Noncommutative Wick polynomials

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May. 8th, 2019 @ Trondheim, Norway

Goals

1. Classical Wick polynomials
2. Moments and cumulants
3. Wick polynomials
    1. Modification of products
4. Relation to power series

Classical Wick polynomials

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For example:

$$W_1(x) = x - \mathbb{E}X, \quad W_2(x) = x^2 - 2x\mathbb{E}X + 2(\mathbb{E}X)^2 - \mathbb{E}X^2, \ldots$$

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Multivariate generalization:

$$\frac{\partial}{\partial x_i} W_n(x_1, \dots, x_n) = W_{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$
$$\mathbb{E}W_n(X_1, \dots, X_n) = 0.$$

Definition

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On $T(A) := \bigoplus_{n>0} A^{\otimes n}$ define $\Delta\colon T(A) \to T(A) \otimes T(A)$ by

$$\Delta^{\sqcup\!\sqcup}(a_1 \cdots a_n) := \sum_{S \subseteq [n]} a_S \otimes a_{[n]\setminus S}.$$

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$$\Delta^{⧢}(a_1 \cdots a_n) := \sum_{S \subseteq [n]} a_S \otimes a_{[n]\setminus S}.$$

This induces a product on $T(A)^*$:

$$\mu ⧢ \nu := (\mu \otimes \nu)\Delta^{⧢}.$$

Define $\phi \colon T(A) \to k$ by $\phi(a_1 \cdots a_n) \coloneqq \varphi(a_1 \cdot_A \cdots \cdot_A a_n)$ and extend to $\overline{T}(A) \coloneqq k1 \oplus T(A)$ by $\phi(1) = 1$.

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There is $c\colon \overline{T}(A) \to k$ with $c(1) = 0$ such that $\phi = \exp^{\sqcup\!\sqcup}(c)$. In particular

$$\phi(a_1 \cdots a_n) = \sum_{\pi \in P(n)} \prod_{B \in \pi} c(a_B).$$

Since ϕ is invertible, we set $W := (\mathrm{id} \otimes \phi^{-1})\Delta^{\sqcup\!\sqcup}$.

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Theorem

The map $W : \overline{T}(A) \to \overline{T}(A)$ is the unique linear map such that $\partial_a \circ W = W \circ \partial_a$ and $\phi \circ W = \varepsilon$. Its inverse is given by $W^{-1} = (\mathrm{id} \otimes \phi)\Delta^{⧢}$.

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Example:

$$W(a) = a - \varphi(a), \quad W(a^{\otimes 2}) = a^{\otimes 2} - 2a\varphi(a) + 2\varphi(a)^2 - \varphi(a \cdot_A a), \ldots$$

$$W(ab) = ab - a\varphi(b) - b\varphi(a) + 2\varphi(a)\varphi(b) - \varphi(a \cdot_A b).$$

Moments and cumulants

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On the double tensor algebra $\overline{T}(T(A))$ consider

$$\Delta(a_1 \cdots a_n) := \sum_{S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S}.$$

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$$\Delta(a_1 \cdots a_n) := \sum_{S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S}.$$

This splits as

$$\Delta_{\prec}(a_1 \cdots a_n) := \sum_{1 \in S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S},$$
$$\Delta_{\succ}(a_1 \cdots a_n) := \sum_{1 \notin S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S}.$$

Therefore, the convolution product $\mu * \nu := (\mu \otimes \nu)\Delta$ also splits:

$$\mu \prec \nu := (\mu \otimes \nu)\Delta_{\prec}, \quad \mu \succ \nu := (\mu \otimes \nu)\Delta_{\succ}.$$

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Consider $\Phi \colon \overline{T}(T(A)) \to k$ the unique character extension of ϕ.

Theorem (Ebrahimi-Fard,Patras; 2014, 2017)

The cumulants κ, β, ρ are the unique infinitesimal characters of $\overline{T}(T(A))$ such that

$$\begin{aligned} \Phi &= \varepsilon + \kappa \prec \Phi \\ &= \varepsilon + \Phi \succ \beta \end{aligned}$$

and $\Phi = \exp_*(\rho)$.

Theorem (Speicher; 1997)

$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{\pi \in NC(n)} \prod_{B \in \pi} \kappa(a_B).$$

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$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{\pi \in Int(n)} \prod_{B \in \pi} \beta(a_B).$$

Theorem (Hasebe, Saigo; 2011)

$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{(\pi,\lambda) \in M(n)} \frac{1}{|\pi|!} \prod_{B \in \pi} \rho(a_B)$$

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Thus, the character group G acts on $\mathrm{End}(T(A))$.

Wick polynomials

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Examples:

$$\begin{aligned}
W(a) &= a - \phi(a)1 \\
W(ab) &= ab - a\phi(b) - b\phi(a) + (2\phi(a)\phi(b) - \phi(a \cdot b))1 \\
W(abc) &= abc - \varphi(c)ab - \varphi(b)ac - \varphi(a)bc \\
&\quad - [\phi(b \cdot c) - 2\phi(b)\phi(c)]a + \phi(a)\phi(c)b - [\phi(a \cdot b) - 2\phi(a)\phi(b)]c \\
&\quad - [\phi(a \cdot b \cdot c) - 2\phi(a)\phi(b \cdot c) - 2\phi(c)\phi(a \cdot b) - \phi(b)\phi(a \cdot c) \\
&\quad + 5\phi(a)\phi(b)\phi(c)]1
\end{aligned}$$

By definition

$$\Phi \circ W = (\Phi \otimes \Phi^{-1})\Delta = \varepsilon$$

that is, $\Phi(W(a_1 \dots a_n)) = 0$ for any $a_1, \dots, a_n \in A$.

