Existence of minimizers in incremental elasto-plasticity with finite strains*

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Abstract. We consider elasto-plastic deformations of a body which is subjected to a time-dependent loading. The model includes fully nonlinear elasticity as well as the multiplicative split of the deformation gradient into an elastic part and a plastic part. Using the energetic formulation for this rate-independent process we derive a time-incremental problem, which is a minimization problem with respect to the deformation and the plastic variables. We provide assumptions on the constitutive laws of the material which guarantee that the incremental problem can be solved for as many time steps as desired. The methods relies on the polyconvexity of the so-called condensed energy functional and on a priori estimates for the plastic variables using the dissipation distance.

Key words. nonlinear elasticity, plasticity, polyconvexity, time incremental minimization problems, energetic formulation

AMS subject classifications. Primary 74C15, 49J40 ; Secondary 74A20, 49J52

1 Introduction

The mathematical theory of linearized elasto-plasticity was developed in the 1970s by J.J. Moreau [Mor74, Mor76] and further developed subsequently up to efficient numerical implementations, see e.g., [Joh76, HR95]. This theory relies on the additive decomposition

$$\varepsilon = \tfrac{1}{2}(\mathrm{D}u + \mathrm{D}u^{\mathsf{T}}) = \varepsilon_{\mathrm{elast}} + \varepsilon_{\mathrm{plast}}$$

of the linearized strain tensor ε, where $u : \Omega \to \mathbb{R}^d$ denotes the displacement. Moreover, the energy is assumed to be a quadratic functional such that the problem takes the form of a quasi-variational inequality. More general approaches with nonlinear hardening laws and viscoplastic effects can be found in [BF96, Alb98, ACZ99, Che01a, Che01b, Nef02].

With this work we want to start a mathematical investigation of elasto-plasticity which allows for large strains and which is based on the multiplicative decomposition

$$F = \mathrm{D}\varphi = F_{\mathrm{elast}} F_{\mathrm{plast}}. \tag{1.1}$$

Here, $\varphi : \Omega \to \mathbb{R}^d$ is the deformation of the body $\Omega \subset \mathbb{R}^d$. The energy $\mathcal{E}$ stored in a deformed body depends only on the elastic part F_{elast} of the deformation tensor and suitable hardening parameters $p \in \mathbb{R}^m$, but not on the plastic part F_{plast},

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which is contained in $\mathrm{SL}(\mathbb{R}^d)$ or another Lie group $\mathfrak{G}$ contained in $\mathrm{GL}_+(\mathbb{R}^d) = \{\, P \in \mathbb{R}^{d\times d} \mid \det P > 0 \,\}$. The energy functional takes the form

$$\mathcal{E}(t, \varphi, (F_{\text{plast}}, p)) = \int_\Omega W(x, \mathrm{D}\varphi(x) F_{\text{plast}}(x)^{-1}, p(x))\,\mathrm{d}x - \langle \ell(t), \varphi\rangle$$

where the external loading $\ell(t)$ is given via

$$\langle \ell(t), \varphi\rangle = \int_\Omega f_{\text{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x + \int_\Gamma g_{\text{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}a.$$

To model the plastic effects one prescribes either a plastic flow law or, equivalently, a dissipation potential $\Delta : \Omega \times \mathrm{T}(\mathfrak{G}\times\mathbb{R}^m) \to [0,\infty]$. We consider $\Delta(x,\cdot,\cdot)$ as an infinitesimal metric which defines the global dissipation distance $D(x,\cdot,\cdot)$ on $\mathfrak{G}\times\mathbb{R}^m$. Thus, the second ingredient to our material model is the dissipation distance between two internal states $z_j = (F^{(j)}_{\text{plast}}, p_j) : \Omega \to \mathrm{SL}(\mathbb{R}^d)\times\mathbb{R}^m$:

$$\mathcal{D}(z_1, z_2) = \int_\Omega D(x, (F^{(1)}_{\text{plast}}(x), p_1(x)),\ (F^{(2)}_{\text{plast}}(x), p_2(x)))\,\mathrm{d}x.$$

Allowing for finite strains one is forced to avoid convexity assumptions on the stored-energy density W, as it has to be frame indifferent (i.e., $W(x,RF,z) = W(x,F,z)$ for $R \in \mathrm{SO}(\mathbb{R}^d)$) and to enforce local invertibility (i.e., $W(F) = \infty$ for $F \notin \mathrm{GL}_+(\mathbb{R}^d)$). It was a major breakthrough in [Bal77] that these conditions are compatible with quasiconvexity and polyconvexity. The aim of this work is to show that it is possible to find constitutive functions W (being polyconvex) and Δ which, one the one hand, satisfy all the above-mentioned natural, physical conditions of finite-strain elasticity as well as the multiplicative plastic decomposition (1.1) (giving raise to the Lie group structure for $P = F_{\text{plast}}$) and, on the other hand, allow for a mathematical existence theory.

We follow the work in [MT99, MTL02, Mie02a, Mie03, MR03] which shows that rate-independent evolution for elastic materials with internal variables ("standard generalized materials") can be formulated by energy principles as follows. A pair $(\varphi, z) : [0,T]\times\Omega \to \mathbb{R}^d\times\mathrm{SL}(\mathbb{R}^d)\times\mathbb{R}^m$ is called a solution of the elasto-plastic process associated with $\mathcal{E}(t,\cdot,\cdot)$ and $\mathcal{D}$, if **stability (S)** and the **energy inequality (E)** holds:

(S) For all $t \in [0,T]$ we have
$\mathcal{E}(t, \varphi(t), z(t)) \le \mathcal{E}(t, \widetilde{\varphi}, \widetilde{z}) + \mathcal{D}(z(t), \widetilde{z})$ for all admissible states $(\widetilde{\varphi}, \widetilde{z})$.

(E) For all $s, t \in [0,T]$ with $s < t$ we have
$\mathcal{E}(s, \varphi(s), z(s)) + \mathrm{Diss}(z, [s,t]) \le \mathcal{E}(t, \varphi(t), z(t)) - \int_s^t \langle \dot{\ell}(\tau, \varphi(\tau)\rangle\,\mathrm{d}\tau.$

So far, we are not able to provide existence results for (S) & (E) in the present elasto-plastic setting. However, analogous models in phase transformations [MTL02, MR03], in delamination [KMR03], in micro-magnetism [Kru02, RK02] and in fracture [FM93, FM98, DMT02] have been treated with mathematical success. In these works two major restrictions had to be made: (i) $\mathcal{E}$ has to be convex in the strains (leading to infinitesimal strains) and (ii) the internal variable z has to lie in a closed convex subset of a Banach space. In finite-strain elasto-plasticity these two assumptions are clearly violated. For a more general nonlinear version we refer to [MM03a], where severe compactness assumptions are used to construct solutions. So far it is not clear how this compactness can be established in elasto-plasticity, however, in [MM03b] first steps are taken by introducing a suitable regularization.

Since most of the above-mentioned existence results are based on time-incremental approximations we devote this work to an existence theory for the following incremental problem (IP). The hope is that after having developed a suitable existence theory for (IP) that the methods in [MM03a] can be adjusted to pass to the limit for step size to tending to 0 and thus find solutions for (S) & (E).

> **(IP) Incremental problem.** For given $t_0 = 0 < t_1 < \ldots < t_N = T$ and z_0 find incrementally, for $k = 1, \ldots, N$,
>
> $$(\varphi_k, z_k) \in \underset{(\varphi,z)}{\arg\min}[\mathcal{E}(t_k, \varphi, z)+\mathcal{D}(z_{k-1}, z)].$$

Here "arg min" denotes the set of all global minimizers. Hence, the (IP) consists of k minimization problems which are coupled via the dissipation distance. The problem in solving (IP) is that the minimization at the k-th step involves the solution z_{k-1} from the previous step. For solving the N minimization problems in (IP) it needs a careful bookkeeping of the properties of the solutions, in particular we have to control the integrability conditions of P_k and P_k^{-1} independently of k. This will be done by the help of the dissipation distance $\mathcal{D}$, whereas the elastic energy $\mathcal{E}$ is used to control the Sobolev norm of φ_k.

Such incremental minimization problems are heavily used in the engineering community, cf. [OR99, OS99, MSS99, ORS00, ML03, MSL02, HH03], which justifies to study (IP) in its own right. In fact, existence and nonexistence for (IP) relates to questions of formation of microstructure, localization or failure, see the discussions in [Mie03, Mie04]. The failure mechanisms in elasto-plasticity are currently an active research area. However, the aim of our work is to provide and examples and to isolate general conditions which excludes these failures. In fact, there are many commercial codes for the numerical simulation of plastic processes (like deep drawing) which are expected to describe nice solutions in regions where no failure arises. We want to contribute to the challenging task of providing a mathematical understanding of these models and hopefully improve the numerical simulation techniques.

The plan of the paper is as follows. In Section 2 we introduce the notions of finite-strain elasto-plasticity in detail and establish the relation between the classical flow rules of elasto-plasticity with our energetic formulation (S) &(E). For a more extensive and mechanical treatment we refer to [Mie03]. In Section 3 we start the mathematical analysis by studying the incremental problem (IP) in specific function spaces $\mathcal{F} \times \mathcal{Z}$. To start with, we establish a rather general result which says that any solution $(\varphi_k, z_k)_{k=1,\ldots,N}$ of (IP) is stable in the sense of (S) and satisfies a two-sided discretized energy inequality replacing (E).

The key feature to the analysis of (IP) is to realize that the internal variables $z = (F_{\text{plast}}, p)$ occur under the integral over the body Ω only in a local fashion. Hence, it is possible to minimize in (IP) with respect to z pointwise in $x \in \Omega$. This leads to the condensed energy density

$$W^{\text{cond}}(z_{\text{old}}; F) = \min\{\, W(FP, p) + D(z_{\text{old}}, (P, p)) \mid (P, p) \in \mathrm{SL}(\mathbb{R}^d) \times \mathbb{R}^m \,\}.$$

In [CHM02, Mie03] it is shown that W^{cond} has also mechanical significance, as it contains the effective information of the interplay between energy storage through W and the dissipation mechanism through D. The first major assumption for our existence theory is that $W^{\text{cond}}((\mathbf{1}, p_*); \cdot) : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ is polyconvex. The second major assumptions is that the condensed energy density W^{cond} and the dissipation distance D are coercive:

$$W^{\text{cond}}((\mathbf{1}, p_*); F) \geq c|F|^{q_F} - C \quad \text{and} \quad D((\mathbf{1}, p_*), (P, p)) \geq c|P|^{q_P} - C.$$

If the growth exponents satisfy $\frac{1}{q_P} + \frac{1}{q_F} \le \frac{1}{q} < \frac{1}{d}$, then existence of solutions $(\varphi_k, F^{(k)}_{\text{plast}}, p_k)$ for (IP) is obtained with $\varphi_k \in \mathrm{W}^{1,q}(\Omega, \mathbb{R}^d)$ and $F^{(k)}_{\text{plast}} \in \mathrm{L}^{q_P}(\Omega, \mathbb{R}^{d\times d})$.

