Modification of branched Rough Paths

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Introduction

Rough paths were introduced by Terry Lyons near the end of the 90's to deal with stochastic integration (and SDEs) in a path-wise sense.

Some years later Massimiliano Gubinelli introduced controlled rough paths, and brached Rough Paths a decade after Lyons' work.

In 2014, Martin Hairer introduced Regularity Structures which generalize branched Rough Paths.

All of these objects consist of a mixture of algebraic and analytic properties.

A crucial tool in Regularity Structures is the renormalization step.

This step relies on knowledge of the group of automorphisms of the space of models.

In this setting, an answer has been given by Bruned, Hairer and Zambotti (2016) for stationary models.

Now we will discuss the same problem for branched Rough Paths.

Some work on this has already been carried by Bruned, Chevyrev, Friz and Preiß (2017).

Geometric rough paths

Geometric rough paths (signatures) have recently found a number of applications in Data Analysis and Statistical Learning.

For a "smooth" path $x : [0,1] \to \mathbb{R}^d$, one defines its signature $S(x) : [0,1]^2 \to T(\mathbb{R}^d)^*$ as

$$\langle S(x)_{s,t}, e_{i_1 \cdots i_n} \rangle = \int_s^t \int_s^{t_{n-1}} \cdots \int_s^{t_1} \mathrm{d}x_{u_1}^{i_1}\, \mathrm{d}x_{u_2}^{i_2} \cdots \mathrm{d}x_{u_n}^{i_n}$$

i.e. $S(x)$ is the collection of all iterated integrals of the components of x. Here, $e_{i_1 \cdots i_n} := e_{i_1} \otimes \cdots \otimes e_{i_n}$ is a basis element of $T(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus \cdots$

For example:

$$\langle S(x)_{s,t}, e_i \rangle = x_t^i - x_s^i$$

$$\langle S(x)_{s,t}, e_{ij} \rangle = \int_s^t (x_u^i - x_s^i)\, \mathrm{d}x_u^j, \quad \langle S(x)_{s,t}, e_{ii} \rangle = \frac{(x_t^i - x_s^i)^2}{2}$$

The family of iterated integrals satisfies the so-called *shuffle relation*, implied by the integration-by-parts formula:

$$\langle S(x)_{s,t}, e_{i_1\cdots i_n} \sqcup\!\sqcup\, e_{i_{i+1}\cdots i_{n+m}}\rangle = \langle S(x)_{s,t}, e_{i_1\cdots i_n}\rangle\langle S(x)_{s,t}, e_{i_{n+1}\cdots i_{n+m}}\rangle.$$

For example, for $n = 1, m = 1$ we recover integration by parts:

$$\int_s^t \int_s^u \mathrm{d}x^i_{u_1}\,\mathrm{d}x^j_{u_2} + \int_s^t \int_s^u \mathrm{d}x^j_{u_1}\,\mathrm{d}x^i_{u_2} = \int_s^t \mathrm{d}x^i_u \int_s^t \mathrm{d}x^j_u.$$

It also satisfies the following identity, called *Chen's rule*, a generalization of $\int_s^u + \int_u^t = \int_s^t$:

$$\begin{aligned}\langle S(x)_{s,t}, e_{i_1\cdots i_n}\rangle &= \langle S(x)_{s,u}, e_{i_1\cdots i_n}\rangle + \langle S(x)_{u,t}, e_{i_1\cdots i_n}\rangle\\ &\quad + \sum_{j=1}^{n-1}\langle S(x)_{s,u}, e_{i_1\cdots i_j}\rangle\langle S(x)_{u,t}, e_{i_{j+1}\cdots i_n}\rangle.\end{aligned}$$

The vector space $T(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus \cdots$ can be made into an algebra in two ways: the tensor (or concatenation) product, and the *shuffle product*.

It also carries two coproducts: the deconcatenation coproduct Δ and the deshuffling coproduct $\Delta_{\sqcup\!\sqcup}$.

In fact, $(T(\mathbb{R}^d), \otimes, \Delta_{\sqcup\!\sqcup})$ and $(T(\mathbb{R}^d), \sqcup\!\sqcup, \Delta)$ are Hopf algebras, dual to one another.

The signature $S(x)$ of a smooth path is a family of linear maps on $T(\mathbb{R}^d)$, i.e. an element of $T(\mathbb{R}^d)^* := T(\!(\mathbb{R}^d)\!)$.

The above properties can be summarized by saying that, for each $s < t$ the element $S(x)$ is an algebra morphism (shuffle relation) satisfying $S(x)_{s,u} \otimes S(x)_{u,t}$ for all $s < u < t$.

