$ and for which $f(x) \neq 0$ for all $x \in X$ (so E is an example of a Riesz space that is not countably generated).
Let $X = (0, 1]$. Consider measure space $((0, 1], \mathcal{B}, \lambda)$ and let E be the Riesz space $\mathcal{L}((0, 1], \mathcal{B}, \lambda)$. Let $f : X \to E$ be given by

$$f(x) = \mathbf{1}_{\{x\}} \qquad (x \in X).$$

Then $f(x) > 0$ for all $x \in X$.
For $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, 2^n\}$ let I_{ni} be the interval $(\frac{i-1}{2^n}, \frac{i}{2^n}]$. And for $n \in \mathbb{N}$ let $\sigma_n : X \to E$ be given by

$$\sigma_n(x) := \mathbf{1}_{I_{ni}} \qquad (x \in I_{ni}, n \in \mathbb{N}, i \in \{1, 2, \ldots, 2^n\}).$$

Then σ is simple and $\sigma_1 \geq \sigma_2 \geq \cdots$ and $\sigma_n \geq f$ for all $n \in \mathbb{N}$. And

$$\int \sigma_n \, \mathrm{d}\mu = \sum_{i\in\{1,2,\ldots,2^n\}} \lambda(I_{ni})\mathbf{1}_{I_{ni}} = \sum_{i\in\{1,2,\ldots,2^n\}} 2^{-n}\mathbf{1}_{I_{ni}} = 2^{-n}\mathbf{1}_X \downarrow 0.$$

Because $f \geq 0$ we conclude f is R-integrable with $\int f \, \mathrm{d}\mu = 0$.

4.21 Theorem. *Let E be a Banach lattice. Let $f : X \to E$ be a strongly measurable function. Then f is order-measurable.*

Proof. By Theorem 1.25 f is Borel measurable and μ-essentially separably valued. The set $\{b \in E : b \leq a\}$ is closed by Lemma 1.6. So because f is Borel measurable, we have $\{x \in X : f(x) \leq a\} = f^{-1}(\{b \in E : b \leq a\}) \in \mathcal{A}$. □

4.22 Theorem. *Let E be a countably generated Banach lattice, with σ-order continuous norm $\|\cdot\|$. Let $f : X \to E$ be order measurable. Then f is strongly measurable (however, see Example 4.23).*

Proof. We show that f^+ is strongly measurable. In the same way f^- then is strongly measurable and therefore so is $f = f^+ - f^-$ (by Theorem 1.24).
f^+ is also order measurable. By Theorem 4.16 there are step functions $(\pi_n)_{n\in\mathbb{N}}$ with $\pi_n \uparrow f^+$ μ-a.e.. Because $\|\cdot\|$ is σ-order continuous, we have $\|\pi_n - f^+\| \to 0$ μ-a.e.. Therefore f is strongly measurable. □

4.23 Example of an order measurable function that is not strongly measurable.
Let $X = L^1(\mathbb{R}, \mathcal{B}, \lambda)$ and let $\mathcal{A}$ be given by

$$A \in \mathcal{A} \iff \begin{cases} A \text{ is meager,} \\ X \setminus A \text{ is meager.} \end{cases}$$

$\emptyset$ is meager, thus an element of $\mathcal{A}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$. If A_n is meager for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n$ is meager. If $X \setminus A_m$ is meager for some $m \in \mathbb{N}$, then $X \setminus \left(\bigcup_{n\in\mathbb{N}} A_n\right)$ is meager since it is a subset of the meager set $X \setminus A_m$. So $\mathcal{A}$ is a σ-algebra on X. Equip X with this σ-algebra. Let $\mu : \mathcal{A} \to [0, \infty)$ be the measure given by

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is meager,} \\ 1 & \text{if } X \setminus A \text{ is meager.} \end{cases}$$

Because a subset of a meager set is meager, $(X, \mathcal{A}, \mu)$ is a complete measure space. We consider this measure space.

Let E be the Banach lattice $L^1(\mathbb{R}, \mathcal{B}, \lambda)$. Let $I : X \to E$ be the identity map, $I(f) = f$. Then $I^{-1}(B) = B \notin \mathcal{A}$, for B a nonempty ball. So I is not Borel measurable and thus not strongly measurable.