In Section 4 we supply a specific two-dimensional example in which all assumption can be checked explicitly and are fulfilled for suitable parameter values. Thus, we provide a first existence theory a for multi-dimensional elasto-plastic incremental problem in the geometric nonlinear case.

In Section 5 we treat a one-dimensional example where again the existence theory for (IP) can be carried out explicitly. Using this example we discuss the difficulties in proving existence of solutions for the time-continuous problem (S) & (E) by letting the step-size of the time discretizations going to 0. In Section 6, using the very specific properties of the one-dimensional case (like $\operatorname{div} \sigma = 0 \implies \sigma = $const.), we finally prove a convergence result for the incremental solution which implies that the time-continuous problem (S) & (E) has a solution as well.

2 Elasto-plasticity at finite strain

We consider an elastic body $\Omega \subset \mathbb{R}^d$ which is bounded and has a Lipschitz boundary $\partial\Omega$. A deformation is a mapping $\varphi : \Omega \to \mathbb{R}^d$ such that the deformation gradient $F(x) = \mathrm{D}\varphi(x)$ exists for a.e. $x \in \Omega$ and satisfies

$$F(x) \in \mathrm{GL}_+(\mathbb{R}^d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0 \,\}.$$

The internal plastic state at a material point $x \in \Omega$ is described by the plastic tensor $P = F_{\text{plast}} \in \mathrm{GL}_+(\mathbb{R}^d)$ and a possibly vector-valued hardening variable $p \in \mathbb{R}^m$. We shortly write $z = (P, p)$ to denote the set of all plastic variables. The major assumption in finite-strain elasto-plasticity is the multiplicative decomposition of the deformation gradient F into an elastic and a plastic part

$$F = F_{\text{elast}} F_{\text{plast}} = F_{\text{elast}} P. \tag{2.1}$$

The point of this decomposition is that the elastic properties will depend only on F_{elast}, whereas previous plastic transformations through P are completely forgotten. However, the hardening variable p will record changes in P and may influence the elastic properties.

The deformation process is governed by two principles. First we have energy storage which gives rise to the equilibrium equations and second we have dissipation due to plastic transformations which give rise to the plastic flow rule. Energy storage is described by the Gibbs energy

$$\mathcal{E}(t, \varphi, z) = \textstyle\int_\Omega W(x, \mathrm{D}\varphi(x), z(x))\,\mathrm{d}x - \langle \ell(t), \varphi \rangle, \tag{2.2}$$

where $\langle \ell(t), \varphi \rangle = \int_\Omega f_{\text{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x + \int_{\Gamma_{\text{Neu}}} g_{\text{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}a(x)$ denotes the loading depending on the process-time $t \in [0,T]$. The major constitutive assumption is the multiplicative decomposition

$$W(x, F, (P,p)) = \widehat{W}(x, FP^{-1}, p). \tag{2.3}$$

From now on we drop the variable x for notational convenience. However, the whole theory and analysis works in the inhomogeneous case as well.

The dissipational effects are usually modeled by prescribing yield surfaces. For our purpose it is more convenient and mathematically clearer to start on the other side, namely the dissipation metric. In mechanics this metric is called dissipation potential, since the dissipational friction forces are obtained from it via differentiation with respect to the plastic rates. We emphasize that the natural setup for the plastic transformation $P \in \mathrm{GL}_+(\mathbb{R}^d)$ is that of an element of a Lie group $\mathfrak{G} \subset \mathrm{GL}_+(\mathbb{R}^d)$. A usual assumption is incompressibility, which gives $\mathfrak{G} = \mathrm{SL}(\mathbb{R}^d) = \{\, P \mid \det P = 1 \,\}$. However, $\mathfrak{G} = \mathrm{GL}_+(\mathbb{R}^d)$ or a single-slip system $\mathfrak{G} = \{\, \mathbf{1} + \gamma e_1 \otimes e_2 \mid \gamma \in \mathbb{R} \,\}$ may also be possible. A dissipation potential is a mapping

$$\Delta : \Omega \times \mathrm{T}(\mathfrak{G} \times \mathbb{R}^m) \to [0, \infty], \tag{2.4}$$

which is called a dissipation metric if it is continuous and $\Delta(x, (P, p), \cdot)$ is convex and positively homogeneous of degree 1:

$$\Delta(x, (P, p), \alpha(\dot{P}, \dot{p})) = \alpha \Delta(x, (P, p), (\dot{P}, \dot{p})) \text{ for } \alpha \geq 0. \tag{2.5}$$

(Again we will drop the variable x for notational convenience.) This condition leads to rate-independent material behavior. Together with the multiplicative decomposition (2.1) one assumes **plastic indifference**

$$\Delta((P\widehat{P}, p), (\dot{P}\widehat{P}, \dot{p})) = \Delta((P, p), (\dot{P}, \dot{p})) \text{ for all } \widehat{P} \in \mathfrak{G}. \tag{2.6}$$

This amounts in the existence of a function $\widehat{\Delta} : \mathbb{R}^m \times \mathbb{R}^m \times \mathfrak{g} \to [0, \infty]$ such that

$$\Delta((P, p), (\dot{P}, \dot{p})) = \widehat{\Delta}(p, \dot{p}, \dot{P}P^{-1}). \tag{2.7}$$

Here $\mathfrak{g} = \mathrm{T}_{\mathbf{1}}\mathfrak{G}$ is the Lie algebra associated with the Lie group $\mathfrak{G}$, and $\dot{P}P^{-1}$ is strictly speaking the right translation of $\dot{P}(t) \in \mathrm{T}_{P(t)}\mathfrak{G}$ to $\mathfrak{g} = \mathrm{T}_{\mathbf{1}}\mathfrak{G}$.

An important feature of our theory is the induced dissipation distance D on $\mathfrak{G}\times\mathbb{R}^m$ defined via (recall $z = (P, p)$)

$$D(z_0, z_1) = \inf\{\, \textstyle\int_0^1 \Delta(z(s), \dot{z}(s))\,\mathrm{d}s \mid z \in \mathrm{C}^1([0,1], \mathfrak{G}\times\mathbb{R}^m), z(0) = z_0, z(1) = z_1 \,\}. \tag{2.8}$$

It is important to note that we didn't assume symmetry (i.e., $\Delta(z, -\dot{z}) = \Delta(z, \dot{z})$) which would contradict hardening. Thus, $D(\cdot, \cdot)$ will not be symmetric either. However, we will often use the triangle inequality

$$D(z_1, z_3) \leq D(z_1, z_2) + D(z_2, z_3), \tag{2.9}$$

which is immediate from the definition. Plastic difference implies that the dissipation distance satisfies

$$D((P_1, p_1), (P_2, p_2)) = D((\mathbf{1}, p_1), (P_2 P_1^{-1}, p_2)). \tag{2.10}$$

Integration over the body Ω gives the total dissipation between two internal states $z_j : \Omega \to \mathfrak{G}\times\mathbb{R}^m$ via

$$\mathcal{D}(z_0, z_1) = \textstyle\int_\Omega D(z_0(x), z_1(x))\,\mathrm{d}x. \tag{2.11}$$

To make the energetic formulation mathematically rigorous we define the set of kinematically admissible deformations via

$$\mathcal{F} = \{\, \varphi \in \mathrm{W}^{1,q}(\Omega; \mathbb{R}^d) \mid \varphi|_{\Gamma_{\mathrm{Dir}}} = \varphi_{\mathrm{Dir}} \,\}, \tag{2.12}$$

where $\Gamma_{\mathrm{Dir}} = \partial\Omega/\Gamma_{\mathrm{Neu}}$ is a part of the boundary with positive surface measure. Moreover, $\varphi_{\mathrm{Dir}} = \widehat{\varphi}|_{\Gamma_{\mathrm{Dir}}}$ where $\widehat{\varphi} \in \mathrm{C}^1(\overline{\Omega}; \mathbb{R}^d)$ with $\mathrm{D}\widehat{\varphi}(x) \in \mathrm{GL}_+(\mathbb{R}^d)$ for all $x \in \overline{\Omega}$. The integrability power q in $\mathrm{W}^{1,q}$ will be chosen larger than the space dimension d

in order to apply the theory of polyconvexity. The loading can then be considered as a function $\ell : [0,T] \to \mathrm{W}^{1,q}(\Omega, \mathbb{R}^d)^*$, where * denotes the dual space (space of all continuous linear forms).

The set of admissible internal states is simply

$$\mathcal{Z} = \{\, z : \Omega \to \mathfrak{G} \times \mathbb{R}^m \mid z \text{ measurable} \,\}. \tag{2.13}$$

Because of the image space, which is a manifold, it is not clear whether it is reasonable to consider $\mathcal{Z}$ as a subset of a Banach space like $L^1(\Omega, \mathbb{R}^{d\times d} \times \mathbb{R}^m)$. It rather seems natural to equip $\mathcal{Z}$ with the metric $\mathcal{D}$ and use arguments of general metric spaces. Nevertheless, our analysis will be based on states $z = (P,p) \in \mathcal{Z}$ with $P \in \mathrm{L}^{q_P}(\Omega, \mathbb{R}^{d\times d})$ for a suitable $q_P > 1$. However, the topology on the set $\mathcal{Z}$ will not be important.

Definition 2.1 *A process $(\varphi, z) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ is called a solution of the elasto-plastic problem defined via $\mathcal{E}(t,\cdot,\cdot)$ and $\mathcal{D}$ if the stability condition* ***(S)*** *and the energy inequality* ***(E)*** *holds:*

(S) *For all $t \in [0,T]$ we have*
$\mathcal{E}(t, \varphi(t), z(t)) \le \mathcal{E}(t, \widetilde{\varphi}, \widetilde{z}) + \mathcal{D}(z(t), \widetilde{z})$ for all $(\widetilde{\varphi}, \widetilde{z}) \in \mathcal{F} \times \mathcal{Z}$.

(E) *For all $s,t \in [0,T]$ with $s < t$ we have*
$\mathcal{E}(t, \varphi(t), z(t)) + \mathrm{Diss}(z, [s,t]) \le \mathcal{E}(s, \varphi(s), z(s)) - \int_s^t \langle \dot{\ell}(r), \varphi(r)\rangle \,\mathrm{d}r.$

(2.14)

Here $-\int_s^t \langle \dot{\ell}, \varphi\rangle \,\mathrm{d}r = \int_s^t \langle \ell, \dot{\varphi}\rangle \,\mathrm{d}r - \langle \ell, \varphi\rangle|_s^t$ is called the reduced work of the external forces, since $\mathcal{E}$ denotes the Gibbs energy instead of the Helmholtz energy. The dissipation is defined as

$$\mathrm{Diss}(z, [s,t]) = \sup\{\, \textstyle\sum_{j=1}^N \mathcal{D}(z(t_{j-1}), z(t_j)) \mid N \in \mathbb{N}, s \le t_0 < \ldots < t_N \le t \,\}$$

for general processes, which equals $\mathrm{Diss}(z, [s,t]) = \int_s^t \int_\Omega \Delta(z(r,x), \dot{z}(r,x)) \,\mathrm{d}x \,\mathrm{d}t$ for differentiable processes.