A classical theorem by Young tells us that the integration operator

$$I(f,g) := \int_0^1 f_s \, \mathrm{d}g_s$$

can be extended continuously from $C^0 \times C^1 \to C^1$ to $C^\alpha \times C^\beta \to C^\beta$ **if and only if** $\alpha + \beta > 1$.

Thus, finding the signature $S(x)$ as above is only possible for paths in C^α for $\alpha > \frac{1}{2}$.

Theorem (Lyons–Victoir (2007))

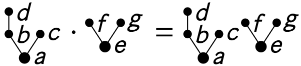
Given $\alpha < \frac{1}{2}$ with $\alpha^{-1} \notin \mathbb{N}$ and $x \in C^\alpha$, there exists a map $X : [0,1]^2 \to T(\!(\mathbb{R}^d)\!)$ such that $X_{s,t}$ is multiplicative, $X_{s,u} \otimes X_{u,t} = X_{s,t}$ and $|\langle X_{s,t}, e_{i_1 \cdots i_k} \rangle| \lesssim |t-s|^{k\gamma}$.

Branched rough paths

Let $(\mathscr{H}, \cdot, \Delta)$ be the Butcher–Connes–Kreimer Hopf algebra.

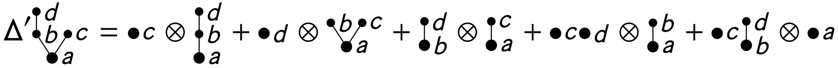
As an algebra, $\mathscr{H}$ is the commutative polynomial algebra over the set of non-planar trees decorated by some alphabet A.

The product is simply the disjoint union of forests, e.g.



The empty forest 1 acts as the unit.

The coproduct Δ is described in terms of admissible cuts. For example



Definition (Gubinelli (2010))

A branched Rough Path is a map $X : [0,1]^2 \to \mathscr{H}^*$ *such that each* $X_{s,t}$ *is an algebra morphism and*

$$X_{su} \star X_{ut} = X_{st}, \quad |\langle X_{st}, \tau\rangle| \lesssim |t-s|^{\gamma|\tau|}.$$

Example: let $(B_t)_{t\geq 0}$ be a Brownian motion, set $\langle X_{st}, \bullet\rangle := B_t - B_s$ and

$$\langle X_{st}, [\tau_1 \cdots \tau_k]\rangle = \int_s^t \langle X_{su}, \tau_1\rangle \cdots \langle X_{su}, \tau_k\rangle \,\mathrm{d}B_u.$$

N.B.: This definition can be *uniquely* extended such that $X_{s,t}$ is an algebra morphism.

Let $\mathscr{C}_k$ be the continuous functions in k variables vanishing when consecutive variables coincide.

Gubinelli (2003) defines an *exact cochain complex*

$$0 \to \mathbb{R} \to \mathscr{C}_1 \xrightarrow{\delta_1} \mathscr{C}_2 \xrightarrow{\delta_2} \mathscr{C}_3 \xrightarrow{\delta_3} \cdots$$

that is $\delta_{k+1} \circ \delta_k = 0$ and $\operatorname{im} \delta_k = \ker \delta_{k+1}$.

Remark

If $F \in \ker \delta_2$ then there exists $f \in \mathscr{C}_1$ such that $F_{st} = f_t - f_s$.
If $C \in \ker \delta_3$ then there exists $F \in \mathscr{C}_2$ such that $C_{sut} = F_{st} - F_{su} - F_{ut}$.

In general, none of these operators are injective: if $F = G + \delta_{k-1} H$ then $\delta_k F = \delta_k G$.

Can do more if we restrict to smaller spaces: let $\mathscr{C}_2^\mu$ be the $F \in \mathscr{C}_2$ such that

$$\|F\|_\mu := \sup_{s<t} \frac{|F_{st}|}{|t-s|^\mu} < \infty.$$

Similarly, $\mathscr{C}_3^\mu$ are the $C \in \mathscr{C}_3$ such that $\|C\|_\mu < \infty$ for some suitable norm.

Theorem (Gubinelli (2004))

There is a unique linear map $\Lambda : \mathscr{C}_3^{1+} \cap \ker \delta_3 \to \mathscr{C}_2^{1+}$ *such that* $\delta_2 \Lambda = \mathrm{id}$. *In each of* $\mathscr{C}_3^\mu$ *for* $\mu > 1$ *it satisfies*

$$\|\Lambda C\|_\mu \le \frac{1}{2^\mu - 2} \|C\|_\mu.$$

Chen's rule reads

$$\langle X_{st}, \tau \rangle = \langle X_{su}, \tau \rangle + \langle X_{ut}, \tau \rangle + \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle.$$

or

$$\delta_2 F^{\tau}_{sut} = \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle$$

where $F^{\tau}_{st} := \langle X_{st}, \tau \rangle$.

