Existence Results for Energetic Models for Rate-Independent Systems

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1 Introduction

We consider mechanical models which are driven by an external loading on a time scale much slower than any internal time scale (like viscous relaxation times) but still much faster than the time needed to find the thermo-dynamical equilibrium. Typical phenomena involve dry friction, elasto-plasticity, certain hysteresis models for shape-memory alloys and quasistatic delamination or fracture. The main feature is the rate-independency of the system response, which means that a loading with twice (or half) the speed will lead to a response with exactly twice (or half) the speed. We refer to [BrS96, KrP89, Vis94, Mon93] for approaches to these phenomena involving either differential inclusions or abstract hysteresis operators. Our method is different, as we avoid time derivatives and use energy principles instead.

As is well-known from dry friction, such systems will not necessarily relax into a complete equilibrium, since friction forces do not tend to 0 for vanishing velocities. One way to explain this phenomenon on a purely energetic basis is via so-called "wiggly energies", where the macroscopic energy functional has a super-imposed fluctuating part with many local minimizers. Only after reaching a certain activation energy it is possible to leave these local minima and generate macroscopic changes, cf. [ACJ96, Jam96, Men02]. Here we use a different approach which involves a dissipation distance which locally behaves homogeneous of degree 1, in contrast to viscous dissipation which is homogeneous of degree 2. This approach was introduced in [MiT99, MiT03, MTL02, GMH02] for models for shape-memory alloys and is now generalized to many other rate-independent systems. See [Mie03a] for a general setup for rate-independent material models in the framework of "standard generalized materials".

To be more specific we consider the following continuum mechanical model. Let $\Omega \subset \mathbb{R}^d$ be the undeformed body and $t \in [0,T]$ the slow process time. The deformation or displacement $\varphi(t) : \Omega \to \mathbb{R}^d$ is considered to lie in the space $\mathcal{F}$ of admissible deformations containing suitable Dirichlet boundary conditions. The internal variable $z(t) : \Omega \to Z \subset \mathbb{R}^m$ describes the internal state which may involve plastic deformations, hardening variables, magnetization or phase indicators. The elastic (Gibbs) stored energy is given via

$$\mathcal{E}(t,\varphi,z) = \int_\Omega W(x,\mathrm{D}\varphi(x),z(x))\,\mathrm{d}x - \langle \ell(t),\varphi\rangle,$$

where $\langle \ell(t),\varphi\rangle = \int_\Omega f_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x + \int_{\partial\Omega} g_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x$ denotes the external loading depending on the process time t.

Changes of the internal variables are associated with dissipation of energy which is given constitutively via a dissipation potential $\Delta : \Omega\times \mathrm{T}Z \to [0,\infty]$, i.e., an internal process $Z : [t_0,t_1]\times\Omega \to Z$ dissipates the energy

$$\mathrm{Diss}(z,[t_0,t_1]) = \int_{t_0}^{t_1}\int_\Omega \Delta(x,z(t,x),\dot z(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

Rate-independency is obtained via homogeneity: $\Delta(x,z,\alpha v) = \alpha\Delta(x,z,v)$ for $\alpha \le 0$. We associate with Δ a global dissipation distance $\widetilde{\mathcal{D}}$ on the set of all internal states:

$$\widetilde{\mathcal{D}}(z_0,z_1) = \inf\{\,\mathrm{Diss}(z[0,1]) \mid z\in \mathrm{C}^1([0,1]\times\Omega,Z),\ z(0)=z_0,\ z(1)=z_1\,\}.$$

In the setting of smooth continuum mechanics the evolution equations associated with such a process are given through the theory of standard generalized materials (cf. [Mie03a] and the references therein). They are the elastic equilibrium and the force balance for the internal variables:

$$\left.\begin{aligned} -\operatorname{div}\tfrac{\partial W}{\partial F}(x,\mathrm{D}_x\varphi(t,x),z(t,x)) &= f_{\mathrm{ext}}(t,x)\\ 0\in \partial^{\mathrm{sub}}_{\dot z}\Delta(x,z(t,x),\dot z(t,x)) + \tfrac{\partial W}{\partial z}(x,\mathrm{D}_x\varphi(t,x),z(t,x))& \end{aligned}\right\}\ \text{in } \Omega,$$

where boundary conditions need to be added and ∂^{sub} denotes the subdifferential of a convex function. Using the functionals this system can be written in abstract form as

$$\mathrm{D}_\varphi\mathcal{E}(t,\varphi(t),z(t)) = 0, \quad 0\in\partial^{\mathrm{sub}}_{z_2}\widetilde{\mathcal{D}}(z(t),\cdot)[\dot z(t)] + \mathrm{D}_z\mathcal{E}(t,\varphi(t),z(t)), \tag{1.1}$$

which has the form of the doubly nonlinear problems studied in [CoV90].

It was realized in [MiT99, MTL02, Mie03a] that this problem can be rewritten in a derivative-free, energetic form which does not require solutions to be smooth in time or space. Hence, it is much more adequate for many mechanical systems. Moreover, the energetic formulation allows for the usage of powerful tools of the modern theory of the calculus of variations, such as lower semi-continuity, quasi- and poly-convexity and nonsmooth techniques. A pair $(\varphi,z) : [0,T] \to \mathcal{F}\times\mathcal{Z}$ is called a solution of the rate-independent problem associated with $\mathcal{E}$ and $\widetilde{\mathcal{D}}$ if (S) and (E) hold:

(S) Stability: *For all $t\in[0,T]$ and all $(\widetilde\varphi,\widetilde z)\in\mathcal{F}\times\mathcal{Z}$ we have*
$\mathcal{E}(t,\varphi(t),z(t)) \le \mathcal{E}(t,\widetilde\varphi,\widetilde z) + \widetilde{\mathcal{D}}(z(t),\widetilde z)$.

(E) Energy equality: *For all $t\in[0,T]$ we have*
$\mathcal{E}(t,\varphi(t),z(t)) + \mathrm{Diss}_{\widetilde{\mathcal{D}}}(z,[0,t]) = \mathcal{E}(0,\varphi(0),z(0)) - \int_0^t\langle\dot\ell(\tau),\varphi(\tau)\rangle\,\mathrm{d}\tau.$

For a simple nontrivial application of the abstract theory we consider the case $\varphi\equiv 0$ and let

$$\mathcal{E}(t,z) = \int_\Omega \tfrac{a(x)}{2}|\mathrm{D}_x z(x)|^2 - g_{\mathrm{ext}}(t,x)z(x)\,\mathrm{d}x \quad \text{on } \mathcal{Z} = \mathrm{H}^1_0(\Omega)$$

and $\widetilde{\mathcal{D}}(z_0,z_1) = \int_\Omega \kappa|z_1(x){-}z_0(x)|\,\mathrm{d}x$ with $\kappa > 0$. Then, $\Delta(x,z,\dot z) = \kappa|\dot z|$ and (1.1) reduces to the partial differential inclusion

$$0 \in \kappa\,\mathrm{Sign}(\dot z(t,x)) - \mathrm{div}\,\Big(a(x)\mathrm{D}_x z(t,x)\Big) - g_{\mathrm{ext}}(t,x), \tag{1.2}$$

where Sign denotes the set-valued signum function. Our general theory using (S) & (E) will provide a generalized solution to this problem which satisfies $z \in \mathrm{BV}([0,T],\mathrm{L}^1(\Omega)) \cap \mathrm{L}^\infty([0,T],\mathrm{H}^1_0(\Omega))$ whenever $g_{\mathrm{ext}} \in \mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{H}^{-1}(\Omega))$, see Theorem 4.6. However, using the uniform convexity of $\mathcal{E}(t,\cdot)$ this result can be considerably improved; the theory in [MiT03, Sect.7] provides uniqueness and $z \in \mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{H}^1_0(\Omega))$.

Under the assumptions that the sets $\mathcal{F}$ and $\mathcal{Z}$ are closed, convex subspaces of a suitable Banach space and that $\widetilde{\mathcal{D}}(z_0,z_1) = \boldsymbol{\Delta}(z_1{-}z_0)$, an existence theory was developed in the above-mentioned work and certain refinements were added in [MiR03, Efe03, KMR03]. The purpose of this work is to provide an abstract framework for constructing solutions to (S) & (E) without relying on any underlying linear structures in $\mathcal{Y} = \mathcal{F}\times\mathcal{Z}$. Thus, we hope to provide a basis for applications in genuinely nonlinear mechanical models such as elasto-plasticity with finite strains, see [OrR99, CHM02, Mie02, Mie03a, LMD03, Mie03b]. Our existence proof is based on the commonly used time-incremental approach which leads to minimization problems. Denoting the pair (φ,z) by y and letting $\mathcal{D}(y_0,y_1) = \widetilde{\mathcal{D}}(z_0,z_1)$ the incremental problem takes the form

(IP) *Given* $y_0 \in \mathcal{Y}$ *and a partition* $0 = t_0 < t_1 < \ldots t_N = T$ *find* $y_1,\ldots,y_k$ *such that* $\mathcal{E}(t_k,y_k) = \inf\{\, \mathcal{E}(t_k,y) + \mathcal{D}(y_{k-1},y) \mid y \in \mathcal{Y}\,\}$.

We equip the space $\mathcal{Y}$ with a Hausdorff topology $\mathcal{T}$ such that the functions $\mathcal{E} : [0,T]\times\mathcal{Y} \to [\mathcal{E}_{\min},\infty]$ and $\mathcal{D} : \mathcal{Y}\times\mathcal{Y} \to [0,\infty]$ are s-lower semicontinuous, where "s-" stands for "sequentially". Moreover, we assume that the reachable sets $\mathcal{R}(t) = \{\, y \in \mathcal{Y} \mid \mathcal{E}(t,y) + \mathcal{D}(y_0,y) \le \mathcal{E}(t,y_0) + C_{\mathcal{E}}t + 1\,\}$ are s-compact. From this we can deduce existence of solutions for (IP).

Solutions to the time-continuous problem (S) & (E) are obtained as limits of incremental solutions for a sequence of nested partitions $P(n) = \{0{=}t_0^{(n)}{<}t_1^{(n)}{<}\cdots{<}t_{N(n)}^{(n)}{=}T\} \subset P(n{+}1)$ whose fineness $\phi(P(n)) = \max\{\, t_j^{(n)} - t_{j-1}^{(n)} \mid j = 1,\ldots,N(n)\,\}$ tends to 0. Under the assumption $|\partial_t\mathcal{E}(t,y)| \le C_{\mathcal{E}}$ all these solutions satisfy the a priori bound

$$\mathrm{Diss}_{\mathcal{D}}(Y^n,[0,T]) = \textstyle\sum_{j=1}^{N(n)} \mathcal{D}(Y^n(t_{j-1}^{(n)}), Y^n(t_j^{(n)})) \le \mathcal{E}(0,y_0) - \mathcal{E}_{\min} + C_{\mathcal{E}}T.$$

Here Y^n denotes the piecewise constant interpolant with $Y^n(t) = y_j^{(n)}$ for $t \in [t_j^{(n)}, t_{j+1}^{(n)})$. Using this bound a generalized, abstract version of Helly's selection principle (see Section 3) allows us to extract a subsequence such that $Y^{n_k}(t) \xrightarrow{\mathcal{T}} Y^\infty(t)$. For this we need an additional compatibility between the topology $\mathcal{T}$ and the dissipation distance $\mathcal{D}$, namely that $\min\{\mathcal{D}(y_k,y),\mathcal{D}(y,y_k)\} \to 0$ implies $y_k \xrightarrow{\mathcal{T}} y$. Using s-continuity of $\partial_t\mathcal{E}(t,\cdot)$ and assuming stability (S) for Y^∞ it is then straightforward to deduce the energy equality (E) for the limit Y^∞.

The major task is to show that Y^∞ satisfies (S). For this we use the set of stable states, shortly called the stable set:

$$\mathcal{S}_{[0,T]} = \cup_{t\in[0,T]}(t,\mathcal{S}(t)) \text{ with } \mathcal{S}(t) = \{\, y \in \mathcal{Y} \mid \mathcal{E}(t,y) \le \mathcal{E}(t,\widetilde y) + \mathcal{D}(y,\widetilde y) \text{ for all } \widetilde y \in \mathcal{Y}\,\}.$$

¿From the incremental problem we obtain $(t_k^{(n)}, y_k^{(n)}) \in \mathcal{S}_{[0,T]}$. Hence s-closedness of the stable set is sufficient to conclude $(t, Y^\infty(t)) \in \mathcal{S}_{[0,T]}$ for all t, which is exactly (S).

In Theorem 4.5 we summarize the main existence result and provide afterwards a typical application to the Banach space setting. In Section 5 we discuss abstract conditions on $\mathcal{E}$ and $\mathcal{D}$ which guarantee the s-closedness of $\mathcal{S}_{[0,T]}$. In Section 6 we discuss a few applications of the abstract theory to continuum mechanics. In particular, we show how the abstract theory lays the basis for the treatment of the delamination problem in [KMR03]. Moreover, we show that the model of brittle fracture introduced in [FrM93] and developed further in [FrM98, DaT02, Cha03, FrL03] is a special case of our theory. We show that the conditions posed there are equivalent to our (S) & (E) which provides a clearer mechanical interpretation to this theory.

2 Abstract setup of the problem

We start with a topological Hausdorff space $(\mathcal{Y}, \mathcal{T})$, and we will write $y_k \xrightarrow{\mathcal{T}} y$ to denote the convergence in this space. In fact, throughout it will be sufficient to consider sequential closedness, compactness and continuity. We will indicate this fact by writing s-closedness, s-compactness and s-continuity.

The first ingredient of the energetic formulation is the *dissipation distance* $\mathcal{D} : \mathcal{Y} \times \mathcal{Y} \to [0, \infty]$ satisfying $\mathcal{D}(y, y) = 0$ and the triangle inequality:

$$\mathcal{D}(y_1, y_3) \leq \mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \quad \text{for all } y_1, y_2, y_3 \in \mathcal{Y}. \tag{A1}$$

We enforce neither strict positivity (i.e., $\mathcal{D}(y_1, y_2) = 0$ for $y_1 \neq y_2$ is allowed) nor symmetry (i.e., we allow for $\mathcal{D}(y_0, y_1) \neq \mathcal{D}(y_1, y_0)$ as is needed in Section 6.2). We call $\mathcal{D}(y_0, y_1)$ the dissipation distance from y_0 to y_1.

