

AN APPROACH TO NONLINEAR VISCOELASTICITY VIA METRIC GRADIENT FLOWS*

ALEXANDER MIELKE[†], CHRISTOPH ORTNER[‡], AND YASEMIN ŞENGÜL[§]

Abstract. We formulate quasistatic nonlinear finite-strain viscoelasticity of rate type as a gradient system. Our focus is on nonlinear dissipation functionals and distances that are related to metrics on weak diffeomorphisms and that ensure time-dependent frame indifference of the viscoelastic stress. In the multidimensional case we discuss which dissipation distances allow for the solution of the time-incremental problem. Because of the missing compactness the limit of vanishing timesteps can be obtained only by proving some kind of strong convergence. We show that this is possible in the one-dimensional case by using a suitably generalized convexity in the sense of geodesic convexity of gradient flows. For a general class of distances we derive discrete evolutionary variational inequalities and are able to pass to the time-continuous limit in a specific case.

Key words. nonlinear viscoelasticity, gradient flow, dissipative distance, generalized geodesics

AMS subject classifications. 74D10, 35A15, 35Q74, 37L05, 53C22

DOI. 10.1137/130927632

1. Introduction. The equation of quasistatic nonlinear viscoelasticity of strain-rate type can be written as

$$(1.1) \quad \operatorname{div} (D_F W(x, \nabla u) + S(x, \nabla u, \nabla \dot{u})) = f(x),$$

where $u(t, x)$ is the deformation of the body, $F = \nabla u(t, x)$ is the deformation gradient, W is the stored-energy density, and f is a given external force (see Section 2.1 for a more detailed discussion). In (1.1), $D_F W(x, F)$ is the elastic part of the Piola–Kirchhoff stress tensor, whereas $S(x, F, \dot{F})$, depending linearly on the strain rate $\nabla \dot{u}(t, x)$, is its viscoelastic part. In addition to classical frame indifference of the stored-energy density, the main modeling postulates are that the viscous stress is derived from a dissipation potential and that the stress is time-dependently frame indifferent.

The latter conditions mean that S takes the form

$$(1.2) \quad S(x, F, \dot{F}) = FG(x, C, \dot{C}),$$

where $C = F^T F$ is the right Cauchy–Green tensor and G is a symmetric matrix-valued function (see [3] for a more general statement about the history of motion). Condition (1.2) is quite difficult to handle analytically due to the fact that it is not compatible

*Received by the editors July 3, 2013; accepted for publication (in revised form) January 14, 2014; published electronically March 20, 2014.

<http://www.siam.org/journals/sima/46-2/92763.html>

[†]Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin 10117, Germany (alexander.mielke@wias-berlin.de). This author was partially supported by DFG (FOR797 MicroPlast) and ERC (AdG 267802 AnaMultiScale).

[‡]Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom (c.ortner@warwick.ac.uk). This author was partially supported by the Leverhulme Trust through a Philip Leverhulme Prize.

[§]Department of Natural and Mathematical Sciences, Ozyegin University, Istanbul 34794, Turkey (yasemin.sengul@ozyegin.edu.tr). This author was supported by EPSRC grants EP/D048400/1 and EP/E035027/1, and TÜBİTAK fellowship 2213 during D.Phil. studies in the University of Oxford, UK.

with some common hypotheses (e.g., monotonicity with respect to the strain rate) on the stress (see [12], [29, Section 2.3]). The existence of a dissipation potential R means that there exists a real-valued function $R(x, F, \dot{F})$ with $R(x, F, \dot{F}) \geq R(x, F, 0) = 0$ such that $\partial_{\dot{F}} R(x, F, \dot{F}) = S(x, F, \dot{F})$. On the level of R the condition of time-dependent frame indifference means that R can be written as $R = \tilde{R}(x, C, \dot{C})$; see Section 2.1.

The fully dynamical equation of nonlinear viscoelasticity of rate type (which corresponds to (1.1) together with the inertia term) has been well-studied by various authors such as [12, 14, 31, 27, 28, 25, 26] for existence and long-time behavior of solutions. The only theory for the existence of solutions for this problem with frame indifferent $S(x, F, \dot{F})$ is that of Potier-Ferry [25, 26], who established global existence and uniqueness of solutions for initial data close to a smooth equilibrium for pure displacement boundary conditions. Demoulini [12], on the other hand, obtained Young measure solutions and also imposed the so-called uniform strict monotonicity condition on the strain rate to prove existence of classical weak solutions. As shown in [29] this condition is not compatible with frame indifference. Similarly, Tvedt [31] proved existence and uniqueness of weak solutions with mixed boundary conditions under a similar uniform strict monotonicity assumption. In the one-dimensional case, one should also mention [9, 24, 22, 18, 17] for the treatment of different types of stresses and [32], where the dependence on temperature is also taken into account. A different approach is adopted in [8], where (1.1) is analyzed in a one-dimensional setting with a specific viscoelastic stress and the existence of solutions is obtained, as well as some results on the asymptotic behavior of solutions to an equilibrium state.

In this work we start from the fact that, on the formal level, (1.1) can be understood as a gradient system, $0 = D_{\dot{u}} \mathcal{R}(u, \dot{u}) + D\phi(u)$, with respect to the energy functional

$$\phi(u) := \int_{\Omega} W(x, \nabla u(x)) - f(x) \cdot u(x) \, dx$$

and the dissipation functional

$$\mathcal{R}(u, \dot{u}) = \int_{\Omega} R(x, \nabla u(x), \nabla \dot{u}(x)) \, dx.$$

Guided by the modern theory of gradient flows in metric spaces (cf. [1]) we propose to consider dissipation metrics d of the form

$$d(u_1, u_2) = \left(\int_{\Omega} D(x, \nabla u_1(x), \nabla u_2(x))^2 \, dx \right)^{1/2},$$

where the dissipation density D^2 has to be connected to R via $\frac{1}{\varepsilon^2} D(x, F, F + \varepsilon \dot{F})^2 \rightarrow R(x, F, \dot{F})$ for $\varepsilon \searrow 0$.

Thus, it is natural to consider the incremental minimization problems

$$(1.3) \quad u_k \text{ minimizes } u \mapsto \frac{1}{2\tau} d(u, u_{k-1})^2 + \phi(u),$$

where $\tau > 0$ is a small timestep. In Sections 2.3 and 2.4 we discuss natural conditions on W and D that are physically admissible and allow for an existence theory for (1.3) and provide some examples. However, the main difficulty lies in the limit passage for $\tau \searrow 0$, since d and W are of the same order (namely, one space derivative), so

there is no direct compactness argument for passing to the limit. Thus, a natural approach is to look for strong convergence results that allow us to pass to the limit in the nonlinear terms directly. For this it will be essential that the two constitutive functions W and D work together nicely.

Presently, the multidimensional case seems out of reach. However, even in the one-dimensional case, to which we restrict ourselves starting from Section 3 (i.e., $\Omega = (0, 1)$), it is productive to follow the approach paved by the theory of metric gradient flows. We consider dissipation distances of the form

$$d_\xi(u, v) = \left(\int_0^1 (\xi(u'(x)) - \xi(v'(x)))^2 dx \right)^{1/2},$$

where ξ is a differentiable and strictly increasing function. The main idea of the paper is that the combined function $W_\xi(x) := W(\xi^{-1}(y))$ must have good properties, e.g., it has to be λ -convex.

In one dimension, there are two different cases, namely, the cases of Dirichlet and Neumann boundary conditions:

$$(\text{Dir}) \ u(0) = 0 \text{ and } u(1) = 1, \quad (\text{Neu}) \ u(0) = 0 \text{ and } W'(u'(1)) + S(u'(1), \dot{u}'(1)) = 0.$$

The Neumann case (Neu) is much simpler than the Dirichlet case (Dir), and it is given in Appendix A. The reader is advised to read that part first. There we see that the metric admits geodesic curves, while this is no longer the case for (Dir). Thus, we have to work with generalized geodesics that do not enjoy all the necessary properties that are needed to apply the abstract theory of metric gradient flows. Nevertheless, in Section 4 we obtain existence results for the time-incremental minimization problem (1.3) if ξ is given in the form $\xi(z) = z^\alpha$ and derive a suitably generalized discrete variational inequality (cf. Theorem 4.4).

In Section 5 we perform the limit $\tau \rightarrow 0$ by establishing strong convergence of the discrete solutions to a solution of the metric evolutionary variational inequality, where we closely follow the ideas in [1, Section 4]. Unfortunately, this step works only for the square-root distance $d_{\text{sq}}(u, v)^2 = \int_0^1 (\sqrt{u'} - \sqrt{v'})^2 dx$, which is also called the Hellinger distance in probability theory. Finally we show that ϕ has a strong upper gradient $|\partial\phi|$ and that all solutions of the evolutionary variational inequality are curves of maximal slope and finally that they are weak solutions of the one-dimensional version of the viscoelastic problem (1.1), namely,

$$(W'(u'(t, x)) + (\xi'(u'(t, x)))^2 \partial_t u'(t, x))' = 0.$$

In the (Dir) case most steps work for general distances d_ξ , except for the strong convergence where $d = d_{\text{sq}}$ or $d(u, v) = \|u - v\|_{L^2}$ is needed, i.e., $\xi(z) = z^\alpha$ with $\alpha = 1/2$ or $\alpha = 1$. In the (Neu) case the abstract theory of metric gradient systems works directly for a large class of ξ ; see Appendix A.

Finally we emphasize that our gradient-flow approach does not use any higher regularity of the solutions than the one induced by the functional ϕ and the metric d_ξ . In particular, we can allow for arbitrary measurable dependence of W and ξ in the material point $x \in (0, 1)$. It is just for notational convenience that we do not write this possible dependence explicitly.

2. Modeling of viscoelasticity as a formal gradient system. In this section we take a formal approach to modeling frame indifferent viscoelastic stress in general space dimension d by assuming that all vector fields and solutions are smooth.

A rigorous analysis will be performed in later sections and will be restricted to the one-dimensional case.

2.1. Energy functional and dissipation potential. We consider a bounded domain $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ with Lipschitz boundary. The deformation of the body is denoted by $u : \Omega \rightarrow \mathbb{R}^d$ and the deformation gradient by $F(x) = \nabla u(x) \in \mathbb{R}^{d \times d}$. The elastic energy in the body is given via a stored-energy density (cf. [4]) $W(x, F(x))$ such that

$$\phi(u) = \int_{\Omega} W(x, \nabla u(x)) dx - \langle \ell, u \rangle \quad \text{with} \quad \langle \ell, u \rangle := \int_{\Omega} f(x) \cdot u(x) dx,$$

where the volume force f satisfies $f \in L^\infty(\Omega; \mathbb{R}^d)$. With

$$\begin{aligned} \text{GL}_+(d) &= \{ F \in \mathbb{R}^{d \times d} \mid \det F > 0 \} \text{ and} \\ \text{SO}(d) &= \{ Q \in \mathbb{R}^{d \times d} \mid Q^T Q = I, \det Q = 1 \}, \end{aligned}$$

our constitutive assumptions on $W : \Omega \times \mathbb{R}^{d \times d} \rightarrow [0, \infty]$ are (suppressing x -dependence)

- (2.1a) $W \in C^0(\text{GL}_+(d)),$
- (2.1b) $W(F) = \infty$ for $F \in \mathbb{R}^{d \times d} \setminus \text{GL}_+(d),$
- (2.1c) $W(F) \rightarrow \infty$ for $|F| + 1/|\det F| \rightarrow \infty,$
- (2.1d) $W(QF) = W(F) \forall F \in \text{GL}_+(d)$ and $Q \in \text{SO}(d),$
- (2.1e) $W(I) = 0$ and $W(F) > 0$ for $F \notin \text{SO}(d),$
- (2.1f) $|\text{DW}(F)F^T| \leq CW(F) + C.$

Hence, the elastic part of the stress is given in terms of the first Piola–Kirchhoff tensor $T(x) = \text{D}_F W(x, \nabla u(x))$. Viscosity is related to strain rates $\nabla \dot{u}(t, x) = \nabla \frac{\partial}{\partial t} u(t, x)$ such that we now consider time-dependent deformations $u : [0, T] \times \Omega \rightarrow \mathbb{R}^d$. The viscous stress $S \in \mathbb{R}^{d \times d}$ also depends on the strain rate $\nabla \dot{u}(t, x)$ in the form $S(t, x) = \widehat{S}(x, \nabla u(t, x), \nabla \dot{u}(t, x))$. The equations of viscoelasticity then read

$$(2.2) \quad \text{div}(\text{D}_F W(x, \nabla u(t, x)) + \widehat{S}(x, \nabla u(t, x), \nabla \dot{u}(t, x))) = f(x) \quad \text{in } [0, T] \times \Omega,$$

where we have to add boundary conditions, which we will mostly impose as $u(t, x) = x$ for all $(t, x) \in [0, T] \times \partial\Omega$.