The norm on $\mathscr{C}_3$ is such that the bound for X implies $\delta_2 F^{\tau} \in \mathscr{C}_3^{\gamma|\tau|}$.

The integer $N := \lfloor \gamma^{-1} \rfloor$ is special. Let G_N denote the multiplicative maps on the *truncated space*

$$\mathscr{H}_N := \mathbb{R}\{\tau : |\tau| \leq N\}$$

Theorem (Gubinelli (2010))

Suppose $X : [0,1]^2 \to G_N$ satisfies $|\langle X_{st}, \tau\rangle| \lesssim |t-s|^{\gamma|\tau|}$. Then there exists a unique multiplicative extension $\hat{X} : [0,1]^2 \to \mathscr{H}^*$ such that $\hat{X}\big|_{\mathscr{H}_N} = X$.

Proof.

Suppose $|\tau| = N+1$ is a tree and set $C^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta'\tau\rangle$.
First one shows that $C^\tau \in \ker \delta_3$ by using the coassociativity of Δ'.
The bound above implies that $C^\tau \in \mathscr{C}_3^{\gamma|\tau|}$.
Therefore C^τ lies in the domain of Λ and we can set

$$\langle X_{st}, \tau\rangle := (\Lambda C^\tau)_{st}.$$

Continue inductively. □

Results

The previous argument works only because $\gamma|\tau| > 1$ i.e. $|\tau| > N$.

If $\gamma|\tau| \leq 1$, for any $g^\tau \in C^{\gamma|\tau|}$ (Hölder space) the function

$$G^\tau_{st} := F^\tau_{st} + \delta_1 g^\tau_{st}$$

also satisfies $\delta_2 G^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle$.

Let X and X' be two BRPs coinciding on $\mathbb{R}\{\bullet_1, \dots, \bullet_d\}$.

Fix a *tree* τ with $|\tau| = 2$ and let $F^\tau_{st} := \langle X_{st}, \tau \rangle$, $G^\tau_{st} := \langle X'_{st}, \tau \rangle$.

Then $\delta_2 F^\tau = \delta_2 G^\tau$ so there is $g^\tau \in \mathscr{C}_1$ such that

$$F^\tau_{st} = G^\tau_{st} + \delta_1 g^\tau_{st}.$$

Moreover $g^\tau \in C^{2\gamma}$.

This suggests that there might be an action of

$$\mathscr{D}^\gamma := \{(g^\tau)_{|\tau|\leq N} : g^\tau \in C^{\gamma|\tau|}, g_0^\tau = 0\}$$

on the space $\mathbf{BRP}^\gamma$ of branched Rough Paths.

Theorem (T.-Zambotti (2018))

Let $\gamma \in (0,1)$ such that $\gamma^{-1} \notin \mathbb{N}$. There is a regular action of $\mathscr{D}^\gamma$ on $\mathbf{BRP}^\gamma$.

This means we have a mapping

$$\mathscr{D}^\gamma \times \mathbf{BRP}^\gamma \ni (g, X) \to gX \in \mathbf{BRP}^\gamma$$

such that

- $g'(gX) = (g' + g)X$ for all $g, g' \in \mathscr{D}^\gamma$ and,
- for every pair $X, X' \in \mathbf{BRP}^\gamma$ there exists a *unique* $g \in \mathscr{D}^\gamma$ such that $X' = gX$.

$\mathbf{BRP}^\gamma$ is a *principal homogeneous space* for $\mathscr{D}^\gamma$.

Coments

1. Lifting of Chen's rule to the Lie algebra $\mathfrak{g}$. If $X_{st} = \exp_\star(\alpha_{st})$ then
$$\alpha_{st} = \mathrm{BCH}(\alpha_{su}, \alpha_{ut}) = \alpha_{su} + \alpha_{ut} + \mathrm{BCH}'(\alpha_{su}, \alpha_{ut}).$$
2. We use an explicit BCH formula due to Reutenauer.
3. However, the action is not unique nor canonical. The construction depends on a finite number of arbitrary choices.
4. We are able to construct branched rough paths over any $x \in C^\gamma(\mathbb{R}^d)$.

Next goals

1. Understand the algebraic picture. The action gX is not very easy to compute.
2. Relation with modification of products as explored in Ebrahimi-Fard, Patras, T. and Zambotti (2017).
3. Actions of an appropriate $\mathscr{D}^\gamma$ for the geometric case.
4. Clarify what the action means for Rough Differential Equations.
5. Further study of the geometrical properties of $\mathbf{BRP}^\gamma$.

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