The major advantage of the energetic formulation via (S) and (E) is that neither derivatives of the constitutive functions W and Δ nor of the solution $(\mathrm{D}\varphi, z)$ are needed. Nevertheless, (S) and (E) are strong enough to determine the physically relevant solutions. We refer to [MT03] for uniqueness results under additional convexity assumptions. Moreover, it is shown in [Mie03] that sufficiently smooth solutions (φ, z) of (S) and (E) satisfy the classical equations of elasto-plasticity, namely the **equilibrium equation**

$$\left.\begin{array}{rcll} -\operatorname{div} T(t,x) &=& f_{\mathrm{ext}}(t,x) & \text{in } \Omega, \\ \varphi(t,x) &=& Y_{\mathrm{Dir}}(x) & \text{on } \Gamma_{\mathrm{Dir}}, \\ T(t,x)\nu(x) &=& g_{\mathrm{ext}}(t,x) & \text{on } \Gamma_{\mathrm{Neu}}, \end{array}\right\} \tag{2.15}$$

where $T(t,x) = \frac{\partial}{\partial F} W(\mathrm{D}\varphi(t,x), z(t,x)) = \frac{\partial}{\partial F_{\mathrm{elast}}} \widehat{W}(\mathrm{D}\varphi(t,x)P(t,x)^{-1}, p(t,x))P(t,x)^{-\mathsf{T}}$, and the **flow rule**

$$0 \in \partial_{\dot{z}}^{\mathrm{sub}} \Delta(z(t,x), \dot{z}(t,x)) - Q(t,x), \tag{2.16}$$

where $\partial_{\dot{z}}^{\mathrm{sub}} \Delta(z, \dot{z})$ denotes the subgradient of the convex function $\Delta(z,\cdot) : \mathrm{T}_z(\mathfrak{G} \times \mathbb{R}^m) \to [0,\infty]$ and Q is the driving force thermodynamically conjugated to z, i.e.,

$$Q = -\tfrac{\partial}{\partial(P,p)} W(F,(P,p)) = (P^{-\mathsf{T}} F^{\mathsf{T}} \tfrac{\partial}{\partial F_{\mathrm{elast}}} \widehat{W}(FP^{-1}, p)P^{-\mathsf{T}}, -\tfrac{\partial}{\partial p} \widehat{W}(FP{-}1, p)).$$

Defining the elastic domain as $\mathbb{Q}(z) = \partial_{\dot z}^{\mathrm{sub}}\Delta(z,0) \subset \mathrm{T}_z^*(\mathfrak{G}\times\mathbb{R}^m)$, the Legendre-Fenchel transform shows that (2.16) is equivalent to

$$\dot z \in \partial\mathcal{X}_{\mathbb{Q}(z)}(Q) = \mathrm{N}_Q\mathbb{Q}(z). \tag{2.17}$$

If $\mathbb{Q}(z)$ is given by a yield function Φ in the form

$$\mathbb{Q}(z) = \{\, Q \mid \Phi(z,Q) \leq 0 \,\}$$

and $\frac{\partial}{\partial Q}\Phi(z,Q) \neq 0$ at $\Phi(z,Q) = 0$, then (2.17) can be reformulated via the Karush-Kuhn-Tucker conditions

$$\dot z = \lambda\tfrac{\partial}{\partial Q}\Phi(z,Q), \quad \lambda \geq 0, \quad \Phi(z,Q) \leq 0, \quad \lambda\Phi(z,Q) = 0.$$

3 Incremental problems

Until now no existence theory for the time continuous problem (S) & (E) is available, except for the case $d = 1$ given in Section 5 below. Following the abstract developments in [MT03] and the applications of the same energetic approach to models for shape-memory alloys [MTL02, MR03] it is clear that for proving existence results for the highly nonlinear problem (S) & (E) it is essential to provide an existence theory for suitable associated time-discretized problems. Moreover, such incremental problems are the basis to all engineering simulations and, hence, provide a first step to the mathematical understanding of elasto-plasticity.

It was realized in [OR99, ORS00, CHM02, Mie03, Mie04] that existence of solutions for the incremental problem is not to be expected in general situations. In fact, nonexistence can be connected either with failure of the material due to localization (e.g. in shear bands) or fracture or with formation of microstructure in material domains of positive measure. Here we present constitutive assumptions which allow us to prove existence of solutions for each incremental step.

We now start with the mathematical analysis and recall that $\mathcal{F}$ and $\mathcal{Z}$ are defined in (2.12) and (2.13), respectively. Consider a time discretization $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ of the interval $[0,T]$. Moreover, assume that an initial state $(\varphi_0, z_0) \in \mathcal{F}\times\mathcal{Z}$ is given which is stable according to (S) at $t = 0$.

(IP) Incremental Problem:
For $k = 1, \ldots, N$ find $(\varphi_k, z_k) \in \mathcal{F}\times\mathcal{Z}$ such that (3.1)
$(\varphi_k, z_k) \in \arg\min\{\, \mathcal{E}(t_k,\varphi,z) + \mathcal{D}(z_{k-1},z) \mid (\varphi,z) \in \mathcal{F}\times\mathcal{Z} \,\}$.

Here "arg min" denotes the set of global minimizers. The main point is to show that this set is nonempty, i.e. there exists $(\varphi_k, z_k) \in \mathcal{F}\times\mathcal{Z}$ such that

$$\mathcal{E}(t_k,\varphi_k,z_k) + \mathcal{D}(z_{k-1},z_k) = \inf\{\, \mathcal{E}(t_k,\varphi,z) + \mathcal{D}(z_{k-1},z) \mid (\varphi,z) \in \mathcal{F}\times\mathcal{Z} \,\}.$$

We say that the minimum of $\mathcal{E}(t_k,\cdot,\cdot)+\mathcal{D}(z_{k-1},\cdot)$ is attained at the minimizer (φ_k, z_k).

Before we start the analysis of (IP) we first establish a result which emphasizes the fact that the given incremental problem is the most natural one. In particular, it illuminates the positive rôle of the dissipation distance $\mathcal{D}$, which is difficult to characterize as it is defined only implicitly via Δ in (2.8). However, replacing $\mathcal{D}(z_{k-1},z)$ in (IP) by some approximation (e.g., $\Delta(z_{k-1}, z_k{-}z)$) would destroy at least one of the three estimates provided in (i) and (ii) below.

Theorem 3.1 *Let $(\varphi_k, z_k)_{k=0,\dots,N}$ be any solution of (IP). Then, the following discrete versions of (S) and (E) hold:*

(i) For $k = 0,\dots,N$ the state (φ_k, z_k) is stable at t_k, i.e.,

$$\mathcal{E}(t_k, \varphi_k, z_k) \le \mathcal{E}(t_k, \widetilde{\varphi}, \widetilde{z}) + \mathcal{D}(z_k, \widetilde{z}) \text{ for all } (\widetilde{\varphi}, \widetilde{z}) \in \mathcal{F} \times \mathcal{Z}.$$

(ii) For all $s, t \in \{\, t_j \mid j = 0, 1, \dots, N \,\}$ with $s < t$ we have

$$\begin{aligned} -\textstyle\int_s^t \langle \dot{\ell}(r), \varphi^{\rm cl}(r)\rangle \,\mathrm{d}r \le \mathcal{E}(t, \varphi^{\rm cr}(t), z^{\rm cr}(t)) &+ \mathrm{Diss}(z^{\rm cr}, [s,t]) \\ -\mathcal{E}(s, \varphi^{\rm cr}(s), z^{\rm cr}(s)) &\le -\textstyle\int_s^t \langle \dot{\ell}(r), \varphi^{\rm cr}(r)\rangle \,\mathrm{d}r. \end{aligned}$$

Here, $\varphi^{\rm cr}$ and $\varphi^{\rm cl}$ are the piecewise constant interpolants which are continuous from the right "cr" and from the left "cl", i.e. $\varphi^{\rm cr}(t) = \varphi_{k-1}$ for $t \in [t_{k-1}, t_k)$ and $\varphi^{\rm cl}(t) = \varphi_k$ for $t \in (t_{k-1}, t_k]$ with $\varphi^{\rm cr}(t_N) = \varphi_N$ and $\varphi^{\rm cl}(t_0) = \varphi_0$. Hence,

$$\textstyle\int_{t_j}^{t_k} \langle \dot{\ell}(r), \varphi^{\rm cr}(r)\rangle \,\mathrm{d}r = \sum_{i=j+1}^{k} \langle \ell(t_i) - \ell(t_{i-1}), \varphi_{i-1}\rangle$$

and with the same notation for $z^{\rm cr}$ we have $\mathrm{Diss}(z^{\rm cr}, [t_j, t_k]) = \sum_{i=j+1}^{k} \mathcal{D}(z_{i-1}, z_i)$.

The proof does not need any specific assumptions on the function space $\mathcal{F} \times \mathcal{Z}$ or on the functionals $\mathcal{E}$ and $\mathcal{D}$, since it assumes the existence of a solution. Essential to the proof are the minimization property and the triangle inequality (2.9) for $\mathcal{D}$.

Proof: To simplify the proof we write $y_k = (\varphi_k, z_k)$ and $\widetilde{y} = (\widetilde{\varphi}, \widetilde{z})$.

ad (i): For arbitrary $\widetilde{y} \in \mathcal{F} \times \mathcal{Z}$ and $k \in \{1, \dots, N\}$ we have

$$\begin{aligned} &\mathcal{E}(t_k, \widetilde{y}) + \mathcal{D}(z_k, \widetilde{z}) = \mathcal{E}(t_k, \widetilde{y}) + \mathcal{D}(z_{k-1}, \widetilde{z}) + \mathcal{D}(z_k, \widetilde{z}) - \mathcal{D}(z_{k-1}, \widetilde{z}) \\ &\ge \mathcal{E}(t_k, y_k) + \mathcal{D}(z_{k-1}, z_k) + \mathcal{D}(z_k, \widetilde{z}) - \mathcal{D}(z_{k-1}, \widetilde{z}) \ge \mathcal{E}(t_k, y_k), \end{aligned}$$

where the first estimate follows since y_k is a minimizer and the second estimate follows from the triangle inequality for $\mathcal{D}$.

ad (ii): The lower estimate follows sine y_{i-1} is stable at t_{i-1}:

$$\begin{aligned} &-\textstyle\int_{t_{i-1}}^{t_i} \langle \dot{\ell}(r), \varphi^{\rm cl}(r)\rangle \,\mathrm{d}r = -\langle \ell(t_i), \varphi_i\rangle + \langle \ell(t_{i-1}), \varphi_i\rangle \\ &= \mathcal{E}(t_i, y_i) - \mathcal{E}(t_{i-1}, y_i) = \mathcal{E}(t_i, y_i) - \mathcal{E}(t_{i-1}, y_{i-1}) + \mathcal{E}(t_{i-1}, y_{i-1}) - \mathcal{E}(t_{i-1}, y_i) \\ &\le \mathcal{E}(t_i, y_i) - \mathcal{E}(t_{i-1}, y_{i-1}) + \mathcal{D}(z_{i-1}, z_i). \end{aligned}$$

