INFINITE–DIMENSIONAL TRAJECTORY ATTRACTORS FOR ELLIPTIC PROBLEMS ON CYLINDERS

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An abstract model of second order elliptic boundary problems in cylindrical domains is studied from the dynamical point of view. It is proved that under natural assumptions the essential solutions of such problems can be described in terms of global attractors of the associated trajectory dynamical systems. It is shown that these attractors may have an infinite fractal dimension and infinite topological entropy. Moreover, sharp upper and lower bounds for Kolmogorovs ε-entropy of such attractors are given.

§0 Introduction

Spatial dynamical systems arise as nonlinear elliptic problems on cylindrical domains where the axial coordinate plays the rôle of time. The use of dynamical–system methods in such situations was initiated in [Kir82] where a local center manifold for a semilinear elliptic equation on a strip was constructed. This method of spatial center–manifold reduction, nowadays also called Kirchgässner reduction, was further developed and applied in many situations, in particular in nonlinear elasticity and hydrodynamical problems, see [Mie88,Mie90,IoM91,IoK92,GrT97]. The special case of elliptic variational problems was studied in [Mie91] where the Hamiltonian structure of the reduced flow on the spatial center manifold was obtained.

The use of global ideas from dynamical–systems theory for elliptic problems was developed in parallel starting with [CMS93,Mi94a,Mi94b]. The general idea is to introduce an auxiliary elliptic problem in a semi-infinite cylinder $\Omega_+ := (0,\infty)\times\omega$ (where $(t,x)\in\Omega_+$), endowed by an additional boundary condition $u\big|_{t=0} = u_0$ at $t=0$ and study the 'evolution' operator

$$S_t : u(0) \to u(t), \tag{0.1}$$

where $u(t,x)$ is a *bounded* solution of this problem, from the dynamical point of view. Then, as known, if the attractor of (0.1) exists it is generated by all essential solutions of the initial problem in a complete cylinder $\Omega = \mathbb{R} \times \omega$, i.e. by all solutions existing and bounded on Ω. This relation admits to study the 'dynamics' of essential solutions investigating the dynamical properties of the evolution operator (0.1) on the attractor.

Unfortunately, a bounded solution of the introduced auxiliary problem usually is not unique and thus, in general, (0.1) can be rigorously defined only as a semigroup of multivalued maps. The usage of multivalued maps can be overcome using the so-called trajectory approach under which one considers the set $\mathcal{K}^+$ of all bounded solutions of the auxiliary problem, equipped by a suitable topology, as a (trajectory) phase space for the semi-flow, defined by the translation semigroup $(\mathcal{T}_h)_{h\geq 0}$ via

$$(\mathcal{T}_h u)(t,x) = u(t+h,x) \text{ for } (t,x) \in \Omega_+,\ \ h \geq 0. \tag{0.2}$$

If an associated attractor exists it is called the *trajectory attractor* of the system. If it can be embedded into a finite–dimensional invariant manifold this manifold is called *essential manifold* as it contains all *essential solutions*, see [Mie94b,ViZ96,CSV97,SVWZ99]. (See also [ChV97] for applications of the described trajectory approach to various *evolutionary* equations of mathematical physics, for which the uniqueness problem is not solved yet, such as 3D Navier-Stokes equations, nonlinear hyperbolic equations with supercritical nonlinearities, etc.)

For the case of second order elliptic systems there is another way to avoid the multivalued maps. We replace the evolution operator (0.1) by the following one:

$$\mathbb{S}_t : (u(0), \partial_t u(0)) \to (u(t), \partial_t u(t)), \quad (u(0), \partial_t u(0)) \in \mathbb{K}^+ \tag{0.3}$$

where $\mathbb{K}^+$ is a set of $(u(0), \partial_t u(0))$ for which the auxiliary problem has a bounded solution. Then under certain assumptions this solution will be unique and (0.3) defines a continuous semigroup on the phase space $\mathbb{K}^+$ (see [CMS93]). Moreover, this semigroup is occurred to be homeomorphic to the semigroup of shifts (0.2), defined on the trajectory space $\mathcal{K}^+$ (see e.g. Section 2 for details).

The alternative approach of a direct study of the 'evolution' operator (0.1) using the proper generalization of the attractor theory to the case of multivalued semigroups was suggested in [Bab95a].

Further ideas from dynamical–systems theory can be applied to elliptic problems on cylinders. In [PSS97] exponential dichotomies are constructed to study bifurcations of solitary waves. A Floquet theory near spatially periodic states is developed in [Mi94a,DFKM96]. In [FSV98] Conley's connection index is employed to obtain heteroclinic front solutions. A structure of attractors for the case where the corresponding elliptic system possesses a global Liapunov function and where solutions of the auxiliary problem are unique is described in [ViZ99].

Note however, that very few is known about the Hausdorff and fractal dimension for attractors of elliptic equations, although there is a highly developed industry of estimating these dimensions for the case of evolutionary equations (see e.g. [Tem88]). In a fact up to the moment we know only two rather restrictive classes of elliptic equations with finite–dimensional attractors. The first of them is the case of uniqueness where the situation is very similar to the case of evolutionary problems (see [ViZ99]) and the second one is the case of an essential manifold existence where the dimension of the attractor is naturally majoranted by the dimension of an essential manifold (see [Mi94b, Ba95b]).

The aim of the present paper is to show that the dimension of the attractor may be infinite for the elliptic equations beyond of the mentioned classes and to give quantitative bounds for the 'size' of such attractors in terms of their Kolmogorov epsilon-entropy.

We consider an abstract semilinear elliptic problem in the form

$$\begin{cases} \ddot{u} - \gamma\dot{u} - Au = F(u,\dot{u}) \quad \text{for } t>0, \\ u\big|_{t=0} = u_0, \end{cases} \tag{0.4}$$

Here $u(t)$ is element of a Hilbert space H with scalar product $\langle\cdot,\cdot\rangle$. The linear operator $A: D(A) \to H$ is selfadjoint; moreover $\langle Au, u\rangle \geq \lambda_0\|u\|^2$ with $\lambda_0 \geq 1$ and A^{-1} is compact. We define $(H^s)_{s\in\mathbb{R}}$, to be the scale of Hilbert spaces generated by A, i.e. $H^s = D(A^{s/2})$, $\|\cdot\|_s \equiv \|\cdot\|_{H^s} = \|A^{s/2}\cdot\|$. We also introduce the Hilbert spaces $\mathbb{H}^s = H^s \times H^{s-1}$ equipped with the natural induced scalar product.

Moreover, γ is a bounded symmetric operator in H and the nonlinearity F is assumed to satisfy the following conditions. There exists a constant C, a small positive exponent δ, and for all $\mu > 0$ there exists a monotone function $Q_\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{cases} (a) \quad F \in C^1(H^{3/2-\delta} \times H^{1/2-\delta}, H) \\ (b) \quad \mathrm{D}_u F(u,v) \geq -C - \frac{1}{2}A, \\ (c) \quad \langle F(u,v), u\rangle \geq -C - 1/2(\|u\|_1^2 + \|v\|^2), \\ (d) \quad \|F(u,v)\|^2 \leq Q_\mu(\|u\|_{1/2}) + \mu\|u\|_2^2 + C(\|u\|_1^2 + \|v\|^2). \end{cases} \tag{0.5}$$

The particular form of the abstract problem (0.4) is motivated by the following elliptic system in a cylindrical domain $\Omega_+ = \mathbb{R}^+ \times \omega$, where $\omega \subset \mathbb{R}^n$ is bounded:

$$\begin{cases} \ddot{u} - \gamma\dot{u} + \Delta_x u = f(u,\dot{u}) + g(x) \quad \text{for } (t,x) \in \Omega_+ \\ u\big|_{\mathbb{R}^+\times\partial\omega} = 0; \qquad u\big|_{t=0} = u_0, \end{cases} \tag{0.6}$$

where $u = (u^1, \dots, u^k) \in \mathbb{R}^k$, $\gamma = \gamma^* \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$, $g \in L^2(\omega)$, which arises for instance under studying traveling wave solutions of the corresponding evolution equation in an unbounded cylindrical domain $\Omega = \mathbb{R} \times \omega$ (see e.g. [CMS93], [Bab95a], or [ViZ96]).

The existence of a bounded solution $u(t)$, $t \geq 0$ of the problem (0.4) for every fixed $u_0 \in H^{3/2}$ is verified in Section 1. Moreover, in this Section we derive a dissipative estimate for *bounded* solutions $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ (see Definition 1.1) of the problem (0.4), which has a fundamental significance for applying the trajectory approach, described above.

In Section 2 we show that the abstract system (0.4) has a trajectory attractor $\mathcal{A} = \mathcal{A}^{\mathrm{traj}}$, i.e. that the semigroup (0.2), defined the space $\mathcal{K}^+$ of all bounded solutions $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$, possesses a global attractor $\mathcal{A}$, which is generated by all essential solutions of the problem (0.4).

$$\mathcal{A}^{\mathrm{traj}} = \Pi_+\mathcal{K},$$

where $\mathcal{K} \subset W^2_{\rm bd}(\mathbb{R})$ is a set of all essential solutions of (0.4) and Π_+ is a restriction operator to the semiaxis $\mathbb{R}^+$.

Moreover, under the additional assumption

$$\|\,\mathrm{D}_u\, F(u,v)\|_{H^1\to H} + \|\,\mathrm{D}_v\, F(u,v)\|_{H\to H} \le Q(\|u\|_{3/2}+\|v\|_{1/2}) \tag{0.7}$$

(where Q is an appropriate monotonic function) on the nonlinear term F we prove that a bounded solution $u(t)$ of (0.4) is uniquely determined by the pair $u(0)$ and $\partial_t u(0)$ and verify thus, that the semigroup (0.3) is well defined on $\mathbb{K}^+$. This result relies on logarithmic convexity estimates for elliptic problems, see [AgN67,CMS93]. We verify also that a natural projection $\Pi_0 : \mathcal{K}^+ \to \mathbb{K}^+$, defined via $\Pi_0 u := (u(0), \partial_t u(0))$ realizes a Holder continuous homeomorphism and consequently the semigroup (0.3) can be defined via

$$\mathbb{S}_h := \Pi_0 \mathcal{T}_h (\Pi_0)^{-1} \tag{0.8}$$

This result guarantees the existence of a global attractor $\mathbb{A} \subset \mathbb{K}^+$ for the semigroup (0.3) as well and gives the relation

$$\mathbb{A} = \Pi_0 \mathcal{A}^{\rm traj} \tag{0.9}$$

In Section 3 we apply the concept of Kolmogorov's ε-entropy to study quantitative properties of the obtained attractor $\mathcal{A}^{\rm traj}$ of the abstract elliptic system (0.4). See [KoT93] for a detailed study of this concept and [ChV98,CoE99,Zel99,Zel99a,CoE00,Zel01] for its application to evolutionary equations of mathematical physics. The main result of Section 3 is the following upper estimate of the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon\left(\mathcal{A}^{\rm traj}\big|_{(0,T)}\right)$ of the restriction $\mathcal{A}^{\rm traj}$ to an arbitrary finite interval $(0,T)$:

$$\mathbf{H}_\varepsilon\left(\mathcal{A}^{\rm traj}|_{(0,T)}\right) \le C\left[T + \ln_+ \frac{R_0}{\varepsilon}\right] \ln_+ \frac{R_0}{\varepsilon}, \tag{0.10}$$

where C and R_0 are positive constants independent of $\varepsilon > 0$ and $T \ge 0$, and $\ln_+ z := \max\{\ln z, 0\}$. The upper estimate (0.10) is not strong enough to conclude that the fractal dimension $d_{\rm fract}(\mathbb{A})$ is finite and, as it shown in Section 4, it is really may be infinite. In fact, we give an example of an operator A and a nonlinear map F satisfying (0.5) and (0.7) such that the Kolmogorov entropy of the corresponding attractor possesses the lower bound

$$\mathbf{H}_\varepsilon\left(\mathcal{A}^{\rm traj}|_{(0,T)}\right) \ge C' T \ln_+ \frac{R_0'}{\varepsilon} \tag{0.11}$$

for a some $C' > 0$, $R_0' > 0$ which are independent of $T \ge 1$ and $\varepsilon > 0$. Moreover, using the logarithmic convexity arguments we derive from it that

$$\mathbf{H}_\varepsilon\left(\mathbb{A}\right) \ge C''\left(\ln_+ \frac{R_0''}{\varepsilon}\right)^{3/2} \tag{0.12}$$

and consequently the fractal dimensions are infinite:

$$\dim_{\text{fract}}\left(\mathcal{A}^{\text{traj}}\big|_{(0,T)}\right) = \dim_{\text{fract}}(\mathbb{A}) = \infty$$

Our example is built around a counterexample for Floquet theory in linear elliptic problems provided in [DFKM96]. This example has the form

$$\ddot{v} - Av = L_1(t)v + L_2(t)\dot{v},$$

where L_1 and L_2 are periodic in $t \in \mathbb{R}$. The example is such that there exists a nontrivial solution $v : \mathbb{R} \to H^2$ which decays like $\|v(t)\|_2 \le ce^{-t^2}$. For the counterexample to work we need

$$L_1 \in C_{\text{per}}(\mathbb{R}, \mathcal{L}(H^{s+r_1}, H^s)) \text{ and } L_2 \in C_{\text{per}}(\mathbb{R}, \mathcal{L}(H^{s+r_2}, H^s))$$

with $r_1 \ge 1$ and $r_2 \ge 0$ and some $s \ge 0$. On the other hand our existence theory for the compact attractor (see (0.5), (0.7)) needs $r_1 \le 1$ and $r_2 \le 0$. Hence, we are exactly in the borderline case.

In Section 5 we give a more precise study of the complexity for the trajectory dynamical system (0.2) associated with the example constructed in Section 4. In particular, we show that in contrast to the case of dynamical systems (DS) generated by ODEs and the many natural evolution PDEs in bounded domains, this system has infinite topological entropy. The chaotic nature of this system is exhibited by a homeomorphic embedding of a Bernoulli shift on an infinite number of symbols. Moreover, this type of chaotic behavior occurs to be very close to the behavior of DS associated with the evolution in PDEs on unbounded domains (see [Zel00,Zel00a]).