One major point of the theory is the interplay between the topology $\mathcal{T}$ and the dissipation distance. To have a typical nontrivial application in mind, one may consider $\mathcal{Y} = \{\, y \in \mathrm{L}^1(\Omega, \mathbb{R}^k) \mid \|y\|_{\mathrm{L}^\infty} \leq 1 \,\}$ equipped with the weak L^1 topology and the dissipation distance $\mathcal{D}(y_1, y_2) = \|y_1 - y_2\|_{\mathrm{L}^1}$.

For a given curve $y : [0, T] \to \mathcal{Y}$ we define the total dissipation on $[s, t]$ via

$$\mathrm{Diss}_{\mathcal{D}}(y; [s, t]) = \sup\{\, \textstyle\sum_1^N \mathcal{D}(y(\tau_{j-1}), y(\tau_j)) \mid N \in \mathbb{N}, s = \tau_0 < \tau_1 < \cdots < \tau_N = t \,\}. \tag{2.1}$$

Further we define the following set of functions:

$$\mathrm{BV}_{\mathcal{D}}([0, T], \mathcal{Y}) := \{\, u : [0, T] \to \mathcal{Y} \mid \mathrm{Diss}_{\mathcal{D}}(u; [0, T]) < \infty \,\}.$$

Note that the functions are defined everywhere and changing it at one point may increase the dissipation. Moreover, the dissipation is additive:

$$\mathrm{Diss}_{\mathcal{D}}(y; [r, t]) = \mathrm{Diss}_{\mathcal{D}}(y; [r, s]) + \mathrm{Diss}_{\mathcal{D}}(y; [s, t]) \quad \text{for all } r < s < t.$$

The second ingredient is the energy-storage functional $\mathcal{E} : [0, T] \times \mathcal{Y} \to [\mathcal{E}_{\min}, \infty]$, which is assumed to be bounded from below by a fixed constant $\mathcal{E}_{\min}$. Here $t \in [0, T]$ plays the rôle of a (very slow) process time which changes the underlying system via changing loading conditions. We assume that for all y with $\mathcal{E}(t,y)<\infty$ the function $t\mapsto\mathcal{E}(t,y)$ is Lipschitz continuous, i.e.,

$$\partial_t\mathcal{E}(\cdot,y):[0,T]\to\mathbb{R} \text{ is measurable and } |\partial_t\mathcal{E}(t,y)|\leq C_{\mathcal{E}}. \tag{A2}$$

Definition 2.1 *A curve $y:[0,T]\to\mathcal{Y}$ is called a **solution** of the rate-independent model $(\mathcal{D},\mathcal{E})$, if **global stability (S)** and **energy equality (E)** holds:*

(S) *For all $t\in[0,T]$ and all $\widehat{y}\in\mathcal{Y}$ we have $\mathcal{E}(t,y(t))\leq\mathcal{E}(t,\widehat{y})+\mathcal{D}(y(t),\widehat{y})$.*
(E) *For all $t\in[0,T]$ we have*
$$\mathcal{E}(t,y(t))+\mathrm{Diss}_{\mathcal{D}}(y;[0,t])=\mathcal{E}(0,y(0))+\int_0^t\partial_t\mathcal{E}(\tau,y(\tau))\,\mathrm{d}\tau.$$

The definition of solutions of (S) & (E) is such that it implies the two natural requirements for evolutionary problems, namely that *restrictions* and *concatenations* of solutions remain solutions. To be more precise, for any solution $y:[0,T]\to\mathcal{Y}$ and any subinterval $[s,t]\subset[0,T]$, the restriction $y|_{[s,t]}$ solves (S) & (E) with initial datum $y(s)$. Moreover, if $y_1:[0,t]\to\mathcal{Y}$ and $y_2:[t,T]\to\mathcal{Y}$ solve (S) & (E) on the respective intervals and if $y_1(t)=y_2(t)$, then the concatenation $y:[0,T]\to\mathcal{Y}$ solves (S) & (E) as well.

We use also the following weakened version of (E):

(E)$_{\text{weak}}$ *For all $t\in[0,T]$ we have*
$$\mathcal{E}(t,y(t))+\mathrm{Diss}_{\mathcal{D}}(y;[0,t])\leq\mathcal{E}(0,y(0))+\int_0^t\partial_t\mathcal{E}(s,y(s))\,\mathrm{d}s.$$

The condition (E)$_{\text{weak}}$ enables *concatenations* of solutions, but doesn't guarantee that *restrictions* remain solutions. It is shown in [MiT03] that (S) and (E)$_{\text{weak}}$ together imply that (E) holds (see our Proposition 4.4 for a simple proof).

Rate-independency manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that y is a solution for $(\mathcal{D},\mathcal{E})$ if and only if the reparametrized curve $\widetilde{y}:t\mapsto y(\alpha(t))$, where $\dot{\alpha}>0$, is a solution for $(\mathcal{D},\widetilde{\mathcal{E}})$ with $\widetilde{\mathcal{E}}(t,y)=\mathcal{E}(\alpha(t),y)$. In particular, the stability (S) is a static concept and the energy balance (E) is rate-independent, since the dissipation defined via (2.1) is scale invariant like the length of a curve.

The major importance of the energetic formulation is that neither the given functionals $\mathcal{D}$ and $\mathcal{E}(t,\cdot)$ nor the solutions $y:[0,T]\to\mathcal{Y}$ need to be differentiable. In particular, applications to continuum mechanics often have low smoothness. Of course, under additional smoothness assumptions on $\mathcal{D}$ and $\mathcal{E}$ the weak energetic form (S) & (E) can be replaced by local formulations in the form of differential inclusions like (1.1) ([CoV90, Vis01]) or variational inequalities. See [MiT03] for a discussion of the implications between these different formulations.

3 An abstract version of Helly's selection principle

Since our existence result is based on incremental approximations and since the rate-independent setting leads to a priori estimates of bounded variation in time, we provide here a generalization of Helly's selection principle, which states that from a sequence $(Y_n)_{n\in\mathbb{N}}$ we can select a subsequence $(Y_{n_k})_{k\in\mathbb{N}}$ which converges pointwise.

The classical selection theorem of Helly states that a bounded sequence of monotone functions on the real line always has a subsequence which converges pointwise everywhere. Our result extends the following generalization which works for Banach-space valued functions (see Thm. 3.5, Ch. 1 in [BaP86]).

Theorem 3.1 *Let $(X, \|\cdot\|)$ be a separable, reflexive Banach space with separable dual X^*. Assume that for the sequence of functions $Y_n : [0,T] \to X$ there exists a constant C such that*

$$\|Y_n\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\|\cdot\|}(Y_n; [0,T]) \le C.$$

Then, there exist a function $Y \in \mathrm{BV}_{\|\cdot\|}([0,T], X)$ and a subsequence $(Y_{n_k})_{k\in\mathbb{N}}$ such that for all $t \in [0,T]$ we have $Y_{n_k}(t) \rightharpoonup Y(t)$ in X (weak convergence).

Our theory completely avoids the setting of vector spaces. Our topology $\mathcal{T}$ on $\mathcal{Y}$ replaces the weak topology on X, and the dissipation $\mathcal{D}$ replaces the norm $\|\cdot\|$.

Theorem 3.2 *Let $\mathcal{R}_{[0,T]} = \cup_{[0,T]}(t, \mathcal{R}(t))$ be a s-compact subset of $[0,T] \times \mathcal{Y}$ and $\mathcal{V}_{[0,T]} = \cup_{[0,T]}(t, \mathcal{V}(t)) \subset \mathcal{R}_{[0,T]}$. Assume that $\mathcal{D}$, $\mathcal{T}$, $\mathcal{R}_{[0,T]}$, and $\mathcal{V}_{[0,T]}$ satisfy the following two compatibility conditions:*

$$\begin{aligned}&\text{For all } t_1, t_2 \in [0,T] \text{ the functional } \mathcal{D}(\cdot,\cdot) : \mathcal{R}(t_1) \times \mathcal{R}(t_2) \to [0,\infty]\\ &\text{is s-lower semi-continuous.}\end{aligned} \tag{A3}$$

$$\begin{aligned}&\text{If } (t_k, y_k) \in \mathcal{V}_{[0,T]} \text{ with } t_k \to t \text{ and } \min\{\mathcal{D}(y_k, y), \mathcal{D}(y, y_k)\} \to 0,\\ &\text{then } y_k \xrightarrow{\mathcal{T}} y.\end{aligned} \tag{A4}$$

Consider a sequence of functions $Y_n : [0,T] \to \mathcal{Y}$ such that there exists a constant $C > 0$ such that $\mathrm{Diss}_{\mathcal{D}}(Y_n; [0,T]) \le C$ for all $n \in \mathbb{N}$. Moreover, for all $t \in [0,T]$ we have

$$Y_n(t) \in \mathcal{R}(t) \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \mathrm{acc}_{\mathcal{T}}(Y_k(t))_{k\in\mathbb{N}} \subset \mathcal{V}(t),$$

where $\mathrm{acc}_{\mathcal{T}}(y_k)_{k\in\mathbb{N}}$ denotes the set of all possible accumulation points, i.e., $\mathcal{T}$-limits of subsequences.

Then, there exist a subsequence $(Y_{n_k})_{k\in\mathbb{N}}$ and functions $\varphi_\infty \in \mathrm{BV}([0,T], \mathbb{R})$, $Y^\infty \in \mathrm{BV}_{\mathcal{D}}([0,T], \mathcal{Y})$ such that the following holds:

(a) $\varphi_{n_k}(t) := \mathrm{Diss}_{\mathcal{D}}(Y_{n_k}, [0,t]) \to \varphi_\infty(t)$ for all $t \in [0,T]$,

(b) $Y_{n_k}(t) \xrightarrow{\mathcal{T}} Y^\infty(t) \in \mathcal{V}(t)$ for all $t \in [0,T]$,

(c) $\mathrm{Diss}_{\mathcal{D}}(Y^\infty, [t_0, t_1]) \le \varphi_\infty(t_1) - \varphi_\infty(t_0)$ for all $0 \le t_0 < t_1 \le T$.

Proof: The functions $\varphi_n : [0,T] \to [0,C];\ t \mapsto \mathrm{Diss}_{\mathcal{D}}(Y_n, [0,t])$ are nondecreasing. The scalar version of Helly's selection principle guarantees the existence of a function $\varphi_\infty : [0,T] \to [0,C]$ and a subsequence $(n_l)_{l\in\mathbb{N}}$ with $\varphi_{n_l}(t) \to \varphi_\infty(t)$ for all $t \in [0,T]$. Thus, we have proved (a).

Since φ_∞ is monotone and bounded, the set J of all its discontinuity points is at most countable. We choose a countable set M with the following properties:

$$J \subset M, \qquad M \text{ is dense in } [0,T], \qquad 0 \in M.$$

Using $Y_n(t) \in \mathcal{R}(t)$ and the s-compactness of $\mathcal{R}(t)$ we select, by the aid of Cantor's diagonal process, a subsequence (n_k) of the sequence (n_l) such that $Y_{n_k}(t)$ converges in $(\mathcal{Y}, \mathcal{T})$ for all $t \in M$. The limit of the sequence $(Y_{n_k}(t))$ is denoted by $Y^\infty(t)$, such that $Y^\infty : M \to \mathcal{Y}$ is defined.

We now show that this subsequence also converges for $t \in [0,T]\backslash M$, which provides the extension of Y^∞ to the whole interval. Fix an arbitrary $t \in [0,T]\backslash M$. The s-compactness of $\mathcal{R}(t)$ guarantees an accumulation point $Y^\infty(t) \in \mathcal{V}(t)$, i.e., $Y_{\widetilde{n}_m}(t) \xrightarrow{\mathcal{T}} Y^\infty(t)$ for a subsequence $(Y_{\widetilde{n}_m})$ of (Y_{n_k}). It remains to show that this accumulation point is unique. For this we use (A3) and (A4).

Take any sequence $(t_i)_{i\in\mathbb{N}}$ such that $t_i \in M$ and $t_i \to t$. Then, if $t_i < t$, (A3) implies

$$\begin{aligned}\mathcal{D}(Y^\infty(t_i), Y^\infty(t)) &\le \liminf_{m\to\infty} \mathcal{D}(Y_{\widetilde{n}_m}(t_i), Y_{\widetilde{n}_m}(t)) \le \liminf_{m\to\infty} \mathrm{Diss}(Y_{\widetilde{n}_m}; [t_i, t]) \\ &= \liminf_{m\to\infty} \varphi_{\widetilde{n}_m}(t) - \varphi_{\widetilde{n}_m}(t_i) = \varphi_\infty(t) - \varphi_\infty(t_i).\end{aligned}$$

Similarly, if $t < t_i$ we obtain $\mathcal{D}(Y^\infty(t), Y^\infty(t_i)) \le \varphi_\infty(t_i) - \varphi_\infty(t)$ and together with the continuity of φ_∞ at t we conclude

$$\min\{\mathcal{D}(Y^\infty(t_i), Y^\infty(t)), \mathcal{D}(Y^\infty(t), Y^\infty(t_i))\} \le |\varphi_\infty(t) - \varphi_\infty(t_i)| \to 0 \text{ for } i \to \infty.$$

Now we employ (A4) which implies $Y^\infty(t_i) \xrightarrow{\mathcal{T}} Y^\infty(t)$. Since $(\mathcal{Y}, \mathcal{T})$ is a Hausdorff space, the limit of a converging sequence is unique, and we conclude that $(Y_{n_k})_{k\in\mathbb{N}}$ has exactly one accumulation point. Thus, we have proved (b).