Summing over i from $j{+}1$ to k gives the lower estimate. The upper estimate follows similarly since y_i is a minimizer at t_i:

$$\mathcal{E}(t_i, y_i) - \mathcal{E}(t_{i-1}, y_{i-1}) + \mathcal{D}(z_{i-1}, z_i) \le \mathcal{E}(t_i, y_{i-1}) - \mathcal{E}(t_{i-1}, y_{i-1}) = -\textstyle\int_{t_{i-1}}^{t_i} \langle \dot{\ell}, \varphi^{\rm cr}\rangle \,\mathrm{d}r.$$

Thus, the result is proved. ∎

We now study the existence of solutions to (IP). For this we need specific properties of the space $\mathcal{F} \times \mathcal{Z}$ and strong conditions on the functionals $\mathcal{E}$ and $\mathcal{D}$. In each time step we have to solve the global minimization problem for the functional $\mathcal{I}_k : \mathcal{F} \times \mathcal{Z} \to \mathbb{R}_\infty$ given as

$$\mathcal{I}_k(\varphi, z) := \textstyle\int_\Omega [W(\mathrm{D}\varphi(x), z(x)) + D(z_{k-1}(x), z(x))] \,\mathrm{d}x - \langle \ell(t_k), \varphi\rangle. \tag{3.2}$$

The special structure here is that $z \in \mathcal{Z}$ occurs under the integral only with its point values and no derivatives appear. We note that $\mathcal{I}_k : \mathcal{F} \times \mathcal{Z} \to \mathbb{R}_\infty$ is not lower semicontinuous because of the geometric nonlinearity coming from the multiplicative decomposition, i.e., $W(F,(P,p))=\widehat{W}(FP^{-1},p)$. It is shown in [FKP94, LDR00] that lower semicontinuity of $\mathcal{I}_k$ implies cross-quasiconvexity of

$$(F,P,p)\mapsto W(F,(P,p))+D(z_{k-1}(x),(P,p)),$$

which in turn implies convexity in $z=(P,p)$. However, this can only be achieved if $F_{\mathrm{elast}}\mapsto\widehat{W}(F_{\mathrm{elast}})$ is convex, but this contradicts the standard axioms of finite-strain elasto-plasticity, see [CHM02] and below.

Of course, lower semi-continuity of $\mathcal{I}_k$ is not necessary and we may obtain minimizers without it. The idea is, that we can minimize with respect to z for each point $x\in\Omega$ separately. To prepare the following result we define the **condensed energy density**

$$W^{\mathrm{cond}}(z_{\mathrm{old}};F)=\min\{\,W(F,z)+D(z_{\mathrm{old}},z)\mid z\in\mathfrak{G}\times\mathbb{R}^m\,\}$$

and the condensed functional

$$\mathcal{I}_k^{\mathrm{cond}}(\varphi)=\int_\Omega W^{\mathrm{cond}}(z_{k-1}(x);\mathrm{D}\varphi(x))\,\mathrm{d}x-\langle\ell(t_k),\varphi\rangle.$$

According to [ET76, Ch.VIII,§1.6] we can choose a measurable **update function**

$$\begin{aligned}&z^{\mathrm{upd}}:(\mathfrak{G}\times\mathbb{R}^m)\times\mathbb{R}^{d\times d}\to\mathfrak{G}\times\mathbb{R}^m \text{ with}\\ &z^{\mathrm{upd}}(z_{\mathrm{old}};F)\in Z(z_{\mathrm{old}};F):=\arg\min\{\,W(F,z)+D(z_{\mathrm{old}},z)\mid z\in\mathfrak{G}\times\mathbb{R}^m\,\},\end{aligned}$$

i.e., $W^{\mathrm{cond}}(z_{\mathrm{old}};F)=(W(F,z)+D(z_{\mathrm{old}},z))|_{z=z^{\mathrm{upd}}(z_{\mathrm{old}};F)}$.

Lemma 3.2 *Let W and D be nonnegative, measurable functions, such that for each $(z_{\mathrm{old}};F)$ the function $z\mapsto W(F,z)+D(z_{\mathrm{old}},z)$ is coercive. Then, W^{cond} and z^{upd} as above are well defined. Moreover, we have:*

(a) For all $(\varphi,z)\in\mathcal{F}\times\mathcal{Z}$ we have $\mathcal{I}_k^{\mathrm{cond}}(\varphi)\le\mathcal{I}_k(\varphi,z)$ with equality if and only if $z(x)\in Z(z_{k-1}(x);\mathrm{D}\varphi(x))$ for a.a. $x\in\Omega$.

(b) A pair $(\varphi,z)\in\mathcal{F}\times\mathcal{Z}$ minimizes $\mathcal{I}_k$ in (3.2) if and only if φ is a minimizer of $\mathcal{I}_k^{\mathrm{cond}}:\mathcal{F}\to\mathbb{R}_\infty$ and $z(x)\in Z(z_{k-1}(x);\mathrm{D}\varphi(x))$ for a.a. $x\in\Omega$.

(c) If $\widetilde{\varphi}\in\mathcal{F}$ minimizes $\mathcal{I}_k^{\mathrm{cond}}$ and $\widetilde{z}\in\mathcal{Z}$ satisfies $\widetilde{z}(x)=z^{\mathrm{upd}}(z_{k-1}(x);\mathrm{D}\widetilde{\varphi}(x))$, then $(\widetilde{\varphi},\widetilde{z})$ minimizes $\mathcal{I}_k$.

Proof: Part (a) is obvious, as $W^{\mathrm{cond}}(z_{k-1};F)\le W(F,z)+D(z_{k-1},z)$.

For part (b) first assume that $(\varphi,z)\in\mathcal{F}\times\mathcal{Z}$ minimizes $\mathcal{I}_k$ and let $A=\{\,x\in\Omega\mid z(x)\in Z(z_{k-1}(x);\mathrm{D}\varphi(x))\,\}$. Outside of A we can change z, while keeping φ fixed, such that the integrand $W{+}D$ becomes strictly smaller. However, decreasing an integrand strictly on a set of positive measure decreases the integral $\mathcal{I}_k$. Hence, A must have measure 0.

Assume that φ minimizes $\mathcal{I}^{\mathrm{cond}}$ and that $z\in\mathcal{Z}$ is given such that A has full measure in Ω. Then, $W^{\mathrm{cond}}=W+D$ on A implies $\mathcal{I}_k^{\mathrm{cond}}(\varphi)=\mathcal{I}_k(\varphi,z)$. With part (a) we conclude that (φ,z) minimizes $\mathcal{I}_k$.

Part (c) is obtained exactly the same way, as now $A=\Omega$. ■

This simple lemma shows that each step in the incremental problem (IP) reduces to a classical variational problem of nonlinear elasticity. Using the multiplicative decomposition (2.3) and the plastic indifference of the dissipation (2.10) we immediately see that W^{cond} satisfies

$$W^{\mathrm{cond}}((P_{\mathrm{old}},p_{\mathrm{old}});F)=W^{\mathrm{cond}}((\mathbf{1},p_{\mathrm{old}});FP_{\mathrm{old}}^{-1}),\tag{3.3}$$

thus it is uniquely determined by $W^{\mathrm{cond}}((\mathbf{1},\cdot);\cdot) : \mathbb{R}^m \times \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$. Similarly, we may choose z^{upd} such that it satisfies

$$z^{\mathrm{upd}}((P_{\mathrm{old}},p_{\mathrm{old}});F) = z^{\mathrm{upd}}((\mathbf{1},p_{\mathrm{old}});FP_{\mathrm{old}}^{-1}) \begin{matrix} P_{\mathrm{old}} & 0 \\ 0 & 1 \end{matrix} . \tag{3.4}$$

We now list all assumptions which are stated in terms of W^{cond} and D. Thus, the assumptions are quite implicit, since in practice the stored-energy density W and the dissipation potential Δ are given. From Δ one has to calculate the dissipation distance $D(\cdot,\cdot)$ and then the condensed energy density W^{cond}. However, up to date, there are no conditions on W and Δ which are known to be sufficient for our conditions. In the next section we provide an example where all these conditions are satisfied.

$$\begin{array}{rl}
\text{(i)} & W^{\mathrm{cond}}((\mathbf{1},\cdot);\cdot) : \mathbb{R}^m \times \mathbb{R}^{d\times d} \to [0,\infty] \text{ and } D(\cdot,\cdot) : (\mathfrak{G}\times\mathbb{R}^m)^2 \to [0,\infty] \\
& \text{are lower semi-continuous.} \\
\text{(ii)} & \text{For each } p \in \mathbb{R}^m \text{ the function } W^{\mathrm{cond}}((\mathbf{1},p),\cdot) : \mathbb{R}^{d\times d} \to [0,\infty] \\
& \text{is polyconvex.} \\
\text{(iii)} & \text{There exist } C, c > 0, p_* \in \mathbb{R}^m \text{ and exponents } q_F, q_P \geq 1 \text{ such that} \\
& \qquad D((\mathbf{1},p_*),(P,p)) \geq c|P|^{q_P} - C \\
& \text{for all } (P,p), \text{and} \\
& \qquad W^{\mathrm{cond}}((\mathbf{1},p);F) \geq c|F|^{q_F} - C \\
& \text{for all } (F,P,p) \text{ with } D((\mathbf{1},p_*),(P,p)) < \infty. \\
\text{(iv)} & z^{\mathrm{upd}}((\mathbf{1},\cdot);\cdot) : \mathbb{R}^m \times \mathbb{R}^{d\times d}_+ \to \mathfrak{G}\times\mathbb{R}^m \text{ is Borel measurable.}
\end{array} \tag{3.5}$$

Note that we do not need any additional assumptions on W or Δ.

Theorem 3.3 *Let the assumptions (3.5) be satisfied such that additionally*

$$\tfrac{1}{q_F} + \tfrac{1}{q_P} \leq \tfrac{1}{q} < \tfrac{1}{d}$$

holds, where q occurs in the definition of $\mathcal{F}$ in (2.12).

Then, for each $z_0 \in \mathcal{Z}$ with $\mathcal{D}((\mathbf{1},p_*),z_0) = \int_\Omega D((\mathbf{1},p_*),(P_0(x),p_0(x)))\,\mathrm{d}x < \infty$ and each $\ell \in \mathrm{C}^0([0,T],\mathrm{W}^{1,q}(\Omega,\mathbb{R}^d)^*)$ the incremental problem (IP), see (3.1), has a solution $((\varphi_k,z_k))_{k=1,\ldots,N}$ with

$$\varphi_k \in \mathcal{F} \subset \mathrm{W}^{1,q}(\Omega,\mathbb{R}^d) \quad \textit{and} \quad z_k = z^{\mathrm{upd}}(z_{k-1};\mathrm{D}\varphi_k(\cdot)) \in \mathcal{Z} \cap \mathrm{L}^{q_P}(\Omega,\mathbb{R}^{d\times d}).$$

Proof: Obviously, the result is proved by induction over $k = 1,2,\ldots,N$.

