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Mathematical Theory of Water Waves

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Abstracts

Deriving modulation equations via Lagrangian and Hamiltonian reduction

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Modulation equations can be seen as effective macroscopic equations describing the evolution of a microscopically period pattern. We discuss general strategies how to pass from the microscopic systems to a macroscopic one by using the Hamiltonian or the Lagrangian structure.

The derivation of macroscopic equations for discrete models (or continuous models with microstructure) can be seen as a kind of reduction of the infinite dimensional system to a simpler subclass. If we choose well-prepared initial conditions, we hope that the solution will stay in this form and evolve according to a slow evolution with macroscopic effects only. We may interpret this as a kind of (approximate) invariant manifold, and the macroscopic equation describes the evolution on this manifold. We refer to [Mie91] for exact reductions of Hamiltonian systems and to [DHM06, GHM06, Mie06, GHM07] for the full details concerning this note.

As the easiest example we consider the one-dimensional Klein-Gordon equation

$$u_{tt} = u_{xx} - au - bu^3, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

The sum of the kinetic and potential energy gives the Hamiltonian

$$H(u,u_t) = \int_{\mathbb{R}} \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{a}{2}u^2 + \frac{b}{4}u^4 \right).$$

As we are interested in modulated waves we embed this system $\mathbb{R}$ into the cylinder $\Xi = \mathbb{R}\times\mathbb{S}^1$, where $\mathbb{S}^1$ contains the additional microscopic phase variable. The continuous Hamiltonian system is

$$\begin{aligned} &\partial_t^2 u = \Delta_{(1,0)}u - au + bu^3 \quad \text{with } a > 0, \quad u \in \mathrm{L}^2(\Xi), \\ &\text{and } \Delta_{(1,0)}u(x,\phi) := u_{xx}(x,\phi). \end{aligned} \tag{1}$$

Introducing $p = \partial_\tau u$, this is a canonical Hamiltonian system with

$$H^{\mathrm{cont}}(u,p) = \textstyle\int_\Xi \frac{1}{2}p^2 + \frac{1}{2}\big(\nabla_{(1,0)}u\big)^2 + \frac{a}{2}u^2 + \frac{b}{4}u^4 \,\mathrm{d}x\,\mathrm{d}\phi.$$

Like the original KG equation the enlarged problem (1) is translationally invariant in the x direction. Moreover, it is invariant under translations in the phase direction ϕ. This leads to the two first integrals $I^{\mathrm{sp}}(u,p) = \int_\Xi p\,\partial_x u \,\mathrm{d}x\,\mathrm{d}\phi$ and $I^{\mathrm{ph}}(u,p) = \int_\Xi p\,\partial_\phi u \,\mathrm{d}x\,\mathrm{d}\phi$. Using the symmetry transformation

$$(\widetilde{u},\widetilde{p}) = T^{\mathrm{sp}}_{ct}T^{\mathrm{ph}}_{(\omega-c\theta)t}(u,p), \qquad \widetilde{\mathcal{H}} = \mathcal{H} - cI^{\mathrm{sp}} - (\omega-c\theta)I^{\mathrm{ph}}$$

the associated canonical Hamiltonian system $\mathbf{\Omega}^{\mathrm{can}}(\widetilde{u}_t,\widetilde{p}_t) = \mathrm{D}\widetilde{\mathcal{H}}(\widetilde{u},\widetilde{p})$ on $\mathrm{L}(\Xi)^2$ is still fully equivalent to a family of uncoupled KG chains.