Let $B = \{f \in L^1(\mathbb{R}, \mathcal{B}, \lambda) : f \geq 0\}$ (notice B is closed). We will show $B \in \mathcal{A}$ by showing that B has empty interior. Let $f \in B$, $\varepsilon > 0$. Let $\Gamma \in \mathcal{B}$ be a subset of $f^{-1}([0, \frac{\varepsilon}{2}])$ with $0 < \lambda(\Gamma) \leq 1$. Then $g = f - \varepsilon \mathbf{1}_\Gamma$ is integrable. And $g(x) \leq -\frac{\varepsilon}{2} < 0$ for all $x \in \Gamma$. So $g \not\geq 0$ in $L^1(\mathbb{R}, \mathcal{B}, \lambda)$. And $\|g - f\|_{L^1} = \|\varepsilon \mathbf{1}_\Gamma\|_{\mathcal{L}^1} = \varepsilon\lambda(\Gamma) \leq \varepsilon$. So for all $f \in B$ and $\varepsilon > 0$ there exists a $g \in L^1(\mathbb{R}, \mathcal{B}, \lambda)$ with $\|f - g\|_{L^1} \leq \varepsilon$ and $g \notin B$. Therefore B has empty interior. And thus B is meager.
Then also the sets $\{f \in L^1(\mathbb{R}, \mathcal{B}, \lambda) : f \geq a\} = B + a$ and $\{f \in L^1(\mathbb{R}, \mathcal{B}, \lambda) : f \leq a\} = a - B$ are elements of $\mathcal{A}$ for all $a \in L^1(\mathbb{R}, \mathcal{B}, \lambda)$. And so I is order measurable.
Notice that by Theorem 4.22 and Theorem 1.13 now follows that $L^1(\mathbb{R}, \mathcal{B}, \lambda)$ is not countably generated.

4.24 Example of an positive order measurable function that is bounded by a σ-simple function, which is not R-integrable.
Recall Example 2.17 (and Example 2.43). f and g are σ-simple functions for which

$$\begin{aligned} f - g &= (e_1, -e_1, e_3, -e_3, \dots), \\ (f - g)^+ &= (e_1, 0, e_3, 0, \dots), \end{aligned}$$

And $(f-g)^+$ is not σ-simple and not R-integrable (this is also mentioned in Example 2.43). But it is order measurable and f is a σ-simple function with $0 \leq (f-g)^+ \leq f$.

4.25 Definition. Let E be a Riesz space. We call a function $f : X \to E$ *R-measurable* if for every positive R-integrable function $h : X \to E^+$ holds that

$$(f \vee -h) \wedge h$$

is R-integrable.

4.26 Comment. The set of R-measurable functions is a lattice, just as the set of order measurable functions. But the question whether a linear combination of R-measurable functions is R-measurable, is still open (also the case for order measurable). But if E is R-complete and the set of R-measurable functions is a Riesz space $X \to E$ (under some assumptions on E), then the space of R-integrable functions is a Riesz ideal in the space of R-measurable functions by definition of R-measurability.

5 Examples

5.1 Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is uniformly continuous if and only if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_n \downarrow 0$ and for which*

$$|s-t| \leq 2^{-n} \Longrightarrow |f(s)-f(t)| \leq \varepsilon_n \qquad (s,t \in \mathbb{R}, n \in \mathbb{N}).$$

Proof. The "if" part is trivial.
Suppose g is uniformly continuous. We first show that $\{|f(s)-f(t)| : s,t \in \mathbb{R}, |s-t| \leq 1\}$ is bounded. Let $\delta > 0$ be such that $|s-t| \leq \delta$ implies that $|f(s)-f(t)| \leq 1$. If $1 \leq \delta$, then the set $\{|f(s)-f(t)| : s,t \in \mathbb{R}, |s-t| \leq 1\}$ is bounded by 1. In case $\delta < 1$, assume that $m \in \mathbb{N}$ is such that $\frac{1}{m} \leq \delta$. Let $s,t \in \mathbb{R}$ with $t > s$ and $t-s \leq 1$. Then there are $s_1, \dots, s_{m+1} \in [s,t]$ with $s = s_1 \leq \dots \leq s_{m+1} = t$ and $s_{i+1} - s_i \leq \frac{1}{m}$ for $i \in \{1,\dots,m\}$. And thus

$$|f(s)-f(t)| = \left| \sum_{i=1}^{m} f(s_{i+1}) - f(s_i) \right| \leq \sum_{i=1}^{m} \big|f(s_{i+1}) - f(s_i)\big| \leq m.$$