For the k-th step we assume that $z_{k-1} \in \mathcal{Z}$ is known to satisfy $\mathcal{D}((\mathbf{1},p_*),z_{k-1}) < \infty$, which certainly holds for $k=1$. With (3.5)(iii) we conclude $P_{k-1} \in \mathrm{L}^{q_P}(\Omega,\mathbb{R}^{d\times d})$. By Lemma 3.2, the k-th minimization problem for $\mathcal{I}_k$ (cf. (3.2)) reduces to minimization of $\mathcal{I}_k^{\mathrm{cond}} : \mathcal{F} \to \mathbb{R}_\infty$, where $\mathcal{I}_k^{\mathrm{cond}}(\varphi) = \int_\Omega W_k(x,\mathrm{D}\varphi(x))\,\mathrm{d}x - \langle \ell(t_k),\varphi\rangle$ with

$$W_k(x,F) = W^{\mathrm{cond}}(z_{k-1}(x);F) = W^{\mathrm{cond}}((\mathbf{1},p_{k-1}(x));FP_{k-1}(x)^{-1}).$$

Clearly, $W_k : \Omega \times \mathbb{R}^{d\times d} \to [0,\infty]$ is measurable in x and lower semi-continuous in F. Moreover, by (3.5)(iii) we have the lower bound

$$\begin{array}{rcl}
W_k(x,F) & \geq & c|FP_{k-1}(x)^{-1}|^{q_F} - C \\
& \geq & \frac{cq_F}{q}|F|^q - c(\frac{q_F}{q}-1)|P_{k-1}(x)|^{q_F q/(q_F-q)} - c
\end{array}$$

where we have used $|FP^{-1}| \geq |F|/|P|$ and $|a/b|^{q_F} \geq ra^{q_F/r} - (r-1)b^{q_F/(r-1)}$ with $r = q_F/q > 1$. Using the assumption $\frac{1}{q_P} \leq \frac{1}{q} - \frac{1}{q_F}$ we conclude $W_k(x,F) \geq \widetilde{c}|F|^q - h(x)$ for $\widetilde{c} > 0$ and $h \in \mathrm{L}^1(\Omega)$. Hence, W_k is coercive.

Moreover, the minors (of order s) of the product FP_{k-1}^{-1} are in fact linear combinations of products of the minors (of order s) of F and P_{k-1}^{-1}. Since by (3.5)(ii) W^{cond} is polyconvex we conclude that $F \mapsto W_k(x,F)$ is polyconvex as well.

The classical existence theory of Ball [Bal76, Bal77] provides $\varphi_k \in \mathcal{F} \subset \mathrm{W}^{1,q}(\Omega,\mathbb{R}^d)$ such that $\mathcal{I}_k^{\mathrm{cond}}(\varphi_k) = \inf\{\,\mathcal{I}_k^{\mathrm{cond}}(\varphi) \mid \varphi \in \mathcal{F}\,\}$. By Lemma 3.2 we see that (φ_k, z_k) with $z_k = z^{\mathrm{upd}}(z_{k-1};\mathrm{D}\varphi_k) \in \mathcal{Z}$ minimizes $\mathcal{I}_k : \mathcal{F}\times\mathcal{Z} \to \mathbb{R}_\infty$.

To finish the induction we have to show $\mathcal{D}((\mathbf{1},p_*),z_k) < \infty$. To see this we use the triangle inequality for $\mathcal{D}$ and the minimization property of (φ_k, z_k) in the form of the energy estimate as in part (ii) of Theorem 3.1. We have

$$\begin{aligned}&\mathcal{D}((\mathbf{1},p_*),z_k) \leq \mathcal{D}((\mathbf{1},p_*),z_{k-1}) + \mathcal{D}(z_{k-1},z_k)\\ &\leq \mathcal{D}((\mathbf{1},p_*),z_{k-1}) + \mathcal{I}_{k-1}^{\mathrm{cond}}(\varphi_{k-1}) - \mathcal{I}_k^{\mathrm{cond}}(\varphi_k) + \langle \ell(t_{k-1}) - \ell(t_k), \varphi_{k-1}\rangle < \infty.\end{aligned}$$

This concludes the induction step, and hence the whole proof. ■

4 A two-dimensional example

The purpose of this section is to supply a multi-dimensional example with $\mathfrak{G} = \mathrm{SL}(\mathbb{R}^d)$ where all assumptions of the previous section can be fulfilled. Unfortunately, our example only works in $d = 2$, since it depends on the fact that everything can be calculated explicitly.

We consider the isotropic elastic energy density

$$W : \left\{ \begin{array}{ccl} \mathbb{R}^{2\times 2} & \to & \mathbb{R}_\infty, \\ F & \mapsto & \frac{1}{\alpha}(\nu_1^\alpha + \nu_2^\alpha) + V(\det F), \end{array}\right. \tag{4.1}$$

where $\nu_1, \nu_2 \geq 0$ are the two singular values of F (i.e., the eigenvalues of $(F^{\mathsf{T}}F)^{1/2}$) and $V : \mathbb{R} \to [0,\infty]$ is convex, continuous and satisfies

$$V(\delta) = \infty \text{ for } \delta \leq 0, \quad V(\delta) \nearrow \infty \text{ for } \delta \searrow 0.$$

For the plastic variables we take $z = (P,p) \in \mathrm{SL}(2)\times\mathbb{R}$ with the dissipation metric

$$\Delta(P,p,\dot P,\dot p) = \left\{ \begin{array}{cl} A'(p)\|\dot P P^{-1}\| & \text{for } \dot p \geq \|\dot P P^{-1}\|, \\ \infty & \text{else.} \end{array}\right. \tag{4.2}$$

Here, $\|\cdot\|$ denotes the classical Euclidean norm on $\mathfrak{g} \subset \mathbb{R}^{2\times 2}$, i.e., $\|\xi\|^2 = \sum_{i,j=1}^2 \xi_{ij}^2$, and $A(p) = \mathrm{e}^{\beta p}$ for $\beta > 0$. The associated dissipation distance D is plastically invariant and isotropic, i.e.

$$D((RP_0\widehat{P},p_0),\,(RP_1\widehat{P},p_1)) = D((P_0,p_0),(P_1,p_1))$$

for all arguments. From the analysis in [Mie02a, HMM03, Mie03] we know that

$$D((\mathbf{1},p_0),(E(s),p_1)) = \left\{ \begin{array}{cl} \mathrm{e}^{\beta(p_0+\sqrt{2}|s|)} - \mathrm{e}^{\beta p_0} & \text{for } p_1 \geq p_0 + \sqrt{2}|s|, \\ \infty & \text{else,} \end{array}\right. \tag{4.3}$$

where $E(s) = \mathrm{diag}(\mathrm{e}^s, \mathrm{e}^{-s})$, and, for all $R, \widehat{R} \in \mathrm{SO}(2)$,

$$D((\mathbf{1},p_0),(RE(s)\widehat{R},p_1)) \geq D((\mathbf{1},p_0),(E(s),p_1)). \tag{4.4}$$

With this information, it is shown in [Mie03] that the condensed stored-energy density takes the form

$$W^{\rm cond}((\mathbf{1},p);F) = \min_{s\in\mathbb{R}} \tfrac{1}{\alpha}((\mathrm{e}^{-s}\nu_1)^\alpha + (\mathrm{e}^{s}\nu_2)^\alpha) + V(\nu_1\nu_2) + \mathrm{e}^{p\beta}(\mathrm{e}^{\sqrt{2}\beta|s|}-1).$$

To see this, one uses the isotropy of W and D together with (4.4) to deduce that the minimum in $W^{\rm cond}$ with $F=\mathrm{diag}(\nu_1,\nu_2)$ is attained for $P=E(s)=\mathrm{diag}(\mathrm{e}^s,\mathrm{e}^{-s})$ for some $s\in\mathbb{R}$.

The minimum over $s\in\mathbb{R}$ can be evaluated explicitly if we choose $\beta=\alpha/\sqrt{2}$. This gives the final form

$$W^{\rm cond}((\mathbf{1},p);F) = V(\nu_1\nu_2) - \mathrm{e}^{\alpha p/\sqrt{2}} + \begin{cases} \frac{2}{\alpha}\sqrt{\nu_1^\alpha(\nu_2^\alpha+b_p)} & \text{for } \nu_1^\alpha \geq \nu_2^\alpha + b_p, \\ \frac{1}{\alpha}(\nu_1^\alpha+\nu_2^\alpha+b_p) & \text{for } |\nu_1^\alpha-\nu_2^\alpha| \leq b_p, \\ \frac{2}{\alpha}\sqrt{\nu_2^\alpha(\nu_1^\alpha+b_p)} & \text{for } \nu_2^\alpha \geq \nu_1^\alpha + b_p, \end{cases}$$

where $b_p=\alpha\mathrm{e}^{\alpha p/\sqrt{2}}$. Moreover, the update functions can be given explicitly as well. With the auxiliary function

$$S(\nu,p) = \begin{cases} -\frac{1}{2\alpha}\log\frac{\nu_1^\alpha}{\nu_2^\alpha+b_p} & \text{for } \nu_1^\alpha \geq \nu_2^\alpha + b_p, \\ 0 & \text{for } |\nu_1^\alpha-\nu_2^\alpha| \leq b_p, \\ \frac{1}{2\alpha}\log\frac{\nu_2^\alpha}{\nu_1^\alpha+b_p} & \text{for } \nu_2^\alpha \geq \nu_1^\alpha + b_p, \end{cases}$$

we find the update functions (for $\det F=\nu_1\nu_2>0$)

$$P^{\rm upd}((\mathbf{1},p_0);F) = R_F^{-1}E(S(\nu,p_0))R_F \quad\text{and}\quad p^{\rm upd}((\mathbf{1},p_0);F) = p_0 + \sqrt{2}|S(\nu,p_0)|,$$

where $\nu_1,\nu_2>0$ and R_F are defined via $F=\widehat{R}\,\mathrm{diag}(\nu_1,\nu_2)R_F$ with $\widehat{R},R_F\in\mathrm{SO}(2)$. Both update functions are locally Lipschitz continuous since R_F is uniquely defined where $S(\nu,p)\neq 0$.

We summarize the properties of $W^{\rm cond}$ and D in the following proposition which establishes the conditions (3.5).

Proposition 4.1 *Let W and Δ be defined as above with $\beta=\alpha/\sqrt{2}$. Then:*

(i) $W^{\rm cond}((\mathbf{1},\cdot);\cdot):\mathbb{R}\times\mathbb{R}^{2\times2}\to\mathbb{R}_\infty$ is continuous and $D(\cdot,\cdot):(\mathrm{SL}(2)\times\mathbb{R})^2\to[0,\infty]$ is lower semi-continuous.

(ii) For $\alpha\geq 2$ and $p\in\mathbb{R}$ the function $W^{\rm cond}((\mathbf{1},p);\cdot):\mathbb{R}^{2\times d}\to\mathbb{R}_\infty$ is polyconvex.