For assertion (c) we consider any discretization $t_0 = \theta_0 < \theta_1 < \ldots < \theta_N = t_1$ of the segment $[t_0, t_1]$. Using $Y_{n_k}(\theta_i) \xrightarrow{\mathcal{T}} Y^\infty(\theta_i)$ for $i = 0, 1, \ldots, N$ and (A3) we obtain

$$\begin{aligned}&\textstyle\sum_{j=1}^N \mathcal{D}(Y^\infty(\theta_{j-1}), Y^\infty(\theta_j)) \le \liminf_{k\to\infty} \sum_{j=1}^N \mathcal{D}(Y_{n_k}(\theta_{j-1}), Y_{n_k}(\theta_j)) \\ &\le \liminf_{m\to\infty} \mathrm{Diss}(Y_{n_k}; [t_0, t_1]) = \liminf_{m\to\infty} \varphi_{n_k}(t_1) - \varphi_{n_k}(t_0) = \varphi_\infty(t_1) - \varphi_\infty(t_0).\end{aligned}$$

Taking the supremum on the left-hand side gives the desired estimate (c). ■

We now collect a few results on functions $Y \in \mathrm{BV}_\mathcal{D}([0,T], \mathcal{Y})$ which will be useful later on.

Theorem 3.3 *Let $\mathcal{V}_{[0,T]}$ be a s-compact subset of $[0,T] \times \mathcal{Y}$ and assume that $\mathcal{D}$ satisfies (A3),(A4). Furthermore assume that $Y \in \mathrm{BV}_\mathcal{D}([0,T], \mathcal{Y})$ satisfies $(t, Y(t)) \in \mathcal{V}_{[0,T]}$ for all $t \in [0,T]$.*
(a) Then, $t \mapsto Y(t)$ is s-continuous (w.r.t. $\mathcal{T}$) at all continuity points of $t \mapsto \mathrm{Diss}_\mathcal{D}(Y, [0,t])$.
(b) For all $t \in [0,T]$ the $\mathcal{T}$-limits from the right $Y_+(t) = \lim_{\tau\searrow t} Y(\tau)$ and from the left $Y_-(t) = \lim_{\tau\nearrow t} Y(\tau)$ are well defined. Moreover, $\lim_{\tau\nearrow t} \mathrm{Diss}_\mathcal{D}(Y, [\tau, t]) = \mathcal{D}(Y_-(t), Y(t))$ and $\lim_{\tau\searrow t} \mathrm{Diss}_\mathcal{D}(Y, [t, \tau]) = \mathcal{D}(Y(t), Y_+(t))$.
(c) If $P(n) = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots t_{N_n-1}^{(n)} < t_{N_n}^{(n)} = T\}$ defines a sequence of partitions of the interval $[0,T]$ such that the fineness $\phi(P(n)) = \max_{1\le k\le N_n} t_k^{(n)} - t_{k-1}^{(n)}$ tends to 0, then the piecewise constant interpolants $Y^{(n)}$ with $Y^{(n)}(t) = Y(t_k^{(n)})$ for $t \in (t_{k-1}^{(n)}, t_k^{(n)}]$ lie in $\mathrm{BV}_\mathcal{D}([0,T], \mathcal{Y})$, and for almost all $t \in [0,T]$ we have $Y^{(n)}(t) \xrightarrow{\mathcal{T}} Y(t)$. In fact, the convergence holds for all t except in the (at most countable) set of jump points of $t \mapsto \mathrm{Diss}_\mathcal{D}(Y, [0,t])$.

Proof: Let t be a continuity point of $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Y,[0,t])$. Then

$$\lim_{\tau \nearrow t} \mathcal{D}(Y(\tau),Y(t)) \leq \lim_{\tau \nearrow t} \mathrm{Diss}_{\mathcal{D}}(Y,[\tau,t]) = \mathrm{Diss}_{\mathcal{D}}(Y,[0,t]) - \lim_{\tau \nearrow t} \mathrm{Diss}_{\mathcal{D}}(Y,[0,\tau]) = 0.$$

Using (A4) we obtain $\lim_{\tau \nearrow t} Y(\tau) = Y(t)$. Similarly we can show that $\lim_{\tau \searrow t} Y(\tau) = Y(t)$. Thus, we have proven (a).

Fix an arbitrary t in $[0,T]$ and consider a monotone increasing sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow t$. All points $(t_n, Y(t_n))$ lie in the set $\mathcal{V}_{[0,T]}$ and therefore we can select a subsequence $(n_k)_{k\in\mathbb{N}}$ with $Y(t_{n_k}) \to y_*$. We let $Y_-(t) := y_*$ and have to show that $Y_-(t) = \lim_{\tau \nearrow t} Y(\tau)$: The function $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Y,[0,t])$ is nondecreasing and bounded. This fact implies the existence of the limit $v_-(t) := \lim_{s \nearrow t} \mathrm{Diss}_{\mathcal{D}}(Y,[0,s])$. Let $\varepsilon > 0$ be arbitrary. There exists a number $s_0 < t$ such that for all $s \in (s_0,t)$ holds

$$v_-(t) - \varepsilon \leq \mathrm{Diss}_{\mathcal{D}}(Y,[0,s]) \leq v_-(t)$$

and therefore $\mathcal{D}(Y(s),Y(s_1)) \leq \varepsilon$ for all s, s_1 with $s_0 < s < s_1 < t$. Using (A3) and $t_{n_k} \to t$ we obtain

$$\mathcal{D}(Y(s),Y_-(t)) \leq \liminf_{k\to\infty} \mathcal{D}(Y(s),Y(t_{n_k})) \leq \varepsilon \tag{3.1}$$

which in conjunction with (A4) implies that $Y_-(t) = \lim_{\tau \nearrow t} Y(\tau)$.

On the one hand, the estimate $\mathcal{D}(Y(s),Y(t)) \leq \mathrm{Diss}_{\mathcal{D}}(Y,[s,t])$ and the additivity of the dissipation give $\mathrm{Diss}_{\mathcal{D}}(Y,[0,s]) + \mathcal{D}(Y(s),Y(t)) \leq \mathrm{Diss}_{\mathcal{D}}(Y,[0,t])$. Taking the limit $s \nearrow t$, using $Y(s) \to Y_-(t)$ and (A3), we obtain

$$v_-(t) + \mathcal{D}(Y_-(t),Y(t)) \leq \mathrm{Diss}_{\mathcal{D}}(Y,[0,t]). \tag{3.2}$$

On the other hand, for each partition $P = \{0 = t_0 < t_1 < \dots t_{N-1} < t_N = t\}$ of the interval $[0,t]$ with $t_{N-1} > s_0$ we obtain by using the triangle inequality and (3.1)

$$\begin{aligned}\sum_{j=1}^{N} \mathcal{D}(Y(t_{j-1}),Y(t_j)) &\leq \mathrm{Diss}_{\mathcal{D}}(Y,[0,t_{N-1}]) + \mathcal{D}(Y(t_{N-1}),Y_-(t)) + \mathcal{D}(Y_-(t),Y(t)) \\ &\leq v_-(t) + \varepsilon + \mathcal{D}(Y_-(t),Y(t)).\end{aligned} \tag{3.3}$$

Taking the supremum over all partitions and using that $\varepsilon > 0$ is arbitrary we infer with (3.2) that $\mathrm{Diss}_{\mathcal{D}}(Y,[0,t]) = \mathcal{D}(Y_-(t),Y(t)) + v_-(t)$. ¿From this we find

$$\lim_{\tau \nearrow t} \mathrm{Diss}_{\mathcal{D}}(Y,[\tau,t]) = \mathcal{D}(Y_-(t),Y(t)) + v_-(t) - \lim_{\tau \nearrow t} \mathrm{Diss}_{\mathcal{D}}(Y,[0,\tau]) = \mathcal{D}(Y_-(t),Y(t)).$$

Likewise we can show the existence of $Y_+(t) = \lim_{\tau \searrow t} Y(\tau)$ and $\lim_{\tau \searrow t} \mathrm{Diss}_{\mathcal{D}}(Y,[t,\tau]) = \mathcal{D}(Y(t),Y_+(t))$. This proves (b).

Since Y is s-continuous at all t except in the at most countable set of jump points of $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Y,[0,t])$, part (c) follows immediately. ∎

4 Existence via time-incremental problems

The major task is now to develop an existence theory for the initial value problem, i.e., to find a solution in the above sense which additionally satisfies $y(0) = y_0$. In general, we should not expect uniqueness without imposing further conditions like smoothness and uniform convexity of $\mathcal{E}(t,\cdot)$ and $\mathcal{D}$, see [MiT03].

Existence of solutions is shown via time-incremental minimization problems. For this we assume that the functionals $\mathcal{E}(t,\cdot) : \mathcal{Y} \to [\mathcal{E}_{\min}, \infty]$ and $\mathcal{D} : \mathcal{Y} \times \mathcal{Y} \to [0,\infty]$ are s-lower semicontinuous. In the standard case $\mathcal{Y}$ is a closed, convex and bounded subset of a reflexive Banach space (like the Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^m)$ with $p \in]1,\infty[$) equipped with weak topology $\mathcal{T}$, and hence $\mathcal{Y}$ is compact. Then, s-lower semicontinuity of $\mathcal{E}$ and $\mathcal{D}$ in $(\mathcal{Y},\mathcal{T})$ is the same as the classical weak sequential lower semicontinuity in the calculus of variations, see e.g. [Dac89].

The stability condition (S) can be rephrased by defining the stable sets

$$\begin{aligned}
&\mathcal{S}(t) := \{\, y \in \mathcal{Y} \mid \mathcal{E}(t,y) < \infty,\ \mathcal{E}(t,y) \le \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y}) \text{ for all } \widehat{y} \in \mathcal{Y} \,\},\\
&\mathcal{S}_{[0,T]} := \{\, (t,y) \in [0,T] \times \mathcal{Y} \mid y \in \mathcal{S}(t) \,\} = \cup_{t\in[0,T]} (t, \mathcal{S}(t)).
\end{aligned}$$

Then, (S) simply means that $y(t) \in \mathcal{S}(t)$ for all $t \in [0,T]$. The properties of the stable sets turn out to be crucial for deriving existence results.

For the time discretizations we choose discrete times $0 = t_0 < t_1 < \ldots < t_N = T$ and seek for a y_k which approximates the solution y at t_k, i.e., $y_k \approx y(t_k)$. Our energetic approach has the major advantage that the values y_k can be found incrementally via minimization problems. Since the methods of the calculus of variations are especially suited for applications in material modeling this will allow for a rich field of applications.

In our general setting the incremental problem takes the following form:

(IP) *For $y_0 \in \mathcal{S}(0) \subset \mathcal{Y}$ find $y_1, \ldots, y_N \in \mathcal{Y}$ such that*

$$y_k \in \arg\min\{\, \mathcal{E}(t_k, y) + \mathcal{D}(y_{k-1}, y) \mid y \in \mathcal{Y} \,\} \quad \text{for } k = 1, \ldots, N. \tag{4.1}$$

Here "arg min" denotes the set of all minimizers. The following result shows that (IP) is intrinsically linked to (S) & (E). Without any smallness assumptions on the time steps, the solutions of (IP) satisfy properties which are closely related to (S) & (E).

Theorem 4.1 *Let (A1) and (A2) hold. Any solution of the incremental problem (4.1) satisfies the following properties:*

(i) y_k *is stable for time* t_k, *i.e.,* $y_k \in \mathcal{S}(t_k)$;

(ii) $\int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_k)\,\mathrm{d}s \le \mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(y_{k-1}, y_k)$
$\le \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_{k-1})\,\mathrm{d}s$ *for* $k = 1, \ldots, N$;

(iii) $\mathcal{E}(t_j, y_j) + \sum_{k=1}^{j} \mathcal{D}(y_{k-1}, y_k) \le \mathcal{E}(0, y_0) + C_{\mathcal{E}} t_j$.

Proof: (i) The stability follows from minimization properties of the solutions and the triangle inequality. For all $\widehat{y} \in \mathcal{Y}$ and we have

$$\begin{aligned}
\mathcal{E}(t_k,\widehat{y}) + \mathcal{D}(y_k,\widehat{y}) &= \mathcal{E}(t_k,\widehat{y}) + \mathcal{D}(y_{k-1},\widehat{y}) + \mathcal{D}(y_k,\widehat{y}) - \mathcal{D}(y_{k-1},\widehat{y})\\
&\ge \mathcal{E}(t_k,y_k) + \mathcal{D}(y_{k-1},y_k) + \mathcal{D}(y_k,\widehat{y}) - \mathcal{D}(y_{k-1},\widehat{y}) \ \ge\ \mathcal{E}(t_k,y_k).
\end{aligned}$$

(ii) The first estimate is deduced from $y_{k-1} \in \mathcal{S}(t_{k-1})$ as follows:

$$\begin{aligned}&\mathcal{E}(t_k, y_k) + \mathcal{D}(y_{k-1}, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) = \\ &\mathcal{E}(t_{k-1}, y_k) + \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_k)\,\mathrm{d}s + \mathcal{D}(y_{k-1}, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) \geq \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_k)\,\mathrm{d}s.\end{aligned}$$

Since $y_k \in \arg\min\{\, \mathcal{E}(t_k, y) + \mathcal{D}(y_{k-1}, y) \mid y \in \mathcal{Y}\,\}$ the second estimate follows via

$$\begin{aligned}&\mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(y_{k-1}, y_k) \\ &\leq \mathcal{E}(t_k, y_{k-1}) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(y_{k-1}, y_{k-1}) \;=\; \int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, y_{k-1})\,\mathrm{d}s.\end{aligned} \tag{4.2}$$

(iii) This estimate is obtained by adding up the second estimate in (ii) for $k = 1, \ldots, j$. ■

For each incremental solution $(y_k)_{k=1,\ldots,N}$ of (IP) we define two piecewise constant functions which attain the values y_k at t_k and are constant in-between: Y^P is continuous from the right and $\widehat{Y}^P$ is continuous from the left. Summing up the estimates (ii) in Theorem 4.1 over $k = j, \ldots, m$ we find the following two-sided energy estimate:

Corollary 4.2 *Let (A1) and (A2) hold and let P be any partition of $[0, T]$. Then, any solution $(y_k)_{k=0,\ldots,N}$ of (IP) satisfies, for $0 \leq j < m \leq N$, the two-sided energy inequality*

$$\begin{aligned}\mathcal{E}(t_j, Y^P(t_j)) + \int_{t_j}^{t_m} \partial_s \mathcal{E}(s, \widehat{Y}^P(s))\,\mathrm{d}s &\leq \mathcal{E}(t_m, Y^P(t_m)) + \mathrm{Diss}_{\mathcal{D}}(Y^P, [t_j, t_m]) \\ &\leq \mathcal{E}(t_j, Y^P(t_j)) + \int_{t_j}^{t_m} \partial_s \mathcal{E}(s, Y^P(s))\,\mathrm{d}s.\end{aligned}$$

So far, we have not yet proved the existence of solutions to (IP). However, the above theorem already indicates that we can use induction arguments to provide compactness and hence existence results. We define first the reachable sets

$$\begin{aligned}&\mathcal{R}_{[0,T]} := \{\, (t, y) \in [0, T] \times \mathcal{Y} \mid \mathcal{E}(t, y) + \mathcal{D}(y_0, y) \leq \mathcal{E}(0, y_0) + C_{\mathcal{E}} t + 1\,\} \\ &\text{and } \mathcal{R}(t) := \{\, y \in \mathcal{Y} \mid (t, y) \in \mathcal{R}_{[0,T]}\,\}.\end{aligned} \tag{4.3}$$

With (A2) we conclude $\mathcal{R}(s) \leq \mathcal{R}(t)$ for $s < t$. As a consequence we have $\mathcal{R}_{[0,T]} \subset [0, T] \times \mathcal{R}(T)$. The following two assumptions will ensure the existence of a solution to (IP).