The elliptic problem (0.6) in the infinite cylinder $\Omega = \mathbb{R} \times \omega$ can be interpreted as an equation for the equilibria of the corresponding reaction–diffusion problem in Ω:

$$\text{(0.13)} \qquad \begin{cases} \partial_\eta u = \ddot{u} - \gamma\dot{u} + \Delta_x u - f(u,\dot{u}) - g(x), \ (t,x) \in \Omega, \ \eta > 0, \\ u\big|_{\eta=0} = u^0, \ u|_{\partial\Omega} = 0, \end{cases}$$

where t is still a spatial variable and $\eta \ge 0$ denotes the physical time. It is known (see [BaV92,MiS95,EfZ99]) that under natural assumptions on the nonlinearity f and on the external force g this equation possesses a global attractor $\mathcal{A}^{\text{glob}} \subset W^2_{\text{bd}}(\mathbb{R})$. Evidently one has an embedding

$$\text{(0.14)} \qquad \mathcal{K} \subset \mathcal{A}^{\text{glob}},$$

where $\mathcal{K}$ is a set of all essential solutions of the elliptic boundary problem (0.6). The analogs of the estimates (0.10) and (0.11) for the ε-entropy of the global attractor $\mathcal{A}^{\text{glob}}$ were obtained in [CoE99, Zel99, EfZ99], see also Section 5 for a deeper discussion of this analogy.

In future work we will study the question under which conditions the fractal dimension $d_{\text{fract}}(\mathbb{A})$ and/or topological entropy $h_{\text{top}}(\mathbb{S}_h, \mathbb{A})$ are finite. By now, such results exist only in the case of spectral gaps allowing for the construction of essential manifolds [Mi94b] or in the case of $\gamma \gg$ id where (0.4) is uniquely solvable, see [CSV97,ViZ99] and Remark 2.3.

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In this section we provide a number of a–priori estimates for solutions of (0.4) and based on these estimates we obtain the existence of bounded solutions. To this end we need to define the corresponding functional spaces.

Definition 1.1. For every $-\infty \le T_1 < T_2 \le +\infty$ and $l \in \mathbb{R}^+$ we define

$$W^l(T_1,T_2) \equiv L^2((T_1,T_2),H^l) \cap W^{l,2}((T_1,T_2),H). \tag{1.1}$$

As a shorthand we will write $W^l(T)$ instead of $W^l(T,T{+}1)$. By $W^l_{\mathrm{loc}}(\mathbb{R}^+)$ we denote the Fréchet space generated by the seminorms $\|\cdot\|_{W^l(T)}$, $T \in \mathbb{R}^+$. Moreover, we define

$$W^l_{\mathrm{bd}}(\mathbb{R}^+) \equiv \{u \in W^l_{\mathrm{loc}}(\mathbb{R}^+) : \|u\|_{l,b} \equiv \sup_{T\in\mathbb{R}^+} \|u\|_{W^l(T)} < \infty\} \tag{1.2}$$

The spaces $W^l_{\mathrm{loc}}(\mathbb{R})$ and $W^l_{\mathrm{bd}}(\mathbb{R})$ are defined analogously by taking $T \in \mathbb{R}$.

The main result of this section is the following theorem.

Theorem 1.2. *Let the assumptions* (0.5) *hold. Then, for every $u_0 \in H^{3/2}$ there exists at least one solution $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ for* (0.4). *Moreover there exist $C_*, \alpha > 0$ and a monotone function $Q : \mathbb{R}^+ \to \mathbb{R}^+$ such that every solution $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ of* (0.4) *satisfies*

$$\|u\|_{W^2_{\mathrm{bd}}(T)} \le Q(\|u_0\|_{3/2})e^{-\alpha T} + C_* \quad \text{for } T \ge 0. \tag{1.3}$$

Proof. We deduce firstly the a–priori estimate (1.3). The existence of a solution will be derived below basing on this estimate.

Following [ViZ99] we multiply the equation (0.4) by $\rho(t)u(t)$ in H and integrate over $t \in [\tau,+\infty)$. Here $\rho : \mathbb{R}^+ \to (0,\infty)$ is a weight function such that $\int_0^\infty \rho(t)\,dt < \infty$ and $|\dot\rho(t)| \le \varepsilon\rho(t)$ for all $t \ge 0$. After integrating by parts twice and using the fact that γ is symmetric we find

$$\begin{aligned}\int_\tau^\infty [\|\dot u\|^2{+}\|u\|_1^2]\,\rho\,dt + \big[\tfrac{\rho}{2}\partial_t\|u\|^2 - \tfrac{\rho}{2}\langle\gamma u,u\rangle\big]_{t=\tau} &= \int_\tau^\infty \big[-\langle u, F(u,\dot u)\rangle\rho - \langle u,\dot u\rangle\dot\rho + \langle\gamma u,u\rangle\dot\rho/2\big]\,dt \\ &\le \int_\tau^\infty \big[C{+}\tfrac12(\|u\|_1^2{+}\|\dot u\|^2){+}\varepsilon\|u\|\|\dot u\|{+}\varepsilon\|\gamma\|\,\|u\|^2\big]\,\rho\,dt.\end{aligned} \tag{1.4}$$

Here we have used the assumption (0.5)(c). Taking $\rho(t) = e^{\varepsilon(t-\tau)}$ and choosing $\varepsilon > 0$ small enough we deduce that

$$\frac{d}{d\tau}\|u(\tau)\|^2 \le C' + C'\|u(\tau)\|^2 \quad \text{for } \tau \ge 0. \tag{1.5}$$

Applying Gronwall's lemma we obtain a first a–priori estimate for the simple H norm:

$$\|u(t)\|^2 \le e^{C't}\|u_0\|^2 + e^{C't} - 1 \quad \text{for } t \ge 0. \tag{1.6}$$

Of course this estimate is only useful for small t due to the exponential growth with $C' > 0$.

Next we derive an analogue of (1.3) for the norm in $W^1(T)$ where the right–hand side does not grow for $t \to \infty$. Therefore we need to take special care of the initial condition $u_0 = u|_{t=0}$. According to the abstract trace theorem there exists a bounded linear operator $\mathbb{T} : H^{3/2} \to W^2(\mathbb{R}^+)$ such that $v = \mathbb{T}u_0$ satisfies

$$v(0) = u_0, \quad \operatorname{supp} v \subset [0,1] \quad \text{and} \quad \|v\|_{W^2(0)} \le C\|u_0\|_{3/2}. \tag{1.7}$$

Rewriting the equation (0.4) with respect to a new unknown function $w = u - v$ we find

$$\begin{cases} \ddot{w} - \gamma\dot{w} - Aw = F(w+v, \dot{w}+\dot{v}) - h(t), \\ w\big|_{t=0} = 0, \end{cases} \tag{1.8}$$

where $h = \ddot{v} - \gamma\dot{v} - Av$ and consequently

$$\operatorname{supp} h \in [0,1], \quad \|h\|_{L^2(0)} \le C_1\|u_0\|_{3/2} \tag{1.9}$$

Taking the scalar product of (1.8) with $\rho(t)w(t)$ we obtain as above by integration over $t \in \mathbb{R}^+$ (but now using $w(0) = 0$)

$$\begin{aligned} &\int_0^\infty [\|\dot{w}\|^2 + \|w\|_1^2]\,\rho\,dt \\ &\qquad = \int_0^\infty \Big[-\langle w, F(v+w, \dot{v}+\dot{w})\rangle - \langle w, h\rangle + \frac{\dot{\rho}}{2\rho}[\langle \gamma w, w\rangle - 2\langle w, \dot{w}\rangle]\Big]\rho\,dt \end{aligned} \tag{1.10}$$

The difficult term to estimate is the first one involving F.

Recall that $v(t) \ne 0$ and $h \ne 0$ only for $t \in [0,1]$. Thus, for $t \in [0,1]$ we have to use the weaker estimates (0.5)(b)+(d) while for $t \ge 1$ we can use the better estimate (0.5)(c). For nontrivial v we obtain

$$\begin{aligned} -\langle w, F(v+w, \dot{v}+\dot{w})\rangle &= -\langle w, F(v+w, \dot{v}+\dot{w}) - F(v, \dot{v}+\dot{w})\rangle - \langle w, F(v, \dot{v}+\dot{w})\rangle \\ &\le C\|w\|^2 + \frac{1}{2}\|w\|_1^2 + \frac{1}{4\alpha}\|w\|^2 + \alpha\|F(v, \dot{v}+\dot{w})\|^2 \\ &\le C(\alpha)\|w\|^2 + \frac{1}{2}\|w\|_1^2 + \alpha Q_\mu(\|v\|_{1/2}) + \alpha\mu\|v\|_2^2 + \alpha C(\|v\|_1^2 + \|\dot{v}+\dot{w}\|^2) \\ &\le C(\alpha)\|w\|^2 + \frac{1}{2}\|w\|_1^2 + Q_1(\alpha)(\|u_0\|_{3/2}) + 2\alpha C\|\dot{w}\|^2. \end{aligned}$$

For $t \ge 1$ we will use the better estimate (0.5)(c) since $v \equiv 0$:

$$-\langle w, F(v+w, \dot{v}+\dot{w})\rangle = -\langle F(w, \dot{w}), w\rangle \le C_2 + 1/2(\|w\|_1^2 + \|\dot{w}\|^2).$$

Inserting these estimates in (1.10) with α and $\varepsilon = \sup|\dot{\rho}|/\rho$ small enough and using the inequality (1.6) for estimating $C(\alpha)\|w(t)\|^2$ for $t \in [0,1]$ we arrive at

$$\int_0^\infty [\|\dot{w}\|^2 + \|w\|_1^2]\,\rho\,dt \le \int_0^1 Q_2(\|u_0\|_{3/2})\rho\,dt + \int_0^\infty \Big[C_2 + \frac{3}{4}[\|\dot{w}\|^2 + \|w\|_1^2]\Big]\rho\,dt.$$

We now employ the special weight function $\rho(t) = e^{-\varepsilon|t-T|}$ for $T \geq 0$ and obtain

$$\|w\|^2_{W^1(T)} = \int_T^{T+1} \left[\|\dot{w}\|^2 + \|w\|_1^2\right] dt \leq e^{\varepsilon} \int_0^{\infty} \left[\|\dot{w}\|^2 + \|w\|_1^2\right] e^{-\varepsilon|t-T|} dt$$
$$\leq 4e^{\varepsilon} \sup_{t\in[0,1]} e^{-\varepsilon|t-T|} Q_2(\|u_0\|_{3/2}) + e^{\varepsilon} \frac{8}{\varepsilon} C_2 \leq 5Q_2(\|u_0\|_{3/2}) e^{-\varepsilon T} + 10C_2/\varepsilon.$$

Together with the estimate in (1.7) for v we find with $u = v+w$

$$\|u\|_{W^1(T)} \leq Q_3(\|u_0\|_{3/2}) e^{-\varepsilon T} + C_3. \tag{1.11}$$

Now we are in position to complete the proof of the theorem by providing an a–priori estimate in $W^2(T)$. To this end we use regularity theory for the linear equation $\ddot{z} - Az = f$ with Dirichlet data, which is an abstract elliptic problem. Introduce the interval $J_T = (\max\{0, T-1\}, T+1\}$ and a cut–off function $\psi : \mathbb{R} \to [0,1]$ such that $\psi(t) = 1$ for $t \in [0,1]$ and $\psi(t) = 0$ for $t \notin [-1,2]$. We let $\psi_T(t) = \psi(t-T)$ and define $w_T = \psi_T w$. With (1.8) this leads to the equation

$$\begin{cases} \ddot{w}_T - Aw_T = f_T \equiv \psi_T(\gamma\dot{w} + F(u,\dot{u}) + h) + 2\dot{\psi}_T\dot{w} + \ddot{\psi}_T w & \text{for } t \in J_T, \\ w_T = 0 \quad \text{for } t \in \partial J_T. \end{cases} \tag{1.12}$$

Applying the linear regularity theory and assumptions (0.5) to (1.12) we will have

$$\|w_T\|^2_{W^2(J_T)} \leq C\Big[\|w\|^2_{W^1(J_T)} + \|h\|^2_{L^2(J_T)} + \|\psi_T F(u,\dot{u})\|^2_{L^2(J_T)} \tag{1.13}$$
$$\leq Q_4(\|u_0\|_{3/2}) e^{-\varepsilon T} + C_4 + \int_{J_T} C\left[Q_\mu(\|u(t)\|_{1/2}) + \mu\psi_T(t)^2\|w(t)\|_2^2\right] dt$$
$$\leq Q_5(\|u_0\|_{3/2}) e^{-\varepsilon T} + C_5 + C\mu\|w_T\|^2_{W^2(J_T)}$$

The first estimate uses (0.5)(d) and the $W^1(T)$ estimate for w; the second estimate uses (1.11). Taking μ in (1.13) small enough, we conclude that

$$\|w\|_{W^2(T)} \leq C\|w_T\|_{W^2(J_T)} \leq 2Q_5(\|u_0\|_{3/2}) e^{-\varepsilon T} + 2C_5. \tag{1.14}$$

Thus, the a–priori estimate stated in Theorem 1.2 is proved. So it remains to verify the existence of a solution $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ of this problem. To this end we construct firstly a solution $u_N(t)$ for the following auxiliary problem of type (0.4) on a finite interval:

$$\begin{cases} \ddot{u}_N - \gamma\dot{u}_N - Au = F(u_N, \dot{u}_N) & \text{for } t \in (0,N), \\ u_N\big|_{t=0} = u_0, \ u_N\big|_{t=N} = 0 \end{cases} \tag{1.15}$$

for every $N \in \mathbb{N}$ and obtain a solution of the initial problem by letting $N \to \infty$.

Repeating word by word the proof of the estimate (1.3) we derive that every solution $u_N \in W^2(0,T)$ of the problem (1.15) satisfies the estimate

$$\|u_N\|_{W^2(T)} \leq Q(\|u_0\|_{H^{3/2}}) e^{-\alpha T} + C_* \quad \text{for } T \in [0, N-1] \tag{1.16}$$

where the function Q and the constants C_* and α are the same as in (1.3) and consequently are independent of $N \in \mathbb{N}$. Observe also that, due to (0.5)(a) and due to the abstract trace theorem, the nonlinear operator $F : u \to F(u, \dot{u})$ is compact and continuous as a map from $W^2(0, N)$ to $W^0(0, N)$ for every *finite* N. Thus, the existence of a solution u_N for the problem (1.15) can be derived from (1.16) in a standard way using the Leray-Schauder fixed point principle, see e.g. [ViZ96,ViZ99].