Therefore $\{|f(s)-f(t)| : s,t \in \mathbb{R}, |s-t| \leq 1\}$ is bounded.
Notice that $\{|f(s)-f(t)| : s,t \in \mathbb{R}, |s-t| \leq 2^{-n}\}$ is bounded for all $n \in \mathbb{N}$. Let

$$\varepsilon_n := \sup\{|f(s)-f(t)| : s,t \in \mathbb{R}, |s-t| \leq 2^{-n}\}.$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $|s-t| \leq 2^{-N}$ implies $|f(s)-f(t)| \leq \varepsilon$, then $n \geq N$ implies $\varepsilon_n \leq \varepsilon$. So $\varepsilon_n \downarrow 0$. □

5.2 Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. We call a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_n \downarrow 0$ and for which

$$|s-t| \leq 2^{-n} \Longrightarrow |f(s)-f(t)| \leq \varepsilon_n \qquad (s,t \in \mathbb{R}, n \in \mathbb{N}).$$

a *continuity sequence* for f.
Notice that if $(\varepsilon_n)_{n\in\mathbb{N}}$ is a continuity sequence for f, and $(\delta_n)_{n\in\mathbb{N}}$ is a sequence in $(0,\infty)$ with $\delta_n \downarrow 0$ and $\delta_n \geq \varepsilon_n$ for all $n \in \mathbb{N}$, then $(\delta_n)_{n\in\mathbb{N}}$ is also a continuity sequence for f.

5.3 Lemma. *Let E be a Riesz space. For $a,b,c,d \in E$ we have*

$$|a \vee b - c \vee d| \leq |a-c| \vee |b-d|.$$

Proof. Let $a,b,c,d \in E$. Then

$$\begin{aligned} a = a-c+c \leq |a-c| + c \vee d \leq |a-c| \vee |b-d| + c \vee d, \\ b = b-d+d \leq |b-d| + c \vee d \leq |a-c| \vee |b-d| + c \vee d. \end{aligned}$$

Therefore we have $a \vee b \leq |a-c| \vee |b-d| + c \vee d$ and thus

$$a \vee b - c \vee d \leq |a-c| \vee |b-d|.$$

And in the same way $c \vee d - a \vee b \leq |a-c| \vee |b-d|$. □

5.4 Lemma. *Let $f,g : \mathbb{R} \to \mathbb{R}$. Suppose f and g are uniformly continuous with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Then $f \vee g$ is uniformly continuous with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$.*

Proof. This is a consequence of Lemma 5.3:

$$|f \vee g(s) - f \vee g(t)| \leq |f(s) - f(t)| \vee |g(s) - g(t)|.$$

$\square$

5.5 Theorem. *Let G be a subset of $\mathbb{R}^{\mathbb{R}}$ that is bounded from above (respectively from below) in $\mathbb{R}^{\mathbb{R}}$, i.e. the set $\{g(s) : g \in G\}$ is bounded from above (respectively from below) for all $s \in \mathbb{R}$. Suppose that G is a set of uniformly continuous functions with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Then $f := \sup_{g\in G} g$ (respectively $f := \inf_{g\in G} g$) is a uniformly continuous function with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$.*

Proof. Assume that G is bounded from above. Let $f := \sup_{g\in G} g$ in $\mathbb{R}^{\mathbb{R}}$. Let $s, t \in \mathbb{R}$. Let $\varepsilon > 0$. Let $g_1, g_2 \in G$ be such that

$$|f(s) - g_1(s)| < \varepsilon, \qquad |f(t) - g_2(t)| < \varepsilon.$$

Suppose G is a collection of uniformly continuous functions with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Let $n \in \mathbb{N}$. Then

$$\begin{aligned} |s - t| < 2^{-n} \implies & |f(s) - f(t)| \leq |f(s) - g_1 \vee g_2(s)| + |g_1 \vee g_2(s) - g_1 \vee g_2(t)| \\ & + |g_1 \vee g_2(t) - f(t)| \\ & \leq 2\varepsilon + \varepsilon_n. \end{aligned}$$

This can be done for all $\varepsilon > 0$ and all $s, t \in \mathbb{R}$. Therefore f is uniformly continuous with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$.