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It's easy to check that W is invertible with $W^{-1} = (\mathrm{id} \otimes \Phi)\Delta$.

In particular

$$a_1 \cdots a_n = \sum_{S \subseteq [n]} W(a_s)\Phi(a_{J_1^S}) \cdots \Phi(a_{J_k^S}).$$

But also $W = (\mathrm{id} \otimes \mathscr{E}_{\succ}(-\kappa))\Delta$, so

Theorem (Anshelevich, 2004)

$$W(a_1 \cdots a_n) = \sum_{S \subseteq [n]} a_S \sum_{\substack{\pi \in \mathrm{Int}([n] \setminus S) \\ \pi \cup S \in NC(n)}} (-1)^{|\pi|} \prod_{B \in \pi} \kappa(a_B).$$

Theorem

The Wick polynomials satisfy the recursion

$$W(a_1 \cdots a_n) = a_1 W(a_2 \cdots a_n) - \sum_{j=0}^{n-1} W(a_{j+1} \cdots a_n)\kappa(a_1 \cdots a_j).$$

Proof.

$$\begin{aligned} W &= (\mathrm{id} \otimes \Phi^{-1})\Delta \\ &= \mathrm{id} \prec \Phi^{-1} + \mathrm{id} \succ \Phi^{-1} \\ &= \mathrm{id} \prec \Phi^{-1} - \mathrm{id} \succ (\Phi^{-1} \succ \kappa) \\ &= \mathrm{id} \prec \Phi^{-1} - W \succ \kappa. \end{aligned}$$

□

Theorem

The Wick polynomials can be expressed in terms of boolean cumulants

$$W = (\mathrm{id} - \mathrm{id} \succ \beta) \prec \Phi^{-1}$$

Proof.

Previous theorem plus the fact that $\kappa = \Phi \succ \beta \prec \Phi^{-1}$. □

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Theorem

Boolean Wick polynomials are centered

Proof.

$\Phi \circ W' = \Phi - \Phi \succ \beta = \varepsilon$ □

Theorem

We have

$$a_1 \cdots a_n = W'(a_1 \cdots a_n) + \sum_{j=1}^{n-1} \Phi(a_1 \cdots a_j) W'(a_{j+1} \cdots a_n).$$

From a previous computation

$$W' = W \prec \Phi,$$

that is

$$W'(a_1 \cdots a_n) = \sum_{1 \in S \subseteq [n]} W(a_S)\Phi(a_{J_1^S}) \cdots \Phi(a_{J_k^S}).$$

Assume we have a second state $\psi : A \to k$.

Definition

Two-state cumulants are defined implicitly by

$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{\pi \in NC(n)} \prod_{B \in \mathrm{Outer}(\pi)} R^{\varphi,\psi}(a_B) \prod_{B \in \mathrm{Inner}(\pi)} \kappa^{\psi}(a_B).$$

Theorem (Ebrahimi-Fard, Patras; 2018)

$R^{\varphi,\psi}$ *is the unique infinitesimal character of* $\overline{T}(T(A))$ *such that*

$$\Phi = \varepsilon + \Phi \succ (\Psi^{-1} \succ R^{\varphi,\psi} \prec \Psi).$$

Directly,

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In particular,

$$\begin{aligned} R^{\varphi,\varphi} &= \Phi \succ \beta^{\varphi} \prec \Phi^{-1} = \kappa^{\varphi}, \\ R^{\varphi,\varepsilon} &= \beta^{\varphi}. \end{aligned}$$

Definition

The conditionally-free Wick polynomials are defined as

$$W^c := W \prec (\Phi * \Psi^{-1}).$$

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This means

$$W^c = \left(\mathrm{id} - \mathrm{id} \succ \Theta_\Psi(R^{\varphi,\psi})\right) \prec \Psi^{-1}$$

where $\Theta_\Psi(\mu) := \Psi^{-1} \succ \mu \prec \Psi$.

Since W is invertible, one can induce a product on $T(A)$ by

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Proposition

The $\bullet$ product admits the closed-form expression: for $x = a_1 \cdots a_n$, $y = a_{n+1} \cdots a_{n+m}$

$$x \bullet y = \sum_{S \subseteq [n+m]} a_S \Phi(a_{J_1^S}) \cdots \Phi(a_{J_k^S}).$$

Power series

The relations between moments and cumulants can also be encoded by power series.

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In the classical case, one uses exponential generating functions:

$$\sum_{n\geq 0} m_n \frac{\lambda^n}{n!} = \exp\left(\sum_{k>0} c_k \frac{\lambda^k}{k!}\right).$$

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Let

$$M(w) := 1 + \sum_{\alpha} \varphi(a_\alpha) w_\alpha, \quad R(w) := \sum_{\alpha} \kappa(a_\alpha) w_\alpha, \quad \eta(w) := \sum_{\alpha} \beta(a_\alpha) w_\alpha.$$

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Considering a new set of variables $z_i = w_i M(w)$ we have

$$M(w) = 1 + R(z), \quad M(w) = 1 + \eta(w) M(w).$$

It turns out that the Hopf-algebraic language above describes two operations on power series.

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For $f, g \in G^p$ define

$$f^g(w) := g(w)f(z), \quad z_i = w_i g(w).$$

Also let

$$(f \curvearrowleft g)(w) := f(z), \quad z_i = w_i g(w).$$

Given $F : T(A) \to k$ let $\Lambda(F) \in k[[w]]$ be given by

$$\Lambda(F)(w) = F(1) + \sum_{\alpha} F(a_\alpha) w_\alpha .$$

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Theorem

$$\Lambda(F \prec G) = \Lambda(F) \curvearrowleft \Lambda(G).$$

Thank you!