Frame indifference for the viscous stress tensor S can be formulated via \widehat{S} and leads to a time-dependent version of frame indifference (cf. [3] and the illuminating discussion in [2]):

$$(2.3) \quad \widehat{S}(x, F, \dot{F}) = F \widetilde{S}(F^T F, F^T \dot{F} + \dot{F}^T F) \quad \text{and} \quad \widetilde{S}(C, \dot{C}) = \widetilde{S}(C, \dot{C})^T,$$

where $C = F^T F$ is the Cauchy–Green tensor.

A potential $R = R(x, F, \dot{F}) \in \mathbb{R}$ is called a dissipation potential for the viscous stress tensor S if $S(x, F, \dot{F}) = \text{D}_{\dot{F}} R(x, F, \dot{F})$. If S depends linearly on \dot{F} , which we always assume in this work, then the existence of R follows from classical arguments in linear irreversible thermodynamics; see, e.g., [3, 23, 21]. The invariance properties (2.3) can be obtained from general dissipation potentials $R(x, F, \dot{F})$ if R satisfies the invariance

$$(2.4) \quad \forall x \in \Omega, F \in \text{GL}_+(d), \dot{F} \in \mathbb{R}^{d \times d}, Q \in \text{SO}(d), A \in \text{Skew}(d) : \\ R(x, QF, Q(\dot{F} + AF)) = R(x, F, \dot{F}),$$

where $\text{Skew}(d) := \{A \in \mathbb{R}^{d \times d} \mid A = -A^T\}$.

The invariance of R can also be written as $R(F, \dot{F}) = \tilde{R}(C, \dot{C})$, which gives $\tilde{S}(C, \dot{C}) = 2\partial_{\dot{C}}\tilde{R}(C, \dot{C})$ in (2.3). Typically R is given in the form $\tilde{R}(x, C, \dot{C}) = \frac{1}{2}\dot{C} : \mathbb{V}(C) : \dot{C}$, and the choice $\mathbb{V}(C) = \nu_1 I + \nu_2 C^{-1}$ leads to

$$S(F, \dot{F}) = 2\nu_1 F(F^\top \dot{F} + \dot{F}^\top F) + 2\nu_2(\dot{F} + F^{-\top} \dot{F}^\top F).$$

Defining the global dissipation potential

$$\mathcal{R}(u, \dot{u}) = \int_{\Omega} R(x, \nabla u(x), \nabla \dot{u}(x)) \, dx,$$

we can rewrite (2.2) as an abstract gradient flow in the form

$$(2.5) \quad 0 = D_{\dot{u}}\mathcal{R}(u(t), \dot{u}(t)) + D\phi(u).$$

Indeed, if we use the variational derivatives we have

$$\begin{aligned} D_{\dot{u}}\mathcal{R}(u, \tilde{u}) &= -\operatorname{div}(D_{\dot{F}}R(x, \nabla u(x), \nabla \tilde{u}(x))) \text{ and} \\ D\phi(u) &= -\operatorname{div}(D_F W(x, \nabla u(x))) - f(x). \end{aligned}$$

2.2. Dissipation distances and incremental minimization problems. To construct solutions to (2.5) defined in terms of the gradient systems (ϕ, \mathcal{R}) it is most efficient to use a time-discretization and define suitable incremental minimization. For this purpose it is useful to replace the dissipation potential R , which has the mathematical structure of a Riemannian metric $R(x, F, \dot{F}) = \frac{1}{2}\dot{F} : \mathbb{V}(x, F) : \dot{F}$, where the fourth-order viscosity tensor \mathbb{V} plays the role of a Riemannian tensor on $\operatorname{GL}_+(d)$ by a global distance $D(x, \cdot, \cdot) : (0, 1) \times \operatorname{GL}_+(d) \times \operatorname{GL}_+(d) \rightarrow [0, \infty]$, which is usually defined by

$$D(x, F_0, F_1)^2 = \inf \left\{ \int_0^1 R(x, F(s), \dot{F}(s)) \, ds \mid F \in \mathcal{C}(F_0, F_1) \right\},$$

where $\mathcal{C}(F_0, F_1) = \{F \in C^1([0, 1]; \operatorname{GL}_+(d)) \mid F(0) = F_0, F(1) = F_1\}$. For quadratic R defined in terms of \mathbb{V} as above, the standard theory of Riemannian manifolds shows that $D(x, \cdot, \cdot)$ defines a Riemannian pseudodistance, i.e., we have nonnegativity, symmetry, and the triangle inequality. We do not have positivity because of the invariance (2.4).

However, from the point of modeling it is much easier to postulate a metric D and calculate the associated R , namely,

$$(2.6) \quad R_D(x, F, \dot{F}) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon^2} D(x, F + \varepsilon \dot{F}, F)^2.$$

Recalling that $D(x, F, F) = 0$ and, typically, $D(x, F, F + \varepsilon \dot{F}) = O(\varepsilon|\dot{F}|)$ we see that $1/\varepsilon^2$ is the proper scaling. The following result shows that separate frame indifference of D implies the time-dependent frame indifference of R .

LEMMA 2.1. *If D satisfies the separate frame indifference*

$$(2.7) \quad \forall Q_1, Q_2 \in \operatorname{SO}(d) \forall F_1, F_2 \in \operatorname{GL}_+(d) : \quad D(Q_1 F_1, Q_2 F_2) = D(F_1, F_2),$$

then R_D satisfies the time-dependent frame indifference (2.4).

Proof. For $Q_1 \in \operatorname{SO}(d)$ and $A \in \operatorname{Skew}(d)$ we define $Q_2 = Q_1 \exp(\varepsilon A) \in \operatorname{SO}(d)$ and consider $D(F, F + \varepsilon \dot{F})^2 = D(Q_1 F, Q_1 \exp(\varepsilon A)(F + \varepsilon \dot{F}))^2$. Dividing by $2\varepsilon^2$ and

taking the limit $\varepsilon \rightarrow 0$ (using $\exp(\varepsilon A) = I + \varepsilon A + O(\varepsilon^2)$) we obtain the desired result for R_D . \square

The global dissipation distance \mathcal{D} between two deformations u_0 and u_1 is defined via

$$\mathcal{D}(u_0, u_1) = \left(\int_{\Omega} D(x, \nabla u_0(x), \nabla u_1(x))^2 dx \right)^{1/2}.$$

The abstract incremental problem for time step $\tau > 0$ is then given in the form

$$(2.8) \quad U_{\tau}^n = \operatorname{argmin}_u \frac{1}{2\tau} \mathcal{D}(U_{\tau}^{n-1}, u)^2 + \phi(u) \text{ for } n \in \mathbb{N}, \quad U_{\tau}^0 = u_0,$$

where U_{τ}^n is hopefully approximating $u(n\tau, \cdot)$ with u being a solution of the viscoelastic problem (2.2) with $u(0) = u_0$.

2.3. Examples of dissipation distances. We discuss possible choices of distances D on $\text{GL}_+(d)$ which satisfy as many of the relevant assumptions as possible. We first collect mathematically and physically desirable assumptions:

- (2.9a) $D(F, G) > 0$ if $F^T F \neq G^T G$,
- (2.9b) $D(F, G) = D(G, F)$,
- (2.9c) $D(F, H) \leq D(F, G) + D(G, H)$,
- (2.9d) D satisfies the separate frame indifference (2.7),
- (2.9e) $\forall F : G \mapsto D(F, G)^2$ is polyconvex,
- (2.9f) $D(F, G)^2 = \Psi(GF^{-1}) \det F$ with $\Psi(G) \geq 0$.

Here we dropped the dependence on $x \in \Omega$ for notational simplicity. Conditions (2.9a) to (2.9c) clearly state that D is a true distance when restricted to symmetric matrices in $\text{GL}_+(d)$. This is the best we can hope for, given the frame indifference (2.9d).

The polyconvexity condition (2.9e) is very useful to obtain the existence of solutions for the incremental minimization problem (2.8), where we may even allow for non-quasi-convex behavior in W if this is compensated by $\frac{1}{2\tau} D(F, \cdot)^2$, where $0 < \tau \ll 1$ is helpful; see [27, 28, 14] for a similar overcoming of nonconvexity in (non frame indifferent) viscoelasticity.

Finally, condition (2.9f) is a special condition that relates to the multiplicative character of diffeomorphisms. For D being independent of the material point $x \in \Omega$ satisfying this condition we obtain a global dissipation distance that is invariant under diffeomorphisms, namely,

$$(2.10) \quad \mathcal{D}(u_0 \circ v, u_1 \circ v) = \mathcal{D}(u_0, u_1)$$

for all diffeomorphisms $v : \Omega \rightarrow \Omega$. Indeed using the chain rule $\nabla(u_j \circ v) = \nabla u_j(v(x)) \nabla v(x)$ the integral transformation rule with $y = v(x)$ gives

$$\begin{aligned} &\mathcal{D}(u_0 \circ v, u_1 \circ v) \\ &= \int_{\Omega} \Psi(\nabla u_1(v(x)) \nabla v(x) (\nabla u_0(v(x)) \nabla v(x))^{-1}) \det(\nabla u_0(v(x)) \nabla v(x)) dx \\ &= \int_{\Omega} \Psi(\nabla u_1(y) \nabla u_0(y)^{-1}) \det \nabla u_0(y) dy = \mathcal{D}(u_0, u_1). \end{aligned}$$

In particular, we conclude that for diffeomorphisms u_0 and u_1 from Ω into itself, such \mathcal{D} satisfy $\mathcal{D}(u_0, u_1) = \mathcal{D}(\text{id}, u_1 \circ u_0^{-1})$.

We remark that if D satisfies (2.9f), then the symmetry (2.9b) is equivalent to the fact that Ψ satisfies the inversion relation

$$(2.11) \quad \Psi(F) = \det F \Psi(F^{-1}).$$

Moreover, the separate frame indifference (2.9d) is now equivalent to frame indifference and isotropy of Ψ , i.e., $\Psi(Q_1 F Q_2) = \Psi(F)$ for all $Q_1, Q_2 \in \text{SO}(d)$. We refer to [30, 20] and the references therein for characterizations of polyconvexity of isotropic functions.

Example 2.2. Additive distances in the one-dimensional case. In one space dimension the frame indifference condition (2.9d) is trivial. We obtain a distance by taking any strictly monotone function $\xi : (0, \infty) \rightarrow \mathbb{R}$ and let $D(F, G) = |\xi(F) - \xi(G)|$.

The polyconvexity condition (2.9e) reduces to convexity of $G \mapsto |\xi(G) - \eta|^2$ for all $\eta \in \text{im}(\xi)$. Considering the family $\xi(G) = G^\alpha$ this holds for $\alpha \in [1/2, 1]$.

Example 2.3. Multiplicative distances in the one-dimensional case. We start from the multiplicative ansatz (2.9f). For $\alpha + \beta = 1/2$ the function $\Psi(z) = z^\beta (z^\alpha - 1)^2$ satisfies the inversion symmetry (2.11) and hence D with

$$D(F, G) = (FG)^{\beta/2} |F^\alpha - G^\alpha|$$

satisfies all conditions in (2.9) except possibly the triangle inequality (2.9c). The latter holds for the case $\beta = 0$ and hence $\alpha = 1/2$, which is a special case of Example 2.2.

In Corollary 3.4 we will show that the validity of the triangle inequality implies that $\Psi(z)$ has upper and lower linear bounds for $z \rightarrow \infty$. Hence, the case

$$D(F, G) = |\sqrt{F} - \sqrt{G}|$$

is distinguished and, in fact, will play the central role in this work.