The set $\mathcal{R}(T)$ is s-compact in $(\mathcal{Y}, \mathcal{T})$. (A5)

For all $t \in [0, T]$ and all $\widehat{y} \in \mathcal{R}(T)$ the mapping $y \mapsto \mathcal{E}(t, y) + \mathcal{D}(\widehat{y}, y)$ is s-lower semi-continuous on $\mathcal{R}(t) \subset \mathcal{Y}$. (A6)

Using (A6) it is easy to see that $\mathcal{R}_{[0,T]}$ is a closed subset of $[0, T] \times \mathcal{Y}$. Hence, together with (A5) we conclude that each $\mathcal{R}(t)$, and $\mathcal{R}_{[0,T]}$ are s-compact.

Theorem 4.3 *Let (A1), (A2), (A5), and (A6) hold. Then, (IP) has a solution.*

Proof: The proof works by induction over $k = 1, \ldots, N$, since y_0 is given.

In step k the value y_{k-1} is given and we have to find $y_k \in \arg\min \mathcal{E}(t_k, y) + \mathcal{D}(y_{k-1}, y)$. Since y_{k-1} was the minimizer in the previous step we have $y_{k-1} \in \mathcal{R}(t_{k-1})$. In fact, by (iii) in Theorem 4.1 we have $\mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}(y_0, y_{k-1}) \leq \mathcal{E}(0, y_0) + C_{\mathcal{E}} t_{k-1}$.

Let $(y^l)_{l\in\mathbb{N}}$ be an infimizing sequence for $\mathcal{E}(t_k,\cdot)+\mathcal{D}(y_{k-1},\cdot)$ with $\mathcal{E}(t_k,y^l)+\mathcal{D}(y_{k-1},y^l) \le \inf_{y\in\mathcal{Y}} \{\mathcal{E}(t_k,y)+\mathcal{D}(y_{k-1},y)\}+1$ for all $l\in\mathbb{N}$. Following the estimate (4.2) in Theorem 4.1 we obtain $y^l\in\mathcal{R}(t_k)$ for all $l\in\mathbb{N}$. Using (A5) and (A6) we conclude the existence of $y^*\in\mathcal{R}(t_k)$ and a subsequence y^{l_m} with $y^{l_m} \xrightarrow{\mathcal{T}} y^*$. Moreover,

$$\mathcal{E}(t_k,y^*)+\mathcal{D}(y_0,y^*) \le \liminf_{m\to\infty} \mathcal{E}(t_k,y^{l_m})+\mathcal{D}(y_{k-1},y^{l_m}) = \inf_{y\in\mathcal{Y}} \{\mathcal{E}(t_k,y)+\mathcal{D}(y_{k-1},y)\}\,.$$

Hence, we let $y_k=y^*$ and the induction step is completed. ∎

The existence of solutions for the time-continuous problem (S) & (E) is now established by extracting a suitable subsequence of approximate solutions obtained from incremental problems and by showing that the limit is a solution. We first give a rough overview of the proof which illuminates the structure and the assumptions needed.

Proposition 4.3 and Theorem 4.1 show that $\mathcal{S}(t)$ is not empty for each $t\in[0,T]$. For all incremental solutions the points (t_k,y_k) lie in the set

$$\mathcal{V}_{[0,T]} := \mathcal{R}_{[0,T]}\cap\mathcal{S}_{[0,T]} = \cup_{t\in[0,T]}(t,\mathcal{V}(t)) \text{ where } \mathcal{V}(t)=\mathcal{R}(t)\cap\mathcal{S}(t).$$

To construct approximate solutions we choose a sequence $(P(n))_{n\in\mathbb{N}}$ of discretizations whose fineness $\phi(P(n)) = \max\{\, t_j^{(n)}-t_{j-1}^{(n)} \mid j=1,\ldots,N_n\,\}$ tends to 0. Moreover, we assume that the sequence is nested, i.e., $P(n)\subset P(n{+}1)$. We write shortly $Y^n=Y^{P(n)}$ ($\widehat{Y}^n=\widehat{Y}^{P(n)}$) for the right (left) continuous, piecewise constant interpolant associated with the partitions $P(n)$.

The dissipation bound (iii) of Theorem 4.1 provides an a priori bound in $\mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Y})$:

$$\mathrm{Diss}_{\mathcal{D}}(Y^n,[0,T]) \le \mathcal{E}(0,y_0)-\mathcal{E}_{\min}+C_{\mathcal{E}}T.$$

Using the abstract version of Helly's selection principle allows us to extract a subsequence $(n_l)_{l\in\mathbb{N}}$ such that for all $t\in[0,T]$ the sequence $Y^{n_l}(t) \xrightarrow{\mathcal{T}} Y^\infty(t)$ with

$$\mathrm{Diss}_{\mathcal{D}}(Y^\infty,[t_0,t_1]) \le \lim_{l\to\infty} \mathrm{Diss}_{\mathcal{D}}(Y^{n_l},[t_0,t_1]).$$

For this result we need to impose the two additional conditions from above: 1) $\mathcal{D}:\mathcal{R}(T)\times\mathcal{R}(T)\to[0,\infty]$ is s-lower semicontinuous (see (A3)), 2) $(t_k,y_k)\in\mathcal{V}_{[0,T]}$ with $t_k\to t$ and $\min\{\mathcal{D}(y_k,y),\mathcal{D}(y,y_k)\}\to 0$ implies $y_k \xrightarrow{\mathcal{T}} y$ (see (A4)).

Now we need to show that Y^∞ is a solution of (S) & (E). Stability is obtained via the stability of the incremental solutions at the discretization points which become dense in the limit of $n\to\infty$. For this we need to assume that

$$\mathcal{V}_{[0,T]} \text{ is s-compact.} \tag{A7}$$

This is certainly the most restrictive assumption and it will be considered in the next section in more detail.

Using Corollary 4.2 it is easy to give conditions which guarantee that Y^∞ satisfies $(\mathrm{E})_{\mathrm{weak}}$. From this we obtain energy equality (E) via using the stability (S). The following result relies on the additional assumption:

$$\begin{aligned}&\text{For almost every } t\in[0,T] \text{ the map } y\mapsto\partial_t\mathcal{E}(t,y)\\ &\text{is s-continuous on } \mathcal{R}(t).\end{aligned} \tag{A8}$$

Theorem 4.4 *Let $y \in \mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Y})$ with $y(t) \in \mathcal{V}(t) \subset \mathcal{S}(t)$ for $t \in [0,T]$. If (A1), (A2) and (A8) hold and if y is continuous for all t except on a set, which is at most countable, then for all $0 \le r < s \le T$ we have the opposite energy inequality*

$$\mathcal{E}(s, y(s)) + \mathrm{Diss}_{\mathcal{D}}(y, [r,s]) \ge \mathcal{E}(r, y(r)) + \int_r^s \partial_t \mathcal{E}(t, y(t))\,\mathrm{d}t. \tag{4.4}$$

Proof: We consider the equidistant partition $P(n)$ with $t_k = r + k(s-r)/n$ of the segment $[r,s]$. Moreover, we set $y_k = y(t_k)$ and $\widehat{Y}^n$ for the piecewise constant interpolant which is continuous from the left. As in Corollary 4.2 (see also the proof of part (ii) of Theorem 4.1), where only the stability was used, we obtain the lower estimate

$$\mathcal{E}(s, y(s)) + \mathrm{Diss}_{\mathcal{D}}(\widehat{Y}^n, [r,s]) - \mathcal{E}(r, y(r)) \ge \int_r^s \partial_t \mathcal{E}(t, \widehat{Y}^n(t))\,\mathrm{d}t.$$

Here the left-hand side is a lower bound for the left-hand side in (4.4). The right-hand side converges by Lebesgue's majorated convergence theorem. For this, use (A8) and $\widehat{Y}^n(t) \to y(t)$, see Proposition 3.3(c). This proves the result. ■

Now we are ready to turn the above construction into a rigorous existence proof.

Theorem 4.5 *Let the conditions (A1)–(A8) be satisfied. Assume additionally that*

$$\mathcal{E} : \mathcal{R}_{[0,T]} \to [\mathcal{E}_{\min}, \infty] \text{ is s-lower semicontinuous.} \tag{A9}$$

Then for each $y_0 \in \mathcal{S}(0)$ there is at least one solution $y \in \mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Y})$ of (S) & (E) with $y(0) = y_0$.

Moreover, for the above incremental approximations there exists a subsequence $(Y^{n_k})_{k\in\mathbb{N}}$ in $\mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Y})$ with the following convergence properties for $k \to \infty$:

(i) For all $t \in [0,T]$ we have $Y^{n_k}(t) \xrightarrow{\mathcal{T}} y(t)$.

(ii) For $0 \le r < s \le T$ we have $\mathrm{Diss}_{\mathcal{D}}(Y^{n_k}, [r,s]) \to \mathrm{Diss}_{\mathcal{D}}(y, [r,s])$.

(iii) For all $t \in [0,T]$ we have $\mathcal{E}(t, Y^{n_k}(t)) \to \mathcal{E}(t, y(t))$.

Remark 1. Assumption (A6) follows immediately from (A9) and (A3).
Remark 2. If $\mathcal{E}(t,\cdot) : \mathcal{R}(t) \to [\mathcal{E}_{\min}, \infty]$ is s-lower semicontinuous for all $t \in [0,T]$, then assumption (A2) implies that (A9) also holds.
Proof: Proposition 4.3 provides the existence of a solution for the incremental problem (4.1) for any partition. We take a sequence of hierarchical partitions $P(n) = \{0 = t_0^n, t_1^n, \ldots, t_{N_n}^n = T\}$ which is nested, i.e., $P(n) \subset P(n+1)$, and whose fineness tends to 0, i.e., $\phi(P(n)) = \max\{\, t_j^n - t_{j-1}^n \mid j = 1, \ldots, N_n \,\} \to 0$. For each partition $P(n)$ we have an incremental solution $(y_k^n)_{k=0,\ldots,N_n}$ and we define the two piecewise constant functions Y^n (continuous from the right) and and $\widehat{Y}^n$ (continuous from the left).

Using $\mathcal{R}(r) \subset \mathcal{R}(s)$ for $r < s$ we conclude that $Y^n(t) \in \mathcal{R}(t)$ for all t and n. To apply our selection result in Theorem 3.2 we have to show that the accumulation points of each sequence $(Y^n(t))_{n\in\mathbb{N}}$ lie in $\mathcal{V}(t) = \mathcal{S}(t) \cap \mathcal{R}(t)$. We fix t and assume $Y^{n_m}(t) \xrightarrow{\mathcal{T}} y$, then we know that $Y^{n_m}(t) = y_k^{n_m}$ with $t \in [t_k^{n_m}, t_{k+1}^{n_m})$. Since $\mathcal{V}_{[0,T]}$ is s-compact, $(t_k^{n_m}, y_k^{n_m}) \in \mathcal{V}_{[0,T]}$, $t_k^{n_m} \to t$ and $y_k^{n_m} \xrightarrow{\mathcal{T}} y$, we conclude $y \in \mathcal{V}(t)$ as desired.

Thus, the selection principle is applicable and we obtain a subsequence $(Y^{n_k})_{k\in\mathbb{N}}$ which converges for all t and its limit Y^∞ satisfies $Y^\infty(t) \in \mathcal{V}(t) \subset \mathcal{S}(t)$ and

$$\mathrm{Diss}_{\mathcal{D}}(Y^\infty, [r,s]) \le \lim_{k\to\infty} \mathrm{Diss}_{\mathcal{D}}(Y^{n_k}, [r,s]) =: \varphi_\infty(s) - \varphi_\infty(r) \tag{4.5}$$

for $0 \leq r < s \leq T$. In order to show that the desired solution y is this particular Y^∞ we have to prove that the stability condition (S) and the energy equality (E) holds.

To prove the energy equality (E) together with the convergence results stated in (ii) and (iii) we introduce the real-valued functions e_k, φ_k, w_k and $\widehat{w}_k$ via

$$\begin{aligned} e_k(t) &:= \mathcal{E}(t, Y^{n_k}(t)), & \varphi_k(t) &:= \mathrm{Diss}_{\mathcal{D}}(Y^{n_k}, [0,t]), \\ w_k(t) &:= \textstyle\int_0^t \partial_t \mathcal{E}(s, Y^{n_k}(s))\,\mathrm{d}s, & \widehat{w}_k(t) &:= \textstyle\int_0^t \partial_t \mathcal{E}(s, \widehat{Y}^{n_k}(s))\,\mathrm{d}s. \end{aligned}$$

Using Corollary 4.2 and (A2) we obtain for all t and all k the two-sided energy estimate

$$\widehat{w}_k(t) - C_{\mathcal{E}}\phi_k \leq e_k(t) + \varphi_k(t) - \mathcal{E}(0, y_0) \leq w_k(t) + 2C_{\mathcal{E}}\phi_k,$$

where $\phi_k = \phi(P(n_k))$ denotes the fineness of the partitions. For grid points $t \in P(n_k)$ the estimate holds without the corrections $\pm 2C_{\mathcal{E}}\phi_k$. For general points we use the fact that Y^{n_k} is piecewise constant and (A2) (i.e., $|\partial_t \mathcal{E}| \leq C_{\mathcal{E}}$).