In case G is bounded from below, then $-G = \{-g : g \in G\}$ is bounded from above and thus $f = \inf_{g\in G} g = -\sup_{g\in -G} g$ is uniformly continuous with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. $\square$

5.6 Lemma. *Let $g_1, \ldots, g_m : \mathbb{R} \to \mathbb{R}$ be uniformly continuous functions with continuity sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Then $\sum_{i=1}^m \lambda_i g_i$ is a uniformly continuous function with continuity sequence $(\alpha\varepsilon_n)_{n\in\mathbb{N}}$, where $\alpha = \sum_{i=1}^m |\lambda_i|$.*

Proof. Let $s, t \in \mathbb{R}$ with $|s - t| \leq 2^{-n}$. Then

$$\left| \sum_{i=1}^m \lambda_i g_i(s) - \sum_{i=1}^m \lambda_i g_i(t) \right| \leq \sum_{i=1}^m |\lambda_i||g_i(s) - g_i(t)| \leq \sum_{i=1}^m |\lambda_i|\varepsilon_n.$$

$\square$

5.7 Definition. Let $g : \mathbb{R} \to \mathbb{R}$. The *left translation* of g with respect to an element $s \in \mathbb{R}$, denoted by $L_s g$ is defined by:

$$L_s g(t) := g(t - s) \qquad (t \in \mathbb{R}).$$

5.8 Comment. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. For a bounded set $A \subset \mathbb{R}$, the set $g(A)$ is bounded. Therefore $\{L_s g(t) : s \in [(i-1)2^{-n}, i2^{-n})\} = \{g(u) : u \in \{t - s : s \in [(i-1)2^{-n}, i2^{-n})\}\}$ is bounded for all $t \in \mathbb{R}$.

5.9 Lemma. *Let $g : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function in $C(\mathbb{R})$ with continuity sequence $(\varepsilon_k)_{k\in\mathbb{N}}$. For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ define (see Comment 5.8)*

$$a_{ni} := \sup_{s\in[(i-1)2^{-n}, i2^{-n})} L_s g,$$
$$b_{ni} := \inf_{s\in[(i-1)2^{-n}, i2^{-n})} L_s g.$$

Then a_{ni} and b_{ni} are uniformly continuous with continuity sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ for all $i \in \mathbb{N}$ and $n \in \mathbb{N}$.
For all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$a_{ni} - b_{ni} \le \varepsilon \mathbf{1}_{\mathbb{R}} \qquad (n \ge N,\ i \in \mathbb{Z}).$$

Proof. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a continuity sequence for g. Then $L_x g$ is uniformly continuous with continuity sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ for all $x \in \mathbb{R}$. By Theorem 5.5 a_{ni} and b_{ni} are uniformly continuous with continuity sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ for all $i \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that $\varepsilon_N \le \varepsilon$. Then for $n \ge N$ and $x \in \mathbb{R}$:

$$\begin{aligned} a_{ni}(x) - b_{ni}(x) &= \sup_{s\in[(i-1)2^{-n}, i2^{-n})} L_s g(x) - \inf_{s\in[(i-1)2^{-n}, i2^{-n})} L_s g(x) \\ &= \sup_{s,t\in[(i-1)2^{-n}, i2^{-n})} L_s g(x) - L_t g(x) \\ &\le \sup_{u,v\in\mathbb{R}:|u-v|\le 2^{-n}} |g(u) - g(v)| \le \varepsilon_n \le \varepsilon_N \le \varepsilon. \end{aligned}$$

□

5.10 Comment. Let E be a Riesz space. Let $f, g : X \to E$ be R-integrable functions. Let $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}}, (\rho_n)_{n\in\mathbb{N}}$ and $(\pi_n)_{n\in\mathbb{N}}$ be sequences of σ-simple functions with

$$\sigma_n \ge f \ge \tau_n \quad \mu\text{-a.e.}, \qquad\qquad \rho_n \ge g \ge \pi_n \quad \mu\text{-a.e.},$$
$$\inf_{n\in\mathbb{N}} \int \sigma_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \,\mathrm{d}\mu, \qquad \inf_{n\in\mathbb{N}} \int \rho_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \pi_n \,\mathrm{d}\mu.$$