Example 2.4. Additive distances in higher dimensions. The simplest dissipation distance, leading to the easiest mathematical structures, is $D(F, G) = |F - G|$ and is used in [27, 28, 14], which obviously satisfies the distance properties (2.9a)–(2.9c) and the polyconvexity (2.9e), but not frame indifference (2.9d). To fulfill the latter, a natural choice is

$$D(F, G) = |\Xi(F^T F) - \Xi(G^T G)|,$$

where $\Xi : \mathbb{R}_{>0}^{d \times d} \rightarrow \mathbb{R}_{>0}^{d \times d}$ should be injective, e.g., $\Xi(C) = C$. However, it seems difficult to satisfy polyconvexity for such Ξ .

Example 2.5. Multiplicative distances in higher dimensions. In higher space dimensions polyconvexity can be satisfied most easily for dissipation distances D satisfying the ansatz (2.9f) by choosing a polyconvex function Ψ . In analogy to Example 2.3 we let

$$\Psi(F) = (\det F)^{-\beta} |F - (\det F)^\alpha (F^{-1})^T|^2.$$

We always have the double frame indifference due to $|Q_1 A Q_2|^2 = |A|^2 = \text{trace}(A^T A)$. Moreover, for $\alpha - \beta = 1/2$ we also have the symmetry (2.9b) via the inversion relation (2.11). For $\alpha = 1$ and $\beta \in [0, 1]$ we use $\det F (F^{-1})^T = \text{cof } F$ and observe that

$$\tilde{\Psi}(F) = (\det F)^{-\beta} |F - \text{cof } F|^2$$

is polyconvex in any space dimension $d \in \mathbb{N}$. Indeed, using convexity of $(x, y) \mapsto x^2/y^\beta$ the potential $\tilde{\Psi}$ is convex in $(F, \text{cof } F, \det F)$. Thus, D defined via $D(F, G)^2 = \tilde{\Psi}(GF^{-1}) \det F$ satisfies the polyconvexity (2.9e), but it does not satisfy the triangle inequality (2.9c).

For $d = 2$ we note that $\tilde{\Psi}(F) = 0$ holds for all conformal F , i.e., $F = \lambda Q$ for $\lambda > 0$ and $Q \in \text{SO}(2)$. Restricting to incompressible elasticity, i.e., $\det F = \det G = 1$, we obtain

$$D_{d=2}^{\text{inc}}(F, G)^2 = |G(\text{cof } F)^\top - (\text{cof } G)F^\top|^2 \quad \text{and} \quad \tilde{\Psi}(F) \geq \text{dist}(F, \text{SO}(2))^2.$$

For $d \geq 3$ we have that $\tilde{\Psi}(F) = 0$ implies $F \in \text{SO}(d)$, i.e., the positivity (2.9a) is satisfied on $\text{GL}_+(d)$. Moreover, we have the dissipation coercivity

$$(2.12) \quad \tilde{\Psi}(F) \geq c \text{dist}(F, \text{SO}(d))^\gamma \quad \text{with } \gamma = 2 - \beta \quad \text{for } F \in \text{GL}_+(d).$$

2.4. Toward a multidimensional existence theory. Because in the multi-dimensional case already the existence of minimizers of the incremental problem

$$(2.13) \quad U_\tau^n = \underset{u}{\text{argmin}} \int_\Omega \frac{1}{2\tau} D(\nabla U_\tau^{n-1}(x), \nabla u(x))^2 + W(x, \nabla u(x)) - u(x) \cdot f(x) \, dx$$

is a major difficulty, the polyconvexity (2.9e) of $G \mapsto D(F, G)^2$ appears unavoidable. One may therefore need to proceed without the use of the triangle inequality, which represents an interesting challenge.

Assuming also polyconvexity of $W(x, \cdot)$ and additional coercivity

$$\exists c, C > 0 \exists p > d \forall F \in \text{GL}_+(d) : \quad W(x, F) \geq c(|F|^p + |F^{-1}|^p) - C,$$

it is standard to obtain existence of minimizers U_τ^n ; cf. [5, 4]. Moreover, using the Dirichlet boundary conditions $u(t, x) = x$ on $\partial\Omega$, the theory of weak diffeomorphisms (cf. [16]) can be applied to conclude that the inverse mapping $(U_\tau^n)^{-1}$ exists, and we have the a priori estimates

$$\|U_\tau^n\|_{W^{1,p}(\Omega)} + \|(U_\tau^n)^{-1}\|_{W^{1,p}(\Omega)} \leq C_*,$$

where C_* depends only on the initial condition u_0 . Additionally, the time increments satisfy

$$\sum_{n=1}^N \frac{1}{\tau} \mathcal{D}(U_\tau^{n-1}, U_\tau^n)^2 \leq \phi(u_0) - \phi(U_\tau^N) \leq \phi(u_0) - C_\phi.$$

For the incremental mappings $V_\tau^n = U_\tau^n \circ U_\tau^{n-1}$, one expects that the $\mathfrak{V}_\tau^n = \frac{1}{\tau}(V_\tau^n - \text{id})$ converges to an Eulerian velocity field \mathfrak{V} such that the limit deformation $u(t, x)$ satisfies $\partial_t u(t, x) = \mathfrak{V}(t, u(t, x))$. However, the composition invariance (2.10) of \mathcal{D} , the coercivity (2.12), and the Dirichlet boundary conditions together with the rigidity estimate in [15] only imply

$$\begin{aligned} \sum_{n=1}^N \frac{1}{\tau} \|V_\tau^n - \text{id}\|_{W^{1,\gamma}(\Omega)}^\gamma &\leq C_1 \sum_{n=1}^N \frac{1}{\tau} \int_\Omega \text{dist}(\nabla V_\tau^n, \text{SO}(d))^\gamma \, dx \\ &\leq C_2 \sum_{n=1}^N \frac{1}{\tau} \mathcal{D}(V_\tau^n, \text{id})^2 \leq C_3, \end{aligned}$$

which is not enough to pass to the limit.

Here the main difficulty is that \mathcal{D} does not satisfy a triangle inequality; otherwise the sublevels of ϕ in the set of weak diffeomorphisms could be considered as a complete metric space equipped with the distance \mathcal{D} . Indeed, this will be the approach in the forthcoming sections for the one-dimensional case.

3. The setup for the one-dimensional Dirichlet case. Here we are confined to the one-dimensional case and set without loss of generality $\Omega = (0, 1) \subset \mathbb{R}$. For most of our analysis we consider general true dissipation distances as described in Example 2.2, namely,

$$(3.1) \quad d_\xi(u, v) = \left(\int_0^1 (\xi(v'(x)) - \xi(u'(x)))^2 dx \right)^{1/2},$$

where ξ will be a smooth and strictly increasing function. As a special family we will consider

$$\xi_\alpha(z) = z^\alpha \quad \text{with } \alpha \in [1/2, 1].$$

The case $\alpha = 1/2$ plays an exceptional role, since our theory becomes most complete. We write

$$d_{\text{sq}}(u, v) = \left(\int_0^1 (\sqrt{v'(x)} - \sqrt{u'(x)})^2 dx \right)^{1/2}.$$

This distance is called the *Hellinger or Hellinger–Kakutani distance* in stochastics. Note that it can be extended to all probability measures as $(a, b) \mapsto (\sqrt{a} - \sqrt{b})^2$ is convex and asymptotically linear.

The aim of the remainder of the paper is to show that solutions obtained from the incremental minimization problem

$$U_\tau^n \in \operatorname{argmin}_{u \in \mathcal{S}} \frac{1}{2\tau} d_\xi(U_\tau^{n-1}, u)^2 + \phi(u),$$

where the admissible set \mathcal{S} is defined below, converge to a solution u of the one-dimensional viscoelastic problem

$$(3.2) \quad (W'(u'(t, x)) + \xi'(u'(t, x))^2 \partial_t u'(t, x))' = 0.$$

We will perform most steps of the proof for general ξ ; however, at one crucial passage we need to restrict to the case $\xi = \xi_{1/2}$, i.e., $d_\xi = d_{\text{sq}}$.

3.1. State space and energy. Throughout we use the general state space \mathcal{S} and define additionally the subset \mathcal{S}_p via

$$\begin{aligned} \mathcal{S} &:= \{ u \in W^{1,1}(0, 1) \mid u(0) = 0, u(1) = 1, u'(x) \geq 0 \text{ a.e.} \}, \\ \mathcal{S}_p &:= \{ u \in \mathcal{S} \mid u \in W^{1,p}(0, 1) \}. \end{aligned}$$

The energy takes the form

$$\phi(u) = \int_0^1 W(x, u'(x)) - f(x)u(x) dx;$$

however, for notational convenience we set $f \equiv 0$ in what follows and omit the dependence on the material point x . The treatment of the general case requires only minor modifications, which are standard.

We will always assume that W satisfies coercivity and lower semicontinuity:

$$(3.3) \quad W(z) \geq c(z^{m_1} + z^{-m_2}) - C \quad \text{for } z > 0 \text{ and } W(z) = \infty \text{ for } z \leq 0,$$

$$(3.4) \quad W : \mathbb{R} \rightarrow [0, \infty] \text{ is lower semicontinuous.}$$

Thus, the sublevels $\{\phi \leq M\} := \{u \in \mathcal{S} \mid \phi(u) \leq M\}$ satisfy that for each $M > 0$ there exists C_M such that $u \in \mathcal{S}$ satisfies

$$\|u\|_{W^{1,m_1}(0,1)} + \|u^{-1}\|_{W^{1,m_2}(0,1)} \leq C_M.$$

Thus, for all $M \in \mathbb{R}$ we have $\{\phi \leq M\} \subset \mathcal{S}_{m_1}$.

The following condition, which was introduced by Ball in [6], is central to exploiting the multiplicative structure via composition of the weak diffeomorphisms:

$$(3.5) \quad |W'(z)z| \leq K(W(z)+1) \quad \forall z > 0.$$

We refer to [7, 13, 19] for applications of the multidimensional version of this estimate in finite-strain elasticity and plasticity. The following elementary result will be needed in Lemma 4.2.

LEMMA 3.1. *If W satisfies (3.5), then we have*

$$(3.6) \quad \forall z, w > 0 : \quad W(wz) \leq \max\{w^K, w^{-K}\}(W(z)+1) - 1.$$

Proof. Fixing $z > 0$ we set $g(a) = W(e^a z) + 1$. Then, $g'(a) = e^a z W'(e^a z)$ and (3.5) implies $|g'(a)| \leq K g(a)$. Now, Grönwall's estimate gives $g(a) \leq e^{K|a|} g(0)$, which is (3.6). \square

3.2. Generalized geodesics for the distance d_ξ . Here we consider the distances

$$d_\xi \text{ with } \xi(z) = z^\alpha, \quad \text{where } \alpha \in [1/2, 1].$$

A key point is that ξ is concave, whereas $z \mapsto \xi(z)^2$ is convex. We will use this without further notice. We choose $p \geq 2\alpha \geq 1$; then for $u, v \in \mathcal{S}_p$ the distance $d_\xi(u, v)$ is well defined.

Given $u_0, u_1 \in \mathcal{S}_p$, we define generalized geodesics $s \mapsto u_s = U_\xi(s; u_0, u_1)$ via

$$(3.7) \quad u_s(x) = \frac{w_s(x)}{w_s(1)} \quad \text{with} \quad w_s(x) = \int_0^x \xi^{-1}((1-s)\xi(u'_0(y)) + s\xi(u'_1(y))) \, dy.$$

For $u_0, u_1 \in \mathcal{S}_p$ we see that $s \mapsto u_s$ is a continuous curve in (\mathcal{S}_p, d_ξ) connecting u_0 and u_1 . The main difficulty is that the prefactor $1/w_s(1)$, which is needed in the definition of u_s to achieve $u_s(1) = 1$, depends on u_0 and u_1 in a nontrivial way, such that $d_\xi(u_r, u_s)$ cannot be calculated in a simple manner. Below we will give more specific results for $\xi(z) = \sqrt{z}$.

To derive the variational inequality for the incremental minimizers we use that $s \mapsto U_\xi(s; u_0, u_1) \in \mathcal{S}$ is differentiable. The following result for the distance d_ξ strongly depends on the fact that d_ξ is defined as an L^2 -norm, namely, $d_\xi(u, v) = \|\xi(u') - \xi(v')\|_{L^2}$.