(iii) For all $F\in\mathbb{R}^{2\times2}, p_*,p\in\mathbb{R}$ and $P\in\mathrm{SL}(2)$ with $D((\mathbf{1},p_*),(P,p))<\infty$ we have

$$\begin{aligned} D((\mathbf{1},p_*),(P,p)) &\geq \tfrac{\mathrm{e}^{\alpha p_*/\sqrt{2}}}{2}\Big(\|P\|^\alpha-1\Big), \\ W^{\rm cond}((\mathbf{1},p);F) &\geq \tfrac{1}{\alpha}\Big(\sqrt{b_p}\,2^{1-\alpha/2}\,\|F\|^{\alpha/2}-b_p\Big). \end{aligned}$$

(iv) The update function $z^{\rm upd}=(P^{\rm upd},p^{\rm upd})$ is continuous.

Proof: Part (i) and (iv) are immediate from the definitions and formulas. Part (ii) is the most difficult part, its proof is given in [Mie02b].

To prove the lower estimates in (iii) we first note that $P\in\mathrm{SL}(2)$ has the form $P=R_1\,\mathrm{diag}(g,1/g)R_2=R_1E(\log g)R_2$. With (4.3) and (4.4) we obtain

$$D((\mathbf{1},p_*),(P,p)) \geq \mathrm{e}^{\alpha p_*/\sqrt{2}}(\mathrm{e}^{\alpha|\log g|}-1).$$

Using $\|P\| = \sqrt{g^2 + 1/g^2} \leq \sqrt{2}\max\{g, 1/g\} = \sqrt{2}\mathrm{e}^{|\log g|}$ gives the first estimate.

For the second estimate we use the explicit form of $W^{\mathrm{cond}}((\mathbf{1}, p); F)$ and $V \geq 0$ to find the lower estimate $\frac{2}{\alpha}\sqrt{b_p}(\max\{\nu_1, \nu_2\})^{\alpha/2}$. With $\|F\| = \sqrt{\nu_1^2 + \nu_2^2} \leq \sqrt{2}\max\{\nu_1, \nu_2\}$ the desired estimate follows. ■

Thus, we have shown that this example satisfies the assumptions (3.5) for $\alpha \geq 2$ with $q_F = \alpha/2$ and $q_P = \alpha$. Hence, Theorem 3.3 is applicable if

$$\tfrac{1}{2} = \tfrac{1}{d} > \tfrac{1}{q} \geq \tfrac{1}{q_F} + \tfrac{1}{q_P} = \tfrac{3}{\alpha}$$

holds. We summarize the existence result for this example in the following statement.

Theorem 4.2 *Let $d = 2$ and $\mathfrak{G} = \mathrm{SL}(2)$. With $\alpha > 6$ and $\beta = \alpha/\sqrt{2}$ let $W : \mathbb{R}^{2\times 2} \to [0, \infty]$ and $\Delta : \mathrm{T}(\mathfrak{G}\times\mathbb{R}) \to [0, \infty]$ be defined via (4.1) and (4.2), respectively. Assume that there exists a $p_* \in \mathbb{R}$, such that the initial condition $z_0 \in \mathcal{Z}$ satisfies $\mathcal{D}((\mathbf{1}, p_*), z_0) < \infty$ and let $q = \alpha/3$.*

Then, for each $\ell : [0, T] \to (\mathrm{W}^{1,\alpha/3}(\Omega, \mathbb{R}^2))^*$ the incremental problem (IP) (see (3.1)) has a solution $((\varphi_k, z_k))_{k=1,\dots,N} \in (\mathcal{F}\times\mathcal{Z})^N$. Moreover, there exists a constant C which depends only on α, ℓ, and z_0, but neither on the partition $t_1, \dots, t_N$ nor on the solution, such that

$$\|\varphi_k\|_{\mathrm{W}^{1,\alpha/3}} + \|P_k\|_{\mathrm{L}^\alpha} + \|\mathrm{e}^{\alpha p_k/\sqrt{2}}\|_{\mathrm{L}^1} \leq C \text{ for } k = 1, \dots, N.$$

5 A one-dimensional example

The one-dimensional case is quite special and much simpler for two reasons. First, polyconvexity is equivalent to convexity, and second, the equilibrium equation is an ordinary differential equation which can be solved easily. Nevertheless this case is interesting, since we will be able to discuss the problems with convergence for step-size going to 0 of the incremental solutions towards a solution of the time-continuous problem (S) & (E), see (2.14). We will see, that general arguments, which are available in higher space dimensions as well, are not sufficient. In Section 6, using the special one-dimensional structure, we then prove convergence (of a subsequence) and obtain finally an existence result for (S) & (E).

Again we treat a special case, but far more general constitutive laws W and Δ could be considered. We let

$$W(F) = \begin{cases} \frac{1}{\alpha}(F^\alpha + F^{-\alpha}) & \text{for } F > 0, \\ \infty & \text{else,} \end{cases}$$

$\mathfrak{G} = \mathrm{GL}_+(1) = (0, \infty)$, $z = (P, p) \in \mathfrak{G} \times \mathbb{R}$, and

$$\Delta((P, p), (\dot{P}, \dot{p})) = \begin{cases} \alpha \mathrm{e}^{\alpha p}\dot{p} & \text{for } \dot{p} \geq |\dot{P}/P|, \\ \infty & \text{else.} \end{cases}$$

As in the previous section (see also [Mie03]), we obtain the dissipation distance

$$\mathcal{D}((P_0, p_0), (P_1, p_1)) = \begin{cases} \mathrm{e}^{\alpha p_1} - \mathrm{e}^{\alpha p_0} & \text{for } p_1 \geq p_0 + |\log P_1/P_0|, \\ \infty & \text{else.} \end{cases}$$

From this we find the condensed stored-energy density

$$W^{\mathrm{cond}}((1, p); F) = \frac{1}{\alpha}\begin{cases} 2\sqrt{1 + b_p F^\alpha} - b_p & \text{for } F^\alpha \geq b_p + F^{-\alpha}, \\ F^\alpha + F^{-\alpha} & \text{for } |F^\alpha - F^{-\alpha}| \leq b_p, \\ 2\sqrt{1 + b_p F^{-\alpha}} - b_p & \text{for } F^{-\alpha} \geq b_p + F^\alpha, \\ \infty & \text{for } F \leq 0 \end{cases} \tag{5.1}$$

where $b_p = \alpha \mathrm{e}^{\alpha p}$. For $F > 0$ the update functions read

$$\begin{aligned} P^{\mathrm{upd}}((1,p);F) &= \begin{cases} F/(1{+}b_pF^\alpha)^{1/(2\alpha)} & \text{for } F^\alpha \geq b_p + F^{-\alpha}, \\ 1 & \text{for } |F^\alpha{-}F^{-\alpha}| \leq b_p, \\ F\,(1{+}b_pF^{-\alpha})^{1/(2\alpha)} & \text{for } F^{-\alpha} \geq b_p + F^\alpha; \end{cases} \\ z^{\mathrm{upd}}((1,p);F) &= p + \left|\log P^{\mathrm{upd}}((1,p);F)\right|. \end{aligned}$$

As in Section 4 we see that the abstract theory of Section 3 applies for $\alpha > 3$ since $q_F = \alpha/2$ and $q_P = \alpha$ in condition (3.5).

We consider the one-dimensional domain $\Omega = (0,1) \subset \mathbb{R}^1$. The space $\mathcal{F}^q$ of admissible deformation may either be $\mathcal{F}^q_{\mathrm{displ}} = \mathrm{W}^{1,q}_0(\Omega) = \{\, \varphi \in \mathrm{W}^{1,q}(\Omega) \mid \varphi(0) = \varphi(1) = 0 \,\}$ or $\mathcal{F}^q_{\mathrm{tract}} = \{\, \varphi \in \mathrm{W}^{1,q}(\Omega) \mid \varphi(0) = 0 \,\}$. The loading takes the form

$$\langle \ell(t), \varphi \rangle = \textstyle\int_0^1 h_{\mathrm{ext}}(t,x)\varphi(x)\,\mathrm{d}x + \sigma_1(t)\varphi(1) = \int_0^1 H_{\mathrm{ext}}(t,x)\varphi'(x)\,\mathrm{d}x$$

where $H_{\mathrm{ext}}(t,x) = \sigma_1(t) + \int_x^1 h_{\mathrm{ext}}(t,\widetilde{x})\,\mathrm{d}\widetilde{x}$ and $\varphi'(x) = \mathrm{D}\varphi(x) \in \mathbb{R}^{1\times 1}$. At this point it suffices to assume $H_{\mathrm{ext}} \in \mathrm{C}^0([0,T] \times \overline{\Omega})$.

Proposition 5.1 *Fix $\alpha > 3$ and $p_* \in \mathbb{R}$. Then, the above one-dimensional model generates the incremental problem*

$$\textbf{\textit{(IP)}} \qquad (\varphi_k, z_k) \in \arg\min\{\, \mathcal{E}(t_k,\varphi,z) + \mathcal{D}(z_{k-1},z) \mid (\varphi,z) \in \mathcal{F}\times\mathcal{Z} \,\},$$

which has, for each $z_0 \in \mathcal{Z}$ with $\mathcal{D}((1,p_*),z_0) < \infty$, a unique solution $(\varphi_k,z_k)_{k=1,\ldots,N}$.

Moreover, there exists $C > 0$, which depends only on α, ℓ and z_0, such that

$$\|\varphi_k\|_{\mathrm{W}^{1,\alpha/3}} + \|P_k\|_{\mathrm{L}^\alpha} + \|P_k^{-1}\|_{\mathrm{L}^\alpha} + \|\mathrm{e}^{\alpha p_k}\|_{\mathrm{L}^1} \leq C \text{ for } k = 1,\ldots,N. \tag{5.2}$$

Proof: Using Lemma 3.2 φ_k is a minimizer of the condensed functional $\mathcal{I}^{\mathrm{cond}}_k$ which is based on W^{cond}, see (5.1). Because of $\alpha > 3$ this density and hence the functional $\mathcal{I}^{\mathrm{cond}}_k$ is strictly convex. Hence, φ_k is uniquely defined for given z_{k-1} and t_k.

For given F and z_{k-1}, the set $\arg\min\{\, W(FP){+}D(z_{k-1},(P,p)) \,|\, (P,p) \in (0,\infty)\times\mathbb{R} \,\}$ contains just one point. Hence, z_k is also uniquely defined. By induction we conclude uniqueness of the whole solution to (IP).

The estimate (5.2) follows the standard energy estimates as given in Section 3. ■

Finally we want to discuss the problem of establishing convergence for the step size $\max\{\, t_k - t_{k-1} \mid k = 1,\ldots,N \,\}$ going to 0. In [MT99, MTL02, MT03, MM03a] conditions are given which guarantee that from the sequence of the piecewise constant interpolants

$$(\varphi^N_{\mathrm{cr}}, z^N_{\mathrm{cr}}) : \begin{cases} [0,T] & \to & \mathcal{F}\times\mathcal{Z}, \\ t & \mapsto & \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})}(t)(\varphi^k,z^k) \end{cases} \tag{5.3}$$

a subsequence can be extracted which converges to a solution $(\varphi,z) : [0,T] \to \mathcal{F}\times\mathcal{Z}$ of the time-continuous problem (S) & (E), see (2.14). The dissipation $\mathcal{D}$ can be used to bound possible oscillations in time yielding temporal compactness. The problem is to control possible spatial oscillation, i.e., in $x \in \Omega$.