In the limit $k \to \infty$ the left-hand and the right-hand side converge to the same limit $w_\infty(t) = \int_0^t \partial_t \mathcal{E}(s, Y^\infty(s))\,\mathrm{d}s$ by (A8) and Proposition 3.3(a). Using $\varphi_k(t) \to \varphi_\infty(t)$ we conclude that the limit $e_\infty(t) := \lim_{k\to\infty} e_k(t)$ exists. Moreover, by (A9) and (4.5) we have

$$\mathcal{E}(t, Y^\infty(t)) + \mathrm{Diss}_{\mathcal{D}}(Y^\infty, [0,t]) \leq e_\infty(t) + \varphi_\infty(t) = \mathcal{E}(0, y_0) + w_\infty(t),$$

which is $(\mathrm{E})_{\mathrm{weak}}$. Together with the opposite inequality derived in Proposition 4.4 we obtain (E).

In particular, this means $\mathcal{E}(t, Y^\infty(t)) + \mathrm{Diss}_{\mathcal{D}}(Y^\infty, [0,t]) = e_\infty(t) + \varphi_\infty(t)$ in addition to $\mathcal{E}(t, Y^\infty(t)) \leq e_\infty(t)$ and $\mathrm{Diss}_{\mathcal{D}}(Y^\infty, [0,t]) \leq \varphi_\infty(t)$. This implies equality in both cases and (ii) and (iii) are established. ∎

Our solution concept is such that solutions are well-defined for all $t \in [0,T]$ in contrast to definitions for almost every $t \in [0,T]$. In particular, both, the left-hand limit $y_-(t)$ and the right-hand limit $y_+(t)$, may differ from $y(t)$. However, if y is a solution of (S) & (E), then also y_- and y_+ are solutions (with a possible change of initial value in the latter case).

Moreover, the energy equality and stability imply that at jump points the following identities hold:

$$\begin{gathered} \mathcal{E}(t, y_-(t)) = \mathcal{E}(t, y(t)) + \mathcal{D}(y_-(t), y(t)), \quad \mathcal{E}(t, y(t)) = \mathcal{E}(t, y_+(t)) + \mathcal{D}(y(t), y_+(t)), \\ \mathcal{D}(y_-(t), y_+(t)) = \mathcal{D}(y_-(t), y(t)) + \mathcal{D}(y(t), y_+(t)). \end{gathered} \tag{4.6}$$

Note that all three points $y(t), y_-(t), y_+(t)$ lie in the stable set $\mathcal{S}(t)$.

We formulate now a special version of Theorem 4.5, which is based on Banach spaces and which is easy to apply to several models in continuum mechanics.

Theorem 4.6 *Let Y_1 and Y be Banach spaces. Suppose that Y_1 is compactly embedded in Y and that $\{\, y \in Y_1 \mid \|y\|_{Y_1} \leq 1 \,\}$ is closed in Y. The dissipation distance $\mathcal{D} : Y \times Y \to \mathbb{R}$ is the Y-norm, i.e., $\mathcal{D}(y_1, y_2) = \|y_1 - y_2\|_Y$. Furthermore the functional $\mathcal{E} : [0,T] \times Y \to [\mathcal{E}_{\min}, \infty]$ has the following properties:*

(a) $\mathcal{E}$ is s-lower semicontinuous on $[0,T] \times Y$ (with respect to the norm topology of Y).

(b) For some real numbers $c_1 > 0$, C_2 and $\alpha > 1$ we have

$$\mathcal{E}(t,y) \geq c_1\|y\|_{Y_1}^{\alpha} - C_2 \quad (\textit{i.e., } \mathcal{E}(t,y) = \infty \textit{ for } y \in Y \setminus Y_1). \tag{4.7}$$

(c) The map $\partial_t\mathcal{E}(t,\cdot) : Y_1 \to \mathbb{R}$ is s-continuous with respect to the norm topology Y.

(d) There exists C_3 such that $|\partial_t\mathcal{E}(t,y)| \leq C_3(1+\|y\|_{Y_1})$ for all $t \in [0,T]$ and $y \in Y_1$.

Then, for each $y_0 \in \mathcal{S}(0)$ there exists at least one solution $y \in \mathrm{BV}_{\mathcal{D}}([0,T],Y) \cap B([0,T],Y_1)$ of (S) & (E) with $y(0) = y_0$ and all the conclusions of Theorem 4.5 also hold.
Here $B([0,T],Y_1)$ denotes the set of mappings y such that $t \mapsto \|y(t)\|_{Y_1}$ is bounded.

Before giving the proof of this result, we show that it provides a generalized solution to (1.2), i.e., to the partial differential inclusion $0 \in \kappa(x)\,\mathrm{Sign}(\dot{z}(t,x)) - \mathrm{div}(a(x)\mathrm{D}_x z(t,x)) - g_{\mathrm{ext}}(t,x)$ with $z(t,\cdot)|_{\partial\Omega} = 0$. To this end take $Y = \mathrm{L}^1(\Omega), Y_1 = \mathrm{H}^1_0(\Omega)$ and $\mathcal{D}$ and $\mathcal{E}$ as defined for (1.2). Since $\mathcal{E}$ is quadratic, the assumptions (a) and (b) hold with $\alpha = 2$. Moreover, with $g_{\mathrm{ext}} \in \mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{H}^{-1}(\Omega))$ we obtain $|\partial_t\mathcal{E}(t,z)| = |\langle\partial_t g_{\mathrm{ext}}(t), z\rangle| \leq C\|z\|_{\mathrm{H}^1}$.

Proof: Since we want to use Theorem 4.5, one putative problem is the absence of an estimate similar to (A2). The main idea of our proof is to solve the problem (S) & (E) on the set $B_R^{Y_1}(0) := \{\, y \in Y_1 \mid \|y\|_{Y_1} \leq R \,\}$ equipped with the Y-topology and to show that the constructed solution doesn't depend on R. In the sequel these problems are called restricted problems.

The proof is done in three steps:
Step 1: Show that the problem (S) & (E) on the set $B_R^{Y_1}(0)$ equipped with the Y-topology has a solution y_R for all $R \geq \|y_0\|_{Y_1}$.
Step 2: Give a number r_{st} such that all solutions of the restricted problems with $R \geq \|y_0\|_{Y_1}$ as well as possible solutions of the problem on Y lie in $B_{r_{\mathrm{st}}}^{Y_1}(0)$.
Step 3: Give a number R_{dist} such that for all y which are stable on the set $B_{r_{\mathrm{st}}}^{Y_1}(0)$ the inequality $\mathcal{E}(t,y) \leq \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y})$ holds for all $\widehat{y} \in Y \setminus B_{R_{\mathrm{dist}}}^{Y_1}(0)$.

If these three steps are completed, it easy to see that each solution obtained for any $R > \max\{r_{\mathrm{st}}, R_{\mathrm{dist}}\}$ remains a solution for any $\widehat{R} > R$. Hence, each such solution is a solution of the full problem. We now work out Step 1 to 3.

Step 1: Let $R > \|y_0\|_{Y_1}$. We define the space $\mathcal{Y}$ as $B_R^{Y_1}(0)$ and equip it with the Y-topology. From the compact embedding of Y_1 in Y and the Y-closedness of the Y_1-balls in Y it follows that $\mathcal{Y}$ is a compact, topological Hausdorff space.

We need to verify all the assumptions of Theorem 4.5.

Each stable point $y \in \mathcal{Y}$ for the problem (S) & (E) on Y is also stable for the restricted problem on $\mathcal{Y}$. Hence, y_0 is stable at the time $t = 0$ for the restricted problem. Using (a) and (b) we infer that conditions (A8) and (A9) hold on $\mathcal{Y}$. Since $\mathcal{D}(y_1,y_2)$ is equal to $\|y_1 - y_2\|_Y$, conditions (A1), (A3) and (A4) follow immediately. Using (d) we obtain that the assumption (A2) holds on $\mathcal{Y}$ with $C_{\mathcal{E}} = C_3(1+R)$.

The map $y \mapsto \mathcal{E}(t,y) + \mathcal{D}(y_0,y)$ is s-lower semicontinuous. Hence the set $\mathcal{R}(T)$ of the restricted problem is s-closed in $\mathcal{Y}$. Since $\mathcal{Y}$ is a compact, condition (A5) holds.

Using Theorem 5.1 from below we obtain condition (A7) from (A5) and the Y-continuity of the dissipation distance. Thus, Step 1 is proved.

Step 2: We give an a priori bound for $\|y\|_{Y_1}$ for all solutions y of the whole or of the restricted problems. If y is a solution, then $y(t)$ is stable for all $t \in [0,T]$. Using $y \in \mathcal{S}(t)$, (4.7) and assumption (d), we get the following estimate

$$c_1\|y\|_{Y_1}^\alpha - C_2 \le \mathcal{E}(t,y) \le \mathcal{E}(t,y_0)+\mathcal{D}(y,y_0) \le \mathcal{E}(0,y_0)+t\,[C_3(1+\|y_0\|_{Y_1})]+\|y-y_0\|_Y. \quad (4.8)$$

Since Y_1 is continuously embedded in Y there exists a $K>0$ such that $\|y\|_Y \le K\|y\|_{Y_1}$ for all $y \in Y_1$. Hence, using (4.8) we obtain

$$c_1\|y\|_{Y_1}^\alpha - C_2 - K\|y\|_{Y_1} \le \mathcal{E}(0,y_0) + t\,[C_3(1+\|y_0\|_{Y_1})] + K\|y_0\|_{Y_1}. \quad (4.9)$$

The last estimate implies that there exists a number $r_{\mathrm{st}} > 0$ such that

$$\|y\|_{Y_1} \le r_{\mathrm{st}} \quad \text{for all } y \in \cup_{t\in[0,T]}\mathcal{S}(t).$$

Therefore all solutions of the whole problem or of the restricted problems lie in $B_{r_{\mathrm{st}}}^{Y_1}(0)$. This proves Step 2.

Step 3: Suppose y is stable on $B_{r_{\mathrm{st}}}^{Y_1}(0)$ at a time $t\in[0,T]$ and that there exists $\widehat{y}\in Y_1$ such that

$$\mathcal{E}(t,y) > \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y}). \quad (4.10)$$

Our aim is to give an a priori upper bound for the Y_1-norm of $\widehat{y}$. Using the stability of the point y on $B_{r_{\mathrm{st}}}^{Y_1}(0)$ and assumption (d), we obtain the following estimate:

$$\begin{aligned}\mathcal{E}(t,y) \le \mathcal{E}(t,y_0) + \mathcal{D}(y,y_0) &\le \mathcal{E}(0,y_0) + t\,[C_3(1+\|y_0\|_{Y_1})] + \mathcal{D}(y,y_0)\\ &\le \mathcal{E}(0,y_0) + T\,[C_3(1+r_{\mathrm{st}})] + 2r_{\mathrm{st}}.\end{aligned}$$

Combining this estimate, (4.10) and (4.7) we obtain

$$\mathcal{E}(0,y_0) + T\,[C_3(1+r_{\mathrm{st}})] + 2r_{\mathrm{st}} \ge \mathcal{E}(t,y) > \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y}) \ge c_1\|\widehat{y}\|_{Y_1}^\alpha - C_2. \quad (4.11)$$

The last estimate allows us to give an a priori upper bound R_{dist} for the norm of $\widehat{y}$, i.e., for all $\widehat{y}$ with $\|\widehat{y}\| > R_{\mathrm{dist}}$ the estimate

$$\mathcal{E}(t,y) \le \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y})$$

holds. Thus, Step 3 is proved. ■

5 Closedness of the stable set

The major assumption of our existence result in Theorem 4.5 is the s-compactness of $\mathcal{V}_{[0,T]}$ stated in (A7). Since $\mathcal{V}_{[0,T]} = \mathcal{S}_{[0,T]} \cap \mathcal{R}_{[0,T]}$ is a subset of the s-compact set $\mathcal{R}_{[0,T]}$, it suffices to show that $\mathcal{S}_{[0,T]}$ (or just $\mathcal{S}_{[0,T]} \cap \mathcal{R}_{[0,T]}$) is s-closed.

Before deriving abstract results in this direction we give two simple nontrivial applications of the theorem and thus highlight that the choice of the topology $\mathfrak{T}$ is crucial. For both examples let $\mathcal{Y} = \mathrm{L}^1(\Omega)$ with $\Omega \subset \mathbb{R}^d$ open and bounded, and choose the dissipation distance $\mathcal{D}(y_0,y_1) = \|y_1-y_0\|_{\mathcal{Y}} = \int_\Omega |y_1(x)-y_0(x)|\,\mathrm{d}x$.

For the first example consider

$$\mathcal{E}_1(t,y) = \int_\Omega a(x)|y(x)|^\alpha - g(t,x)y(x)\,\mathrm{d}x,$$

where $a(x) \geq a_0 > 0$, $\alpha > 1$, and $g \in \mathrm{C}^1([0,T], \mathrm{L}^\infty(\Omega))$. Since $\mathcal{E}_1(t,\cdot)$ is convex and lower semi-continuous, the set $\mathcal{R}(T)$ is a closed convex set which lies in the intersection of an L^1-ball and an L^α-ball. Hence, taking $\mathcal{T}$ to be the weak topology on $\mathcal{Y}$, we obtain that the s-compactness condition (A5) holds. Note that $\mathcal{R}(T)$ is not compact in the norm topology of $\mathrm{L}^1(\Omega)$. The stable sets for $\mathcal{E}_1$ are given by

$$\mathcal{S}_1(t) = \{\, y \in \mathrm{L}^1(\Omega) \mid |y(x)|^{\beta-2}y(x) \in [\tfrac{g(t,x)-c_\mathcal{D}}{a(x)\alpha}, \tfrac{g(t,x)+c_\mathcal{D}}{a(x)\alpha}] \text{ for a.a. } x \in \Omega \,\},$$

which shows that they are s-closed with respect to $\mathcal{T}$, since they are convex and closed in the norm topology. Hence, with $\mathcal{T}$ as weak topology in $\mathcal{Y} = \mathrm{L}^1(\Omega)$ all conditions of Theorem 4.5 can be satisfied.