We will show that if there exists an $A \in \mathcal{A}$ such that

$$\sigma_n = \sigma_n \mathbf{1}_A \quad \mu\text{-a.e.}, \qquad f = \mathbf{1}_A \quad \mu\text{-a.e.} \qquad \tau_n = \tau_n \mathbf{1}_A \quad \mu\text{-a.e.},$$
$$\rho_n = \rho_n \mathbf{1}_{X\setminus A} \quad \mu\text{-a.e.}, \qquad g = \mathbf{1}_{X\setminus A} \quad \mu\text{-a.e.} \qquad \pi_n = \pi_n \mathbf{1}_{X\setminus A} \quad \mu\text{-a.e.},$$

then $f + g$ is R-integrable and $\int f + g \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$.
Suppose there exists such an $A \in \mathcal{A}$ as above. We may assume $\sigma_n = \sigma_n \mathbf{1}_A$, $\tau_n = \tau_n \mathbf{1}_A$, $\rho_n = \rho_n \mathbf{1}_{X\setminus A}$ and $\pi_n = \pi_n \mathbf{1}_{X\setminus A}$ by Theorem 2.22. By Theorem 2.23 $\sigma_n + \rho_m$ and $\tau_n + \pi_m$ are σ-simple with $\sigma_n + \rho_m \ge f + g \ge \tau_n + \pi_m$ μ-a.e. for all $n, m \in \mathbb{N}$,

$$\int \sigma_n + \rho_m \,\mathrm{d}\mu = \int \sigma_n \,\mathrm{d}\mu + \int \rho_m \,\mathrm{d}\mu \qquad (n, m \in \mathbb{N}),$$
$$\int \tau_n + \pi_m \,\mathrm{d}\mu = \int \tau_n \,\mathrm{d}\mu + \int \pi_m \,\mathrm{d}\mu \qquad (n, m \in \mathbb{N}).$$

Then $\inf_{n,m\in\mathbb{N}} \int \sigma_n + \rho_m \,\mathrm{d}\mu = \inf_{n\in\mathbb{N}} \int \sigma_n \,\mathrm{d}\mu + \inf_{m\in\mathbb{N}} \int \rho_m \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int \tau_n \,\mathrm{d}\mu + \sup_{m\in\mathbb{N}} \int \pi_m \,\mathrm{d}\mu = \sup_{n,m\in\mathbb{N}} \int \tau_n + \pi_m \,\mathrm{d}\mu$. So $f + g$ is R-integrable with $\int f + g \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$.

5.11 Definition. Consider the measure space $(X, \mathcal{B}, \lambda)$ where X is a Lebesgue measurable subset of $\mathbb{R}$. A measurable function $f : X \to \mathbb{R}$ is called *μ-essentially compactly supported* if there exists a compact set $K \subset X$ such that $f(x) = 0$ for μ-almost all $x \in X \setminus K$.

5.12 Lemma. *Let $g : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function in $C(\mathbb{R})$ with continuity sequence $(\varepsilon_k)_{k\in\mathbb{N}}$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a λ-essentially compactly supported σ-simple function. Then $\rho Lg : \mathbb{R} \to C(\mathbb{R})$ given by $x \mapsto \rho(x)L_x g$ is R-integrable. Also $\int \rho Lg \,\mathrm{d}\lambda$ is uniformly continuous with continuity sequence $((\int |\rho| \,\mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$ and*

$$\Big((\mathrm{R})\text{-}\int \rho Lg \,\mathrm{d}\lambda\Big)(s) = \int \rho(x)L_x g(s) \,\mathrm{d}x \qquad (s \in \mathbb{R}).$$

Proof. Let K be a compact set such that $\rho(x) = 0$ for λ-almost all $x \in \mathbb{R} \setminus K$. We may assume that $\rho(x) = 0$ for all $x \in X \setminus K$, i.e. $\rho = \rho\mathbf{1}_K$.
Let $A = \{x \in \mathbb{R} : \rho(x) > 0\}$. Then $\rho^+ = \rho^+\mathbf{1}_A$.
Let a_{ni} and b_{ni} be defined as in Lemma 5.9 for all $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. Define

$$\sigma_n := \sum_{i\in\mathbb{Z}} a_{ni}\mathbf{1}_{[(i-1)2^{-n},i2^{-n})}, \qquad \tau_n := \sum_{i\in\mathbb{Z}} b_{ni}\mathbf{1}_{[(i-1)2^{-n},i2^{-n})}.$$