A crucial tool developed there (see also [Efe03, MM03a, MR03]) is the set of stable states

$$\mathcal{S}_{[0,T]} = \{\, (t,\varphi,z) \in [0,T]{\times}\mathcal{F}{\times}\mathcal{Z} \mid \forall\, \widetilde{\varphi}, \widetilde{z} : \mathcal{E}(t,\varphi,z) \leq \mathcal{E}(t,\widetilde{\varphi},\widetilde{z}) + \mathcal{D}(z,\widetilde{z}) \,\}.$$

The most important condition in the abstract theory developed in the above-mentioned papers is that any limit $(\varphi, z) : [0,T] \to \mathcal{F}\times\mathcal{Z}$ of the subsequence $(\varphi^{N_m}(t), z^{N_m}(t)) \to (\varphi(t), z(t))$ occurs in a topology in which the stable set $\mathcal{S}_{[0,T]}$ is closed. We want to study this question in our explicit one-dimensional example now.

For simplicity, we restrict ourselves to the traction case $\mathcal{F} = \mathcal{F}^{\alpha/3}_{\mathrm{tract}}$ which allows us to characterize $\mathcal{S}_{[0,T]}$ explicitly. A similar result was obtained already in [Mie03].

Lemma 5.2 *In the above one-dimensional example* $(t, \varphi, P, p) \in \mathcal{S}_{[0,T]}$ *if and only if for almost all* $x \in \Omega$ *we have*

$$|(\varphi'/P)^{\alpha} - (\varphi'/P)^{-\alpha}| \leq \alpha \mathrm{e}^{\alpha p} \text{ and } ((\varphi'/P)^{\alpha-1} - (\varphi'/P)^{-\alpha-1})/P = H_{\mathrm{ext}}(t,\cdot). \quad (5.4)$$

Proof: Stability of (t, φ, z) is equivalent to the fact that (φ, z) is a global minimizer of $J : (\widetilde{\varphi}, \widetilde{z}) \mapsto \mathcal{E}(t, \widetilde{\varphi}, \widetilde{z}) + \mathcal{D}(z, \widetilde{z})$. Minimizing with respect to $\widetilde{z} \in \mathcal{Z}$ leads to the condensed functional

$$J^{\mathrm{cond}} : \widetilde{\varphi} \mapsto \textstyle\int_{\Omega} W^{\mathrm{cond}}(z(x); \widetilde{\varphi}'(x))\,\mathrm{d}x - \langle \ell(t), \widetilde{\varphi} \rangle.$$

For $\widetilde{\varphi} = \varphi$ we know that this minimum is attained for $\widetilde{z} = z$, hence we know

$$W^{\mathrm{cond}}(z(x); \varphi'(x)) = W(\varphi'(x)/P(x)) \text{ for a.a. } x \in \Omega. \quad (5.5)$$

This gives the first condition in (5.4).

Since φ minimizes J^{cond} we have $\mathrm{D}J^{\mathrm{cond}}(\varphi) = 0$ which implies the second condition in (5.4), after using (5.5) once again. Thus, we conclude that (5.4) is necessary. The sufficiency follows from the convexity. ∎

Defining the two-dimensional subsets $M(t, x)$ of $\mathbb{R}^3$ via

$$M(t,x) = \Big\{ (F,P,p) \in (0,\infty)^2 \times \mathbb{R} \;\Big|\; \begin{array}{l} \big|(\frac{F}{P})^{\alpha} - (\frac{F}{P})^{-\alpha}\big| \leq \alpha \mathrm{e}^{\alpha p}, \\ (\frac{F}{P})^{\alpha-1} - (\frac{F}{P})^{-\alpha-1} = P\,H_{\mathrm{ext}}(t,x) \end{array} \Big\} \subset \mathbb{R}^3,$$

the stability condition (5.4) can be reformulated as

$$(\varphi'(x), P(x), p(x)) \in M(t,x) \text{ for a.a. } x \in \Omega.$$

We note that the sets $M(t,x)$ are closed but not convex in $\mathbb{R}^3$. Hence, $\mathcal{S}_{[0,T]}$ is closed in the strong topology of $[0,T] \times \mathcal{F} \times \mathcal{Z} \subset \mathbb{R} \times \mathrm{W}^{1,\alpha/3}(\Omega) \times \mathrm{L}^{\alpha}(\Omega) \times \mathrm{L}^{\alpha}(\Omega)$.

However, $\mathcal{S}_{[0,T]}$ is not closed in the weak topology of this Banach space. Yet, so far the a priori estimate (5.2) is the only one available and from it we obtain just weak convergence (at fixed times $t \in [0,T]$):

$$\begin{array}{rcll} \varphi^{N_m}(t) & \rightharpoonup & \varphi(t) & \text{in } \mathrm{W}^{1,\alpha/3}(\Omega), \\ \frac{\partial}{\partial x}(\varphi^{N_m}(t)) P^{N_m}(t)^{-1} & \rightharpoonup & F_{\mathrm{elast}}(t) & \text{in } \mathrm{L}^{\alpha}(\Omega), \\ P^{N_m}(t) & \rightharpoonup & P(t) & \text{in } \mathrm{L}^{\alpha}(\Omega), \\ P^{N_m}(t)^{-1} & \rightharpoonup & K(t) & \text{in } \mathrm{L}^{\alpha}(\Omega), \\ z^{N_m}(t) & \rightharpoonup & p(t) & \text{in } \mathrm{L}^{\alpha}(\Omega). \end{array} \quad (5.6)$$

However, this does not imply $\varphi'(t,x)/P(t,x) = F_{\mathrm{elast}}(t,x)$ or $P(t,x)^{-1} = K(t,x)$ for a.a. $x \in \Omega$, which would be needed to conclude from $((\varphi^{N_m})', P^{N_m}, p^{N_m}) \in M(t,x)$ the desirable condition $(\varphi', P, p) \in M(t,x)$.

Thus, the convergence of the incremental solutions can only be shown by establishing convergence in stronger topologies. Below we will show that the solutions $(\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k, P_k, p_k)$ converge pointwise in $[0,T] \times \Omega$.

Before providing this result we want to mention another abstract approach to obtain strong convergence which is implemented in Section 7 of [MT03]. It relies on the reduced problem where only the internal variable z is kept, whereas the deformation φ is minimized out. We define

$$\mathcal{I}^{\rm red}(t,z)=\min\{\, \mathcal{E}(t,\varphi,z) \mid \varphi\in\mathcal{F}\,\}.$$

In the case of $\mathcal{F}=\mathcal{F}^{\alpha/3}_{\rm tract}$ this minimization can be made explicit, since $\mathcal{E}$ contains φ only via φ'. We denote by W^* the Legendre-Fenchel transform of W, i.e.,

$$W^*(\sigma)=\sup\{\, \sigma F-W(F) \mid F\in\mathbb{R}\,\}. \tag{5.7}$$

Then, $W^*:\mathbb{R}\to\mathbb{R}$ is convex and satisfies $W^*(\sigma)\sim\frac{1}{\alpha_+}\sigma^{\alpha_+}$ for $\sigma\to+\infty$ and $W^*(\sigma)\sim-\frac{1}{\alpha_-}(-\sigma)^{\alpha_-}$ for $\sigma\to-\infty$ where $\alpha_\pm=\frac{\alpha}{\alpha\mp1}$. Moreover, a simple calculation gives

$$\mathcal{I}^{\rm red}(t,z)=-\int_0^1 W^*(H_{\rm ext}(t,x)P(x))\,{\rm d}x.$$

Unfortunately, this functional is concave in P. Hence, the strong convergence theory in the uniformly convex case is not applicable.

6 Convergence in the one-dimensional case

To derive a convergence result we use the very specific structure of the one-dimensional traction problem with $\mathcal{F}=\mathcal{F}^{\alpha/3}_{\rm tract}$. As already used in Lemma 5.2 the incremental problem has the special property that it can be solved independently for each point $x\in\Omega$ to obtain $(F_k,P_k,p_k)=(\frac{\rm d}{{\rm d}x}\varphi_k(x),P_k(x),p_k(x))$ as solution of the finite-dimensional, x-dependent minimization problem

$$(F_k(x),P_k(x),p_k(x))\in\underset{(F,P,p)\in\mathbb{R}^3}{\arg\min}\; W(F/P)-H_{\rm ext}(t_k,x)F+D((P_{k-1}(x),p_{k-1}(x)),(P,p)),$$

which has a unique solution.

We now additionally assume $z_0=(P_0,p_0)\in{\rm C}^0(\overline{\Omega},\mathbb{R}^2)$ with $P_0(x)>0$ for all $x\in\overline{\Omega}$. Moreover, the loading should satisfy $H_{\rm ext}\in{\rm C}^1([0,T]\times\overline{\Omega})$. Using energy estimates as for Proposition 5.1, we find a constant $C>0$, which is independent of $x\in\overline{\Omega}$ and the time discretization, such that all incremental solutions satisfy

$$|F_k(X)|+|P_k(x)|+|1/P_k(x)|+|p_k(x)|\le C \tag{6.1}$$

for all $x\in\overline{\Omega}$ and $k=0,1,\dots,N$.

From now on we omit the x-dependence in most cases and use the short-hand $H_k=H_{\rm ext}(t_k,x)$. Introducing the logarithm $\gamma=\log P$ and eliminating F we are left with the following incremental problem in $\mathbb{R}^2$:

$$(\gamma_k,p_k)\in\arg\min\{\, D(({\rm e}^{\gamma_{k-1}},p_{k-1}),({\rm e}^{\gamma},p))-W^*({\rm e}^{\gamma}H_k) \mid \gamma,p\in\mathbb{R}\,\},$$

Because of the special form of D this reduces to a scalar problem

$$\begin{array}{rcl}\gamma_k &\in& \arg\min\{\, {\rm e}^{\alpha(p_{k-1}+|\gamma-\gamma_{k-1}|)}-W^*({\rm e}^{\gamma}H_k) \mid \gamma\in\mathbb{R}\,\}\\ p_k &=& p_{k-1}+|\gamma_k-\gamma_{k-1}|.\end{array} \tag{6.2}$$

This problem can be solved almost explicitly by using monotonicity arguments relying on the total ordering of the real line.