PROPOSITION 3.2. *For $\xi(z) = z^\alpha$ with $\alpha \in [1/2, 1]$ and $u, v \in \mathcal{S}_{2\alpha}$ we set*

$$(3.8) \quad A_\xi(u, v) := \int_0^1 (\xi(u') - \xi(v')) u' \xi'(u') \, dy \text{ and } B_\xi(u, v) := \int_0^1 \frac{\xi(u') - \xi(v')}{\xi'(u')} \, dy.$$

Then, for $u_0, u_1, w \in \mathcal{S}_p$ with $p \geq 2\alpha$, we have the relations

$$(3.9) \quad \partial_s U'_\xi(s; u_0, u_1)(x)|_{s=0} = \frac{\xi(u'_1(x)) - \xi(u'_0(x))}{\xi'(u'_0(x))} + B_\xi(u_0, u_1)u'_0(x) \text{ and}$$

$$(3.10) \quad \frac{d}{ds} d_\xi(U_\xi(s; u_0, u_1), w)^2|_{s=0} = d_\xi(u_1, w)^2 - d_\xi(u_0, w)^2 - d_\xi(u_0, u_1)^2 + 2A_\xi(u_0, w)B_\xi(u_0, u_1).$$

Proof. For the first relation we simply differentiate using the fact that $\tilde{u}_s := (1-s)u'_0 + su'_1 \in L^{2\alpha}(0, 1)$, and hence $1/\xi'(\tilde{u}_s) \in L^2(0, 1)$.

The second relation follows by the chain rule and the quadratic nature of the distance d_ξ . Indeed, letting $a_j = \xi(u'_j)$ and $b = \xi(w')$ we have

$$\begin{aligned} \frac{d}{ds} d_\xi(u_s, w)^2 \Big|_{s=0} &= \int_0^1 2(a_0 - b)\xi'(u'_0)\partial_s u'_s \Big|_{s=0} dx \\ &= \int_0^1 2(a_0 - b)(a_1 - a_0) dx + 2A(u_0, w)B_\xi(u_0, u_1) \\ &= \int_0^1 (a_1 - w)^2 - (a_0 - w)^2 - (a_1 - a_0)^2 dx + 2A_\xi(u_0, w)B_\xi(u_0, u_1), \end{aligned}$$

which gives the desired result. \square

3.3. One-dimensional distances derived via composition. As an alternative to distance functions of the form (3.1), we now consider distance functions d of the form

$$(3.11) \quad d(u, v)^2 = \|\psi((v \circ u^{-1})'(\cdot))\|_2^2 = \int_0^1 \psi((v \circ u^{-1})'(z))^2 dz,$$

where $\psi \in C^1(0, +\infty)$ is to be chosen. This form is motivated in particular by (2.9f).

It is quite straightforward to see that $d : \mathcal{S} \times \mathcal{S} \rightarrow [0, +\infty]$, and we now investigate under which conditions it is symmetric and satisfies the triangle inequality.

LEMMA 3.3. *Let d be defined by (3.11). Then the following statements are true:*

(i) *d is invariant under composition, that is,*

$$d(u, v) = d(\text{id}, v \circ u^{-1}) = d(u \circ v^{-1}, \text{id}) \quad \forall u, v \in \mathcal{S}.$$

(ii) *d is symmetric on $W^{1,1}(0, 1)$ if and only if*

$$(3.12) \quad |\psi(z)| = \sqrt{z} |\psi(1/z)| \quad \forall z > 0.$$

(iii) *d satisfies the triangle inequality on $W^{1,1}(0, 1)$ if and only if*

$$(3.13) \quad |\psi(zw)| \leq |\psi(z)| + \sqrt{z} |\psi(w)| \quad \forall z, w > 0.$$

Proof. (i) This immediately follows from the property of composition of maps that $(v \circ u^{-1})^{-1} = u \circ v^{-1}$.

(ii) Consider two homogeneous deformations $u(x) = x$ and $v(x) = zx$; then $(v \circ u^{-1})'(u) = z$ and $(u \circ v^{-1})'(v) = 1/z$. The statement $d(u, v) = d(v, u)$, written in terms of ψ , reads

$$d^2(u, v) = \psi(z)^2 = z\psi(1/z)^2 = d^2(v, u),$$

which shows that (3.12) is necessary.

Conversely, (3.12) implies that d is symmetric. Let $u, v \in \mathcal{S}$ such that $\psi((v \circ u^{-1})'(u))$ is integrable; then by (3.12) and the change of variables formula,

$$\begin{aligned} d(u, v)^2 &= \int_0^1 \psi \left(\frac{dv}{du} \circ u^{-1} \right)^2 du = \int_0^1 \psi \left(\frac{du}{dv} \circ u^{-1} \right)^2 \left(\frac{dv}{du} \circ u^{-1} \right) du \\ &= \int_0^1 \psi \left(\frac{du}{dv} \circ v^{-1} \right)^2 dv. \end{aligned}$$

This shows that (3.12) implies symmetry of d .

(iii) Let $u(x) = x, v(x) = zx$ and $w(x) = rzx$, for any $z, r > 0$; then

$$d(u, w) = |\psi(zr)| \quad \text{and} \quad d(u, v) + d(v, w) = |\psi(z)| + \sqrt{z}|\psi(r)|.$$

This shows that (3.13) is necessary.

To prove the converse, we can assume without loss of generality that $u = x$; then by (3.13), for any $v, w \in \mathcal{S}$ we have

$$\begin{aligned} d(u, w) &= \left\| \psi \left(\frac{dw}{du} \circ u^{-1} \right) \right\|_2 = \left\| \psi \left(\left[\frac{dw}{dv} \circ u^{-1} \right] \left[\frac{dv}{du} \circ u^{-1} \right] \right) \right\|_2 \\ &\leq \left\| \psi \left(\frac{dv}{du} \circ u^{-1} \right) \right\|_2 + \left\| \left(\frac{dv}{du} \circ u^{-1} \right)^{1/2} \psi \left(\frac{dw}{dv} \circ u^{-1} \right) \right\|_2, \end{aligned}$$

and a coordinate transformation in the second integral on the right-hand side (similarly as in the proof of (ii)) yields the triangle inequality. \square

As a simple consequence of the foregoing lemma we obtain upper and lower bounds on ψ . We only give bounds for $z \geq 1$; the corresponding bounds for $z < 1$ are obtained from (3.12). We note in particular that the (maximal) choice $\psi(z) \propto (\sqrt{z} - 1)$, which corresponds to $d = d_{sq}$, again appears naturally.

COROLLARY 3.4. *Suppose that d is a metric, $\psi \in C^1((0, +\infty))$, and $\psi(z) = 0$ if and only if $z = 1$. Then, assuming without loss of generality that $\psi(z) > 0$ for all $z > 1$, there exists a constant $c > 0$ such that*

$$(3.14) \quad c(\sqrt{z} - 1) \leq \psi(z) \leq 2\psi'(1)(\sqrt{z} - 1) \quad \forall z \geq 1.$$

Proof. Since $\psi(z) > 0$ for $z > 1$, (3.13) becomes

$$\psi(zw) \leq \psi(z) + \sqrt{z}\psi(w) \quad \forall z, w \geq 1.$$

Let $w = 1 + \varepsilon$ for $\varepsilon > 0$. This implies

$$\frac{\psi(z(1 + \varepsilon)) - \psi(z)}{z\varepsilon} \leq \frac{1}{\sqrt{z}} \frac{\psi(1 + \varepsilon) - \psi(1)}{\varepsilon} \quad \forall \varepsilon > 0.$$

Taking the limit as $\varepsilon \rightarrow 0$ gives

$$\psi'(z) \leq \frac{\psi'(1)}{\sqrt{z}}.$$

Integrating this inequality yields the upper bound in (3.14).

Now suppose, for contradiction, that the lower bound is false. Then there exist $z_j \rightarrow \infty$ such that $\psi(z_j) \ll \sqrt{z_j}$, and consequently,

$$|\psi(1/z_j)| = \frac{\psi(z_j)}{\sqrt{z_j}} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

This clearly contradicts the assumption that $\psi(0) \neq 0$. \square

3.4. The square-root distance. We continue to call the Hellinger distance *the square-root distance*

$$d_{\text{sq}}(u, v) = \left(\int_0^1 (\sqrt{u'(x)} - \sqrt{v'(x)})^2 dx \right)^{1/2}$$

to emphasize its role in the family d_ξ studied in Section 3.2 as well as the composition distance studied in Section 3.3 with $\psi(z) = \sqrt{z} - 1$ or in Examples 2.2 and 2.3 with $\Psi(z) = (\sqrt{z} - 1)^2$.

LEMMA 3.5. *We have the elementary estimates*

$$(3.15) \quad \forall u, v \in \mathcal{S} : \quad \frac{1}{2}d_{\text{sq}}(u, v)^2 \leq \frac{1}{2}\|u' - v'\|_{L^1} \leq d_{\text{sq}}(u, v) \leq \sqrt{2}.$$

Moreover, $(\mathcal{S}, d_{\text{sq}})$ is a complete metric space.

Proof. The first estimate follows from the simple estimate $(\sqrt{a} - \sqrt{b})^2 \leq |a - b|$. For the second estimate we use $\int_0^1 u' dx = 1 = \int_0^1 v' dx$ to obtain

$$\begin{aligned} \int_0^1 |u' - v'| dx &= \int_0^1 |\sqrt{u'} - \sqrt{v'}| |\sqrt{u'} + \sqrt{v'}| dx \\ &\leq d_{\text{sq}}(u, v) \left(\int_0^1 (u' + 2\sqrt{u'v'} + v') dx \right)^{1/2} \leq 2d_{\text{sq}}(u, v) \end{aligned}$$

and $d_{\text{sq}}(u, v)^2 = \int_0^1 (u' - 2\sqrt{u'v'} + v') dx \leq 2$. Since \mathcal{S} is a closed subspace of $W^{1,1}(0, 1)$ and d_{sq} dominates the norm in $W^{1,1}(0, 1)$, the completeness of $(\mathcal{S}, d_{\text{sq}})$ follows. \square

The main advantage of the square-root distance is that the generalized geodesic curves $u_s = U_{\text{sq}}(s; u_0, u_1)$ can be studied more precisely:

$$(3.16) \quad U_{\text{sq}}(s; u_0, u_1) = \frac{1}{w_s(1)} \int_0^x \left((1-s)\sqrt{u'_0(y)} + s\sqrt{u'_1(y)} \right)^2 dy,$$

where $w_s(1) = 1 - s(1-s)d_{\text{sq}}(u_0, u_1)^2$.

To see the form of $w_s(1)$ given above, we use $u_0, u_1 \in \mathcal{S}$ and find

$$w_s(1) = \int_0^1 \left((1-s)\sqrt{u'_0(y)} + s\sqrt{u'_1(y)} \right)^2 dy = (1-s)^2 + s^2 - 2s(1-s) \int_0^1 \sqrt{u'_0 u'_1} dy$$

and use the identity

$$(3.17) \quad d_{\text{sq}}(u, v)^2 = \int_0^1 (\sqrt{u'} - \sqrt{v'})^2 dx = \int_0^1 (u' - 2\sqrt{u'v'} + v') dx = 2 - 2 \int_0^1 \sqrt{u'v'} dx.$$

The next result specializes Proposition 3.2 for the case $\xi = \sqrt{\cdot}$, i.e., $\alpha = 1/2$, and provides the derivative of d_{sq} along $s \mapsto U_{\text{sq}}(s; u_0, u_1)$.

PROPOSITION 3.6. *For $u, v, w \in \mathcal{S} = \mathcal{S}_1$ and A and B defined in (3.8) we have*

$$(3.18) \quad A_{\text{sq}}(u, v) = \frac{1}{4}d_{\text{sq}}(u, v)^2 \quad \text{and} \quad B_{\text{sq}}(u, v) = d_{\text{sq}}(u, v)^2.$$

Hence, we find the relations

$$(3.19a) \quad \left. \frac{d}{ds} d_{\text{sq}}(U_{\text{sq}}(s; u_0, u_1), w)^2 \right|_{s=0} = d_{\text{sq}}(u_1, w)^2 - d_{\text{sq}}(u_0, w)^2 - d_{\text{sq}}(u_0, u_1)^2 + \frac{1}{2} d_{\text{sq}}(u_0, w)^2 d_{\text{sq}}(u_0, u_1)^2,$$

$$(3.19b) \quad \left. \frac{d}{ds} d_{\text{sq}}(U_{\text{sq}}(s; u_0, u_1), u_0)^2 \right|_{s=0} = 0, \quad \text{and}$$

$$(3.19c) \quad \left. \frac{d^2}{ds^2} d_{\text{sq}}(U_{\text{sq}}(s; u_0, u_1), u_0)^2 \right|_{s=0} = 2d_{\text{sq}}(u_0, u_1)^2 + \frac{1}{2} d_{\text{sq}}(u_0, u_1)^4.$$

Proof. The identities (3.18) and (3.19a) are special cases of Proposition 3.2. The identity (3.19b) is a special case of (3.19a).