In order to construct now a solution u of the initial problem (0.4) we note that due to (1.16) the sequence u_N is uniformly bounded in $W^2(0, T)$ for every $T > 0$. Consequently, due to the reflexivity of $W^2(0, T)$ and using Cantor's diagonal procedure, we may assume without loss of generality that u_N converges weakly in $W^2_{\text{loc}}(\mathbb{R}_+)$ to a some function $u \in W^2_{\text{bd}}(\mathbb{R}^+)$. Passing to the limit $N \to \infty$ in the equations (1.15) we derive that u is a solution of (0.4). Indeed, passing to the limit in linear terms of (1.15) is evident and passing to the limit in the nonlinear term also gives no problems since

$$(u_N, \dot{u}_N) \to (u, \dot{u}) \text{ strongly in } C_{\text{loc}}(H^{3/2-\delta} \times H^{1/2-\delta})$$

for every $\delta > 0$ and since the assumption (0.5)(a) holds. Theorem 1.2 is proved.

§2 The attractor

This section is devoted to study the behavior of the solutions of (0.4) for $t \to \infty$. Note that the conditions (0.5) guarantee only the existence of a solution u but not the uniqueness of it, consequently we cannot apply the attractor technique in a direct way. To overcome this difficulty we will use the trajectory attractor approach, developed in [ChV97] for evolutionary equations without uniqueness and in [ViZ96] for elliptic problems.

First we define the phase space $\mathcal{K}^+$ of our DS to be the set of all solutions on $\mathbb{R}^+$ which stay bounded for $t \to \infty$, i.e., we set

$$\mathcal{K}^+ \equiv \{u \in W^2_{\text{bd}}(\mathbb{R}^+) : u \text{ solves (0.4) for some } u_0 \in H^{3/2}\}.$$

Since our equation does not depend explicitly on t, we can define a semigroup $(\mathcal{T}_h)_{h \geq 0}$ of translations along the t–axis:

$$\mathcal{T}_h : \mathcal{K}^+ \to \mathcal{K}^+, \quad (\mathcal{T}_h u)(t) \equiv u(t+h), \quad h \geq 0 \tag{2.1}$$

We endow the set $\mathcal{K}^+$ by the *local* topology, induced by embedding $\mathcal{K}^+$ into the Fréchet space $W^2_{\text{loc}}(\mathbb{R}^+)$. Since $W^2_{\text{loc}}(\mathbb{R}^+)$ is metrizable, the subset $\mathcal{K}^+$ is also a metrizable topological space.

Definition 2.1. The set $\mathcal{K}^+$ endowed by the *local* topology is called the *trajectory phase space* of (0.4); the semigroup $(\mathcal{T}_h)_{h \geq 0}$ defined in (2.1) is called the *trajectory dynamical system* generated by (0.4); the (global) attractor $\mathcal{A}$ of $(\mathcal{T}_h)_{h \geq 0}$ in $\mathcal{K}^+$ is called the *trajectory attractor* of (0.4) and denoted by $\mathcal{A}^{\text{traj}}$.

Remark 2.2. Recall that by the definition of the global attractor $\mathcal{A}$ of $\mathcal{T}_h$ in $\mathcal{K}^+$ attracts all *bounded* subsets of $\mathcal{K}^+$ but *boundedness* is a metric concept which a–priori may depend on the choice of metric in $W^2_{\rm loc}(\mathbb{R}^+)$. But it is not difficult to prove, using the estimate (1.3), that in our situation any set $B \subset \mathcal{K}^+$ is bounded in $W^2_{loc}(\mathbb{R})$ if and only if B is bounded in $W^2_{\rm bd}(\mathbb{R}^+)$ and consequently the concept of 'bounded' set in $\mathcal{K}^+$ has a well–defined meaning.

Remark 2.3. It is worth to emphasize here that the topology in $\mathcal{K}^+$ is chosen in such a way that in the case when the problem (0.4) has unique solution which continuously depends on u_0 (see [CSV97,ViZ99] for sufficient conditions) the semigroup $\mathcal{T}_h$ coincides up to homeomorphism (even up to C^1–diffeomorphism under the assumptions of [ViZ99]) with the 'ordinary' semigroup $S_h : H^{3/2} \to H^{3/2}$, $S_h u_0 = u(h)$.

To formulate the next result we introduce the notion of *essential solutions.* These are solutions of (0.4) defined for $t \in \mathbb{R}$ and lying in $W^2_{\rm bd}(\mathbb{R})$. By $\mathcal{K} \subset W^2_{\rm bd}(\mathbb{R})$ we denote the *essential set* of (0.4) which is the union of all essential solutions ([Mi94b]).

Theorem 2.4. *Under the above assumptions equation* (0.4) *possesses a unique trajectory attractor* $\mathcal{A} = \mathcal{A}^{\rm traj}$ *which can be described as*

$$\mathcal{A} = \Pi_+\mathcal{K}, \tag{2.2}$$

where $\mathcal{K} \subset W^2_{\rm bd}(\mathbb{R})$ *is the essential set of* (0.4) *and* Π_+ *is a restriction operator to the semi–axis* $\mathbb{R}^+$.

Proof. According to the existence theory for attractors of abstract semigroups (see for example [BaV92]) it is sufficient to verify that:

(i). The set $\mathcal{K}^+$ is a complete metric space.

(ii). The semigroup $\mathcal{T}_h : \mathcal{K}^+ \to \mathcal{K}^+$ is continuous for every fixed h.

(iii). The semigroup $\mathcal{T}_h$ possesses a precompact absorbing set B_0 in $\mathcal{K}^+$, i.e., for every bounded subset $B \subset \mathcal{K}^+$ there is $\tau = \tau(B)$ such that $\mathcal{T}_h B \subset B_0$ if $h \geq \tau$.

ad (i). Since the space $W^2_{\rm loc}(\mathbb{R}^+)$ is complete the first assertion follows from the closedness of $\mathcal{K}^+$ in $W^2_{\rm loc}(\mathbb{R}^+)$. The latter fact is easily seen as the topology is generated by uniform convergence on compact subsets of $\mathbb{R}^+ = [0, \infty)$ in W^2. Clearly, the limit of solutions of (0.4) is again a solution with the corresponding limit initial value u_0 (see the end of the proof of Theorem 1.2).

ad (ii). The continuity of $\mathcal{T}_h$ is also evident since the map $\mathcal{T}_h$ is a simple translation.

ad (iii). It remains to construct a precompact absorbing set $B_0 \subset \mathcal{K}^+$. Using the estimate (1.3) we see that the set

$$B_* = \{u \in W^2_{\rm loc}(\mathbb{R}^+) \; : \; \|u\|_{W^2_{\rm bd}(\mathbb{R}^+)} \leq 2C_*\} \cap \mathcal{K}^+ \neq \varnothing \tag{2.3}$$

is an absorbing set for the semigroup $\mathcal{T}_h$ on $\mathcal{K}^+$. Hence $B_0 = \mathcal{T}_1 B_*$ is also an absorbing set of $\mathcal{T}_h$.

To complete the proof of the theorem it suffices to show that B_0 is precompact in $W^2_{\rm loc}(\mathbb{R}^+)$. For this we use elliptic regularity theory. Using Cantor's diagonal procedure we have to prove that for every $T \geq 1$ the set $B_*\big|_{[T,T+1]} = \{u|_{[T,T+1]} \; : \; u \in B_*\}$ is precompact in $W^2(T)$. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in B_*. Using the cut–off function ψ_T as defined above of (1.12) we see that $z_{T,n} = \psi_T u_n$ solves the equation

$$\text{(2.4)} \qquad \begin{cases} \ddot z_{T,n} - A z_{T,n} = h_{T,n} \equiv \psi_T(\gamma \dot u_n + F(u_n, \dot u_n)) + 2\dot\psi_T \dot u_n + \ddot\psi_T u_n, \\ z_{T,n}(T-1) = z_{T,n}(T+2) = 0. \end{cases}$$

Using the boundedness of the sequence $(u_n)_{n\in\mathbb{N}}$ in $W^2_{\mathrm{bd}}(\mathbb{R}^+)$, the compactness of the embedding $W^2((T-1,T+2)) \subset W^1((T-1,T+2))$ and the compactness of the nonlinear function from assumption (0.5)(a), we may assume (after extracting a subsequence if necessary) that $h_{T,n} \to h_T$ in $L^2((T-1,T+2))$. Then according to the abstract regularity theorem applied to (2.4) $z_{T,n} = \psi_T u_n \to u_T$ in $W^2((T-1,T+2))$, and because of $\psi_T(t) = 1$ of $[T, T+1]$ we have $u_n \to u$ in $W^2(T)$. Thus, Theorem 2.4 is proved.

In the second part of this section we give another interpretation of the DS generated by (0.4) which clarifies the nature of the nonuniqueness for the problem (0.4). To this end we need the additional assumption (0.7) on the nonlinear term F. To study uniqueness and continuity properties of the semiflow we consider two solutions $u_1, u_2 \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ and want to estimate their difference $v(t) = u_2(t) - u_1(t)$ which satisfies the abstract linear elliptic equation

$$\text{(2.5)} \qquad \ddot v - Av = L_1(t)v + L_2(t)\dot v$$

where the operators L_j are given via $L_1(t) = \int_0^1 \mathrm{D}_u\, F(u_1(t) + sv(t), \dot u_1(t) + s\dot v(t))ds$ and $L_2(t) = \gamma + \int_0^1 \mathrm{D}_v\, F(u_1(t) + sv(t), \dot u_1(t) + s\dot v(t))ds$. By the assumptions (0.7) and Theorem 1.2 we conclude that

$$\text{(2.6)} \qquad \|L_1(t)\|_{H^1\to H} + \|L_2(t)\|_{H\to H} \le M, \quad t \in \mathbb{R}^+$$

where the constant M depends only on $\|u_1(0)\|_{H^{3/2}}$ and $\|u_2(0)\|_{H^{3/2}}$, and consequently is uniformly bounded on bounded subsets of $\mathcal{K}^+$. For equation (2.5) we have the following abstract result.

Theorem 2.5. *Let $J = (0, T)$ and $v \in W^2(J)$. Assume there exists $M > 0$ such that*

$$\text{(2.7)} \qquad \sup_{t\in J} \|L_1(t)\|_{H^1\to H} \le M, \qquad \sup_{t\in J} \|L_2(t)\|_{H\to H} \le M.$$

(Recall $\|L_1\|_{H^1\to H} = \|LA^{-1/2}\|_{H\to H}$.) Define $y(t) = \|v(t)\|_1^2 + \|\dot v(t)\|^2$, then $y : J \to \mathbb{R}$ satisfies the following two estimates. For any $t \in J$ we have

$$\text{(2.8)} \qquad y(t) \ge y(0)e^{-2M^2t^2 - bt} \quad \text{where } b = 4M - 4\langle A^{3/4}v(0), A^{1/4}\dot v(0)\rangle / y(0)$$

and

$$\text{(2.9)} \qquad y(t) \le \big[y(0)\big]^{1-t/T} \big[y(T)\big]^{t/T} e^{2M(M+4/T)t(T-t)}.$$

The continuous embedding $W^2((0,T)) \to C([0,T], H^{3/2}) \cap C^1([0,T], H^{1/2})$ makes all the terms in the above theorem well–defined.

Proof. The proof of this theorem uses the theory of [AgN67] which is based on logarithmic convexity. Introduce the function

$$\xi(t) = \begin{pmatrix} \dot v(t) + A^{1/2} v(t) \\ \dot v(t) - A^{1/2} v(t) \end{pmatrix} \quad \text{with } \xi \in W^1_{\mathrm{bd}}(J)^2 \subset C_{\mathrm{bd}}(J, H^{1/2} \times H^{1/2}). \tag{2.10}$$

It can be easily derived from equation (2.5) that ξ satisfies the linear equation

$$\dot\xi(t) - \mathcal{B}\xi(t) = \mathcal{C}(t)\xi(t) \text{ with } \mathcal{B} = \begin{pmatrix} -A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix}, \ \mathcal{C}(t) = \frac{1}{2}\begin{pmatrix} \mathcal{C}_1(t) & \mathcal{C}_2(t) \\ \mathcal{C}_1(t) & \mathcal{C}_2(t) \end{pmatrix}, \tag{2.11}$$

where $\mathcal{C}_1 = L_2 + L_1 A^{-1/2}$ and $\mathcal{C}_2 = L_2 - L_1 A^{-1/2}$. A simple calculation gives $2y(t) = \|\xi(t)\|_*^2 \equiv \|\xi_1(t)\|^2 + \|\xi_2(t)\|^2$. According to assumption (2.7) the operator $\mathcal{C}(t)$ satisfies

$$\|\mathcal{C}(t)\xi\|_* \le 2M\|\xi\|_* \quad \text{for all } t \in J. \tag{2.12}$$

Assume that $y(\tau) > 0$ for some $\tau \in J$ (otherwise there is nothing to prove). With (2.10) we conclude that there is a maximal relatively open interval J_τ in $[0,T]$ such that $\tau \in J_\tau$ and $y(t) > 0$ for $t \in J_\tau$. On J_τ define the function

$$\alpha(t) = \ln y(t) - \int_\tau^t \Phi(s)\,ds \ \text{ with } \Phi(t) = \frac{\langle \mathcal{C}(t)\xi(t), \xi(t)\rangle_*}{y(t)}, \tag{2.13}$$

where $\langle\cdot,\cdot\rangle_*$ is the scalar product in $H \times H$.

Simple computations involving the definition of $\Phi(t)$ and (2.11) gives the formulae

$$\dot\alpha = \langle \mathcal{B}\xi, \xi\rangle_*/y, \qquad \ddot\alpha = \frac{2}{y}\left[\langle \eta, \eta\rangle_* + \langle \eta, \mathcal{C}\xi\rangle_*\right] \quad \text{with } \eta = \mathcal{B}\xi - \frac{\langle \mathcal{B}\xi, \xi\rangle_*}{2y}\xi.$$

Applying the Cauchy–Schwarz inequality and estimate (2.12) we find

$$\ddot\alpha(t) \ge -\frac{\|\mathcal{C}(t)\xi(t)\|_*^2}{2y(t)} \ge -4M^2 \quad \text{for } t \in J_\tau.$$

This shows that the function $J_\tau \ni t \mapsto \tilde\alpha(t) = \alpha(t) + 2M^2t^2$ is convex. As $y : J \to \mathbb{R}$ is bounded from above so is $\tilde\alpha$. Together with convexity this implies that J_τ has to equal $[0,T]$.

Now the estimate (2.8) follows by exponentiating $\tilde\alpha(t) \ge \tilde\alpha(0) + \dot{\tilde\alpha}(0)t$:

$$y(t) \ge y(0)e^{-2M^2t^2 + \dot\alpha(0) + \int_0^t \Phi(s)\,ds}.$$

With $|\Phi(s)| \le 4M$ and $\dot\alpha = 4\langle A^{3/4}v, A^{1/4}\dot v\rangle$ we obtain estimate (2.8).