For the second example consider the nonconvex energy functional

$$\mathcal{E}_2(t,y) = \textstyle\int_\Omega \frac{1}{2}|\mathrm{D}y(x)|^2 + f(t,x,y(x))\,\mathrm{d}x \text{ for } y \in \mathrm{H}^1(\Omega) \quad \text{and } +\infty \text{ else,}$$

where $f : [0,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$ and $\partial_t f$ are continuous and bounded. Now, $\mathcal{R}(T)$ is already compact in the norm topology of $Y = \mathrm{L}^1(\Omega)$, since it is closed and contained in a Y_1-ball, where $Y_1 = H^1(\Omega)$ is compactly embedded in Y. With these properties, it can be shown that with $\mathcal{T}$ as strong topology of $\mathrm{L}^1(\Omega)$ all conditions of Theorem 4.6 can be satisfied.

As a first abstract result we show that s-continuity of $\mathcal{D}$ on $\mathcal{V}_{[0,T]}$ leads to s-closedness, since then $\mathcal{E}$ is also s-continuous.

Theorem 5.1 *Let (A2) hold. Assume that $\mathcal{E}$ is s-lower semicontinuous on $[0,T]\times\mathcal{Y}$ and that $\mathcal{D}$ is s-continuous on $\mathcal{Y}\times\mathcal{Y}$. Then, $\mathcal{E} : \mathcal{S}_{[0,T]} \to [\mathcal{E}_{\min},\infty)$ is s-continuous as well and the set $\mathcal{S}_{[0,T]}$ is s-closed.*

Proof: For $(s,y_s),(t,y_t) \in \mathcal{S}_{[0,T]}$ we have by stability

$$-C_\mathcal{E}|t-s| - \mathcal{D}(y_s,y_t) \leq \mathcal{E}(t,y_t) - \mathcal{E}(s,y_s) \leq C_\mathcal{E}|t-s| + \mathcal{D}(y_t,y_s).$$

This estimate together with the s-continuity of $\mathcal{D}$ implies the s-continuity of $\mathcal{E}$.

Now, consider a sequence $(t_k,y_k)_{k\in\mathbb{N}}$ in $\mathcal{S}_{[0,T]}$ with $t_k \to t^*$ and $y_k \xrightarrow{\mathcal{T}} y^*$. It remains to show that $y^* \in \mathcal{S}(t^*)$. For an arbitrary $y \in \mathcal{Y}$ we have $\mathcal{E}(t_k,y_k) \leq \mathcal{E}(t_k,y) + \mathcal{D}(y_k,y)$ for all $k \in \mathbb{N}$. Taking the limit $k \to \infty$ the s-continuities yield $\mathcal{E}(t^*,y^*) \leq \mathcal{E}(t^*,y) + \mathcal{D}(y^*,y)$. Since $y \in \mathcal{Y}$ is arbitrary, it follows that $y^* \in \mathcal{S}(t^*)$. ■

The next result is a strengthened version of the previous one.

Theorem 5.2 *Let (A2) hold. Assume that for each sequence $(t_k,y_k)_{k\in\mathbb{N}}$ with $(t_k,y_k) \in \mathcal{S}_{[0,T]}$, $t_k \to t^*$ and $y_k \xrightarrow{\mathcal{T}} y^*$ in $\mathcal{Y}$ the following condition holds:*

$$\forall\, y \in \mathcal{Y} : \liminf_{k\to\infty} \big[\mathcal{E}(t_k,y_k) - \mathcal{D}(y_k,y)\big] \geq \mathcal{E}(t^*,y^*) - \mathcal{D}(y^*,y). \tag{5.1}$$

Then, the set $\mathcal{S}_{[0,T]}$ is s-closed.

Proof: Let $y \in \mathcal{Y}$ be arbitrary. We have to show that $\mathcal{E}(t^*,y^*) \leq \mathcal{E}(t^*,y) + \mathcal{D}(y^*,y)$. Since $(t_k,y_k) \in \mathcal{S}_{[0,T]}$ we have the following estimates

$$\begin{aligned}\mathcal{E}(t^*,y^*) &= \mathcal{E}(t^*,y^*) - \mathcal{E}(t_k,y_k) + \mathcal{E}(t_k,y_k) \leq \mathcal{E}(t^*,y^*) - \mathcal{E}(t_k,y_k) + \mathcal{E}(t_k,y) + \mathcal{D}(y_k,y) \\ &= \mathcal{E}(t^*,y) + \mathcal{D}(y^*,y) + (\mathcal{E}(t_k,y) - \mathcal{E}(t^*,y)) - [\mathcal{E}(t_k,y_k) - \mathcal{D}(y_k,y) - \mathcal{E}(t^*,y^*) + \mathcal{D}(y^*,y)].\end{aligned}$$

Taking the limit $k \to \infty$, using (A2) (i.e., $|\partial_t \mathcal{E}| \le C_{\mathcal{E}}$) and condition (5.1) we obtain the desired result. ■

For an application to the delamination problem we use the following result, which uses s-continuity of $\mathcal{E}$ and some approximation property for $\mathcal{D}$. This approximation property is weaker than the continuity assumed in Theorem 5.1.

Theorem 5.3 *Let (A1), (A2), (A3), (A5), and (A9) hold and assume that there exists an s-closed set $\mathcal{M}_{[0,T]}$ with $\mathcal{S}_{[0,T]} \subset \mathcal{M}_{[0,T]} \subset [0,T]\times\mathcal{Y}$ such that $\mathcal{E} : \mathcal{M}_{[0,T]} \to [\mathcal{E}_{\min}, \infty]$ is s-continuous. Moreover, assume that $\mathcal{D}$ satisfies the following condition:*

$$\begin{aligned}&\text{For all } (t,\widehat{y}), (t_k, y_k) \in \mathcal{S}_{[0,T]} \text{ with } (t_k, y_k) \xrightarrow{\mathcal{T}} (t,y)\\ &\text{there exists } \widehat{y}_k \in \mathcal{M}(t_k) \text{ such that}\\ &\widehat{y}_k \xrightarrow{\mathcal{T}} \widehat{y} \quad \text{and} \quad \liminf_{k\to\infty} \mathcal{D}(y_k, \widehat{y}_k) \le \mathcal{D}(y, \widehat{y}).\end{aligned} \tag{5.2}$$

Then, the set $\mathcal{S}_{[0,T]}$ is s-closed.

Remark: For the case that $\mathcal{Y}$ is a Banach space, $\mathcal{M}_{[0,T]} = [0,T]\times\mathcal{Y}$, and $\mathcal{D}(y,\widehat{y}) = \Delta(\widehat{y}-y)$ with $c_1\|y\| \le \Delta(y) \le c_2\|y\|$, we simply choose $\widehat{y}_k = \widehat{y}-y+y_k$. Then $\mathcal{D}(y_k,\widehat{y}_k) = \Delta(\widehat{y}-y) = \mathcal{D}(y,\widehat{y})$, and the assumption holds trivially.

Proof: Take any sequence $(t_k, y_k) \in \mathcal{S}_{[0,T]}$ with $(t_k, y_k) \xrightarrow{\mathcal{T}} (t,y)$. We have to show that $y \in \mathcal{S}(t)$. Obviously, we have $(t,y) \in \mathcal{M}_{[0,T]}$.

For arbitrary $\widehat{y} \in \mathcal{S}(t)$ we choose $\widehat{y}_k \in \mathcal{M}(t_k) \supset \mathcal{S}(t_k)$ according to condition (5.2). Using the s-continuity of $\mathcal{E}$ and $y_k \in \mathcal{S}(t_k)$ we obtain

$$\begin{aligned}\mathcal{E}(t,y) &= \lim_{k\to\infty} \mathcal{E}(t_k,y_k) \le \liminf_{k\to\infty} \mathcal{E}(t_k,\widehat{y}_k) + \mathcal{D}(y_k,\widehat{y}_k)\\ &= \lim_{k\to\infty} \Big(\mathcal{E}(t_k,\widehat{y}_k) + \mathcal{D}(y,\widehat{y})\Big) + \liminf_{k\to\infty} \Big(\mathcal{D}(y_k,\widehat{y}_k) - \mathcal{D}(y,\widehat{y})\Big)\\ &\le \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y}).\end{aligned}$$

Thus, the function $J_y : \mathcal{Y} \to [\mathcal{E}_{\min},\infty]; \widetilde{y} \mapsto \mathcal{E}(t,\widetilde{y})+\mathcal{D}(y,\widetilde{y})$ assumes values in $[\mathcal{E}(t,y),\infty]$ only, if it is restricted to $\mathcal{S}(t)$.

Define $\beta_y := \inf\{\, J(\widetilde{y}) \mid \widetilde{y} \in \mathcal{Y}\,\}$, then obviously $y \in \mathcal{S}(t)$ if and only if $\beta_y \ge \mathcal{E}(t,y) = J_y(y)$. Arguing as in the proof of Theorems 4.1 and 4.3 we find that the infimum β_y is attained at a point $y^* \in \mathcal{Y}$ and that $y^* \in \mathcal{S}(t)$. Hence, we conclude $\beta_y = J_y(y^*) \ge \mathcal{E}(t,y) = J_y(y)$. This implies $y \in \mathcal{S}(t)$ as desired. ■

In [FrL03] an even weaker sufficient condition for s-closedness of the stable sets is used. The set $\mathcal{M}_{[0,T]}$ and the s-continuity of $\mathcal{E}$ are needed no longer but (5.2) is replaced by

$$\begin{aligned}&\text{For all } (t,\widehat{y}), (t_k, y_k) \in \mathcal{S}_{[0,T]} \text{ with } (t_k, y_k) \xrightarrow{\mathcal{T}} (t,y)\\ &\text{there exists } \widehat{y}_k \in \mathcal{Y} \text{ such that}\\ &\widehat{y}_k \xrightarrow{\mathcal{T}} \widehat{y} \quad \text{and} \quad \liminf_{k\to\infty} \mathcal{E}(t_k,\widehat{y}_k)+\mathcal{D}(y_k, \widehat{y}_k) \le \mathcal{E}(t,\widehat{y})+\mathcal{D}(y, \widehat{y}).\end{aligned} \tag{5.3}$$

The corresponding result is not formulated in the given abstract way, but is called ‘transfer of jump sets’ in that specific context, see Thm. 2.1 in [FrL03] and our Section 6.3.

6 Applications to continuum mechanics

The flexibility of the energetic formulation allows for several applications in continuum mechanics. Concerning the notations we refer to the introduction. Recall that $\mathcal{F}$ denotes the set of admissible deformations $\varphi : \Omega \to \mathbb{R}^d$ and $\mathcal{Z}$ denotes the set of internal states $z : \Omega \to Z$. In the abstract setting we used $y = (\varphi, z) \in \mathcal{Y} = \mathcal{F} \times \mathcal{Z}$ and $\mathcal{D}(y_1, y_2) = \widetilde{\mathcal{D}}(z_1, z_2)$.

The energetic formulation is as above: A pair $(\varphi, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ is called a **solution** of the rate-independent problem associated with $\mathcal{D}$ and $\mathcal{E}$ if the **global stability (S)** and the **energy equality (E)** hold:

(S) *For all $t \in [0, T]$ and all $(\widehat{\varphi}, \widehat{z}) \in \mathcal{F} \times \mathcal{Z}$ we have*

$$\mathcal{E}(t, \varphi(t), z(t)) \leq \mathcal{E}(t, \widehat{\varphi}, \widehat{z}) + \widetilde{\mathcal{D}}(z(t), \widehat{z}).$$

(E) *For all $t \in [0, T]$ we have*

$$\mathcal{E}(t, \varphi(t), z(t)) + \mathrm{Diss}_{\mathcal{D}}(z; [0, t]) = \mathcal{E}(0, \varphi(0), z(0)) + \int_0^t \partial_t \mathcal{E}(s, \varphi(s), z(s))\,\mathrm{d}s.$$

The associated incremental problem reads

$$(\varphi_k, z_k) \in \arg\min\{\, \mathcal{E}(t_k, \widehat{\varphi}, \widehat{z}) + \mathcal{D}(z_{k-1}, \widehat{z}) \mid (\widehat{\varphi}, \widehat{z}) \in \mathcal{F} \times \mathcal{Z} \,\}. \tag{6.1}$$

A specific feature occurs if $\mathcal{E}(t, \varphi, z)$ depends only locally on z, in the sense that at $x \in \Omega$ the integral over Ω uses z only through its point value $z(x)$. Hence, z can be eliminated pointwise. We define the *condensed energy density* Ψ^{cond} and the *update function* Z^{update} for the internal variable via

$$\begin{aligned} &\Psi^{\mathrm{cond}}(z_{\mathrm{old}}; x, F) := \min\{\, W(x, F, z) + D(x, z_{\mathrm{old}}, z) \mid z \in Z \,\}, \\ &Z^{\mathrm{update}}(z_{\mathrm{old}}; x, F) \in \arg\min\{\, W(x, F, z) + D(x, z_{\mathrm{old}}, z) \mid z \in Z \,\}. \end{aligned} \tag{6.2}$$

With this we obtain a functional $\mathcal{E}^{\mathrm{cond}}(z_{\mathrm{old}}; t, \varphi) = \int_\Omega \Psi^{\mathrm{cond}}(z_{\mathrm{old}}; \mathrm{D}\varphi)\,\mathrm{d}x - \langle \ell_{\mathrm{ext}}(t), \varphi \rangle$ and the solution of (6.1) is equivalent to finding $\varphi_k \in \arg\min\{\, \mathcal{E}^{\mathrm{cond}}(z_{k-1}; t_k, \widehat{\varphi}) \mid \widehat{\varphi} \in \mathcal{F} \,\}$ and then letting $z_k = Z^{\mathrm{update}}(z_{k-1}; \mathrm{D}\varphi_k)$. For more details we refer to [Mie03a, Mie03b].