To prove (3.19c), we first note that a straightforward calculation using (3.17) shows

$$d_{\text{sq}}(U_{\text{sq}}(s; u_0, u_1), u_0)^2 = 2 - \frac{2 - s d_{\text{sq}}(u_0, u_1)^2}{\sqrt{w_s(1)}},$$

where $w_s(1)$ is given in (3.16). This expression can be explicitly differentiated and evaluated at $s = 0$ to obtain (3.19c). \square

The next result shows that d_{sq}^2 is locally approximately 2-convex (as already suggested by (3.19c)) along the generalized geodesics. This result relies on the L^2 -structure of the norm and will be used to show strong convergence of minimizing sequences for the incremental minimization problems; see Proposition 4.3.

LEMMA 3.7. *Given $u_0, u_1, w \in \mathcal{S}$ the midpoint $u_{1/2} = U_{\text{sq}}(1/2; u_0, u_1)$ satisfies*

$$d_{\text{sq}}(u_{1/2}, w)^2 = \frac{\rho}{2} (d_{\text{sq}}(u_0, w)^2 + d_{\text{sq}}(u_1, w)^2) + 2 - 2\rho,$$

$$\text{where } \rho = \left(1 - \frac{1}{4} d_{\text{sq}}(u_0, u_1)^2\right)^{-1/2} \in [1, \sqrt{2}).$$

Moreover, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_{\text{sq}}(u_0, u_1), d_{\text{sq}}(u_0, w), d_{\text{sq}}(u_1, w) \leq \delta$ implies

$$(3.20) \quad d_{\text{sq}}(u_{1/2}, w)^2 \leq \frac{1}{2} d_{\text{sq}}(u_0, w)^2 + \frac{1}{2} d_{\text{sq}}(u_1, w)^2 - \frac{2 - \varepsilon}{2} \frac{1}{2} \left(1 - \frac{1}{2}\right) d_{\text{sq}}(u_0, u_1)^2.$$

In particular, for $\varepsilon = 1/2$ it suffices to choose $\delta = 1/2$.

Proof. The general identity $d_{\text{sq}}(u_{1/2}, w)^2$ follows from the definition of U_{sq} and (3.17).

To obtain the estimate let $\sigma = d_{\text{sq}}(u_0, u_1)^2$ and $\alpha_j = d_{\text{sq}}(u_j, w)^2$. Then, for $\sigma \leq \delta \leq 1/4$ we have $\rho = (1 - \sigma/4)^{-1/2} \in [1 + \sigma/8, 1 + \sigma/4]$. Hence,

$$d_{\text{sq}}(u_{1/2}, w)^2 \leq \left(1 + \frac{\sigma}{4}\right) \left(\frac{\alpha_0}{2} + \frac{\alpha_1}{2}\right) + 2 - 2 \left(1 + \frac{\sigma}{8}\right)$$

$$= \frac{1}{2}(\alpha_0 + \alpha_1) - \left(\frac{1}{4} - \frac{\alpha_0}{8} - \frac{\alpha_1}{8}\right) \sigma \leq \frac{1}{2}(\alpha_0 + \alpha_1) - \left(\frac{1}{4} - \frac{\delta^2}{4}\right) \sigma,$$

which is the desired result for $\delta = \sqrt{\varepsilon/2}$. \square

4. Time-incremental minimization problem. In this section we keep the time step $\tau > 0$ fixed and study the existence of minimizers for the time-incremental minimization problem

$$(4.1) \quad u_\tau = \operatorname{argmin}_{v \in \mathcal{S}_p} \frac{1}{2\tau} d_\xi(u, v)^2 + \phi(v).$$

By inserting the definition of d_ξ we have to minimize the functional

$$v \mapsto \int_0^1 \frac{1}{2\tau} (\xi(v'(x)) - \xi(u'(x)))^2 + W(x, v'(x)) - f(x)v(x) dx$$

under the constraint $v(0) = 0$, $v(1) = 1$, and $v'(x) \geq 0$ a.e. in $(0, 1)$.

To make the calculations easier, we simplify the energy function in this and the following section by assuming

$$(4.2) \quad \phi(u) = \int_0^1 W(u'(x)) dx \quad \text{and} \quad \inf_{z > 0} W(z) \geq 0,$$

i.e., we omit the x -dependence of W , and the loading $\langle \ell, u \rangle = \int_0^1 f(x)u(x) dx$ is set to $\ell = 0$. It can be easily checked that the whole theory works in the general case as well. The normalization $W(z) \geq 0$ implies $\phi(u) \geq 0$ and hence the solutions u_τ of (4.1) satisfy

$$\frac{1}{2\tau} d_\xi(u, u_\tau)^2 \leq \phi(u) - \phi(u_\tau) \leq \phi(u).$$

Moreover, we will derive a discrete variational inequality $(\text{DVI})_\lambda$ that will allow us to pass to the limit $\tau \rightarrow 0_+$ in the next section. We recall that we do not assume convexity of the energy density $F \mapsto W(F)$ to allow for the modeling of phase transformations. Nevertheless we will use suitable generalized convexity conditions. They will be especially important when studying the slope of ϕ with respect to the metric d_ξ .

4.1. Convexity of the energy ϕ . There are two possible approaches to obtain existence. The first result uses the classical convexity of $z \mapsto W(z)$. The given assumptions (4.3) will only be used for this result and are not needed in the remainder of this work.

PROPOSITION 4.1. *Assume that $\xi(z) = z^\alpha$ for $\alpha \in [1/2, 1]$ and that the stored-energy density W satisfies (3.3) with $m_1 \geq 2\alpha$ and the conditions*

$$(4.3a) \quad \exists \lambda^W \in \mathbb{R} \ \forall z_0, z_1 \geq 0, \ s \in (0, 1) :$$

$$W((1-s)z_0 + sz_1) \leq (1-s)W(z_0) + sW(z_1) - \frac{\lambda^W}{2} s(1-s) |z_1 - z_0|^2,$$

$$(4.3b) \quad \exists R > 1 : \quad W|_{[0, 1/R]} \text{ and } W|_{[R, \infty)} \text{ are convex.}$$

Then, there exists $\tau_* > 0$ such that for all $\tau \in (0, \tau_*)$ and all $u \in \mathcal{S}_{2\alpha}$ there exists a unique minimizer u_τ for (4.1).

Proof. For $z_0 \geq 0$ let $g_{z_0}(z) = (z^\alpha - z_0^\alpha)^2$. Using $\alpha \in [1/2, 1]$ one sees that the mapping g_{z_0} is strictly convex on $[0, \infty)$. Hence the mapping $z \mapsto \frac{1}{2\tau} (z^\alpha - (u'(x))^\alpha)^2 + W(z)$ is strictly convex on $[0, 1/R]$ and on $[R, \infty)$.

For $z, z_0 \in [1/(2R), 2R]$ we have $g''_{z_0}(z) \geq 2\alpha/(2R)^{2-2\alpha} > 0$. Choosing $\tau_* > 0$ such that $\frac{1}{2\tau_*}2\alpha/(2R)^{2-2\alpha} + \lambda^W > 0$, for all $\tau \in (0, \tau_*]$ the sum $\frac{1}{2\tau}g_{z_0}(z) + W(z)$ is strictly convex on $[1/(2R), 2R]$. Together with the above convexity on $[0, 1/R]$ and on $[R, \infty)$, we have strict convexity on $[0, \infty)$ and conclude the existence of a unique minimizer. \square

We now establish existence and uniqueness of minimizers by employing a notion of convexity of ϕ with respect to the metric d_ξ . This notion of convexity is more readily combined with convexity of d_ξ along generalized geodesics and can serve as a basis for studying the time-continuous limit $\tau \rightarrow 0_+$. For general strictly increasing ξ we define

$$W_\xi(y) = W(\xi^{-1}(y)) \text{ for } y \in \text{im}(\xi)$$

and impose a λ -convexity condition for W_ξ :

$$(4.4) \quad \exists \lambda^{W_\xi} \forall y_0, y_1 > 0 \forall s \in [0, 1] :$$

$$W_\xi((1-s)y_0 + sy_1) \leq (1-s)W_\xi(y_0) + sW_\xi(y_1) - \frac{\lambda^{W_\xi}}{2}s(1-s)|y_0 - y_1|^2.$$

The following lemma shows that this condition implies a kind of λ -convexity of ϕ along the generalized geodesics $u_s = U_\xi(s; u_0, u_1)$.

LEMMA 4.2. *Let $\xi(z) = z^\alpha$ with $\alpha \in [1/2, 1]$ and let W satisfy (3.5) and (4.4). Consider $u_0, u_1 \in \mathcal{S}$ with $\phi(u_j) < \infty$ and define $u_s = U_\xi(s; u_0, u_1)$ via (3.7). Then, for all $s \in [0, 1]$ we have*

$$(4.5) \quad \phi(u_s) \leq w_s^{-K} \left((1-s)\phi(u_0) + s\phi(u_1) - \frac{\lambda^{W_\xi}}{2}s(1-s)d_\xi(u_0, u_1)^2 \right) + w_s^{-K} - 1,$$

where $w_s = w_s(1)$ is defined in (3.7) and K in (3.5). If $\alpha = 1/2$, i.e., $d = d_{\text{sq}}$, then

$$(4.6) \quad \phi(u_s) \leq (1-s)\phi(u_0) + s\phi(u_1) - \frac{\lambda^\phi}{2}s(1-s)d_{\text{sq}}(u_0, u_1)^2$$

with $\lambda^\phi = (1 + C_K/2)\lambda^{W_\xi} - C_K(M + 1)$, $M = \max\{\phi(u_0), \phi(u_1)\}$, and $C_K = 2(2^K - 1)$.

Proof. We first show $w_s(1) \in [1/2, 1]$. The upper bound follows from convexity of ξ^{-1} via

$$w_s(1) = \int_0^1 \xi^{-1}((1-s)\xi(u'_0) + s\xi(u'_1)) \, dx \leq \int_0^1 ((1-s)u'_0 + su'_1) \, dx = 1.$$

For the lower bound we use $\frac{d}{d\alpha}((1-s)z_0^\alpha + sz_1^\alpha)^{1/\alpha} \geq 0$ for $\alpha \in [1/2, 1]$, so that we get

$$(4.7) \quad w_s(1) \geq \int_0^1 \left((1-s)\sqrt{u'_0} + s\sqrt{u'_1} \right)^2 \, dx \geq 1 - 2s(1-s) \geq \frac{1}{2},$$

where the last inequality follows from an explicit computation as in Section 3.4.