Similarly, estimate (2.9) follows from the estimate $\tilde\alpha(t) \le (1 - t/T)\tilde\alpha(0) + (t/T)\tilde\alpha(T)$. This proves Theorem 2.5.

Recall $\mathbb{H}^s = H^s \times H^{s-1}$, introduce the continuous linear (trace) mapping

$$\Pi_0 : W^2_{\mathrm{loc}}(\mathbb{R}^+) \to \mathbb{H}^{3/2}; \quad \Pi_0 u = (u(0), \dot u(0)). \tag{2.14}$$

and define the set $\mathbb{K}^+ = \Pi_0\mathcal{K}^+ \subset \mathbb{H}^{3/2}$. Then, $\mathbb{K}^+$ is closed and Theorem 2.5 implies that $\Pi_0\big|_{\mathcal{K}^+} : \mathcal{K}^+ \to \mathbb{K}^+$ is one–to–one. This means that the Cauchy problem

$$\begin{cases} \ddot{u} - \gamma\dot{u} - Au = F(u,\dot{u}), \\ u(0) = u_0, \quad \dot{u}(0) = v_0, \end{cases} \tag{2.15}$$

has a unique solution $u \in W^2_{\mathrm{bd}}(\mathbb{R}^+)$ for every $(u_0, v_0) \in \mathbb{K}^+$ and consequently the semigroup $(\mathbb{S}_h)_{h\ge 0}$ on $\mathbb{K}^+$ is well defined via

$$\mathbb{S}_h : \mathbb{K}^+ \to \mathbb{K}^+; \quad \mathbb{S}_h(u(0), \dot{u}(0)) = (u(h), \dot{u}(h)) \tag{2.16}$$

or, which is the same, via $\mathbb{S}_h = \Pi_0\mathcal{T}_h(\Pi_0)^{-1}$. The following corollary of Theorem 2.5 shows that the semigroup (2.16) is Hölder continuous on $\mathbb{K}^+$.

Corollary 2.6. *Let the assumptions of Theorem 1.2 hold and let in addition condition* (0.7) *be satisfied. Then the semigroup* (2.16) *is Hölder continuous with Hölder exponent* α *for every* $0 < \alpha < 1$, *i.e.*

$$\|\mathbb{S}_h z_1 - \mathbb{S}_h z_2\|_{\mathbb{H}^{3/2}} \le C_\alpha e^{M_\alpha h^2}\|z_1 - z_2\|^{1-\alpha}_{\mathbb{H}^{3/2}}, \quad z_1, z_2 \in \mathbb{K}^+ \tag{2.17}$$

where the constants C_α *and* M_α *depend only on* α *and* $\|z_i\|_{\mathbb{H}^{3/2}}$, $i = 1, 2$.

Proof. Indeed, fixing in (2.9) $t = h$ and $T = h/\alpha$ and taking into the account that u_1 and u_2 are bounded as $t \to \infty$ we derive that

$$\|\mathbb{S}_h z_1 - \mathbb{S}_h z_2\|_{\mathbb{H}^1} \le C'_\alpha e^{M'_\alpha h^2}\|z_1 - z_2\|^{1-\alpha}_{\mathbb{H}^1}, \quad z_1, z_2 \in \mathbb{K}^+ \tag{2.18}$$

In order to derive (2.17) from (2.18) it is sufficient to note that the abstract regularity theorem applied to the equation (2.5) together with the trace theorem gives

$$\|v(h)\|_{H^{3/2}} + \|\dot{v}(h)\|_{H^{1/2}} \le C\|v\|_{W^2(h)} \le C_1\|v(0)\|_{H^{3/2}} + C_2\|v\|_{W^1(h,h+2)} \tag{2.19}$$

where the constants C and C_1 depend only on the operator A and the constant C_2 depends also on the constant M from (2.6). Combining (2.18) and (2.19) we derive (2.17). Corollary 2.6 is proved.

Our next task is to show that the DS $(\mathbb{S}_h)_{h\ge 0}$ on $\mathbb{K}^+$ and the trajectory DS $(\mathcal{T}_h)_{h\ge 0}$ on $\mathcal{K}^+$ are topologically conjugated by a homeomorphism Π_0. To this end we fix in the space $\mathcal{K}^+$ the following metric:

$$d(u_1, u_2) := \sup_{T\in\mathbb{R}^+} e^{-T^4}\|u_1 - u_2\|_{W^2(T)} \tag{2.20}$$

From the one side it is not difficult to verify that the topology induced on $\mathcal{K}^+$ by metric (2.20) coincides with the topology induced by the embedding $\mathcal{K}^+ \subset W^2_{\mathrm{loc}}(\mathbb{R}^+)$ (due to the fact that $\mathcal{K}^+ \subset W^2_{\mathrm{bd}}(\mathbb{R}^+)$ and from the other side the estimate (2.17) implies that

$$d(u_1, u_2) \le C''_\alpha \left(\|u_1(0) - u_2(0)\|_{H^{3/2}} + \|\dot{u}_1(0) - \dot{u}_2(0)\|_{H^{1/2}}\right)^{1-\alpha} \tag{2.21}$$

holds for every $u_1, u_2 \in \mathcal{K}^+$ and every $0 < \alpha < 1$. Here we have used also the estimate

$$\|v\|_{W^2(h)} \le C\|v\|_{C(h,h+1;H^{3/2})\cap C^1(h,h+1;H^{1/2})} \tag{2.22}$$

for the solutions of (2.5), which is a simple corollary of the abstract regularity theorem and of the estimate (2.6).

Thus, we have proved that the semigroups (2.16) and (2.1) are really topologically conjugated and consequently Theorem 2.4 implies the following result.

Theorem 2.7. *Let the assumptions of Theorem 1.2 hold and let in addition* (0.7) *be valid. Then the semigroup $\mathbb{S}_h$ possesses a global attractor $\mathbb{A}$ in $\mathbb{K}^+$, and*

$$\mathbb{A} = \Pi_0\mathcal{A}^{\text{traj}} \tag{2.23}$$

§3 The entropy of the attractor: the upper bounds.

In this section we will study the trajectory attractor of the equation (0.4) using the concept of Kolmogorov's ε–entropy.

First we recall briefly the definition of ε–entropy. For the detailed study of this concept see [KoT93]. The applications of this concept to the evolutionary equations of mathematical physics are given in [ChV98,CoE99,Zel99,Zel99a,Zel01].

Definition 3.1. Let M be a metric space and let K be a precompact subset of it. For a given $\varepsilon > 0$ let $N_\varepsilon(K) = N_\varepsilon(K, M)$ be the minimal number of ε–balls in M which cover the set K (this number is evidently finite by compactness). By definition, Kolmogorov's ε–entropy of K in M is the following number

$$\mathbf{H}_\varepsilon(K) = \mathbf{H}_\varepsilon(K, M) \equiv \ln N_\varepsilon(K) \tag{3.1}$$

Example 3.2. Let K be compact n–dimensional Lipschitz manifold in M. Then the evident estimates imply that $C_1\left(\frac{1}{\varepsilon}\right)^n \le N_\varepsilon(K) \le C_2\left(\frac{1}{\varepsilon}\right)^n$ and consequently $\mathbf{H}_\varepsilon(K) = (n + o(1)_{\varepsilon\to0})\ln\frac{1}{\varepsilon}$.

This example justifies the following definition.

Definition 3.3. The *fractal (box–counting) dimension* of the set $K \subset\subset M$ is defined to be the following number:

$$\dim_{\text{fract}}(K) = \dim_{\text{fract}}(K, M) = \limsup_{\varepsilon\to0}\frac{\mathbf{H}_\varepsilon(K)}{\ln\frac{1}{\varepsilon}} \tag{3.4}$$

Note that we have constructed the attractor $\mathcal{A}$ for (0.4) which is compact only in the local topology of $W^2_{\text{loc}}(\mathbb{R}^+)$, that is why we will consider the ε–entropy of restrictions $\mathcal{A}\big|_{[0,T)}$ and study its dependence on two parameters T and ε. To this end we need weighted analogues for $W^l_{\text{bd}}(\mathbb{R}^+)$.

Definition 3.4. For weight functions $\phi \in C(\mathbb{R}, (0, \infty))$ we define the spaces

$$W^l_{\text{bd},\phi}(\mathbb{R}^+) \equiv \{u \in W^l_{\text{loc}}(\mathbb{R}^+) : \|u\|_{l,\text{bd},\phi} \equiv \sup\nolimits_{T\in\mathbb{R}^+}\{\phi(T)\|u\|_{W^l(T)}\} < \infty\} \tag{3.5}$$

Following to [MiS95,Mi97,EfZ01,Mi00,Zel99a] we introduce also a class of admissible weight functions.

Definition 3.5. A function $\phi \in L^\infty_{\text{loc}}(\mathbb{R})$ is called a weight function with the growth rate $\mu \ge 0$ if there exists $C > 0$ such that the conditions

$$\phi(x+y) \le Ce^{\mu|x|}\phi(y), \quad \phi(x) > 0 \tag{3.6}$$

are satisfied for every $x, y \in \mathbb{R}$.

It is not difficult to deduce from (3.6) that $\phi(x+y) \ge C^{-1}e^{-\mu|x|}\phi(y)$ is also satisfied for every $x, y \in \mathbb{R}$.

The main result of this section is the following theorem.

Theorem 3.6. *Let the assumptions of Theorem 1.2 hold. Then the following estimates are valid*

$$\mathbf{H}_\varepsilon(\mathcal{A}^{\mathrm{traj}}\big|_{(0,T)}, W^2_{\mathrm{bd}}((0,T))) \le C\big[T + \ln_+ \frac{R_0}{\varepsilon}\big] \ln_+ \frac{R_0}{\varepsilon} \tag{3.7}$$

where $\ln_+ \gamma = \max\{0, \ln \gamma\}$ *and the constants* C *and* R_0 *are independent of* T *and* ε.

The proof of this Theorem is based on a series of intermediate results and will be finished after Lemma 3.9.

Proposition 3.7. *Let* $\mathcal{K} \subset W^2_{\mathrm{bd}}(\mathbb{R})$ *be defined as in Theorem 2.4 such that* $\mathcal{A}^{\mathrm{traj}} = \Pi_+\mathcal{K}$ *is the trajectory attractor of* (0.4). *Then for every* $\alpha > 0$ *there exists a constant* $C = C(\alpha)$ *such that for any solutions* $u_1, u_2 \in \mathcal{K}$ *and any* $T \in \mathbb{R}$ *we have*

$$\|u_1 - u_2\|_{W^2(T)} \le C \sup_{t\in\mathbb{R}} \left\{ e^{-\alpha|T-t|} \|u_1 - u_2\|_{W^0(t)} \right\}. \tag{3.8}$$

Proof. Indeed, define $v(t) = u_2(t) - u_1(t)$ as in the proof of Theorem 2.5. Then, v satisfies (2.5). The abstract interior estimate applied to (2.5) implies that

$$\|v\|_{W^2(T)} \le C\|h_v\|_{W^0(T-1,T+2)}, \quad \text{where } h_v := L_1(t)v + L_2(t)\dot{v} \tag{3.9}$$

and C is independent of $T \in \mathbb{R}$. Multiplying (3.9) by $e^{-\alpha|T-M|}$ and taking supremum over $T \in \mathbb{R}$ from both sides of the obtained inequality we derive the following version of regularity theorem in weighted space $W^2_{\mathrm{bd},e^{-\alpha|T-M|}}$:

$$\sup_{T\in\mathbb{R}} \left\{ e^{-\alpha|T-M|} \|v\|_{W^2(T)} \right\} \le C_1 \sup_{T\in\mathbb{R}} \left\{ e^{-\alpha|T-M|} \|h_v\|_{W^0(T)} \right\} \tag{3.10}$$

where the constant C_1 is independent of M. So it remains to estimate only the right–hand side of (3.10). To this end we observe that the assumption (0.5)(a) together with the boundedness of $\mathcal{K}$ in $W^2_{\mathrm{bd}}(\mathbb{R})$ implies that

$$\|L_1(t)\|_{H^{3/2}\to H} + \|L_2(t)\|_{H^{1/2}\to H} \le R_* \tag{3.11}$$

where R_* is independent of t and of $u_1, u_2 \in \mathcal{K}$. Thus, according to (3.11) and to interpolation inequalities (see [Tri78]) we obtain that

$$\|h_v(t)\|_{W^0(T)} \le C_2 R_* \|v\|_{W^{3/2}(T)} \le \mu\|v\|_{W^2(T)} + C_\mu \|v\|_{W^0(T)} \tag{3.12}$$

where $\mu > 0$ is arbitrary and $C_\mu > 0$ is a positive constant depending on μ but is independent of T and $u_1, u_2 \in \mathcal{K}$. Fixing $\mu = 1/(2C_1)$ in (3.12) and inserting it to (3.10) we finally derive that

$$\sup_{T\in\mathbb{R}} \left\{ e^{-\alpha|T-M|} \|v\|_{W^2(T)} \right\} \le 2C_1 C_\mu \sup_{T\in\mathbb{R}} \left\{ e^{-\alpha|T-M|} \|v\|_{W^0(T)} \right\} \tag{3.13}$$

The evident inequality

$$\|v\|_{W^2(M)} \le e^{-\alpha} \sup_{T\in\mathbb{R}} \left\{ e^{-\alpha|T-M|}\|v\|_{W^2(T)} \right\}$$

completes the proof of Proposition 3.7.

Now consider the family of weight functions

$$\phi_R(t) = \begin{cases} 1 & \text{for } |t| < R, \\ e^{R-|t|} & \text{for } |t| \ge R, \end{cases} \tag{3.14}$$

where $R \in \mathbb{R}^+$. These functions satisfy (3.6) with $\mu = 1$ and $C_\phi = 1$ which are both independent of R.

Corollary 3.8. *Let $\mathcal{K} \subset W^2_{\text{bd}}(\mathbb{R})$ be defined as above. Then there exists a constant $C_{\mathcal{K}} > 0$ such that for all $u_1, u_2 \in K$ and all $R > 0$ the following estimate is valid:*

$$\|u_1-u_2\|_{W^2_{\text{bd},\phi_R}(\mathbb{R})} \le C_{\mathcal{K}}\|u_1-u_2\|_{L^2_{\text{bd},\phi_R}(\mathbb{R})} \tag{3.15}$$

Indeed, taking $\alpha = 2$ in (3.8), multiplying it by $\phi_R(T)$ and applying $\sup_{T\in\mathbb{R}}$ to the both sides of the obtained inequality we obtain after simple transformations the estimate (3.15) (see [Zel99a]). Moreover, since (3.6) holds for ϕ_R uniformly with respect to $R > 0$ then the constant $C_{\mathcal{K}}$ in (3.15) is also independent of R.