The above condensation is very useful for computational purposes and it also allows for an existence theory for (IP) in the case of finite-strain elasto-plasticity, see [Mie03b]. However, for the mathematical theory associated with the time-continuous problem (S) & (E) it seems advantageous to reduce the problem to the z-variable alone. The major difficulty in considering the pair $y = (\varphi, z)$ is that $\varphi \in \mathcal{F}$ does not appear in the dissipation. Hence, by (S), $\varphi(t)$ will always be a global minimizer of $\mathcal{E}(t, \cdot, z(t))$. But otherwise we have no control over the temporal oscillations in the approximate functions $\varphi^N : [0, T] \to \mathcal{F}$.

A first possible approach to tackle this difficulty is to introduce the reduced energy functional

$$\mathcal{E}^{\mathrm{red}}(t, z) = \min\{\, \mathcal{E}(t, \varphi, z) \mid \varphi \in \mathcal{F} \,\}.$$

However, in general we will lose the exact control, since $\mathcal{E}^{\mathrm{red}}$ is no longer explicit. In particular, the differentiability of $t \mapsto \mathcal{E}^{\mathrm{red}}(t, z)$ is no longer valid in general. At the moment there is only one way out, which is not always acceptable: We simply restrict ourselves to problems where the minimizer $\varphi = \Phi(t, z)$ of $\mathcal{E}(t, \cdot, z)$ is unique and depends continuously on (t, z). Then, $\mathcal{E}^{\mathrm{red}}(t, z) = \mathcal{E}(t, \Phi(t, z), z)$ and $\partial_t \mathcal{E}^{\mathrm{red}}(t, z) = \partial_t \mathcal{E}(t, \Phi(t, z), z)$.

The same assumption is needed if we keep the φ-variable. In this second approach the bottleneck is the assumption (A4) which states that the dissipation controls convergence in $\mathcal{Y} = \mathcal{F} \times \mathcal{Z}$. Of course, this has to be true only on $\mathcal{V}_{[0,T]} = \mathcal{R}_{[0,T]} \cap \mathcal{S}_{[0,T]}$. Note that $(\varphi, z) \in \mathcal{S}(t)$ already implies $\varphi = \Phi(t, z)$. Hence, (A4) can be satisfied by assuming that (A4) holds for $\widetilde{\mathcal{D}} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ and that Φ is continuous.

This uniqueness assumption will be used in Sections 6.1 and 6.2, see also [MTL02, MiR03, KMR03]. However, in Section 6.3 this uniqueness can be dispensed off.

6.1 Phase transformations in shape-memory alloys

We assume that, in each microscopic point $x \in \Omega$, an elastic material is free to choose one of p crystallographic phases and that the elastic energy density W is then given by $W_j(\mathrm{D}\varphi)$. If the model is made on the mesoscopic level, then the internal variables are phase portions $z^{(j)} \in [0, 1]$ for the j-th phase. We set $Z = \{\, z \in [0, 1]^p \subset \mathbb{R}^p \mid \sum_1^p z^{(j)} = 1 \,\}$ and $\mathcal{Z} = \mathrm{L}^1(\Omega, Z) \subset \mathrm{L}^1(\Omega, \mathbb{R}^p)$. The material properties are given via a mixture function $W : \mathbb{R}^{d\times d} \times Z \to [0, \infty]$, see [MTL02, GMH02]. The dissipation can be shown to have the form $D(z_0, z_1) = \psi(z_1 - z_0)$ with $\psi(v) = \max\{\, \sigma_m \cdot v \mid m = 1, \ldots, M \,\} \geq C_\psi |v|$, where $\sigma_m \in \mathbb{R}^p$ are thermodynamically conjugated threshold values.

So far we are unable to prove existence results for this model in its full generality. However, the case with only two phases ($p = 2$) has been treated in [MTL02] under the additional assumption that the elastic behavior is linear and both phases have the same elastic tensor. In this case, one sets $z = (\theta, 1-\theta)$ with $\theta \in [0, 1]$. It can be shown that $\mathcal{E}$ is a quadratic functional in $\theta \in \mathrm{L}^1(\Omega, [0, 1]) \subset \mathrm{L}^2(\Omega)$. It then follows that the compactness condition (A5) holds for $\mathcal{Y} = \mathcal{F} \times \mathrm{L}^1(\Omega, [0, 1])$ equipped with the weak topology where $\mathcal{F}$ is a suitable (affine) subspace of $H^1(\Omega)$ equipped with the weak topology. The compactness of $\mathcal{V}_{[0,T]}$ (see (A7)) involves a careful analysis using H-measures to show that the non convex sets $\mathcal{S}(t)$ are weakly closed.

In a microscopic model there are no phase mixtures allowed, i.e., we assume $z \in Z_p := \{e_1, e_2, \ldots, e_p\} \subset \mathbb{R}^p$, where e_j is the j-th unit vector. Thus, the functions $z \in \mathcal{Z}$ are like characteristic functions which indicate exactly one phase at each material point. The dissipation is assumed as above, but now the elastic energy contains an additional term measuring the surface area of the interfaces between the different regions:

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W(\mathrm{D}\varphi, z)\,\mathrm{d}x + \sigma \int_\Omega |\mathrm{D}z| - \langle \ell_{\mathrm{ext}}(t), \varphi \rangle,$$

where σ is a positive constant and $\int_\Omega |\mathrm{D}z|$ is $\sqrt{2}$ times the area of all interfaces. Here $\mathcal{Z} = \{\, z : \Omega \to Z_p \mid \int_\Omega |\mathrm{D}z| < \infty \,\}$ and we set $\mathcal{E}(t, \varphi, z) = +\infty$ for $z \notin \mathcal{Z}$.

Hence, after minimization with respect to φ we still have $\widehat{\mathcal{E}}(t, z) \geq \gamma + \sigma \int_\Omega |\mathrm{D}z|$. This term provides for $\mathcal{R}(T)$ (cf. (A5)) an a priori bound in $\mathrm{BV}(\Omega, \mathbb{R}^p)$ and hence we conclude compactness in $Y = \mathrm{L}^1(\Omega, Z)$ equipped with the norm topology. Under the usual additional conditions for the elastic stored-energy densities W_j we obtain for each $z_0 \in \mathcal{Z}$ a solution (φ, z) with $\varphi \in \mathrm{B}([0, T], \mathrm{W}^{1,2}(\Omega, \mathbb{R}^d))$ and $z \in \mathrm{BV}([0, T], \mathrm{L}^1(\Omega, \mathbb{R}^p)) \cap \mathrm{B}([0, T], \mathrm{BV}(\Omega, \mathbb{R}^p))$ with $z(t) \in \mathcal{Z}$ for all $t \in [0, T]$. We refer to [Mai03] for more details.

6.2 A delamination problem

In this section we provide a simple model for rate-independent delamination and refer to [KMR03] for a better model and the detailed analysis. The simplicity of the model is chosen to remove all unnecessary distractions and to focus the attention to the intricate interplay of the different continuity properties in the suitable topologies.

Consider a body $\Omega \subset \mathbb{R}^d$ which is given by an open, bounded, and path-connected domain and let all admissible deformations $\varphi : \Omega \to \mathbb{R}^d$ be equal to $\varphi_{\rm Dir}$ at a boundary part $\Gamma_{\rm Dir}$ of the interior of the closure of Ω, i.e., of $\mathrm{int}(\mathrm{cl}(\Omega))$, of positive surface measure. Assume that $\mathrm{int}(\mathrm{cl}(\Omega))$ differs from Ω by a finite set of sufficiently smooth hypersurfaces Γ_j, $j = 1, \ldots, n$, along which the body is glued to itself. This means that with $\Gamma := \bigcup_{j=1}^n \Gamma_j$ we have $\mathrm{int}(\mathrm{cl}(\Omega)) = \Omega \cup \Gamma$. We assume that the two sides of the body are glued together along these surfaces and that the glue is softer than the material itself. Upon loading, some parts of the glue may break and thus lose their effectiveness. The remaining fraction of the glue which is still effective is denoted by the internal state function $z : \Gamma \to [0, 1]$.

We let $\mathcal{Z} = \{\, z : \Gamma \to [0, 1] \,|\, z \text{ measurable} \,\} \subset \mathrm{L}^1(\Gamma)$. The dissipation distance $\widetilde{\mathcal{D}}(z_0, z_1)$ is proportional to the amount of glue that is broken from state z_0 to state z_1:

$$\widetilde{\mathcal{D}}(z_0, z_1) = c_{\mathcal{D}} \int_\Gamma z_0(y) - z_1(y) \,\mathrm{d}a(y) \text{ for } z_0 \geq z_1 \text{ on } \Gamma \quad \text{and } +\infty \text{ else.}$$

Here we explicitly forbid the healing of the glue by setting $\widetilde{\mathcal{D}}$ equal ∞, if $z_0 \not\geq z_1$.

The energy is given by the elastic energy in the body, the elastic energy in the glue, and the potential of the external loadings:

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W(\mathrm{D}\varphi) \,\mathrm{d}x + \int_\Gamma z(y) Q(y, [\![\varphi]\!]_\Gamma(y)) \,\mathrm{d}a(y) - \langle \ell_{\rm ext}(t), \varphi \rangle,$$

where for $y \in \Gamma$ the vector $[\![\varphi]\!]_\Gamma(y)$ denotes the jump of the deformation φ across the interface Γ and $Q(y, \cdot)$ is the potential defining the elastic properties of the glue.

For simplicity we assume further that W provides linearized elasticity and that Q is quadratic as well. Then there is a unique minimizer $\varphi = \Phi(t, z) \in \mathcal{F} := \{\, \phi \in \mathrm{H}^1(\Omega, \mathbb{R}^d) \mid \phi|_{\Gamma_{\rm Dir}} = \varphi_{\rm Dir} \,\}$ of $\mathcal{E}(t, \cdot, z)$. We let $\mathcal{Y} = \mathcal{Z} \times (\mathcal{F} \cap B_R^{\mathrm{H}^1}(\varphi_{\rm Dir}))$ and choose for $\mathcal{T}$ the trace of the weak topology in $\mathrm{L}^1(\Gamma) \times \mathrm{H}^1(\Omega)$. Instead of $\xrightarrow{\mathcal{T}}$ we then use $\rightharpoonup$. It is immediate that (A1), (A3) and (A5) hold. We also assume $\ell_{\rm ext} \in \mathrm{C}^{\rm Lip}([0, T], \mathrm{H}^{-1}(\Omega))$ which implies that (A2) and (A8) hold.

The two important facts here are (i) that the linear mapping $\varphi \mapsto [\![\varphi]\!]_\Gamma \in \mathrm{L}^2(\Gamma)$ is compact (i.e., $\varphi \rightharpoonup \varphi^*$ in $\mathrm{H}^1(\Omega)$ implies $[\![\varphi_k]\!]_\Gamma \to [\![\varphi^*]\!]_\Gamma$ in $\mathrm{L}^2(\Gamma)$), because of the compact embedding of $\mathrm{H}^{1/2}(\Gamma)$ into $\mathrm{L}^2(\Gamma)$, and (ii) that $z_k \rightharpoonup z^*$ in $\mathcal{Z}$ implies $\Phi(t, z_k) \to \Phi(t, z)$ in $\mathrm{H}^1(\Omega)$. We refer to [KMR03, Lem.2.1] for the proof of this delicate continuity result. Clearly, $y = (\varphi, z) \in \mathcal{S}(t)$ implies $\varphi = \Phi(t, z)$; hence, assumption (A4) holds, since convergence in $\widetilde{\mathcal{D}}$ implies strong L^1-convergence of the z-component and by continuity of Φ also of the φ-component.

The remaining conditions are the s-lower semicontinuity of $\mathcal{E}$ in (A9) and the s-closedness of $\mathcal{V}_{[0,T]}$ in (A7), where the weak topologies have to be used. The former condition is obtained, as $\mathcal{E}$ is convex and quadratic in φ and linear in z. Note that for a weakly convergent sequence (φ_k, z_k) the trilinear term $\int_\Gamma z_k Q([\![\varphi_k]\!]_\Gamma) \,\mathrm{d}a$ converges, since $Q([\![\varphi_k]\!]_\Gamma)$ converges strongly in $\mathrm{L}^1(\Gamma)$ and $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{L}^\infty(\Gamma)$, as $\mathcal{Z} \subset \mathrm{L}^\infty(\Gamma)$.

The latter condition (A7) is obtained using Theorem 5.3 where we use the set $\mathcal{M}_{[0,T]} = \{ (\Phi(t,z), z) \mid z \in \mathcal{Z} \}$ which is s-closed in $[0,T]{\times}\mathcal{Y}$ by the weak continuity of Φ. Moreover, the construction of the sequences $\widehat{y}_k$ in condition (5.2) is obtain in the form $\widehat{y}_k = (\Phi(t_k, \widehat{z}_k), z_k)$ where $\widehat{z}_k$ is constructed in Lemma 6.1 below. The weak continuity of $\mathcal{E}$ on $\mathcal{M}_{[0,T]}$ again follows from (ii) above, since weak convergence of z_k implies strong convergence of $\Phi(t_k, z_k)$ in $\mathrm{H}^1(\Omega)$.

Lemma 6.1 *Let* $\widehat{z}, z, z_k \in \mathcal{Z} = \{ z : \Gamma \to [0,1] \,|\, z \text{ measurable} \} \subset \mathrm{L}^1(\Gamma)$ *be given such that* $z \leq \widehat{z}$ *and* $z_k \rightharpoonup z$. *Then, there exists a sequence* $(\widehat{z}_k)_{k\in\mathbb{N}}$ *satisfying* $z_k \leq \widehat{z}_k \in \mathcal{Z}$, $\widehat{z}_k \rightharpoonup \widehat{z}$ *and* $\lim_{k\to\infty} \widetilde{\mathcal{D}}(z_k, \widehat{z}_k) = \widetilde{\mathcal{D}}(z, \widehat{z})$.