The estimate of $\phi(u_s)$ relies on the multiplicative estimate (3.6), which follows from (3.5) (cf. Lemma 3.1), and the λ -convexity of W_ξ by applying (4.4) with $y_j = \xi(u'_j(x))$:

$$\begin{aligned} \phi(u_s) &= \int_0^1 W\left(\frac{1}{w_s}\xi^{-1}((1-s)\xi(u'_0)+s\xi(u'_1))\right) dx \\ &\stackrel{(3.6)}{\leq} w_s^{-K} \int_0^1 W(\xi^{-1}((1-s)\xi(u'_0)+s\xi(u'_1))) dx + w_s^{-K} - 1 \\ &= w_s^{-K} \int_0^1 W_\xi((1-s)\xi(u'_0)+s\xi(u'_1)) dx + w_s^{-K} - 1 \\ &\stackrel{(4.4)}{\leq} w_s^{-K} \int_0^1 (1-s)W_\xi(\xi(u'_0))+sW_\xi(\xi(u'_1)) \\ &\quad - \frac{\lambda^{W_\xi}}{2}s(1-s)|\xi(u'_0)-\xi(u'_1)|^2 dx + w_s^{-K} - 1 \\ &= w_x^{-K} \left((1-s)\phi(u_0) + s\phi(u_1) - \frac{\lambda^{W_\xi}}{2}s(1-s)d_\xi(u_0, u_1)^2 \right) + w_s^{-K} - 1, \end{aligned}$$

which is (4.5). For $\alpha = 1/2$ we have $w_s(1) = 1 - s(1-s)d_{\text{sq}}(u_0, u_1)^2$ from (3.16), hence, $w_s^{-K} \leq 1 + C_K s(1-s)d_{\text{sq}}(u_0, u_1)^2 \leq 1 + \frac{1}{2}C_K d_{\text{sq}}(u_0, u_1)^2$ with $C_K = 2(2^K - 1)$. Inserting this into (4.5), and employing the bound $\phi(u_i) \leq M$, we obtain (4.6). \square

Using the approximate 2-convexity of d_{sq} established in Lemma 3.7 and the λ^ϕ -convexity of ϕ from (4.6), we can now obtain uniqueness of minimizers for the time-incremental problem.

PROPOSITION 4.3. *Assume $\xi(z) = \sqrt{z}$ and that W satisfies (3.4), (3.5), and (4.4). Then, for all $M > 0$ there exists $\tau_* = \tau_*(M) > 0$ such that for all $\tau \in (0, \tau_*)$ and all $u \in \mathcal{S}$ with $\phi(u) \leq M$ there exists a unique minimizer $u_\tau \in \mathcal{S}$ for the time-incremental minimization problem (4.1).*

Proof. Let λ^ϕ be defined by (4.6) but with M as prescribed in the hypothesis. Let $\Phi(v) := \frac{1}{2\tau}d_{\text{sq}}(u, v)^2 + \phi(v)$. If $\Phi(v) \leq \Phi(u)$, then, since $\phi \geq 0$, $d_{\text{sq}}(u, v)^2 \leq 2\tau M$ and hence we only need to consider $v \in \mathcal{S}' := \{v \in \mathcal{S} \mid d_{\text{sq}}(u, v) \leq \sqrt{2\tau M}\}$.

Suppose that τ is sufficiently small so that $\sqrt{2\tau M} \leq 1/2$; then, for $v_0, v_1 \in \mathcal{S}'$ and $v_{1/2} = U_{\text{sq}}(1/2; v_0, v_1)$, (4.6) and (3.20) imply that

$$\Phi\left(\frac{1}{2}v_0 + \frac{1}{2}v_1\right) \leq \frac{1}{2}\phi(v_0) + \frac{1}{2}\phi(v_1) - \left(\lambda^\phi + \frac{1}{\tau}\right)\frac{1}{8}d_{\text{sq}}^2(v_0, v_1).$$

If τ is so small that $\lambda^\phi + \frac{1}{\tau} > 0$, then one can readily prove the stated result, following, for example, the proof of Lemma 4.1.1 in [1]. \square

4.2. The discrete variational inequality. By the previous subsection we can assume that the time-incremental minimization problems have (unique) solutions. We will now show that solutions satisfy a variational inequality that can be used to derive strong convergence (at least in the case of d_{sq}) and to pass to the limit $\tau \rightarrow 0_+$. The idea is to compare the incremental energy at the minimizer u_τ and at $U_\xi(s; u_\tau, v)$ for s small. For this argument, we do not need geodesic convexity properties along the whole curve $[0, 1] \ni s \mapsto U(s; u_\tau, v)$, but rather the derivative $\frac{d}{ds}\Big|_{s=0_+}$.

THEOREM 4.4. *Assume $\xi(z) = z^\alpha$ for $\alpha \in [1/2, 1]$ and that W satisfies (3.3) with $m_1 > 2\alpha$, (3.4), (3.5), and (4.4). For $\tau > 0$ take $u \in \mathcal{S}_{2\alpha}$ with $\phi(u) < \infty$ and assume*

that $u_\tau \in \mathcal{S}$ satisfies the time-incremental minimization problem (4.1). Then, for all $v \in \mathcal{S}_{2\alpha}$ we have the generalized discrete variational inequality

$$\begin{aligned} \text{(gDVI}_\lambda) \quad \forall v \in \mathcal{S} : \quad & \frac{1}{2\tau}(d_\xi(u_\tau, v)^2 - d_\xi(u, v)^2) + \frac{\lambda^{W_\xi}}{2}d_\xi(u_\tau, v)^2 \\ & \leq \phi(v) - \phi(u_\tau) - \frac{1}{2\tau}d_\xi(u, u_\tau)^2 + C_\xi(\tau, u, u_\tau)B_\xi(u_\tau, v), \end{aligned}$$

where $C_\xi(\tau, u, w) := \int_0^1 (\frac{1}{\tau}(\xi(w') - \xi(u'))\xi'(w') + W'(w'))w' dx$.

Proof. We consider the functional $w \mapsto \Phi(\tau, u; w) = \frac{1}{2\tau}d_\xi(u, w)^2 + \phi(w)$, which satisfies

$$\begin{aligned} 0 \leq \frac{1}{s}(\Phi(\tau, u; U_\xi(s; u_\tau, v)) - \Phi(\tau, u; u_\tau)) &= \frac{1}{2\tau}T_1(s) + T_2(s) \quad \text{with} \\ T_1(s) = \frac{1}{s}(d_\xi(U_\xi(s; u_\tau, v), u)^2 - d_\xi(u_\tau, u)^2) \quad \text{and} \quad T_2(s) &= \frac{1}{s}(\phi(U_\xi(s; u_\tau, v)) - \phi(u_\tau)) \end{aligned}$$

for $s \in (0, 1]$. Using Proposition 3.2 with $u_0 = u_\tau$, $u_1 = v$, and $w = u$ the limit $s \rightarrow 0_+$ gives

$$T_1(s) \rightarrow T_1(0) := d_\xi(v, u)^2 - d_\xi(u, u_\tau)^2 - d_\xi(u_\tau, v)^2 + 2A_\xi(u_\tau, u)B_\xi(u_\tau, v).$$

We decompose T_2 according to the definition of $U_\xi(s; u_0, u_1) = \frac{1}{w_s(1)}w_s(x)$:

$$\begin{aligned} T_2(s) &= \int_0^1 \frac{1}{s}(W(u'_s(x)) - W(u'_0(x))) dx = T_3(s) + T_4(s) \quad \text{with} \\ T_3(s) &:= \int_0^1 \frac{1}{s} \left(W\left(\frac{1}{w_s(1)}w'_s(x)\right) - W(w'_s(x)) \right) dx \quad \text{and} \\ T_4(s) &:= \int_0^1 \frac{1}{s}(W(w'_s(x)) - W(u'_0(x))) dx. \end{aligned}$$

For $T_4(s)$ we use $w'_s = \xi^{-1}((1-s)\xi(u'_0) + s\xi(u'_1))$ and the λ -convexity of W_ξ , namely,

$$\begin{aligned} W(w'_s) &= W_\xi((1-s)\xi(u'_0) + s\xi(u'_1)) \\ &\leq (1-s)W(u'_0) + sW(u'_1) - \frac{\lambda^{W_\xi}}{2}s(1-s)(\xi(u'_0) - \xi(u'_1))^2. \end{aligned}$$

Hence, we conclude $\liminf_{s \rightarrow 0_+} T_4(s) \leq \tilde{T}_4 = \phi(u_1) - \phi(u_0) - \frac{\lambda^{W_\xi}}{2}d_\xi(u_0, u_1)^2$.

For $T_3(s)$ we use $\frac{d}{ds}\Big|_{s=0_+} w_s(1) = -B_\xi(u_0, u_1)$ so that by (3.5) we obtain

$$T_3(s) \rightarrow T_3(0) := B_\xi(u_0, u_1) \int_0^1 W'(u'_0(x))u'_0(x) dx.$$

From $T(s) \geq 0$ we have $T_1(0) + T_3(0) + \tilde{T}_4(0) \geq 0$. Inserting $u_0 = u_\tau$ and $u_1 = v$ into $T_3(0)$ and $\tilde{T}_4(0)$, we obtain the generalized discrete evolutionary inequality (gDVI $_\lambda$). \square

To turn (gDVI $_\lambda$) into the more useful discrete evolutionary inequality, we need to control the new term $C_\xi(\tau, u, u_\tau)B_\xi(u_\tau, v)$. For the first factor we note that the quantity C_ξ is closely related to the Euler–Lagrange equation for the minimizers. If w

minimizes $v \mapsto \frac{1}{2\tau}d_\xi(u, v)^2 + \phi(v)$, then the weak form of the Euler–Lagrange equation for w reads

$$\int_0^1 \left(\frac{1}{\tau}(\xi(w') - \xi(u'))\xi'(w') + W'(w') \right) \eta' \, dx = 0,$$

where $\eta \in C_c^\infty((0, 1))$. Thus, C_ξ is obtained by choosing $\eta = w$ (which, strictly speaking, is not an admissible test function, because $w(1) = 0$). However, by the lemma of Du Bois-Reymond $\frac{1}{\tau}(\xi(w') - \xi(u'))\xi'(w') + W'(w')$ is constant. Using $\int_0^1 w' \, dx = 1$ we conclude that C_ξ must equal to this constant. We also see that the term C_ξ will be in general not small even if τ is small. In fact, in the formal limit $\tau \rightarrow 0$ the time-dependent constant C_ξ converges to the constant stress $\Sigma(t) = \xi'(u'(t, x))^2 u'(t, x) + W'(u'(t, x))$.

Thus, to control the additional term $C_\xi B_\xi$ it is crucial to control B_ξ , which can be done in two cases. First consider $\xi(z) = z$ (i.e., $\alpha = 1$); then $B_\xi \equiv 0$ and we are in the situation of classical convexity. Second, the square-root distance d_{sq} (i.e., $\alpha = 1/2$) can be treated because of the identities (3.18) for $A_\xi = A_{\text{sq}}$ and $B_\xi = B_{\text{sq}}$.

COROLLARY 4.5. *Under the same assumptions as in Theorem 4.4 assume now $\alpha = 1/2$, i.e., $d_\xi = d_{\text{sq}}$. Then, for each $M > 0$ there exists $\lambda_M > 0$ such that for any $u \in \mathcal{S}$ with $\phi(u) \leq M$, any minimizer u_τ of (4.1) satisfies the discrete variational inequality*

$$\begin{aligned} \text{(DVI}_\lambda) \quad \forall v \in \mathcal{S} : \quad & \frac{1}{2\tau} (d_{\text{sq}}(u_\tau, v)^2 - d_{\text{sq}}(u, v)^2) + \frac{\lambda^{W_\xi} - \lambda_M}{2} d_{\text{sq}}(u_\tau, v)^2 \\ & \leq \phi(v) - \phi(u_\tau) - \frac{1}{2\tau} d_{\text{sq}}(u, u_\tau)^2. \end{aligned}$$

Proof. We estimate C_{sq} by exploiting its specific form, namely, $C_{\text{sq}}(\tau, u, w) = \frac{1}{\tau} A_{\text{sq}}(u, w) + \int_0^1 W'(w')w' \, dx$. Using (3.18), (3.5), and (4.1) we obtain the estimate

$$\begin{aligned} |C_{\text{sq}}(\tau, u, u_\tau)B_{\text{sq}}(u_\tau, v)| & \leq \left(\frac{1}{4\tau} d_{\text{sq}}(u, u_\tau)^2 + K\phi(u_\tau) + K \right) d_{\text{sq}}(u_\tau, v)^2 \\ & \leq \left(\max \left\{ \frac{1}{2}, K \right\} \phi(u) + K \right) d_{\text{sq}}(u_\tau, v)^2 \leq \lambda_M d_{\text{sq}}(u_\tau, v)^2 \end{aligned}$$

with $\lambda_M = \max\{\frac{1}{2}, K\}M + K$. Hence the term $C_{\text{sq}}B_{\text{sq}}$ can be moved to the left-hand side and the result is established. \square

Remark 4.6. For general ξ a good estimate B_ξ is still missing. For $\xi(z) = z^\alpha$ we find

$$B_\xi(u, v) = \frac{1}{\alpha} \int_0^1 u' - (v')^\alpha (u')^{1-\alpha} \, dx = \int_0^1 1 - \left(\frac{dv}{du} \right)^\alpha \, du \geq 0.$$

For $a > 1$, $x_* \in (0, 1)$, and $0 < \delta \ll 1$ define piecewise affine functions $u_\delta, v_\delta \in \mathcal{S}$ with

$$u'_\delta(x) = \delta, \quad v'_\delta(x) = a\delta, \quad u'_\delta(y) = \frac{1 - \delta x_*}{1 - x_*}, \quad v'_\delta(y) = \frac{1 - a\delta x_*}{1 - x_*} \quad \text{for } 0 < x < x_* < y < 1.$$

This gives the expansions $d_\xi(u_\delta, v_\delta)^2 = x_*(1 - a^\alpha)^2 \delta^{2\alpha} + O(\delta^2)$ and $B_\xi(u_\delta, v_\delta) = (1 - \alpha + \alpha a - a^\alpha)\delta + O(\delta^2)$. Hence, for $\alpha \in (\frac{1}{2}, 1)$ we cannot estimate B_ξ in terms of d_ξ^2 for general choices of a, x_* , even when restricting to sublevels of ϕ .