The functions $\mathbf{1}_A\sigma_n$ and $\mathbf{1}_A\tau_n$ are simple functions, because A is a subset of the compact set K. We have $\rho^+\tau_n \le \rho^+ Lg \le \rho^+\sigma_n$. By Lemma 4.5 $\rho^+\sigma_n$ and $\rho^+\tau_n$ are σ-simple functions (because $\rho^+\sigma_n = \rho^+\mathbf{1}_A\sigma_n$ and $\rho^+\tau_n = \rho^+\mathbf{1}_A\tau_n$).
Let $n \in \mathbb{N}$. There exists a $M \in \mathbb{N}$ such that $K \subset \bigcup_{i=-M}^{M}[(i-1)2^{-n}, i2^{-n})$ and

$$\sigma_n = \sum_{i=-M}^{M} a_{ni}\mathbf{1}_{[(i-1)2^{-n},i2^{-n})}.$$

Then

$$\int \rho^+\sigma_n \,\mathrm{d}\lambda = \sum_{i=-M}^{M} \Big(\int \rho^+\mathbf{1}_{[(i-1)2^{-n},i2^{-n})} \,\mathrm{d}\lambda\Big)a_{ni}.$$

Note that $(\int \rho^+\sigma_n \,\mathrm{d}\lambda)(s) = \int \rho^+\sigma_n(s) \,\mathrm{d}\lambda$ for all $s \in \mathbb{R}$. By Lemma 5.6 $\int \rho^+\sigma_n \,\mathrm{d}\lambda$ is uniformly continuous with continuity sequence $((\int \rho^+ \,\mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$, because

$$\sum_{i=-M}^{M} \Big|\int \rho^+\mathbf{1}_{[(i-1)2^{-n},i2^{-n})} \,\mathrm{d}\lambda\Big| = \int \rho^+ \,\mathrm{d}\lambda.$$

In the same way $\int \rho^+\tau_n \,\mathrm{d}\lambda$ is uniformly continuous with continuity sequence $((\int \rho^+ \,\mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$.
By Theorem 5.5 the functions

$$s \mapsto \inf_{n\in\mathbb{N}}\Big(\int \rho^+\sigma_n \,\mathrm{d}\lambda\Big)(s) = \inf_{n\in\mathbb{N}} \int \rho^+(x)\sigma_n(x)(s) \,\mathrm{d}x,$$
$$s \mapsto \sup_{n\in\mathbb{N}}\Big(\int \rho^+\tau_n \,\mathrm{d}\lambda\Big)(s) = \sup_{n\in\mathbb{N}} \int \rho^+(x)\tau_n(x)(s) \,\mathrm{d}x.$$

are the infimum and supremum, of the sets $\{\int \rho^+\sigma_n \,\mathrm{d}\lambda : n \in \mathbb{N}\}$ and $\{\int \rho^+\tau_n \,\mathrm{d}\lambda : n \in \mathbb{N}\}$, respectively. In particular those functions are uniformly continuous with

continuity sequence $((\int \rho^+ \, \mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$.
We prove that $\rho^+ Lg$ is R-integrable by showing that $\int \rho^+(\sigma_n - \tau_n) \, \mathrm{d}\lambda \downarrow 0$ (see Comment 2.39). We have (by Lemma 4.5)

$$\int \rho^+(\sigma_n - \tau_n) \, \mathrm{d}\lambda = \sum_{i=-M}^{M} \Big(\int \rho^+ \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})} \, \mathrm{d}\lambda \Big)(a_{ni} - b_{ni}).$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ as in Lemma 5.9. Then $n \geq N$ implies

$$\Big(\int \rho^+(\sigma_n - \tau_n) \, \mathrm{d}\lambda\Big)(x) \leq \Big(\int \rho^+ \, \mathrm{d}\lambda\Big)\varepsilon \qquad (x \in \mathbb{R}).$$