The essential scalar variable is $\zeta_{k-1}^{\pm} = \gamma_{k-1} \mp p_{k-1} + \log(\pm H_k)$ which allows us to write the iteration (6.2) in the form

$$\begin{matrix}\gamma_k \\ p_k\end{matrix} = \begin{cases} \begin{matrix}\Gamma_+(\zeta_{k-1}^+) - \log H_k \\ \Gamma_+(\zeta_{k-1}^+) - \zeta_{k-1}^+\end{matrix}\Bigg) & \text{if } \Gamma_+(\zeta_{k-1}^+) > \gamma_{k-1} + \log H_k, \\ \begin{matrix}\gamma_{k-1} \\ p_{k-1}\end{matrix} & \text{if } \Gamma_+(\zeta_{k-1}^+) \le \gamma_{k-1} + \log|H_k| \le \Gamma_-(\zeta_{k-1}^-), \\ \begin{matrix}\Gamma_-(\zeta_{k-1}^-) - \log(-H_k) \\ \zeta_{k-1}^- - \Gamma_-(\zeta_{k-1}^-)\end{matrix}\Bigg) & \text{if } \Gamma_-(\zeta_{k-1}^-) < \gamma_{k-1} + \log(-H_k), \end{cases} \tag{6.3}$$

where $\Gamma_\pm(\zeta) = \arg\min\{ \mathrm{e}^{\pm\alpha(\gamma-\zeta)} - W^*(\pm\mathrm{e}^\gamma) \mid \gamma \in \mathbb{R} \}$.

We call the first case, where $\gamma_k > \gamma_{k-1}$, plastic loading and the third case, where $\gamma_k < \gamma_{k-1}$, plastic unloading. In the second case the plastic variables do not change. The major observation is that, if in a time interval the solution stays either always in case one and two or always in the case two and three, then the solution can be calculated directly from the initial data when entering this time interval and the loading history, but one does not need to know the solution in between. In particular, the number of steps done in between is irrelevant. We make this now precise.

With $\Gamma_\pm(\zeta) \sim \alpha_\pm \zeta$ for $\zeta \to -\infty$, $\alpha_- < 1 < \alpha_+$, and the a priori estimate (6.1) we find a constant $H^* > 0$ such that $|H_k| \le H^*$ implies that the second case (no change in the plastic variables) occurs. We now decompose the time interval $[0,T]$ into a finite number of subintervals $J_m = [\tau_{m-1}, \tau_m]$ with $0 = \tau_0 \le \tau_1 < \tau_2 < \cdots < \tau_M = T$ such that $H^* + (-1)^m H(t) \ge 0$ for all $t \in J_m$. For the given time discretization $0 = t_0 < t_1 < \cdots < t_N = T$ we define, for $m = 1, \ldots, M$, the exit times $t_{j_m} \in J_m$ of the subintervals J_m via

$$j_0 = 0 \text{ and } j_m = \max\{ k \mid t_k \le \tau_m \}.$$

On the subintervals J_m we change the loading H_k into a monotone version $\widetilde{H}_k$, which is defined for $t_k \in J_m$ via

$$\widetilde{H}_k = (-1)^m \max\{ (-1)^m H(t_n) \mid n \in \{j_{m-1}, \ldots, k\} \}. \tag{6.4}$$

Hence $(-1)^m \widetilde{H}_k$ is nondecreasing for $k = j_{m-1}, \ldots, j_m$.

By induction over the subintervals and by induction over the number of steps inside each subinterval we obtain the following representation formula.

Proposition 6.1 *Let m be even and $t_k \in J_m$, then the solution takes the form*

$$\begin{pmatrix} \gamma_k \\ p_k \end{pmatrix} = \begin{pmatrix} \Gamma_+(\gamma_{j_{m-1}} - p_{j_{m-1}} + \log \widetilde{H}_k) - \log \widetilde{H}_k \\ \Gamma_+(\gamma_{j_{m-1}} - p_{j_{m-1}} + \log \widetilde{H}_k) - \gamma_{j_{m-1}} + p_{j_{m-1}} - \log \widetilde{H}_k \end{pmatrix}. \tag{6.5}$$

A similar formula using Γ_- holds for m odd, cf. (6.3).

Finally, we obtain the desired convergence result, which is formulated in terms of functions over $x \in \Omega = (0,1) \subset \mathbb{R}^1$.

Theorem 6.2 *Consider the one-dimensional traction problem of Section 5 with $\alpha > 2$ and $H_{\mathrm{ext}} \in \mathrm{C}^1([0,T] \times \overline{\Omega})$. Then, there exists a function $(\varphi, P, p) \in \mathrm{C}^0([0,T], \mathrm{W}^{1,\infty}(\Omega) \times \mathrm{L}^\infty(\Omega)^2)$, which is a solution of (S) & (E) (cf. (2.14)). Moreover, there exists a constant $C > 0$ such that for each time discretization $0 = t_0 < t_1 < \cdots < t_N = T$ the unique solution $(\varphi_k, P_k, p_k)_{k=0,\ldots,N}$ of the incremental problem (3.1) satisfies, for $k = 1, \ldots, N$,*

$$\|\varphi(t_k,\cdot) - \varphi_k\|_{\mathrm{W}^{1,\infty}} + \|P(t_k,\cdot) - P_k\|_{\mathrm{L}^\infty} + \|p(t_k,\cdot) - p_k\|_{\mathrm{L}^\infty} \le C \max\{ t_n - t_{n-1} | n=1,\ldots,k \}.$$

Proof: The use the fact that Proposition 6.1 can be applied in a uniform manner for $x \in \overline{\Omega}$.

Firstly, consider the division into subintervals J_m. Since $H_{\rm ext}$ is continuous, the sets Σ_+ and Σ_- with

$$\Sigma_\pm = \{ (t,x) \in [0,T] \times \overline{\Omega} \mid \pm H_{\rm ext}(t,x) \geq H^* \}$$

are strictly separated. Since, the only restrictions to the subintervals are $J_m(x) \supset \Sigma_+ \cap ([0,T]\times\{x\})$ for even m and $J_m(x) \supset \Sigma_- \cap ([0,T]\times\{x\})$ for odd m. Hence, it is possible to choose the intervals piecewise constant on a finite number of subintervals $\Omega_l = (x_{l-1}, x_l)$. In particular, the number of time intervals $J_m(x)$, $m = 1, \ldots, M_l$, is bounded from above.

Secondly, we apply the formula (6.5). To show convergence we define the function $\widetilde{H}_{\rm ext}$ as in (6.4):

$$\widetilde{H}_{\rm ext}(t,x) = (-1)^m \max\{ (-1)^m H_{\rm ext}(s,x) \mid s \in J_m(x) \cap [0,t] \}.$$

By Lipschitz continuity of $H_{\rm ext}(\cdot, x)$ we obtain

$$|\widetilde{H}_k(x) - \widetilde{H}_{\rm ext}(t_k, x)| \leq C_1 \delta_k \quad \text{with } \delta_k = \max\{ t_n - t_{n-1} \mid n{=}1,...,k \},$$

for a constant C_1 independent of $x \in \Omega$ and the partition.

Now, we may take a sequence of partitions $0 < t_1^{N_l} < \cdots < t_{N_l}^{N_l}$ such that the fineness $\widetilde{\delta}_l := \delta_{N_l}^{N_l}$ tends to 0. Now, the exit points $t_{j_m^l(x)}^{N_l}$ have a distance to the end points $\tau_m(x)$ of the intervals $J_m(x)$ of at most $\widetilde{\delta}_l$. Moreover, by induction over m we find that $(\gamma_{j_m^l(x)}(x), p_{j_m^l(x)}(x))$ converges for $l \to \infty$. The limits, called $(\widetilde{\gamma}_m(x), \widetilde{p}_m(x))$ satisfy the recursion

$$\begin{pmatrix} \widetilde{\gamma}_m \\ \widetilde{p}_m \end{pmatrix} = \begin{pmatrix} \Gamma_+(\widetilde{\gamma}_{m-1} - \widetilde{p}_{m-1} + \log \widetilde{H}_{\rm ext}(\tau_m)) - \log \widetilde{H}_{\rm ext}(\tau_m) \\ \Gamma_+(\widetilde{\gamma}_{m-1} - \widetilde{p}_{m-1} + \log \widetilde{H}_{\rm ext}(\tau_m)) - \widetilde{\gamma}_{m-1} + \widetilde{p}_{m-1} - \log \widetilde{H}_{\rm ext}(\tau_m) \end{pmatrix},$$

for even m and similarly for odd m. The error is bounded by $C_2 \widetilde{\delta}_l$, since $\Gamma_\pm$ are Lipschitz continuous.

Thirdly, we define the function $(\gamma, p) : [0,T] \times \Omega \to \mathbb{R}^2$ via

$$\begin{matrix} \gamma(t,x) \\ p(t,x) \end{matrix} \quad = \quad \begin{matrix} \Gamma_+(\widetilde{\gamma}_{m-1}(x) - \widetilde{p}_{m-1}(x) + \log \widetilde{H}_{\rm ext}(t,x)) - \log \widetilde{H}_{\rm ext}(t,x) \\ \Gamma_+(\widetilde{\gamma}_{m-1}(x) - \widetilde{p}_{m-1}(x) + \log \widetilde{H}_{\rm ext}(t,x)) - \widetilde{\gamma}_{m-1}(x) + \widetilde{p}_{m-1}(x) - \log \widetilde{H}_{\rm ext}(t,x) \end{matrix}$$

for $t \in J_m(x)$. By our construction the incremental solutions $(t_k^{N_l}, \gamma_k^{N_l}(x), p_k^{N_l}(x))$ converge to $(t, \gamma(t,x), p(t,x))$ with an error bounded by $C_3 \delta_l$, uniformly on $[0,T] \times \Omega$.

Finally, it remains to show that (γ, p) define a solution of (S) & (E). Let $\widehat{F}(P,H)$ be the unique minimizer of $F \mapsto W(F/P) - HF$, then the desired function (φ, P, p) is obtained from (γ, p) via

$$P(t,x) = \mathrm{e}^{\gamma(t,x)} \text{ and } \varphi(t,x) = \textstyle\int_0^x \widehat{F}(P(t,\xi), H_{\rm ext}(t,\xi))\,\mathrm{d}\xi.$$

Since the function $\widehat{F}$ is also Lipschitz continuous, we obtain uniform convergence of the (unique) incremental solutions towards this limit function. Now we use the abstract theorem 3.1 which guarantees that the incremental solutions are stable and satisfy the discrete version of the energy inequality. The characterization of the stable sets in Lemma 5.2 show that uniform limits (with pointwise convergence almost everywhere) are stable again, i.e., $(t,\varphi(t),P(t),p(t)) \in \mathcal{S}_{[0,T]}$ for each $t \in [0,T]$. Thus, (S) is established.

Similarly, we start from the discrete energy inequality (ii) in Theorem 3.1 for the incremental solutions (φ^{N_l}, z^{N_l}). For $l \to \infty$ the uniform convergence guarantees that all terms converge:

$$\mathcal{E}(t,\varphi(t),z(t)) + \mathrm{Diss}(z,[s,t]) = \mathcal{E}(s,\varphi(s),z(s)) - \int_s^t \int_\Omega \partial_t H_{\mathrm{ext}}(\tau,\xi)\partial_x\varphi(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

For the convergence of the dissipation, uniform convergence is not sufficient. There we use that the piecewise constant interpolants $P^{\mathrm{cr}}(\cdot,x)$ are monotone in t when restricted to the subintervals $J_m(x)$ and that $p^{\mathrm{cr}}(\cdot,x)$ is always monotone. This together with the uniform convergence implies convergence of the dissipation as well. This establishes (E) as an energy equality. ■

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