5. The time-continuous case. In this section we first give the limit passage $\tau \rightarrow 0_+$ from the $(\text{DVI})_\lambda$ to the *evolutionary variational inequality*, namely,

$$(EVI)_\lambda \quad \frac{1}{2} \frac{d^+}{dt} d_\xi(u(t), v)^2 + \frac{\lambda}{2} d_\xi(u(t), v)^2 \leq \phi(v) - \phi(u(t)) + C_M B_\xi(u(t), v),$$

where $u : [0, \infty) \rightarrow \mathcal{S}$ is absolutely continuous in (\mathcal{S}, d_ξ) , $u(0) = u_0$, and it satisfies $\sup\{\phi(u(t)) \mid t \geq 0\} \leq \phi(u_0) \leq M$. Afterward we show that solutions for the $(EVI)_\lambda$ are in fact curves of maximal slope and finally that they satisfy the PDE (3.2).

5.1. Strong convergence in the case d_{sq} . In the case of the square-root distance d_{sq} the discrete variational inequality is exactly of the type studied in [1, Section 4]. Thus, we can employ the same arguments and obtain strong convergence. Let $t^n := n\tau$ and $\overline{U}_\tau, \underline{U}_\tau$ denote, respectively, the backward and forward piecewise constant interpolants of u_τ . Then, we have the following result.

PROPOSITION 5.1. *Let $\tau \in (0, \tau_*)$ and u_τ be the solution of (4.1) with $d_\xi = d_{\text{sq}}$. Then the family \overline{U}_τ of discrete solutions is convergent to a function $u(t) \in C([0, \infty); \mathcal{S})$ as $\tau \rightarrow 0$, uniformly in each bounded interval $[0, T]$. In fact,*

$$(5.1) \quad \forall T > 0 \exists C_T : \sup_{t \in [0, T]} d_{\text{sq}}(u(t), \overline{U}_\tau(t)) \leq C_T \sqrt{\tau}.$$

Proof. One can essentially follow the proof in [1], so we only give a sketch. To begin, we recall some notation. For $\tau, \eta \in (0, \tau_*)$, let

$$\ell_\tau(t) := \frac{t - t^{n-1}}{\tau} \quad \text{for } t \in (t^{n-1}, t^n] \quad \text{and } \ell_\tau(0) = 0,$$

which we use to define the following piecewise affine interpolants:

$$\begin{aligned} \varphi_\tau(t) &:= (1 - \ell_\tau(t))\phi(\underline{U}_\tau(t)) + \ell_\tau(t)\phi(\overline{U}_\tau(t)), \\ d_\tau(t; V)^2 &:= (1 - \ell_\tau(t))d_{\text{sq}}(\underline{U}_\tau(t), V)^2 + \ell_\tau(t)d_{\text{sq}}(\overline{U}_\tau(t), V)^2, \\ d_{\tau\eta}(t, s)^2 &:= (1 - \ell_\eta(s))d_\tau(t; \underline{U}_\eta(s))^2 + \ell_\tau(s)d_\tau(t; \overline{U}_\eta(s))^2, \quad t, s \geq 0. \end{aligned}$$

All these interpolants are defined for all $t, s \geq 0$ and are differentiable everywhere except on a discrete set.

With $\gamma := \lambda^{W_{\text{sq}}} - \lambda_M$ in $(\text{DVI})_\lambda$, we apply [1, Theorem 4.1.4] directly in each interval (t^{n-1}, t^n) and obtain

$$\begin{aligned} \frac{d}{dt} d_\tau(t; V)^2 + 2\gamma d_{\text{sq}}(\overline{U}_\tau, V)^2 + 2(\varphi_\tau(t) - \phi(V)) &\leq \mathcal{R}_\tau(t), \\ \text{where } \mathcal{R}_\tau(t) &:= 2\tau(1 - \ell_\tau)(\phi(\underline{U}_\tau) - \phi(\overline{U}_\tau)) + (1 - 2\ell_\tau(t))\frac{1}{\tau}d_{\text{sq}}(\underline{U}_\tau, \overline{U}_\tau)^2. \end{aligned}$$

To simplify the subsequent notation, we estimate

$$\mathcal{R}_\tau(t) \leq \mathcal{R}'_\tau(t) := 2(\phi(\underline{U}_\tau) - \phi(\overline{U}_\tau)) + \frac{1}{\tau}d_{\text{sq}}(\underline{U}_\tau, \overline{U}_\tau)^2.$$

Next, following proofs of [1, Corollaries 4.1.5 and 4.1.7] (since the argument applies the inequalities for U_τ and U_η separately, it can again be repeated verbatim) we obtain

$$(5.2) \quad \partial_t d_{\tau\eta}(t, s)^2 + 2\gamma d_{\tau\eta}(t, s)^2 + 2\varphi_\tau(t) - 2\varphi_\eta(s) \leq \mathcal{R}'_\tau(t) + |\gamma| \mathcal{D}_\tau^2(t) \text{ and}$$

$$(5.3) \quad \partial_s d_{\tau\eta}(t, s)^2 + 2\gamma d_{\tau\eta}(t, s)^2 + 2\varphi_\eta(s) - 2\varphi_\tau(t) \leq \mathcal{R}'_\eta(s) + |\gamma| \mathcal{D}_\eta^2(s),$$

where $\mathcal{D}_\tau^2(t) := (1 - \ell_\tau(t))^2 d_{\text{sq}}(\overline{U}_\tau, \underline{U}_\tau)^2 \leq d_{\text{sq}}(\overline{U}_\tau, \underline{U}_\tau)^2$. Adding (5.2) and (5.3), and setting $s = t$, we obtain

$$(5.4) \quad \frac{d}{dt} d_{\tau\eta}^2(t, t) + 2\gamma d_{\tau\eta}^2(t, t) \leq \mathcal{E}_\tau(t) + \mathcal{E}_\eta(t),$$

where the residual \mathcal{E}_τ reads $\mathcal{E}_\tau(t) := 2[\phi(\underline{U}_\tau) - \phi(\overline{U}_\tau)] + (|\gamma| + \tau^{-1})d_{\text{sq}}(\underline{U}_\tau, \overline{U}_\tau)^2$. In particular, applying Grönwall's inequality we obtain

$$(5.5) \quad d_{\tau\eta}^2(T, T) \leq e^{4\Lambda(T+\tau)} \int_0^T [\mathcal{E}_\tau(t) + \mathcal{E}_\eta(t)] dt \quad \forall T > 0,$$

where $\Lambda \geq 0$ is a constant.

The first group in the expression for \mathcal{E}_τ is a telescope sum, and hence we get

$$\int_0^{\tau N} 2[\phi(\underline{U}_\tau) - \phi(\overline{U}_\tau)] dt = 2\tau \sum_{n=1}^N [\phi(U_\tau^{n-1}) - \phi(U_\tau^n)] \leq 2\tau\phi(u_0).$$

To estimate the second group we note that $\frac{1}{2\tau}d_{\text{sq}}(\overline{U}_\tau, \underline{U}_\tau)^2 + \phi(\overline{U}_\tau) \leq \phi(\underline{U}_\tau)$, so that

$$(|\gamma| + \tau^{-1})d_{\text{sq}}(\underline{U}_\tau, \overline{U}_\tau)^2 \leq 2(\tau|\gamma| + 1)(\phi(\underline{U}_\tau) - \phi(\overline{U}_\tau)).$$

From the first estimate, we deduce that

$$\int_0^{N\tau} (|\gamma| + \tau^{-1})d_{\text{sq}}(\underline{U}_\tau, \overline{U}_\tau)^2 dt \leq 2(\tau|\gamma| + 1)\tau\phi(u_0).$$

Therefore, there exists a constant $C_\mathcal{E}$, depending only on γ and $\phi(u_0)$ such that

$$(5.6) \quad \int_0^{N\tau} \mathcal{E}_\tau(t) dt \leq C_\mathcal{E}\tau \quad \forall N \in \mathbb{N}, \quad \forall \tau \in (0, \tau_*).$$

We wish to prove that $(\overline{U}_\tau(t))_{\tau \in (0, \tau_*)}$ is a Cauchy sequence in \mathcal{S} for every $t \in [0, T]$. Combining (5.5) and (5.6) we obtain

$$d_{\tau\eta}(t, t)^2 \leq e^{4\Lambda(t+\tau)} C_T(\tau + \eta) \quad \forall t \in [0, T],$$

hence we only need to bound $d_{\text{sq}}(\overline{U}_\tau, \overline{U}_\eta)^2$ in terms of $d_{\tau\eta}^2$. To that end we can again use a result of [1], where it is shown, in the proof of Theorem 4.2.2, that

$$d_{\text{sq}}(\overline{U}_\tau, \overline{U}_\eta)^2 \leq 3d_{\tau\eta}(t, t)^2 + C(\tau + \eta).$$

By completeness of $(\mathcal{S}, d_{\text{sq}})$ (see Lemma 3.5) this shows that there exists a limit curve $u : [0, T] \rightarrow \mathcal{S}$ such that $\overline{U}_\tau(t) \rightarrow u(t)$ in \mathcal{S} for all $t > 0$. Since the constants C and C_T do not depend on $t \in [0, T]$ the convergence is in fact uniform:

$$\max_{t \in [0, T]} d_{\text{sq}}(\overline{U}_\tau(t), u(t)) \leq \widehat{C}_T \tau.$$

In particular, it follows that the piecewise affine interpolant converges, uniformly in $[0, T]$, to the same limit, and hence $u \in C([0, T]; \mathcal{S})$. \square

Having the strong convergence of \bar{U}_τ to an absolutely continuous u , we can now pass to the limit in an integrated version of $(DVI)_\lambda$ to an integrated version of $(EVI)_\lambda$, which is then equivalent to the above differential form of the $(EVI)_\lambda$ [10, 11].

THEOREM 5.2. *Assume that $d_\xi = d_{sq}$ and that W satisfies (3.3) with $m_1 > 1$, (3.4), (3.5), and (4.4). Then, for each initial condition $u_0 \in \mathcal{S}$ with $\phi(u_0) \leq M < \infty$ there exists a unique solution of $(EVI)_\lambda$, where $\lambda \leq \lambda^{W_{sq}} - \lambda_M$.*

Proof. Integrating $(DVI)_\lambda$ from 0 to T , for any $v \in \mathcal{S}$ we obtain

$$\begin{aligned} & \int_0^T \frac{1}{2\tau} \left(d_{sq}(\bar{U}_\tau, v)^2 - d_{sq}(u, v)^2 \right) dt + \int_0^T \left(\phi(\bar{U}_\tau) + \frac{\lambda}{2} d_{sq}(\bar{U}_\tau, v)^2 \right) dt \\ & \leq \int_0^T \phi(v) dt - \int_0^T \frac{1}{2\tau} d_{sq}(u, \bar{U}_\tau)^2 dt, \end{aligned}$$

where $\lambda \leq \lambda^{W_{sq}} - \lambda_M$. Since

$$\frac{1}{2\tau} d_{sq}(u, u_\tau)^2 + \phi(u_\tau) \leq \phi(u),$$

by Proposition 5.1, passing to the limit as $\tau \rightarrow 0_+$ gives

$$\frac{1}{2} \left(d_{sq}(u(T), v)^2 - d_{sq}(u(0), v)^2 \right) + \int_0^T \left(\phi(u(t)) + \frac{\lambda}{2} d_{sq}(u(t), v)^2 \right) dt \leq T \phi(v).$$

By [1, Remark 4.0.5] there exists at most one integral solution to this formulation with prescribed initial datum and it corresponds to

$$\frac{1}{2} \frac{d^+}{dt} d_{sq}(u(t), v)^2 + \frac{\lambda}{2} d_{sq}(u(t), v)^2 \leq \phi(v) - \phi(u(t)),$$

which is equivalent to $(EVI)_\lambda$. \square

5.2. The slope. To connect the evolutionary inequality formulation $(EVI)_\lambda$ with curves of maximal slope, we first study properties of the slope. Under some of our previous assumptions for general $\xi(z) = z^\alpha$ with $\alpha \in [1/2, 1]$ we show that the slope can be characterized and has useful properties, such as lower semicontinuity on $(\mathcal{S}_{2\alpha}, d_\xi)$. For the sake of simplicity, we restrict our result to $\xi(z) = z^\alpha$, but the proof reveals that it is in fact valid under more general conditions involving the regularity and the growth of ξ and ξ^{-1} .