Proof: Given $\widehat{z}, z, z_k \in \mathcal{Z}$ we define $a_k = \max\{0, z_k{-}\widehat{z}\}$ and $b_k = \min\{0, z_k{-}\widehat{z}\}$, which satisfy $0 \leq a_k \leq 1{-}\widehat{z}$, $-\widehat{z} \leq b_k \leq 0$, and $a_k{+}b_k = z_k{-}\widehat{z}$. After choosing a subsequence we have

$$a_k \rightharpoonup a, \quad b_k \rightharpoonup b \quad \text{with } a{+}b = z{-}\widehat{z} \leq 0.$$

We let $c(x) = a(x)/(-b(x))$ where $b(x) < 0$ and $c(x) = 0$ elsewhere. Then, $0 \leq c \leq 1$, $a{+}cb = 0$, and the functions $\widehat{z}_k := \widehat{z}{+}a_k{+}cb_k$ satisfy $\widehat{z}_k \geq \widehat{z}{+}a_k{+}b_k = z_k$ and $\widehat{z}_k \leq \widehat{z}{+}a_k \leq 1$. Hence, $\widehat{z}_k \in \mathcal{Z}$ and $\widehat{z}_k \rightharpoonup \widehat{z}{+}a{+}cb = \widehat{z}$.

Since for $z_k \leq \widehat{z}_k$ we have $\widetilde{\mathcal{D}}(z_k, \widehat{z}_k) = c_{\mathcal{D}} \int_\Gamma \widehat{z}_k{-}z_k \,\mathrm{d}a$, we immediately find

$$\lim_{k\to\infty} \widetilde{\mathcal{D}}(z_k, \widehat{z}_k) = c_{\mathcal{D}} \int_\Gamma \widehat{z}{-}z \,\mathrm{d}a = \widetilde{\mathcal{D}}(z_k, \widehat{z}_k)$$

by using weak convergence. This proves the result. ■

Altogether, we have shown that, for each initial state z_0 and each loading $\ell_{\text{ext}} \in \mathrm{C}^{\text{Lip}}([0,T], \mathrm{H}^{-1}(\Omega))$ such that z_0 is stable, this simplified delamination problem (S) & (E) has a solution (φ, z) with $\varphi \in \mathrm{L}^\infty([0,T], \mathrm{H}^1(\Omega))$ and $z \in \mathrm{BV}_{\widetilde{\mathcal{D}}}([0,T], \mathcal{Z})$. Recall that the condition on z is equivalent to the monotonicity $z(s) \geq z(t)$ on Γ for $s < t$ and that then $\mathrm{Diss}_{\widetilde{\mathcal{D}}}(z, [s,t]) = \widetilde{\mathcal{D}}(z(s), z(t))$.

6.3 A rate-independent model for brittle fracture

In [FrM93, FrM98, DaT02, Cha03] the following fracture model is developed and analyzed. There, also the existence of solutions is established. Here, we want to show that this model is a special case of our abstract formulation. We do not give all the details which can be found in the above-mentioned papers. In this example the interesting point is that the internal variable z is the crack itself, which is considered to be a closed subset of the body $\Omega \subset \mathbb{R}^2$. Hence, the underlying space $\mathcal{Y} = \mathcal{Z}$ will be a highly nonlinear set.

The undeformed body is given by the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$. The dissipative variable $z \in \mathcal{Z}$ is taken to be a closed subset K of $\overline{\Omega}$, which has at most N connected components:

$$\mathcal{Y} = \mathcal{Z} = \mathcal{K}_N := \{ K \subset \overline{\Omega} \mid K \text{ closed}, \, \pi_0(K) \leq N \}.$$

The topology $\mathcal{T}$ on this set is defined via the Hausdorff metric

$$d_{\mathcal{H}}(K_0, K_1) = \max\{\mathrm{dist}(K_0, K_1), \mathrm{dist}(K_1, K_0)\},$$

which turns $(\mathcal{K}_N, \mathcal{T})$ into a compact metric space. The dissipation is defined to be proportional (with constant 1 here) to the opening length of the crack. Using the one-dimensional Hausdorff measure $\mathcal{H}^1$ it takes the form

$$\mathcal{D}(K_0, K_1) = \begin{cases} \mathcal{H}^1(K_1 \backslash K_0) & \text{for } K_0 \subset K_1, \\ +\infty & \text{else.} \end{cases}$$

This definition of $\mathcal{D}$ satisfies the triangle inequality (A1), however, it is highly unsymmetric and takes the value $+\infty$ whenever parts of a crack want to close again. Our lower semi-continuity property (A3) of $\mathcal{D} : \mathcal{K}_N \times \mathcal{K}_N \to [0, \infty]$ now easily follows from Gołąb's theorem, see [DaT02, Cha03]. Our assumption (A4) can be replaced by the monotonicity properties arising from the definition of $\mathcal{D}$.

To avoid confusion with the notations in [Cha03] we denote our energy functional by $\mathbb{E}(t, K)$ instead of $\mathcal{E}(t, z)$ as above. It is defined via minimization of the elastic energy

$$\mathbb{I}_{g,K}(u) := \int_{\Omega \backslash K} \mathbf{A}e(u){:}e(u) \, \mathrm{d}x$$

where $\mathbf{A} \in \mathrm{Lin}(\mathbb{R}^{2\times2})$ is a positive definite tensor and $e(u) = \frac{1}{2}(\mathrm{D}u + (\mathrm{D}u)^\mathsf{T})$. As candidates one has to consider all elastic displacements $u \in \mathrm{LD}(\Omega \backslash K)$ satisfying the Dirichlet boundary condition $u = g(t)$ on $\Gamma^\mathrm{D} \backslash K$:

$$\mathbb{E}(t, K) := \inf\{\, \mathbb{I}_{g(t),K}(u) \mid u \in \mathrm{LD}(\Omega \backslash K, \mathbb{R}^2),\ (u - g(t))|_{\Gamma^\mathrm{D} \backslash K} = 0 \,\},$$

where $\mathrm{LD}(\widetilde{\Omega}) := \{\, u \in \mathrm{L}^2_{\mathrm{loc}}(\widetilde{\Omega}, \mathbb{R}^2) \mid e(u) \in \mathrm{L}^2(\widetilde{\Omega}, \mathbb{R}^{2\times2}_{\mathrm{sym}}) \,\}$. It is shown in [Cha03] that the minimum is always attained at a minimizer $u = U(g(t), K)$ (not necessarily unique), that $\mathbb{E} : [0, T] \times \mathcal{K}_N \to [0, \infty]$ is lower semi-continuous (cf. (A9)), and that the derivative $\partial_t \mathbb{E}$ takes the form

$$\partial_t \mathbb{E}(t, K) = 2 \int_{\Omega \backslash K} \mathbf{A}e(U(g(t), K)){:}e(\dot{g}(t)) \, \mathrm{d}x. \tag{6.3}$$

In [FrM98] the fracture problem is formulated as **continuous monotone evolution** satisfying the following three axioms. We use the notations from [Cha03] but we replace $\mathcal{E}(g(t), K)$ here with our notation $\mathbb{E}(t, K) + \mathcal{D}(\emptyset, K)$.

Given $K_0 \in \mathcal{K}_N$ and $g \in \mathrm{W}^{1,1}([0, T], \mathrm{H}^1(\Omega, \mathbb{R}^2))$ find $K : [0, T] \to \mathcal{K}_N$ such that
(i) $K_0 \subset K(s) \subset K(t)$ for $0 \le s \le t \le T$,
(ii) for $t \in (0, T]$: $\mathbb{E}(t, K(t)) + \mathcal{D}(\emptyset, K(t)) \le \mathbb{E}(t, \widetilde{K}) + \mathcal{D}(\emptyset, \widetilde{K})$ for $\widetilde{K} \supset \cup_{s<t} K(s)$,
(iii) for $0 \le s < t \le T$ we have $\mathbb{E}(t, K(t)) + \mathcal{D}(\emptyset, K(t)) \le \mathbb{E}(t, K(s)) + \mathcal{D}(\emptyset, K(s))$.

Using the suggestion from [FrM98], the problem is solved in [Cha03] via the incremental problem (IP) and it is shown that the limit functions satisfy (i), (ii), (ii)$_0$ and the energy balance (iii)*:

(ii)$_0$ $\quad \mathbb{E}(0, K(0)) + \mathcal{D}(\emptyset, K(0)) \le \mathbb{E}(0, \widetilde{K}) + \mathcal{D}(\emptyset, \widetilde{K})$ for $K_0 \subset \widetilde{K} \in \mathcal{K}_N$,

(iii)* $\quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbb{E}(t, K(t)) + \mathcal{D}(\emptyset, K(t))\Big) = 2 \int_{\Omega \backslash K} \mathbf{A}e(U(g(t), K(t))){:}e(\dot{g}(t)) \, \mathrm{d}x.$

The conditions (iii) and (iii)* are equivalent for so-called monotone loadings $g(t) = tg^*$.

We show that this formulation is equivalent to our energetic formulation (S) & (E), if we assume that $K_0 = K(0)$. The case $K_0 \neq K(0)$ will be discussed afterwards.

Theorem 6.2 *Assume that $K : [0,T] \to \mathcal{K}_N$ satisfies $K(0) = K_0$. Then, the conditions (i), (ii), (ii)$_0$ and (iii)* hold if and only if (S) & (E) hold:*

(S) for all $t \in [0,T]$ and all $\widetilde{K} \in K_N$: $\mathbb{E}(t, K(t)) \le \mathbb{E}(t, \widetilde{K}) + \mathcal{D}(K(t), \widetilde{K})$,

(E) for $0 \le s < t \le T$: $\mathbb{E}(t, K(t)) + \mathrm{Diss}_{\mathcal{D}}(K, [s,t]) = \mathbb{E}(s, K(s)) + \int_s^t \frac{\partial}{\partial t}\mathbb{E}(r, K(r))\,\mathrm{d}r$.

Proof: First note that the monotonicity (i) is equivalent to $\mathrm{Diss}_{\mathcal{D}}(K, [0,T]) < \infty$. This implies that $\mathrm{Diss}_{\mathcal{D}}(K, [s,t]) = \mathcal{D}(K(s), K(t))$. Thus, integrating (iii)* and using (6.3) gives the energy balance (E):

$$\begin{aligned}&\mathbb{E}(t, K(t)) + \mathrm{Diss}_{\mathcal{D}}(K, [s,t]) = \\ &\mathbb{E}(t, K(t)) + \mathcal{D}(K(s), K(t)) = \mathbb{E}(s, K(s)) + \textstyle\int_s^t \frac{\partial}{\partial t}\mathbb{E}(r, K(r))\,\mathrm{d}r.\end{aligned}$$

To see the connections of (ii) and (ii)$_0$ with our stability concept (S) we introduce the left-hand limit $K_-(t) = \overline{\cup_{s<t} K(s)}$. Of course, $K_-(0) = K(0) = K_0$. Hence the condition (ii)$_0$ is contained in (ii) if we allow for $t = 0$ there as well. From our definition of $\mathcal{D}$, (ii) and (ii)$_0$ take the form

$$(\mathrm{ii})_{\mathcal{D}} \quad \text{for } t \in [0,T]: \ \mathbb{E}(t, K(t)) + \mathcal{D}(K_-(t), K(t)) \le \mathbb{E}(t, \widetilde{K}) + \mathcal{D}(K_-(t), \widetilde{K}) \text{ for } \widetilde{K} \in \mathcal{K}_N,$$

since "$\emptyset$" can be replaced by any set $K_* \subset K_-(t)$. It is shown in [DaT02], Prop. 6.1, that there exists a countable jump set $J \subset [0,T]$ such that $K_-(t) = K(t)$ for all $t \in [0,T] \setminus J$. For these t this condition is simply our stability condition $K(t) \in \mathcal{S}(t)$. At jump points with $K_-(t) \ne K(t)$ we first note that $K_-(t)$ is stable as well and that the energy balance implies $\mathbb{E}(t, K_-(t)) = \mathbb{E}(t, K(t)) + \mathcal{D}(K_-(t), K(t))$, see (4.6). Thus, for $t \in J$, we conclude stability of $K(t)$ as follows:

$$\begin{aligned}\mathbb{E}(t, K(t)) &= \mathbb{E}(t, K_-(t)) - \mathcal{D}(K_-(t), K(t)) \\ &\overset{(1)}{\le} \mathbb{E}(t, \widetilde{K}) + \mathcal{D}(K_-(t), \widetilde{K}) - \mathcal{D}(K_-(t), K(t)) \\ &\overset{(2)}{\le} \mathbb{E}(t, \widetilde{K}) + \mathcal{D}(K_-(t), \widetilde{K}) \quad \text{for all } \widetilde{K} \in \mathcal{K}_N,\end{aligned}$$

where (1) uses the stability of $K_-(t)$ and (2) uses the triangle inequality (which is in fact an equality due to monotonicity). Thus, (S) is established.

The opposite conclusion that (S) & (E) implies (i) to (iii)* is now immediate. ∎

In the case $K_0 \ne K(0)$ condition (ii)$_0$ is more complicated. This problem is due to the fact that the state K_0 was not assumed to be stable. Assuming stability of K_0 it is easy to see that $K(0) = K_0$ holds. If this is not the case, the state $K(0)$ is stable and there is an energy drop right before $t = 0$ with $\mathbb{E}(0, K_0) > \mathbb{E}(0, K(0)) + \mathcal{D}(K_0, K(0))$ such that the energy balance does not hold.

More recently, an alternative approach to the fracture problem was given in [FrL03], where the restriction on the number of components of the cracks can be dropped completely. There $\mathcal{Y}$ is the set of *all* closed subsets of $\overline{\Omega}$ and the deformations u are now allowed to lie in the set $\mathrm{SBV}(\Omega)$, the set of special functions of bounded variation.

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