Introducing a suitable scaling, which anticipates the desired microscopic and macroscopic behavior, will expose the desired limit. For this we let

$$(\widetilde{u}(x,\phi),\widetilde{p}(x,\phi)) = (\varepsilon U(\varepsilon x,\phi{-}\theta x),\varepsilon P(\varepsilon x,\phi{-}\theta x)),$$

which keeps the canonical structure (after moving a factor ε arising from $\mathrm{d}y=\varepsilon\,\mathrm{d}x$ into the time parametrization $\tau=\varepsilon^2 t$). We obtain the new Hamiltonian

$$\begin{aligned}\mathcal{H}_\varepsilon(U,P) = \int_\Xi \tfrac{1}{2\varepsilon^2}\Big(\big[P{-}\omega U_\phi{-}\varepsilon c U_y\big]^2 + \big(\nabla_{(\varepsilon,\theta)}U\big)^2 \\ +aU^2 - \big[\omega PU_\phi{+}\varepsilon cPU_y\big]^2\Big){+}\tfrac{b}{4}U^4\,\mathrm{d}y\,\mathrm{d}\phi,\end{aligned}$$

where $\nabla_{(\varepsilon,\theta)}=\varepsilon U_y+\theta U_\phi$. The modulation ansatz now reads

$$(U(y,\phi),P(y,\phi)) = R_\varepsilon(A)(y,\phi) = (\operatorname{Re}A(y)\mathrm{e}^{\mathrm{i}\phi},\omega\operatorname{Re}A(y)\mathrm{e}^{\mathrm{i}\phi})+O(\varepsilon),$$

and leads to $\mathcal{H}_\varepsilon(R_\varepsilon(A))=\mathbb{H}_{\mathrm{nlS}}(A){+}O(\varepsilon)$ and $\mathrm{D}R_\varepsilon(A)^*\mathbf{\Omega}^{\mathrm{can}}\mathrm{D}R_\varepsilon(A)=\mathbf{\Omega}^{\mathrm{red}}{+}O(\varepsilon)$ with

$$\mathbb{H}_{\mathrm{nlS}}(A)=\int_{\mathbb{R}}\omega\omega''|A_y|^2+\tfrac{3b}{8}|A|^4\,\mathrm{d}y \qquad\text{and}\qquad \mathbf{\Omega}^{\mathrm{red}}=2\mathrm{i}\omega.$$

Thus, the macroscopic limit is the one-dimensional nonlinear Schrödinger equation

$$2\mathrm{i}\omega A_\tau = -2\omega\omega'' A_{yy}+\tfrac{3}{2}b|A|^2A.$$

Of course, a mathematically rigorous justification of the nonlinear Schrödinger equation as a modulation equation was known long before (see [KSM92, Sch98, GM04, GM06]). However, the emphasis here is to see how the Hamiltonian and Lagrangian structures need to be transformed to converge to the desired limits.

References

[DHM06] W. Dreyer, M. Herrmann, and A. Mielke. Micro–macro transition for the atomic chain via Whitham's modulation equation. *Nonlinearity*, 19, 471–500, 2006.

[GHM06] J. Giannoulis, M. Herrmann, and A. Mielke. Continuum description for the dynamics in discrete lattices: derivation and justification. In A. Mielke, editor, *Analysis, Modeling and Simulation of Multiscale Problems*, pages 435–466. Springer-Verlag, Berlin, 2006.

[GHM07] J. Giannoulis, M. Herrmann, and A. Mielke. Lagrangian and Hamiltonian two-scale reduction. *In preparation*, 2007.

[GM04] J. Giannoulis and A. Mielke. The nonlinear Schrödinger equation as a macroscopic limit for an oscillator chain with cubic nonlinearities. *Nonlinearity*, 17, 551–565, 2004.

[GM06] J. Giannoulis and A. Mielke. Dispersive evolution of pulses in oscillator chains with general interaction potentials. *Discr. Cont. Dynam. Systems Ser. B*, 6, 493–523, 2006.

[KSM92] P. Kirrmann, G. Schneider, and A. Mielke. The validity of modulation equations for extended systems with cubic nonlinearities. *Proc. Roy. Soc. Edinburgh Sect. A*, 122, 85–91, 1992.

[Mie91] A. Mielke. *Hamiltonian and Lagrangian flows on center manifolds. With applications to elliptic variational problems*, volume 1489 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[Mie06] A. Mielke. Weak convergence methods for the micro-macro transition from from disrcete to continuous systems. *In preparation*, 2006.

[Sch98] G. Schneider. Justification of modulation equations for hyperbolic systems via normal forms. *NoDEA Nonlinear Differential Equations Appl.*, 5(1), 69–82, 1998.

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