Exercise Sheet 3

Exercise 7: Nonuniqueness of curves of maximal slope. Consider the classical metric gradient system $(\mathbb{R}^2, \mathcal{F}, \mathcal{D})$ with

$$\mathcal{F}(u) = u_1 + u_2 \quad \text{and} \quad \mathcal{D}(u,w) = |u{-}w|_1 = |u_1{-}w_1| + |u_2{-}w_2|$$

(a) Calculate $|\partial\mathcal{F}|_{\mathcal{D}}$.

(b) Show that the metric speed is given by $|\dot{u}|_{\mathcal{D}}(t) = |\dot{u}(t)|_1$.

(c) Show that a curve is characterized by the conditions

$$\dot{u}_1(t) + \dot{u}_2(t) = -1 \quad \text{and} \quad |\dot{u}(t)|_1 = 1 \qquad \text{for a.a. } t \in [0,T].$$

(d) Characterized all curves of maximal slope starting at a point $u^0 \in \mathbb{R}^2$ and conclude that uncountably many solutions exist.

Exercise 8: ψ-curves of maximal slope. Consider the generalized metric gradient system $(M, \mathcal{F}, \mathcal{D}, \psi)$ with $M = \mathbb{R}^d$, $\mathcal{F}(u) = \frac{1}{2}|u|^2_{\mathrm{Eucl}}$, and $\mathcal{D}(u,w) = |u{-}w|^\theta$ for a $\theta \in \,]0,1[$.

(a) Characterize all absolutely continuous curves.

(b) Determine $|\partial\mathcal{F}|_{\mathcal{D}}$.

(c) Discuss the applicability of our existence result and describe set of all curves of maximal slope.

Exercise 9: Semiglobal slopes for semiconvex functionals. Consider a geodesic metric space $(M, \mathcal{D})$ and a geodesically λ-convex functional $\mathcal{F} : M \to \mathbb{R}_\infty$.

(a) Show that $|\partial\mathcal{F}|_{\mathcal{D}} = |\partial^{\mathrm{gl}}_\lambda \mathcal{F}|_{\mathcal{D}}$.

(b) Consider $\mathbb{S}^d := \big\{\, u \in \mathbb{R}^{d+1} \,\big|\, |u|_{\mathrm{Eucl}} = 1 \,\big\}$. Show that the arclength distance $\mathcal{D}$ makes $(\mathbb{S}^d, \mathcal{D})$ into a geodesic space.

(c) For the example in (b) fix $w \in \mathbb{S}^d$ and check whether $u \mapsto \mathcal{F}_p(u) = \frac{1}{p}\mathcal{D}(w,u)^p$ is geodesically semiconvex for $p \in [1,\infty]$.

(d) For the example in (b) with $d = 1$ give a function $\mathcal{F}$ that is geodesically 1-convex.

Exercise 10: Metric versus geodesic spaces. In $(M, \mathcal{D})$ set $\mathcal{F}_w : M \to \mathcal{D}(w,u)$.

(a) Show $|\partial\mathcal{F}_w|_{\mathcal{D}}(u) \leq 1$ and provide an example for $(M, \mathcal{D})$ where $|\partial\mathcal{F}_w|_{\mathcal{D}}(u) < 1$ for all $u, w \in M$.

(b) For the case that (M, D) is a geodesic space, show that $|\partial\mathcal{F}_w|_{\mathcal{D}}(u)$ is either 0 or 1.

(c) Assume further that $\overline{B}_R(w) = \big\{\, u \in M \,\big|\, \mathcal{D}(w,u) \leq R \,\big\}$ is compact for all $R > 0$ and $w \in M$. Show that $(M, \mathcal{D})$ is a geodesic space if and only if $|\partial\mathcal{F}_w|_{\mathcal{D}}(u) = 1$ for all $u \neq w$. (*Hint: Use an existence theorem to show that* $\mathrm{Geod}(w{\to}u)$ *is nonempty.*)