Note that the weight functions ϕ_R are chosen such that

$$\|v\big|_{(0,R)}\|_{W^2_{\text{bd}}((0,R))} \le \|v\|_{W^2_{\text{bd},\phi_R}(\mathbb{R})}$$

for all $v \in W^2_{\text{bd}}(\mathbb{R})$; hence

$$\mathbf{H}_\varepsilon(\mathcal{A}^{\text{traj}}\big|_{(0,R)}, W^2_{\text{bd}}((0,R))) \le \mathbf{H}_\varepsilon(K, W^2_{\text{bd},\phi_R}(\mathbb{R})). \tag{3.16}$$

So, following [Zel99a], we will estimate the entropy of the set $\mathcal{K}$ of all bounded solutions $u \in W^2_{\text{bd}}(\mathbb{R})$ for equation (0.4) in the weighted space $W^2_{\text{bd},\phi_R}(\mathbb{R})$ instead of the estimating the entropy of restriction $\mathcal{A}\big|_{(0,R)}$. To this end we need the following lemma. It makes essential use of the estimate (3.15) and the fact that the embedding of $W^2_{\text{bd}}(\mathbb{R})$ into $L^2_{\text{bd},\phi_R}(\mathbb{R})$ is compact.

Lemma 3.9. *The entropy of $\mathcal{K}$ satisfies the recurrence formula*

$$\mathbf{H}_{\varepsilon/2}(\mathcal{K}, W^2_{\text{bd},\phi_R}(\mathbb{R})) \le L\big[R+1+\ln_+ \frac{R_0'}{\varepsilon}\big] + \mathbf{H}_\varepsilon(\mathcal{K}, W^2_{\text{bd},\phi_R}(\mathbb{R})), \tag{3.17}$$

where the constants L and R_0' in (3.17) are independent of $R > 0$ and $\varepsilon > 0$.

Proof. Let $\{B(u^i, \varepsilon, W^2_{\text{bd},\phi_R}) : i = 1, \dots, N_\varepsilon\}$ be an ε–covering of $\mathcal{K}$ (here and below we denote by $B(v, \mu, X)$ the ball of radius μ with center v in the metric space X). Note that $\mathcal{K} \cap B(u^i, C\varepsilon, W^2_{\text{bd},\phi_R}(\mathbb{R}))$ is compact in $L^2_{\text{bd},\phi_R}(\mathbb{R})$, consequently every such set can be covered by a finite number of $\varepsilon/(2C_{\mathcal{K}})$–balls $\{B(u^{i,j},\varepsilon/(2C_{\mathcal{K}}),L^2_{\mathrm{bd},\phi_R}(\mathbb{R})),\ j=1,\cdots,\mathcal{M}_i(\varepsilon)\}$, where $C_{\mathcal{K}}$ is the same as in (3.15) and

$$\mathcal{M}_i(\varepsilon):=N_{\varepsilon/(2C_{\mathcal{K}})}(\mathcal{K}\cap B(u^i,\varepsilon,W^2_{\mathrm{bd},\phi_R}(\mathbb{R})),L^2_{\mathrm{bd},\phi_R}(\mathbb{R}))). \tag{3.18}$$

Then, due to (3.15), $\{B(u^{i,j},\varepsilon/2,W^2_{\mathrm{bd},\phi_R}(\mathbb{R}))\ :\ i=1,\dots,N_\varepsilon,\ j=1,\dots,\mathcal{M}_i(\varepsilon)\}$ is a $\varepsilon/2$–covering of $\mathcal{K}$. This leads to the recurrence formula

$$\mathbf{H}_{\varepsilon/2}(\mathcal{K},W^2_{\mathrm{bd},\phi_R}(\mathbb{R}))\le\max_{i=1,\cdots,N_\varepsilon}\ln\mathcal{M}_i(\varepsilon)+\mathbf{H}_\varepsilon(\mathcal{K},W^2_{\mathrm{bd},\phi_R}(\mathbb{R})). \tag{3.19}$$

So it remains to estimate $\mathcal{M}_i(\varepsilon)$ defined in (3.18). To this end we note that, according to (1.3), $\|u\|_{W^2_{\mathrm{bd}}(\mathbb{R})}\le C_*$ for each $u\in\mathcal{K}$, consequently

$$\|u\|_{W^2_{\phi_R}(T)}\le\varepsilon/(4C_{\mathcal{K}})\text{ if }|T|\ge T_\varepsilon\equiv R+\ln_+\frac{4C_*C_{\mathcal{K}}}{\varepsilon}. \tag{3.20}$$

Thus, we can estimate

$$\begin{aligned}\mathcal{M}_i(\varepsilon)&\le N_{\varepsilon/(4C_{\mathcal{K}})}(\mathcal{K}\cap B(u^i,\varepsilon,W^2_{\mathrm{bd},\phi_R}(\mathbb{R})),L^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon)))\\&\le N_{\varepsilon/(4C_{\mathcal{K}})}(B(u^i,\varepsilon,W^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon))),L^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon)))\\&\le N_{1/(4C_{\mathcal{K}})}(B(0,1,W^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon))),L^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon))).\end{aligned} \tag{3.21}$$

In the first estimate we use the 'tale's' estimate (3.20). In the second one we just omitted '$\mathcal{K}\cap$' making $N_{\varepsilon/(4C_{\mathcal{K}})}(\dots)$ larger. In the third estimate we use the scaling and translation invariance of balls in Banach spaces.

Thus, it remains to estimate the entropy of the embedding operator

$$W^2_{\mathrm{bd},\phi_R}((-T,T))\subset L^2_{\mathrm{bd},\phi_R}((-T,T)). \tag{3.22}$$

To this end we introduce smooth analogs $\psi_R\in C^\infty(\mathbb{R})$ of the weight functions ϕ_R in such a way that

$$\max\{|\dot\psi_R(t)|,|\ddot\psi_R(t)|\}\le\psi_R(t)\text{ and }C'\phi_R(t)\le\psi_R(t)\le C''\phi_R(t)$$

where C' and C'' are independent of R. Then it is not difficult to verify that the map $\mathbb{F}_R\ :\ u\to\psi_R^{1/2}u$ realizes the linear isomorphism between the Banach pairs $\mathbb{B}(T)=(W^2_{\mathrm{bd}}((-T,T)),L^2_{\mathrm{bd}}((-T,T)))$ and $\mathbb{B}_R(T)=(W^2_{\mathrm{bd},\phi_R}((-T,T)),L^2_{\mathrm{bd},\phi_R}((-T,T)))$. Moreover,

$$\|\mathbb{F}_R\|_{\mathbb{B}(T)\to\mathbb{B}_R(T)}+\|\mathbb{F}_R^{-1}\|_{\mathbb{B}_R(T)\to\mathbb{B}(T)}\le C_2 \tag{3.23}$$

where C_2 is independent of T and R (see [Zel99a] for details). Consequently,

$$\begin{aligned}\ln\mathcal{M}_i(\varepsilon)&\le\mathbf{H}_{1/(4C_{\mathcal{K}})}(B(0,1,W^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon))),L^2_{\mathrm{bd},\phi_R}((-T_\varepsilon,T_\varepsilon)))\\&\le\mathbf{H}_{1/(4C_{\mathcal{K}}C_2^2)}(B(0,1,W^2_{\mathrm{bd}}((-T_\varepsilon,T_\varepsilon))),L^2_{\mathrm{bd}}((-T_\varepsilon,T_\varepsilon))).\end{aligned} \tag{3.24}$$

The evident assertion

$$\mathbf{H}_\mu(B(0,1,W^2_{\mathrm{bd}}((-T,T))),L^2_{\mathrm{bd}}((-T,T)))\le \\ \le (2T{+}1)\mathbf{H}_{\mu/2}(B(0,1,W^2_{\mathrm{bd}}((-1,1))),L^2_{\mathrm{bd}}((-1,1)))$$

completes now the proof of Lemma 3.9.

Final step of the proof of Theorem 3.6. Note that, due to (1.3), $\mathbf{H}_{C_*}(\mathcal{K},W^2_{\mathrm{bd},\phi_R}(\mathbb{R}))=0$ for every $R\ge 0$. Iterating the estimate (3.17) we deduce that

$$\mathbf{H}_{2^{-k}C_*}(\mathcal{K},W^2_{\mathrm{bd},\phi_R}(\mathbb{R}))\le L(R{+}1{+}\ln_+\frac{R_0'2^{k-1}}{C^*})k \quad \text{for all } k\in\mathbb{N}. \tag{3.25}$$

For given $\varepsilon>0$ we choose k such that $2^{-k}C_*\le\varepsilon<2^{-k+1}C_*$ and thus the estimate (3.7) is obtained. Theorem 3.6 is proved.

We conclude this Section applying Theorem 3.6 to study the elliptic boundary problem (0.6) in a cylindrical domain $\Omega_+=\mathbb{R}^+\times\omega$, where $\omega\subset\subset\mathbb{R}^n$ is a smooth bounded domain. For simplicity we formulate the assumptions for the nonlinear function $f(u,\dot u)$ only for the case $n\le 3$. These assumptions are

$$\begin{cases} 1.\ f\in C^1(\mathbb{R}^k\times\mathbb{R}^k,\mathbb{R}^k),\\ 2.\ f(u,v)\cdot u\ge -C;\quad \mathrm{D}_u f(u,v)\ge -C,\\ 3.\ |f(u,v)|+|\,\mathrm{D}_u f(u,v)|\le C(1+|u|^{k_1})(1+|v|^{k_2}),\\ 4.\ |\,\mathrm{D}_v f(u,v)|\le C(1+|u|^{k_1}),\end{cases} \tag{3.26}$$

where $0\le k_2<1$ and $0\le k_1<\frac{n+3}{n-1}(1-k_2)$. Note that for the case where the nonlinear term is independent of $\partial_t u$ (and consequently $k_2=0$) we obtain the growth conditions on $f(u)$ introduced in [Bab95a].

Corollary 3.10. *Let $\gamma=\gamma^*\in\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)$, $g\in L^2(\omega)$, $n\le 3$, and the nonlinearity f satisfy* (3.26)*. Then the equation* (0.6) *satisfies all assumptions of Theorem 3.6 and consequently possesses a trajectory attractor $\mathcal{A}^{\mathrm{traj}}$ the entropy of which can be estimated by* (3.7)*.*

Proof. Let us rewrite the problem (0.6) in an abstract form (0.4). Indeed, in this case $H=L^2(\omega)$, $A=-\Delta_x$ (with Dirichlet boundary conditions) and $F(u,v):=f(u,v)+g$. So it remains to verify that the operator $F(u,v)$ thus defined satisfies the conditions (0.5). To this end we recall that the spaces $H^s:=D((-\Delta_x)^{-s/2})$ satisfy for $s\ge 0$ the embedding $H^s\subset H^s(\omega)$, where $H^s(\omega)$ is a classical Sobolev space (see e.g. [Tri78]). Note also that since $n\le 3$ we have the embedding $H^{3/2}\subset L^p$ for every $p<\infty$ and consequently for every fixed $p<\infty$ there is $\delta=\delta(p)>0$ such that $H^{3/2-\delta}\subset L^p$.

Let us verify now the condition (a) of (0.5). Indeed, according to (3.26)(3) and due to Hölder's inequality we have

$$\|\,\mathrm{D}_u F(u,v)\theta\|^2_{L^2}\le C\|(1+|u|^{2k_1})(1+|v|^{2k_2})|\theta|^2\|_{L^1}\le \\ \le (1+\|v\|^{2k_2}_{L^2})(1+\|u\|^{2k_1}_{L^{4k_1/(1-k_2)}})\|\theta\|^2_{L^{4/(1-k_2)}}\le Q(\|u\|_{H^{3/2-\delta}}+\|v\|_{L^2})\|\theta\|^2_{H^{3/2-\delta}}$$

for a sufficiently small $\delta > 0$. The condition

$$\| \mathrm{D}_v F(u,v)\theta\| \le Q(\|u\|_{H^{3/2-\delta}} + \|v\|_{H^{1/2-\delta}})\|\theta\|_{H^{1/2-\delta}}$$

can be verified analogously. The continuity of F, $\mathrm{D}_u F$ and $\mathrm{D}_v F$ also can be deduced in a standard way using (3.26)(a). The assumption (0.5)(a) is verified. The assumptions (0.5)(b) and (0.5)(c) are immediate corollaries of (3.26)(2). So it remains only to derive (0.5)(d) from (3.26)(3). Indeed, due to Hölder inequality and Sobolev's embedding theorem, we have

$$\|F(u,v)\|^2_{L^2} \le C(1+\|v\|^2_{L^2}) + \|u\|^p_{L^p} \le C(1+\|v\|^2_{L^2}) + C_1\|u\|^p_{H^s(\omega)} \tag{3.27}$$

where $p := 2k_1/(1-k_2)$ and $\frac{1}{p} = \frac{1}{2} - \frac{s}{n}$. If $s \le 1/2$ then (3.27) implies (0.5)(d). Assume that $s \ge 1/2$. Then, according to the interpolation inequality

$$\|u\|_{H^s(\omega)} \le C_2\|u\|^{(2s-1)/3}_{H^2(\omega)}\|u\|^{(4-2s)/3}_{H^{1/2}(\omega)} \tag{3.28}$$

Note that (3.27) and (3.28) implies (0.5)(d) if $p(2s-1)/3 < 2$. Let us verify this inequality. Recall that due to our assumptions $p < 2(n+3)/(n-1)$, consequently $s < 2n/(n+3)$ and therefore $p(2s-1)/3 < 2(n+3)/3(n-1) \cdot 3(n-1)/(n+3) = 2$. Thus, the assumption (0.5)(d) is also verified. Corollary 3.10 is proved.

§4 The entropy of the attractor: an example with sharp lower bounds.

In this section we construct an example of equation (0.4) for which the estimate (3.7) is in a sense sharp. Particularly, the fractal dimension of $\mathbb{A}$ will be infinite for this example.

The example is based on a counterexample in the Floquet theory for abstract elliptic problems in cylinders which was considered in [Mi94a,DFKM96]. Consider the linear elliptic equation in the strip $\mathbb{R} \times \omega \equiv \mathbb{R} \times (0,\pi)$:

$$\ddot{u} + \partial_x^2 u = L_1(t)u + L_2(t)\dot{u}, \quad u\big|_{\mathbb{R}\times\partial\omega} = 0 \tag{4.1}$$

Here $L_1(t)$ and $L_2(t)$ are linear operators which depends T–periodically on t. The point of the counterexample is to construct L_1 and L_2 in such a way that equation (4.1) has a solution u, which decays faster than exponential for $t \to \pm\infty$. This clearly contradicts the applicability of the Floquet theory where solutions must be linear combinations of products of exponential functions with periodic functions. The following result is proved in Appendix A of [DFKM96], pp. 261–262.