So $\inf_{n\in\mathbb{N}}(\int \rho^+(\sigma_n - \tau_n) \, \mathrm{d}\lambda)(x) = 0$ for all $x \in \mathbb{R}$. Therefore $\int \rho^+(\sigma_n - \tau_n) \, \mathrm{d}\lambda \downarrow 0$ in $C(\mathbb{R})$. By Theorem 5.5, $\int \rho^+ Lg \, \mathrm{d}\lambda$ is uniformly continuous with continuity sequence $((\int \rho^+ \, \mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$.
$C(\mathbb{R})$ is countably generated by Theorem 4.13. Therefore by Theorem 4.18 we have $\sup_{n\in\mathbb{N}} \rho^+\tau_n = \rho^+ Lg = \inf_{n\in\mathbb{N}} \rho^+\sigma_n$ λ-a.e.. Therefore for all $s \in \mathbb{R}$:

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\Big(\int \rho^+\tau_n \, \mathrm{d}\lambda\Big)(s) = \sup_{n\in\mathbb{N}} \int \rho^+(x)\tau_n(x)(s) \, \mathrm{d}x \leq \int \rho^+(x) L_x g(s) \, \mathrm{d}\lambda \\
&\leq \inf_{n\in\mathbb{N}} \int \rho^+(x)\sigma_n(x)(s) \, \mathrm{d}x = \inf_{n\in\mathbb{N}}\Big(\int \rho^+\sigma_n \, \mathrm{d}\lambda\Big)(s).
\end{aligned}$$

So we conclude

$$\Big((\mathrm{R})\text{-}\int \rho^+ Lg \, \mathrm{d}\lambda\Big)(s) = \int \rho^+(x) L_x g(s) \, \mathrm{d}x \qquad (s \in \mathbb{R}). \tag{1}$$

In the same way $\rho^- Lg$ is R-integrable and $\int \rho^- Lg \, \mathrm{d}\lambda$ is uniformly continuous with continuity sequence $((\int \rho^- \, \mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$ and

$$\Big((\mathrm{R})\text{-}\int \rho^- Lg \, \mathrm{d}\lambda\Big)(s) = \int \rho^-(x) L_x g(s) \, \mathrm{d}x \qquad (s \in \mathbb{R}). \tag{2}$$

By Comment 5.10 then $\rho^+ Lg - \rho^- Lg$ is integrable and $\int \rho Lg \, \mathrm{d}\mu = \int \rho^+ Lg \, \mathrm{d}\mu - \int \rho^- Lg \, \mathrm{d}\mu$ is uniformly continuous with continuity sequence $((\int |\rho| \, \mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$ by Lemma 5.6. And by (1) and (2) we have

$$\Big((\mathrm{R})\text{-}\int \rho Lg \, \mathrm{d}\lambda\Big)(s) = \int \rho(x) L_x g(s) \, \mathrm{d}x \qquad (s \in \mathbb{R}).$$

□

5.13 Theorem. *Let $g : \mathbb{R} \to [0, \infty)$ be a positive uniformly continuous function in $C(\mathbb{R})$. Let $f : \mathbb{R} \to \mathbb{R}$ be a λ-essentially compactly supported integrable function. Then $fLg : \mathbb{R} \to C(\mathbb{R})$ given by $x \mapsto f(x)L_x g$ is R-integrable. Moreover:*

$$\Big((\mathrm{R})\text{-}\int fLg \, \mathrm{d}\lambda\Big)(s) = \int f(x) L_x g(s) \, \mathrm{d}x = \int_{-\infty}^{\infty} f(x) g(s - x) \, \mathrm{d}x \qquad (s \in \mathbb{R}).$$

Proof. As in the proof of Lemma 5.12 we may assume there exists a compact set $K \subset \mathbb{R}$ such that $f = f\mathbf{1}_K$.
Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a continuity sequence for g.

Because f is integrable (by Theorem 2.66 and Theorem 2.66) there exist a sequence of simple functions $(s_n)_{n\in\mathbb{N}}$ and a σ-simple function $\pi : \mathbb{R} \to [0,\infty)$ such that

$$s_n - \frac{1}{n}\pi \le f \le s_n + \frac{1}{n}\pi \qquad \lambda\text{-a.e.} \qquad (n \in \mathbb{N}).$$