Theorem 4.1. *There exist T–periodic operators $L_j(t)$ with*

$$L_1 \in C^\infty_{\mathrm{bd}}(\mathbb{R}, \mathcal{L}(H^{s+1}, H^s)) \text{ and } L_2 \in C^\infty_{\mathrm{bd}}(\mathbb{R}, \mathcal{L}(H^s, H^s)) \quad \text{for } s \in \mathbb{R} \tag{4.2}$$

with $L_j(t+T) = L_j(t)$ for all $t \in \mathbb{R}$ such that equation (4.1) possesses a solution $u \in W^2(\mathbb{R})$ which satisfies the estimate

$$C_1 e^{-\beta t^2} \le \|u(t)\|_{L^2(\omega)} \le C_2 e^{-t^2} \quad \text{for } t \in \mathbb{R}, \tag{4.3}$$

with positive constants β, C_1 *and* C_2*. Moreover, every Fourier coefficient* $u_n(t) = \int_0^\pi u(t,x)\sin(nx)\,dx$ *has finite support in* $\mathbb{R}$*. (We denote by* H^s *here the scale, generated by the Laplacian in* ω *with Dirichlet boundary conditions.)*

Note that together with u all the functions obtained by shifts with kT, $k \in \mathbb{Z}$, (i.e., $\mathcal{T}_{kT}u : t \mapsto u(t-kT)$) are linearly independent bounded solutions of (4.1) and consequently, the linear elliptic operator on $L^2(\mathbb{R}\times\omega)$ defined by (4.1) has an infinite dimensional kernel. We define the set

$$\mathbb{L} \equiv \left\{ v = \sum_{k\in\mathbb{Z}} a_k \mathcal{T}_{kT} u \;:\; a_k \in \mathbb{R},\ \sup_{k\in\mathbb{Z}} |a_k| < \infty \right\} \tag{4.4}$$

and obtain the following result.

Lemma 4.2. *Let* u *be the solution constructed in Theorem 4.1 and* $\mathbb{L}$ *as in* (4.4)*. For* $u_1 : t \mapsto \int_0^\pi u(t,x)\sin x\,dx$ *assume also that there exists* $r \in \mathbb{R}$ *and* $N \in (0,T)$ *such that* $\varnothing \neq \operatorname{supp} u_1 \subset [r, r+N]$ *. Then*

$$C_1 \sup_{k\in\mathbb{Z}} |a_k| \le \|v\|_{W^2_{\mathrm{bd}}(\mathbb{R})} \le C_2 \sup_{k\in\mathbb{Z}} |a_k| \tag{4.5}$$

holds for every $v \in \mathbb{L}$*.*

Proof. Indeed, the left inequality is follows from the fact that the sets $\operatorname{supp}\mathcal{T}_{kT}u_1$ does not intersect for a different k.

For a proof of the right inequality we note that from (4.3) and from the regularity theory we obtain the decay estimate $\|u\|_{W^2(\tau)} \le C_2 e^{-\tau^2/2}$ for the given solution u. For elements $v \in \mathbb{L}$ this implies that

$$\|v\|_{W^2(\tau)} \le \sum_{k\in\mathbb{Z}} |a_k| \|u\|_{W^2(\tau-kT)} \le \sum_{k\in\mathbb{Z}} |a_k| C_2 e^{-(\tau-kT)^2/2} \le C \sup_{k\in\mathbb{Z}} |a_k| \tag{4.7}$$

Thus, Lemma 4.2 is proved.

Note that taking if necessary lT instead of T we may assume without loss of generality that the assumption on $\operatorname{supp} u_1$ holds for the solution constructed in Theorem 4.1.

Lemma 4.3. *Let the assumptions of Lemma 4.2 hold and let*

$$\mathbb{L}_R = \{ v \in \mathbb{L} : \|v\|_{W^2_{\mathrm{bd}}(\mathbb{R})} \le R \}. \tag{4.8}$$

Then, there exist positive $C(R), M_0$ *and* ε_0 *such that for* $M \ge M_0$ *and* $\varepsilon \in (0, \varepsilon_0)$ *the lower estimate*

$$\mathbf{H}_\varepsilon(\mathbb{L}_R\big|_{(0,M)}, W^2_{\mathrm{bd}}((0,M))) \ge C(R) M \ln\frac{1}{\varepsilon} \tag{4.9}$$

holds.

Proof. Without loss of generality we assume $\operatorname{supp} u_1 \subset [0,T]$. Let $\mathbb{L}^k_R = \{ v \in \mathbb{L}_R : v(t) = \sum_{i=1}^k a_k u(t-iT)\}$, then for every $v^1, v^2 \in \mathbb{L}^k_R$ we have

$$\|v^1 - v^2\|_{W^2_b((0,(k+1)T))} \ge \|v^1 - v^2\|_{C([0,(k+1)T],H)} \ge K \sup_{i=1,\dots,k} |a^1_i - a^2_i| \tag{4.10}$$

with $K = \|u_1\|_{C([0,T])} > 0$.

For sufficiently large M chose $k = k_M$ in such a way that $(k+1)T > M \geq kT$. Then the upper bound in (4.5) says that a function $v \in \mathbb{L}$ belongs to $\mathbb{L}_R$ if all $|a_k| \leq R/C_2$. According to (4.10) two functions v_1 and v_2 from $\mathbb{L}_R^k$ are ε separated if $|a_i^1 - a_i^2| \geq \varepsilon/K$ for at least one $i \in \{0, \dots, k\}$. Consequently,

$$N_{\varepsilon/2}(\mathbb{L}_R^k, W^2_{\mathrm{bd}}([0,M])) \geq \left(2\left[\frac{KR}{C_2\varepsilon}\right] + 1\right)^k \tag{4.11}$$

and Lemma 4.3 is proved as $k \sim M$.

Now we are in position to construct the equation of the type (0.4) which satisfy the assumptions of Theorem 3.6 in such a way that its attractor $\mathcal{A}$ contains the set $\Pi_+\mathbb{L}_R$ for a sufficiently small R. Then, according to Lemma 4.3 we obtain the lower bounds for its ε–entropy.

Let the operators L_1 and L_2 are such as in Theorem 4.1. Since these operators are T periodic (for simplicity we assume below that $T = 2\pi$) then there exists smooth operator families $\widehat{L}_1 \in C^\infty(\mathbb{R}^2, \mathcal{L}(H^{s+1}, H^s))$ and $\widehat{L}_2 \in C^\infty(\mathbb{R}^2, \mathcal{L}(H^s, H^s))$ such that

$$\widehat{L}_1(w_1, w_2) = \widehat{L}_2(w_1, w_2) = 0 \text{ for } |w_1|^2 + |w_2|^2 \geq 2 \tag{4.12}$$

and

$$L_1(t) = \widehat{L}_1(\cos t, \sin t), \quad L_2(t) = \widehat{L}_2(\cos t, \sin t). \tag{4.13}$$

However, the pair $w(t) = (\cos t, \sin t)$ can be obtained as a solution of the following second order system of ordinary differential equations

$$\ddot{w} - w = \frac{2(|w|^2 w - 3w)}{1 + |w|^4} \tag{4.14}$$

Let $\phi_R(z) : \mathbb{R} \to \mathbb{R}$ be a cutoff function such that $\phi_R = 1$ if $|z| \leq R^2$ and $\phi_R = 0$ if $|z| \geq 2R^2$. Consider the following system:

$$\begin{cases} \ddot{w} - w = 2(|w|^2 w - 3w)/(1 + |w|^4), \\ \ddot{u} + \partial_x^2 u = \phi_R\big(\|u\|^2_{H^1(\omega)} + \|\dot{u}\|^2_{L^2(\omega)}\big)\big[\widehat{L}_1(w_1, w_2)u + \widehat{L}_2(w_1, w_2)\dot{u}\big]. \end{cases} \tag{4.15}$$

Then, on the one side, due to the embedding $W^2_{\mathrm{bd}}(\mathbb{R}) \subset C_{\mathrm{bd}}(\mathbb{R}, H^1) \cap C^1_{\mathrm{bd}}(\mathbb{R}, H)$, we have the inequality

$$\|u(t)\|^2_{H^1} + \|\dot{u}(t)\|^2_{L^2} \leq P^2 \|u\|^2_{W^2_{\mathrm{bd}}(\mathbb{R})} \tag{4.16}$$

for the appropriate positive constant P, and consequently the essential set $\mathcal{K}$ of this system contains $\{(\cos t, \sin t)\} \times \mathbb{L}_{R/P}$ as a large subset. On the other side this equation satisfies assumptions (0.5) and (0.7) with $H = \mathbb{R}^2 \times L^2(\omega)$, $A = \mathrm{diag}\{1, 1, -\partial_x^2\}$, $\gamma = 0$ and

$$F = \begin{pmatrix} 2(|w|^2 w - 3w)/(1 + |w|^4) \\ \phi_R\big(\|u\|^2_{H^1(\omega)} + \|\dot{u}\|^2_{L^2(\omega)}\big)\big[\widehat{L}_1(w_1, w_2)u + \widehat{L}_2(w_1, w_2)\dot{u}\big] \end{pmatrix}$$

Indeed, it follows from the definition of operators $\widehat{L}_i$, $i=1,2$, that

$$\|\widehat{L}_1(w_1,w_2)u\|_{L^2}+\|\widehat{L}_2(w_1,w_2)\dot u\|_{L^2}\le C\left(\|u\|_{H^1}+\|\dot u\|_{L^2}\right)$$

where the constant C is independent of w, and consequently $\|F(w,u,\dot u)\|_{L^2}\le C'_R$, where C'_R is independent of w, u and $\dot u$. Thus, the conditions (0.5)(c) and (0.5)(d) are verified. Analogously, using the smoothness of $\widehat{L}_i$ with respect to w and the fact that ϕ_R has a finite support we derive that

$$\|\operatorname{D}_w F(w,u,\dot u)\|_{\mathbb{R}^2\to H}+\|\operatorname{D}_u F(w,u,\dot u)\|_{H^1\to H}+\|\operatorname{D}_{\dot u} F(w,u,\dot u)\|_{H^0\to H}\le C''_R,$$

where C''_R is also independent of w, u and $\dot u$. Therefore, the condition (0.5)(a) and (0.7) are also fulfilled. It remains to note that the condition (0.5)(b) is also an immediate corollary of the last estimate.

Thus, we have verified that the nonlinearity $F(w,u,\dot u)$ satisfies assumptions (0.5) and (0.7) and consequently the assertions of Theorem 2.7 and 3.6 holds for the equation (4.15). Particularly, combining results of Theorem 3.6 and Lemma 4.3, we obtain the following two-sided estimate for the ε-entropy of its attractor.

Theorem 4.4. *The equation* (4.15) *possesses the trajectory attractor* $\mathcal{A}^{\mathrm{traj}}$ *and there exist positive constants* $T_0,\varepsilon_0,C_1,$ *and* C_2 *such that the* ε*–entropy of* $\mathcal{A}^{\mathrm{traj}}$ *satisfies*

$$C_1T\ln\frac{1}{\varepsilon}\le \mathbf{H}_\varepsilon(\mathcal{A}^{\mathrm{traj}}\big|_{(0,T)},W^2_{\mathrm{bd}}((0,T)))\le C_2(T+\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon}\tag{4.17}$$

for $T\ge T_0$ *and* $\varepsilon\in(0,\varepsilon_0)$.

Note that the left-hand side of estimate (4.17) is sharp only for $T\ge\ln\frac{1}{\varepsilon}$ and is far from optimal for $T\ll\ln\frac{1}{\varepsilon}$. Particularly, (4.17) gives no information on the entropy of a global attractor $\mathbb{A}$ on a cross section. The following theorem gives a lower bound for the entropy of the global attractors in the case when the trajectory attractor satisfies estimate (4.17).

Theorem 4.5. *Let the assumptions of Theorem 2.7 hold and let the trajectory attractor* $\mathcal{A}^{\mathrm{traj}}$ *of the problem* (0.4) *satisfy* (4.17)*. Then there exist positive constants* C *and* ε_0 *such that the entropy of the global attractor* $\mathbb{A}$ *constructed in Theorem 2.7 satisfies*

$$\mathbf{H}_\varepsilon(\mathbb{A},\mathbb{H}^{3/2})\ge C\left(\ln\frac{1}{\varepsilon}\right)^{3/2},\qquad \textit{for } \varepsilon\in(0,\varepsilon_0).\tag{4.18}$$

Proof. The proof of this result uses the logarithmic convexity estimates from Theorem 2.5. Since the solutions in $\mathcal{A}^{\mathrm{traj}}$ are bounded by $\|(u(t),\dot u(t))\|_{\mathbb{H}^1}\le B_*$, estimate (2.9) gives

$$\|(u_1-u_2,\dot u_1-\dot u_2)\|_{C([0,t],\mathbb{H}^1)}\\ \le\|(u_1(0)-u_2(0),\dot u_1(0)-\dot u_2(0))\|_{\mathbb{H}^1}^{1-t/T}B_*^{t/T}e^{2M(M+4/T)t(T-t)}$$

for any $t < T$. Choosing $T = 2t$ this implies Hölder continuity with exponent $1/2$. Assuming without loss of generality that $t, M \geq 1$ we find the estimate

$$\mathbf{H}_{\mu(\varepsilon)}(\mathcal{A}^{\mathrm{traj}}, C([0,t],\mathbb{H}^1)) \leq \mathbf{H}_\varepsilon(\mathbb{A},\mathbb{H}^1) \quad \text{with } \mu(\varepsilon) = \varepsilon^{1/2}B_*^{1/2}e^{20M^2t^2}.$$

After renaming t into T the lower estimate (4.17) applied to the left–hand side gives

$$\mathbf{H}_\varepsilon(\mathbb{A},\mathbb{H}^1) \geq C_1 T \ln \mu(\varepsilon) = C_1 T\Big(\frac{1}{2}\ln\frac{1}{\varepsilon} - \frac{1}{2}\ln B_* - 20M^2T^2\Big).$$

Maximizing the right–hand side with respect to T (or choosing $T = \frac{1}{10M}\left(\ln\frac{1}{\varepsilon}\right)^{1/2}$) we obtain the assertion of the theorem.