Then $\sigma_n := (s_n - \frac{1}{n}\pi)\mathbf{1}_K$ and $\tau_n := (s_n + \frac{1}{n}\pi)\mathbf{1}_K$ are σ-simple functions for all $n \in \mathbb{N}$. Then $\tau_n \le f \le \sigma_n$ λ-a.e. for all $n \in \mathbb{N}$. By Lemma 5.12 $\sigma_n Lg$ and $\tau_n Lg$ are integrable, and $\int \sigma_n Lg \,\mathrm{d}\lambda$ and $\int \tau_n Lg \,\mathrm{d}\lambda$ are uniformly continuous with continuity sequence $((\int |\sigma_n| \,\mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$ respectively $((\int |\tau_n| \,\mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$ for all $n \in \mathbb{N}$. Because

$$s_1 - \pi \le \tau_n \le \sigma_n \le (s_1 + \pi) \qquad (n \in \mathbb{N}),$$

$\int \sigma_n Lg \,\mathrm{d}\lambda$ and $\int \tau_n Lg \,\mathrm{d}\lambda$ are uniformly continuous with continuity sequence $(2(\int |s_1| + \pi \,\mathrm{d}\lambda)\varepsilon_k)_{k\in\mathbb{N}}$ for all $n \in \mathbb{N}$. And because g (and thus Lg) is positive,

$$\tau_n Lg \le \sigma_n Lg \qquad (n \in \mathbb{N}).$$

Therefore the sets $\{\int \sigma_n Lg \,\mathrm{d}\lambda : n \in \mathbb{N}\}$ and $\{\int \tau_n Lg \,\mathrm{d}\lambda : n \in \mathbb{N}\}$ have an infimum respectively a supremum in $C(\mathbb{R})$ by Theorem 5.5, and:

$$\Big(\inf_{n\in\mathbb{N}} \int \sigma_n Lg \,\mathrm{d}\lambda\Big)(s) = \inf_{n\in\mathbb{N}} \Big(\int \sigma_n Lg \,\mathrm{d}\lambda\Big)(s) = \inf_{n\in\mathbb{N}} \int \sigma_n(x) L_x g(s) \,\mathrm{d}x,$$
$$\Big(\sup_{n\in\mathbb{N}} \int \tau_n Lg \,\mathrm{d}\lambda\Big)(s) = \sup_{n\in\mathbb{N}} \Big(\int \tau_n Lg \,\mathrm{d}\lambda\Big)(s) = \sup_{n\in\mathbb{N}} \int \tau_n(x) L_x g(s) \,\mathrm{d}x.$$

Because

$$0 \le \sigma_n - \tau_n \le \frac{2}{n}\pi \qquad \lambda\text{-a.e.},$$

we have $\int (\sigma_n - \tau_n) Lg \,\mathrm{d}\lambda \le \frac{2}{n} \int \pi Lg \,\mathrm{d}\lambda \downarrow 0$. Then

$$0 \le \inf_{n,m\in\mathbb{N}} \int (\sigma_n - \tau_m) Lg \,\mathrm{d}\mu \le \inf_{n\in\mathbb{N}} \int (\sigma_n - \tau_n) Lg \,\mathrm{d}\mu = 0,$$

i.e. $\inf_{n,m\in\mathbb{N}} \int (\sigma_n - \tau_m) Lg \,\mathrm{d}\mu = 0$.
By Theorem 2.49 we conclude that fLg is R-integrable. By the Monotone Convergence Theorem (or by Lebesgue's Dominated Convergence Theorem) we have $\sup_{n\in\mathbb{N}} \tau_n = f = \inf_{n\in\mathbb{N}} \sigma_n$ λ-a.e.. Therefore for all $s \in \mathbb{R}$:

$$\Big(\sup_{n\in\mathbb{N}} \int \tau_n Lg \,\mathrm{d}\lambda\Big)(s) = \sup_{n\in\mathbb{N}} \int \tau_n(x) L_x g(s) \,\mathrm{d}x \le \int f(x) L_x g(s) \,\mathrm{d}x$$
$$\le \inf_{n\in\mathbb{N}} \int \sigma_n(x) L_x g(s) \,\mathrm{d}x = \Big(\inf_{n\in\mathbb{N}} \int \sigma_n Lg \,\mathrm{d}\lambda\Big)(s).$$

We conclude

$$\Big(\int fLg \,\mathrm{d}\lambda\Big)(s) = \int f(x) L_x g(s) \,\mathrm{d}x \qquad (s \in \mathbb{R}).$$

□

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