§5 Chaos in spatial dynamical systems

In this concluding section we give a more comprehensive interpretation of the results obtained above in the spirit of DS theory. To this end we need to recall some quantitative characteristics which measure the complexity of a DS. We start with the classical concept of topological entropy (see, e.g., [KaH95])

Definition 5.1. Let (M,d) be a compact metric space and $S_h : M \to M$, $h \in \mathbb{R}^+$, be a (continuous) semiflow on it. For every $R > 0$ define a new metric d_R on M via the expression

$$d_R(m_1,m_2) := \sup\nolimits_{h\leq R} d(S_h m_1, S_h m_2), \quad m_1,m_2 \in M, \tag{5.1}$$

Then, evidently, (M,d_R) is also a compact metric space. The topological entropy of the semiflow S_h on M is defined to be the following number:

$$h_{\mathrm{top}}(S_h, M) := \lim_{\varepsilon\to 0}\limsup_{R\to\infty}\frac{1}{R}\mathbf{H}_\varepsilon(M,d_R), \tag{5.2}$$

where $\mathbf{H}_\varepsilon(M,d_R)$ means the ε-entropy of the set M in the metric (5.1).

It is well known (see [KaH95]) that the topological entropy (5.2) is independent of the concrete choice of the initial metric d on M and depends only on the topology on M.

In order to apply the general Definition 5.1 to our trajectory DS (2.1) one should fix a metric on $\mathcal{A}^{\mathrm{traj}}$. To this end we need the following simple proposition.

Proposition 5.2. *Let the weight function $\phi \in C_{\mathrm{bd}}(\mathbb{R})$ satisfy $\lim_{|t|\to\infty}\phi(t) = 0$ and $\phi(t) > 0$. Then the topology, induced on $\mathcal{A}^{\mathrm{traj}}$ by the embedding $\mathcal{A}^{\mathrm{traj}} \subset W^2_{\mathrm{bd},\phi}(\mathbb{R}^+)$ coincides with the local topology of $W^2_{\mathrm{loc}}(\mathbb{R}_+)$ on $\mathcal{A}^{\mathrm{traj}}$.*

Indeed, the assertion of the proposition is an immediate corollary of the boundedness of $\mathcal{A}^{\mathrm{traj}}$ in $W^2_{\mathrm{bd}}(\mathbb{R}^+)$.

Fix now an arbitrary weight function ϕ, satisfying the assumptions of Proposition 5.2 and define a metric on $\mathcal{A}^{\mathrm{traj}}$ via

$$d_\phi(u_1,u_2) := \|u_1 - u_2\|_{W^2_{\mathrm{bd},\phi}(\mathbb{R}^+)}, \quad u_1,u_2 \in \mathcal{A}^{\mathrm{traj}}. \tag{5.3}$$

Then, due to Theorem 2.4 and Proposition 5.2, $(\mathcal{A}^{\mathrm{traj}}, d_\phi)$ is a compact metric space and we may define the topological entropy $h_{\mathrm{top}}(\mathcal{T}_h, \mathcal{A}^{\mathrm{traj}})$ of the trajectory dynamical system (2.1) via (5.2). The following proposition gives a more convenient formula for its computation.

Proposition 5.3. *The topological entropy $h_{\text{top}}(\mathcal{T}_h, \mathcal{A}^{\text{traj}})$ is independent of the choice of the weight function ϕ and can be computed by the following expression:*

$$h_{\text{top}}(\mathcal{T}_h, \mathcal{A}^{\text{traj}}) = \lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T}\mathbf{H}_\varepsilon\left(\mathcal{A}^{\text{traj}}\big|_{(0,T)}, W^2_{\text{bd}}(0,T)\right). \tag{5.4}$$

The proof of the formula (5.4) is more or less evident and given e.g. in [Zel00].

Note now, that in contrast to classical DS generated by ODEs or by the major part of natural evolution PDEs in bounded domains (see, e.g. [Tem88]), the topological entropoy may be infinite in our case. Particularly, it is so in the case of system (4.15).

Corollary 5.4. *The topological entropy of the DS $(T_h, \mathcal{A}^{\text{traj}})$ generated by equation* (4.15) *is infinite:*

$$h_{\text{top}}(\mathcal{T}_h, \mathcal{A}^{\text{traj}}) = \infty. \tag{5.5}$$

Indeed, (5.5) is an immediate corollary of (5.4) and (4.17).

Note also that DS with infinite topological entropy naturally arise in studying the spatial and temoral complexity of global attractors $\mathcal{A}^{\text{glob}}$ for evolution PDEs in *unbounded* domains, particularly for equations in the form (0.13) (see, e.g. [Zel00,Zel00a]). Thus, keeping in mind the embedding (0.14), it seems natural to apply the methods developed there to study our trajectory DS (2.1). We start by introducing one of the possible generalizations of topological entropy (see [LiW00], [Zel00]).

Definition 5.5. Let (M, d) be a compact metric space and $S_h : M \to M$ be a semiflow on it. Then the modified topological entropy of S_h is defined to be the following number:

$$\widehat{h}_{\text{top}}(S_h, (M,d)) := \limsup_{\varepsilon\to 0}\left(\ln\frac{1}{\varepsilon}\right)^{-1}\limsup_{R\to\infty}\frac{1}{R}\mathbf{H}_\varepsilon(M, d_R), \tag{5.6}$$

where the metric d_R is defined by (5.1). Moreover, following to [LiW00], introduce a mean topological dimension $\dim_{\text{top}}$ by the following expression:

$$\dim_{\text{top}}(S_h, M) := \inf_{\widehat{d}}\widehat{h}_{\text{top}}(S_h, (M, \widehat{d})), \tag{5.7}$$

where the infinum is taken over all metrics $\widehat{d}$ on M which generate the same topology as d on M.

In contrast to the topological entropy, the value (5.6) is not preserved under general homeomorphisms, but only under Lipschitz continuous ones (like a fractal dimension). This is the reason to introduce the value (5.7), which is a topological invariant of a DS, in complete analogy with classical topological entropy.

Thus, fixing as before an arbitrary weight function ϕ satisfying the assumptions of Proposition 5.2 and defining the metric d_ϕ on $\mathcal{A}^{\text{traj}}$ via (5.3) one may define the modified topological entropy $\widehat{h}_{\text{top}}(\mathcal{T}_h, (\mathcal{A}^{\text{traj}}, d_\phi))$ and the mean topological dimension $\dim_{\text{top}}(\mathcal{T}_h, \mathcal{A}^{\text{traj}})$ of the trajectory DS (2.1) by (5.6) and (5.7), respectively. (We emphasize that the infinum in (5.7) has to be taken not only over metrics in the form of (5.3) but over *all* metrics in $\mathcal{A}^{\text{traj}}$, which generate the local topology of W^2_{loc} on it.)

The following assertion is in complete analogy to Proposition 5.3.

Proposition 5.6. *The modified topological entropy* $\widehat{h}_{\mathrm{top}}(\mathcal{T}_h,(\mathcal{A}^{\mathrm{traj}},d_\phi))$ *is independent of the choice of the weight function* ϕ *and can be computed by the following expression:*

$$\widehat{h}_{\mathrm{top}}(\mathcal{T}_h,(\mathcal{A}^{\mathrm{traj}},d_\phi)) = \limsup_{\varepsilon\to 0}\left(\ln\frac{1}{\varepsilon}\right)^{-1}\limsup_{T\to\infty}\frac{1}{T}\mathbf{H}_\varepsilon\left(\mathcal{A}^{\mathrm{traj}}\big|_{(0,T)},W^2_{\mathrm{bd}}(0,T)\right). \tag{5.8}$$

The proof of (5.8) is analogous to the proof of (5.4) (see [Zel00]).

The value in the right-hand side of (5.8) has been interpreted in [CoE99] as a (fractal) dimension per unit volume of the corresponding attractor (compare with (3.4)). The finiteness and positiveness of this characteristic for the global attractor $\mathcal{A}^{\mathrm{glob}}$ of a large class of evolution PDEs in unbounded domains has been established in [CoE99,EfZ99,Zel99a,Zel00,Zel01]. The following corollary shows that this value is finite and may be strictly positive for the attractors of equations (0.4) as well.

Corollary 5.7. *Under the assumptions of Theorem 2.4 the modified topological entropy of the trajectory DS* (2.1) *is finite:*

$$\widehat{h}_{\mathrm{top}}(\mathcal{T}_h,\mathcal{A}^{\mathrm{traj}}) \le C < \infty. \tag{5.9}$$

Moreover, this value is strictly positive for the case of equation (4.15)*:*

$$0 < C_1 \le \widehat{h}_{\mathrm{top}}(\mathcal{T}_h,\mathcal{A}^{\mathrm{traj}}) \le C < \infty. \tag{5.10}$$

Indeed, the estimate (5.9) is an immediate corollary of (5.8) and (3.7) and the estimate (5.10) follows from (4.17).

Recall that in the classical theory of DS the chaotic behavior is usually demonstrated and explained constructing a homeomorphic embedding of the Bernoulli shift dynamics into the DS under consideration (see [KaH95] and references therein). Note, however, that classical Bernoulli shifts with a *finite* number of symbols have a *finite* topological entropy and consequently cannot be considered as an adequate model for the case of infinite topological entropy. In this case it seems natural to use Bernoulli shifts with *infinitely* many symbols.

Definition 5.8. Let $\mathcal{M} := [-1,1]^{\mathbb{Z}}$ be the compact topological space endowed with the Tikhonov topology. Recall that $\mathcal{M}$ consists of all functions $v:\mathbb{Z}\to[-1,1]$ and the topology can be generated e.g. by the following standard metric:

$$d(v_1,v_2) := \sum_{i=-\infty}^{\infty} 2^{-|i|}|v_1(i)-v_2(i)|. \tag{5.11}$$

Define the model DS $(\mathcal{T}_l,\mathcal{M})$ of shifts on $\mathcal{M}$ via

$$(\mathcal{T}_l v)(i) := v(i+l),\quad i,l\in\mathbb{Z},\quad v\in\mathcal{M}. \tag{5.12}$$

It is well–known that $h_{\mathrm{top}}(\mathcal{T}_l,\mathcal{M}) = \infty$ and $\widehat{h}_{\mathrm{top}}(\mathcal{T}_l,(\mathcal{M},d)) = \dim_{\mathrm{top}}(\mathcal{T}_l,\mathcal{M}) = 1$.

The following theorem constructs an embedding of the DS $(\mathcal{T}_l,\mathcal{M})$ into the trajectory DS associated with equation (4.15).

Theorem 5.9. *Let $\mathcal{K}$ be the set of all essential solutions of the equation* (4.15). *Then there is a homeomorphic embedding $\kappa : \mathcal{M} \to \mathcal{K}$, such that*

$$\mathcal{T}_{Tl}\kappa(v) = \kappa(\mathcal{T}_l v), \quad l \in \mathbb{Z},\ v \in \mathcal{M} \tag{5.13}$$

where $T > 0$ is the period introduced in Theorem 4.1 (and fixed to equal 2π in (4.13)*). Moreover, this embedding is Lipschitz continuous in the following sense:*

$$C_1 \sum_{i=-\infty}^{\infty} e^{-T|i|}|v_1(i) - v_2(i)| \le \tag{5.14}$$

$$\le \|\kappa(v_1) - \kappa(v_2)\|_{W^2_{\mathrm{bd},e^{-|t|}}(\mathbb{R})} \le C_2 \sum_{i=-\infty}^{\infty} e^{-T|i|}|v_1(i) - v_2(i)|$$

Proof. According to our construction of equation (4.15), the set $\mathbb{L}_R$, defined by (4.4) and (4.8), is contained in $\mathcal{K}$ for a some $R > 0$. Therefore, it is sufficient to construct an embedding $\kappa : \mathcal{M} \to \mathbb{L}_R$. We claim that such an embedding is given by the following formula:

$$\kappa(v)(t) := R \sum_{i=-\infty}^{\infty} v(i)u(t - iT), \quad v \in \mathcal{M} \tag{5.15}$$

where $u(t)$ is the solution defined in Theorem 4.1. Indeed, assertion (5.13) is an immediate corollary of definition (5.15). So it remains to verify the continuity (5.14). Analogously to (4.7) we derive that

$$\|\kappa(v)\|_{W^2(\tau)} \le C \sum_{i=\infty}^{\infty} |v(i)|e^{-(\tau - iT)^2/2} \text{ for } \tau \in \mathbb{R}.$$

The right–hand side estimate of (5.14) is an immediate corollary of this inequality. In order to obtain the left part of (5.14) we recall that due to Theorem 4.1, the first Fourier coefficient $u_1(t)$ of the function $u(t)$ has a finite support $\operatorname{supp} u_1 \subset [0, T]$. Consequently,

$$\|\kappa(v)\|_{W^2(\tau)} \ge \|\langle\kappa(v), e_1\rangle\|_{W^2_2(\tau,\tau+1)} = |v([\tau/T])| \cdot \|u_1\|_{W^2_2(0,1)} \ge C|v([\tau/T])|$$

The left part of (5.14) follows immediately from this inequality. Theorem 5.9 is proved.

Recall that equation (4.15) satisfies the assumptions of Theorem 2.7, consequently the trajectory DS (2.1) is topologically conjugated with the DS $(\mathbb{S}_h, \mathbb{K})$ defined via (2.17) on the cross–section. The following corollary reformulates the assertion of Theorem 5.9 in terms of this DS.

Corollary 5.10. *Let $\mathbb{A} = \Pi_0\mathcal{A}^{\mathrm{traj}}$ be the global attractor associated with equation* (4.15). *Then, there is a homeomorphic embedding $\widehat{\kappa} : \mathcal{M} \to \mathbb{A}$ such that*

$$\mathbb{S}_{Tl}\widehat{\kappa}(v) = \widehat{\kappa}(\mathcal{T}_l v) \text{ for all } l \in \mathbb{Z} \text{ and } v \in \mathcal{M}. \tag{5.16}$$

Moreover, this homeomorphism preserves the quantities (5.6) *and* (5.7), *i.e.*

$$\widehat{h}_{\mathrm{top}}(\mathbb{S}_h, \widehat{\kappa}(\mathcal{M})) = T^{-1}\widehat{h}_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}), \quad \dim_{\mathrm{top}}(\mathbb{S}_h, \widehat{\kappa}(\mathcal{M})) = T^{-1}\dim_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}) \tag{5.17}$$

Proof. Indeed, according to Corollary 2.6 the projection operator $\Pi_0 : \mathcal{K} \to \mathbb{A}$ realizes a homeomorphism, which is Hölder continuous with Hölder exponent α arbitrarily close to 1 under a certain choice of the weight function in $\mathcal{K}$ (see (2.21) and (2.22)). Define now the homeomorphism $\widehat{\kappa} := \Pi_0 \circ \kappa$. Then (5.16) is an immediate corollary of (5.13). Since mean topological dimension is a topological invariant then the second equality of (5.17) is evident. The factor T^{-1} appears due to time rescaling (the semigroup $(\mathcal{T}_l, \mathcal{M})$ is conjugated to $(\mathcal{T}_{Tl}, \mathcal{K})$ via κ). In order to verify the first equaliy of (5.17) we recall that κ is Lipschitz continuous in the sense (5.14). Consequently, using the invariance of the modified topological entropy under Lipschitz continuous transformations and the fact that it is independent of the concrete choice of the weight function ϕ satisfying Proposition 5.2 in the metric (5.3), we derive that

$$\widehat{h}_{\mathrm{top}}(\mathcal{T}_h, \kappa(\mathcal{M})) = T^{-1}\widehat{h}_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}) \tag{5.18}$$

Analogously, since Π_0 is a Hölder continuous homeomorphism with Hölder constant α arbitrarily close to 1, then $\widehat{h}_{\mathrm{top}}(\mathcal{T}_h, \kappa(\mathcal{M})) = \widehat{h}_{\mathrm{top}}(\mathbb{S}_h, \widehat{\kappa}(\mathcal{M}))$. Corollary 5.10 is proved.

Corollary 5.11. *The mean topological dimension of the DS associated with equation* (4.15) *is strictly positive:*

$$\dim_{\mathrm{top}}(\mathcal{T}_h, \mathcal{A}^{\mathrm{traj}}) = \dim_{\mathrm{top}}(\mathbb{S}_h, \mathbb{A}) \geq T^{-1} > 0 \tag{5.19}$$

Indeed, it is known (see [LiW00]) that $\dim_{\mathrm{top}}(\mathcal{T}_l, \mathcal{M}) = 1$. Estimate (5.19) now is an immediate corollary of (5.17).

The following corollary shows that every finite dimensional dynamics can be realized up to a homeomorphism by restricting $\mathbb{S}_h$ to an appropriate invariant subset of the attractor $\mathbb{A}$ of equation (4.15).

Corollary 5.12. *Let* $\mathbb{A}$ *be the global attractor of equation* (4.15) *and let* $K \subset \mathbb{R}^N$ *be a compact set and* $F : K \to K$ *be a homeomorphism. Then, there is a homeomorphic embedding* $\tau : K \to \mathbb{A}$ *such that*

$$\mathbb{S}_{NT} \circ \tau(k) = \tau(Fk) \text{ for all } k \in K. \tag{5.20}$$

Proof. Indeed, due to Corollary 5.10, it is sufficient to embed the DS (F, K) into $(\mathcal{T}_l, \mathcal{M})$. Moreover, without loss of generality we may assume that $K \subset [-1, 1]^N$. Then the desired embedding can be defined via

$$\tilde{\tau}(k)(i) := (F^{(m)}(k))_j, \quad i = mN + j, \quad m, j \in \mathbb{Z}, \quad 0 \leq j \leq N - 1 \tag{5.21}$$

where $F^{(m)}$ means the mth iteration of the map F and $(k)_j$ is a jth coordinate of the point $k \in [-1, 1]^N$. Then the map $\tau := \widehat{\kappa} \circ \tilde{\tau}$ evidently satisfies the assertions of the corollary.

Remark 5.13. Recall that the abstract elliptic equation (0.4) can be formally interpreted as a second order evolution equation. Moreover, this interpretation is partially justified under the assumptions of Theorem 2.7 by considering the continuous DS $(\mathbb{S}_h, \mathbb{K})$ associated with the equation (0.4). But in contrast to the case of natural evolution equations, the DS associated with elliptic equations *are not Lipschitz continuous* in general (one cannot take $\alpha = 0$ in (2.18)), but only Hölder continuous with Hölder constant arbitrarily close to 1 (as in the case of equation (4.15)). The absence of Lipschitz continuity allows such systems to have infinite dimensional attractors with infinite topological entropy. Namely, it can be proved in a standard way that if some equation, satisfying the assumptions of Theorem 2.7, possesses also Lipschitz continuity property in the form

$$\|\mathbb{S}_h(z_1)-\mathbb{S}_h(z_2)\|_{\mathbb{H}^1} \le Q_h(\|z_1\|_{\mathbb{H}^{3/2}}+\|z_2\|_{\mathbb{H}^{3/2}})\,\|z_1-z_2\|_{\mathbb{H}^1}, \quad z_i \in \mathbb{K}$$

then the associated global attractor has finite fractal dimension and consequently the topological entropy is also finite (as in the case of evoulution equations).

Remark 5.14. Note also that as in the case of classical DS, our embedding of Bernoulli shifts dynamics to the trajectory DS associated with equation (4.15) is based on finding appropriate homoclinic orbits. Indeed, the solution $t \mapsto (\sin t, \cos t, u(t))$ where $u(t)$ is the function constructed in Theorem 4.1 is a homoclinic orbit with respect to the 2π–periodic solution $t \mapsto (\sin t, \cos t, 0)$. But in contrast to usual constructions of multibump solutions, in our situation we may sum shifted versions of this homoclinic solution not only with coefficients from $\{0, 1\}$ but from the interval $[-1, 1]$ (see (5.15)). Moreover, Theorem 5.9 shows that the periodic orbit $(\sin t, \cos t, 0)$ is 'infinitely degenerate'. Indeed, there is a one–parameter family of other 2π–periodic orbits near it, which can be parametrized by constant functions $v_\varepsilon \in \mathcal{M}$, $\varepsilon \in [-1, 1]$ (i.e., $v_\varepsilon(i) \equiv \varepsilon$ for $i \in \mathbb{Z}$). Analogously, there is a two–parameter family of 4π–periodic solutions, a three–parameter family of 6π–periodic solutions and so on. The closure of this huge amount of periodic orbits gives us the embedding of the Bernoulli shifts $(\mathcal{T}_l, \mathcal{M})$ constructed in Theorem 5.9.

Remark 5.15. For a large class of reaction-diffusion systems of type (0.13) homeomorphic embeddings of the model DS $(\mathcal{T}_l, \mathcal{M})$ into the associated spatial DS $(\mathcal{T}_h, \mathcal{A}^{\mathrm{glob}})$ on global attractor $\mathcal{A}^{\mathrm{glob}}$ have been constructed in [Zel00,Zel00a]. According to the embedding constructed there the image of $\mathcal{M}$ is contained in the strongly unstable manifold (with respect to temporal direction) of a certain spatially homogeneuous equilibria point of the evolution equation under consideration. Theorem 5.9 shows that for some evolution equations in unbounded domains this type of spatial chaos can be found even inside of the set $\mathcal{K}$ of equilibria points, which is much smaller than $\mathcal{A}^{\mathrm{glob}}$.

References

[AgN67] S. Agmon, L. Nirenberg, *Lower bounds and uniqueness theorems for solutions of differential equations in a Hilbert space*, Comm. Pure Appl. Math. **20** (1967), 207–229.

[Bab95a] A.V. Babin, *Attractor of the generalized semigroup generated by an elliptic equation in a cylindrical domain*, Russian Acad. Sci. Izv. Math. **44** (1995), 207–223.

[Bab95b] A.V. Babin, *Inertial manifolds for traveling-wave solutions of reaction-diffusion systems*, Comm. Pure Appl. Math. **48** (1995), 167–198.

[BaV92] A.V. Babin, M.I. Vishik, *Attractors of evolutionary equations*, Nauka, Moscow, 1989; English transl. Stud. Math. Appl., 25, North Holland, Amsterdam, 1992.

[BrM96] T.J. Bridges, A. Mielke, *Instability of spatially–periodic states for a family of semilinear PDEs on an infinite strip*, Math. Nachr. **179** (1996), 5–25.

[ChV98] V.V. Chepyzhov, M.I. Vishik, *Kolmogorov's ε–entropy for the attractor of reaction–diffusion equation*, Math. Sbornik **189(2)** (1998), 81–110.

[ChV97] V.V. Chepyzhov, M.I. Vishik, *Evolution equations and their trajectory attractors*, J. Math. Pures Appl. **76** (1997), 913–964.

[CMS93] À. Calsina, X. Mora, J. Solà–Morales, *The dynamical approach to elliptic problems in cylindrical domains, and a study of their parabolic singular limit*, J. Diff. Eqns. **102** (1993), 244–304.

[CoE99] P. Collet, J.–P. Eckmann, *Extensive properties of the complex Ginzburg–Landay equation.*, Commun. Math. Phys. **200** (1999), 699–722.

[CoE00] P. Collet, J.–P. Eckmann, *Topological entropy and ε–entropy for damped hyperbolic equations.*, Ann. Henri Poincaré **1** (2000), 715–752.

[CSV97] À. Calsina, J. Solà–Morales, M. València, *Bounded solutions of some nonlinear elliptic equations in cylindrical domains*, J. Dynam. Diff. Eqns. **9** (1997), 343–372.

[DFKM96] G. Dangelmayr, B. Fiedler, K. Kirchgässner, A. Mielke, *Dynamics of nonlinear waves in dissipative systems: reduction, bifurcation and stability. Pitman Research Notes in Mathematics Series, 352.*, Longman, Harlow, 1996.

[EfZ99] M.A. Efendiev, S.V. Zelik, *Upper and Lower Bounds for the Kolmogorov Entropy of the Attractor for an RDE in an Unbounded domain.*, DANSE, Preprint 42/99 (1999).

[EfZ01] M.A. Efendiev, S.V. Zelik, *The attractor for a nonlinear reaction-diffusion system in an unbounded domain*, Comm. Pure Appl. Math. **54** (2001), no. 6, 625–688.

[FSV98] B. Fiedler, A. Scheel, M.I. Vishik, *Large patterns of elliptic systems in infinite cylinders*, J. Math. Pures Appl. **77 (9)** (1998), 879–907.

[GrT97] M.D. Groves, J.F. Toland, *On variational formulations for steady water waves*, Arch. Rational Mech. Anal. **137** (1997), 203–226..

[IoK92] G. Iooss, K. Kirchgässner, *Water waves for small surface tension: an approach via normal form*, Proc. Roy. Soc. Edinburgh Sect. A **122** (1992), 267–299.

[IoM91] G. Iooss, A. Mielke, *Bifurcating time-periodic solutions of Navier-Stokes equations in infinite cylinders.*, J. Nonlinear Sci. **1** (1991), 107–146.

[KaH95] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

[Kir82] K. Kirchgässner, *Wave–solutions of reversible systems and applications*, J. Diff. Eqns. **45** (1982), 113–127.

[KoT93] A.N. Kolmogorov, V.M. Tikhomirov, ε–entropy and ε–capacity of sets in functional spaces, In: Selected works of A.N. Kolmogorov **Vol III**, ed., Dordrecht: Kluver (1993).

[LiW00] E. Lindenstraus, B. Weiss, *Mean topological dimension*, Israel J. Math **115** (2000), 1–24.

[Mie88] A. Mielke, *Reduction of quasilinear elliptic equations in cylindrical domains with applications*, Math. Methods Appl. Sci. **10** (1988), 51–66.

[Mie90] A. Mielke, *Normal hyperbolicity of center manifolds and Saint–Venant's principle*, Arch. Rational Mech. Anal. **110** (1990), 353–372.

[Mie91] A. Mielke, *Hamiltonian and Lagrangian flows on center manifolds. With applications to elliptic variational problems. Lecture Notes in Mathematics Vol.* **1489**, Springer–Verlag, Berlin, 1991.

[Mie92] A. Mielke, *On nonlinear problems of mixed type: a qualitative theory using infinite–dimensional center manifolds*, J. Dynamics Diff. Eqns. **4** (1992), 419–443.

[Mi94a] A. Mielke, *Floquet theory for, and bifurcations from spatially periodic patterns*, Tatra Mt. Math. Publ. **4** (1994), 153–158.

[Mi94b] A. Mielke, *Essential manifolds for an elliptic problem in an infinite strip*, J. Diff. Eqns. **110** (1994), 322–355.

[Mi97] A. Mielke, *The complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors*, Nonlinearity **10** (1997), 199–222.

[Mi00] A. Mielke, *Exponentially weighted* L^∞*–estimates and attractors for parabolic systems on unbounded domains*, In "International Conference on Differential Equations (EQUADIFF 99) Berlin 1999, B. Fiedler, K. Gröger, J. Sprekels (eds), World Scientific 2000", 641–646.

[MiS95] A. Mielke, G. Schneider, *Attractors for modulation equations on unbounded domains — existence and comparison*, Nonlinearity **8** (1995), 743–768.

[PSS97] D. Peterhof, B. Sandstede, A. Scheel, *Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders*, J. Diff. Eqns **140** (1997), 266–308.

[SVWZ99] B.–W. Schulze, M.I. Vishik, I. Witt, S.V. Zelik, *The trajectory attractor for a nonlinear elliptic system in a cylindrical domain with piecewise smooth boundary*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **23** (1999), 125–166.

[Tem88] R. Temam, *Infinite–dimensional dynamical systems in mechanics and physics*, Springer–Verlag, New-York, 1988.

[Tri78] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam-New York, 1978.

[ViZ96] M.I. Vishik, S.V. Zelik, *The trajectory attractor for a nonlinear elliptic system in a cylindrical domain*, Russian Acad. Sci. Sb. Math. **187** (1996), 1755–1789.

[ViZ99] M.I. Vishik, S.V. Zelik, *The regular attractor for a nonlinear elliptic system in a cylindrical domain*, Math. Sbornic **190 (6)** (1999), 23–58 (english 803–834).

[Zel99] S.V. Zelik, *The attractor for an nonlinear reaction–diffusion system in* $\mathbb{R}^n$ *and the estimation of its* ε*–entropy*, Math. Notes **65 (6)** (1999), 941–943.

[Zel99a] S.V. Zelik, *The attractor for an nonlinear reaction–diffusion system in an unbounded domain and Kolmogorov's* ε*–entropy*, DANSE, Preprint 34/99 (1999), to appear in Mathem. Nachr..

[Zel00] S.V. Zelik, *The attractors of reaction–diffusion systems in unbounded domains and their spatial complexity*, DANSE Preprint 32/00, (2000).

[Zel00a] S.V. Zelik, *Spatial and dynamical chaos generated by reaction diffusion systems in unbounded domains*, DANSE Preprint 38/00, (2000).

[Zel01] S.V. Zelik, *The attractor for a nonlinear hyperbolic equation in an unbounded domain*, Disc. Cont. Dyn. Sys. Ser. A **7** (2001), no. 3, 593–641.