Following [1] the (local) slope $|\partial\phi|$ of ϕ at $u \in D(\phi)$ is defined by

$$(5.7) \quad |\partial\phi|(u) := \limsup_{v \rightarrow u} \frac{(\phi(u) - \phi(v))^+}{d(u, v)}.$$

THEOREM 5.3 (Slope). *Let $\xi(z) = z^\alpha, \alpha \in [1/2, 1]$ and assume that (3.4), (3.5), (4.2), and (4.4) hold. Then for $u \in D(\phi) \cap \mathcal{S}_{2\alpha}$ the slope $|\partial\phi|(u)$ is given by*

$$|\partial\phi|(u) = \begin{cases} \|(W'(u') - C_u)/\xi'(u')\|_2 & \text{for } W'(u')/\xi'(u') \in L^2(0, 1), \\ \infty & \text{otherwise,} \end{cases}$$

where $C_u \in \mathbb{R}$ is such that $\int_0^1 (W'(u') - C_u)/\xi'(u')^2 dx = 0$.

Proof. We first consider the case $W'(u')/\xi'(u') \in L^2(0, 1)$. Since (4.4) is equivalent to saying that $z \mapsto W_\xi(z) - \frac{\lambda^{W_\xi}}{2}|z|^2$ is convex, we have

$$W_\xi(\xi(v')) - \frac{\lambda^{W_\xi}}{2}|\xi(v')|^2 \geq W_\xi(\xi(u')) - \frac{\lambda^{W_\xi}}{2}|\xi(u')|^2 + \partial_{\xi(u')} \left[W_\xi(\xi(u')) - \frac{\lambda^{W_\xi}}{2}|\xi(u')|^2 \right] (\xi(v') - \xi(u')).$$

Using $W_\xi(z) = W(\xi^{-1}(z))$ and hence $W'_\xi(\xi(z)) = W'(z)/\xi'(z)$, one obtains

$$W(v') \geq W(u') + W'(u')(\xi(v') - \xi(u'))/\xi'(u') + \frac{\lambda^{W_\xi}}{2}|\xi(v') - \xi(u')|^2.$$

Using that $W'(u')/\xi'(u')$, $\xi(u')$, and $\xi(v')$ lie in $L^2(0, 1)$, we can integrate and use the Dirichlet boundary conditions to obtain for all $C \in \mathbb{R}$ that

$$\begin{aligned} & \int_0^1 (W(u') - W(v') - C(u' - v')) dx \\ & \leq \int_0^1 \left(\frac{W'(u')}{\xi'(u')} - C \left(\frac{u' - v'}{\xi(u') - \xi(v')} \right) \right) (\xi(u') - \xi(v')) dx + \frac{|\lambda^{W_\xi}|}{2} d_\xi(u, v)^2 \\ & \leq \left\| \frac{W'(u')}{\xi'(u')} - C \left(\frac{u' - v'}{\xi(u') - \xi(v')} \right) \right\|_2 d_\xi(u, v) + \frac{|\lambda^{W_\xi}|}{2} d_\xi(u, v)^2. \end{aligned}$$

Definition (5.7) implies

$$\begin{aligned} |\partial\phi|(u) & \leq \limsup_{v \rightarrow u} \left\| \frac{W'(u')}{\xi'(u')} - C \left(\frac{u' - v'}{\xi(u') - \xi(v')} \right) \right\|_2 \\ & \leq \|(W'(u') - C)/\xi'(u')\|_2 + |C| \limsup_{v \rightarrow u} \left\| \frac{1}{\xi'(u')} - \frac{u' - v'}{\xi(u') - \xi(v')} \right\|_2. \end{aligned}$$

For $\xi(z) = z^\alpha$ with $\alpha \in [1/2, 1]$ the second term on the right-hand side vanishes. Indeed, setting $a = (u')^\alpha$ and $b = (v')^\alpha$ we have $v \rightarrow u$ in (\mathcal{S}, d_ξ) if and only if $b \rightarrow a$ in $L^2(0, 1)$. The case $\alpha = 1$ is trivial. With $\beta = 1/\alpha \in (1, 2]$ we have to show $\frac{a^\beta - b^\beta}{a - b} \rightarrow \beta a^{\beta-1}$ in $L^1(0, 1)$. From $|y^{\beta-1} - \xi^{\beta-1}| \leq |y - \xi|^{\beta-1}$ and $\frac{y^\beta - z^\beta}{y - z} = \beta \xi^{\beta-1}$ with ξ between y and z we obtain the elementary estimate $|\frac{y^\beta - z^\beta}{y - z} - \beta y^{\beta-1}| \leq \beta |y - z|^{\beta-1}$. Now we have

$$\begin{aligned} \left\| \frac{1}{\xi'(u')} - \frac{u' - v'}{\xi(u') - \xi(v')} \right\|_2 & = \left\| \beta a^{\beta-1} - \frac{a^\beta - b^\beta}{a - b} \right\|_2 \\ & \leq \beta \| |a - b|^{\beta-1} \|_2 \leq \beta \| a - b \|_2^{\beta-1} = \beta d_\xi(u, v)^{\beta-1}. \end{aligned}$$

We therefore obtain that

$$|\partial\phi|(u) \leq \|(W'(u') - C)/\xi'(u')\|_2 \quad \forall C \in \mathbb{R}.$$

Minimizing with respect to C we obtain the minimizer C_u as stated above.

To prove the lower bound, consider $v_s \in \mathcal{S}$ where

$$v_s(x) = (\text{id} + s\varphi)(u(x)), \quad \varphi \in C^1([0, 1]), \quad \varphi(0) = \varphi(1) = 0,$$

such that $|\varphi'| \leq K$. We assume throughout that $s < 1/K$. Since $|\xi(v'_s)| \leq (1 + Ks)^\alpha |\xi(u')|$, Lebesgue's dominated convergence theorem yields $d_\xi(u, v_s) \rightarrow 0$ as $s \rightarrow 0$, which implies

$$|\partial\phi|(u) \geq \limsup_{s \rightarrow 0} \frac{\phi(u) - \phi(v_s)}{d_\xi(u, v_s)}.$$

For the denominator we use $\xi(z) = z^\alpha$ and $v'_s = u'(1 + s\varphi')$ and obtain

$$\begin{aligned} d_\xi(u, v_s)^2 &= \int_0^1 ((u')^\alpha(1 + s\varphi')^\alpha - (u')^\alpha)^2 dx = \int_0^1 (u')^{2\alpha} ((1 + s\varphi')^\alpha - 1)^2 dx \\ &= s^2 \int_0^1 \alpha^2 (u')^{2\alpha} (\varphi')^2 dx + O(s^3) = s^2 \|\xi'(u')u'(\varphi' \circ u)\|_2^2 + O(s^3). \end{aligned}$$

For the numerator we use $\int_0^1 (u' - v'_s) dx = 0$ and the special form of v_s and find

$$\begin{aligned} \limsup_{s \rightarrow 0} \frac{1}{s} \int_0^1 (W(u') - W(v'_s)) dx &= \limsup_{s \rightarrow 0} \frac{1}{s} \int_0^1 (W(u') - W(u'(1 + s\varphi'(u)))) dx \\ &\geq \int_0^1 \liminf_{s \rightarrow 0} \left(\frac{W(u') - W(u'(1 + s\varphi'(u)))}{s} \right) dx \\ &= \int_0^1 (-W'(u'(x)))u'(x)\varphi'(u(x)) dx. \end{aligned}$$

Changing variables, we obtain that the slope admits the lower bound

$$|\partial\phi|(u) \geq \frac{\int_0^1 a(r)\varphi'(r) dr}{\|\varphi'/b\|_2} \quad \forall \varphi \in C_0^1([0, 1]) \quad \text{with}$$

$$a(r) := -W'(u'(u^{-1}(r))) \quad \text{and} \quad b(r) := \frac{1}{(\sqrt{u'}(\xi' \circ u')) \circ u^{-1}(r)}.$$

We emphasize that due to (3.5) and $u \in D(\phi)$, $a \in L^1$ and hence the numerator is well-defined. Moreover, using the fact that $u' \in L^{2\alpha}$ it is easy to see that $0 < b, 1/b \in L^2$ and hence the denominator is also well-defined.

Thus, we can apply the following lemma, and the desired result follows. \square

LEMMA 5.4. Consider $a, b : (0, 1) \rightarrow \mathbb{R}$ with $a \in L^1(0, 1)$ and $0 < b, 1/b \in L^2(0, 1)$. Let $H(a, b, \varphi) = \int_0^1 a\varphi' dr / \|\varphi'/b\|_2$ and $c_{a,b} = \int_0^1 ab^2 dr / \int_0^1 b^2 dr$; then

$$\sup \{H(a, b, \varphi) \mid \varphi \in C_c^\infty(0, 1)\} = M(a, b) := \begin{cases} \|b(a - c_{a,b})\|_2 & \text{if } ab \in L^2(0, 1), \\ \infty & \text{else.} \end{cases}$$

Proof. Step 1. We first note that the supremum can also be taken over $\varphi \in W_0^{1,\infty}(0, 1) = C_0^{\text{Lip}}([0, 1])$, because $C_c^\infty(0, 1)$ is weakly* dense and $H(a, b, \cdot)$ is weakly* continuous.

Step 2. For $ab \in L^2(0, 1)$ the Cauchy–Schwarz inequality gives

$$H(a, b, \varphi) = \frac{\int_0^1 b(a - c) (\varphi'/b) dr}{\|\varphi'/b\|_2} \leq \|b(a - c)\|_2 \quad \forall c \in \mathbb{R}.$$

Minimizing with respect to c yields the upper bound $\sup_\varphi H(a, b, \varphi) \leq M(a, b)$.

Step 3. For $k \in \mathbb{N}$ we define $\chi_k(x) = 1$ if $|a(x)| < k$ and $b(x) < k$ and $\chi_k(x) = 0$ otherwise. Then, $\chi_k a, \chi_k b \in L^\infty(0, 1)$. We define $\varphi_k \in W_0^{1,\infty}(0, 1)$ via

$$\varphi'_k = \chi_k b^2(a - c_k) \quad \text{with } c_k = \int_0^1 \chi_k ab^2 dx \Big/ \int_0^1 \chi_k b^2 dx.$$

Using $\chi_k^2 = \chi_k$ we easily find $H(a, b, \varphi_k) = \int_0^1 (a - c_k) \varphi'_k dx / \|\varphi'_k/b\|_2 = \|\chi_k b(a - c_k)\|_2$. If $ab \in L^2(0, 1)$, then $c_k \rightarrow c_{a,b}$ and $\chi_k ab \rightarrow ab$ and $\chi_k b \rightarrow b$ in $L^2(0, 1)$ strongly. Hence we find the lower bound

$$\sup_\varphi H(a, b, \varphi) \geq \lim_{k \rightarrow \infty} H(a, b, \varphi_k) = \lim_{k \rightarrow \infty} \|\chi_k b(a - c_k)\|_2 = \|b(a - c_{a,b})\|_2 = M(a, b).$$

Step 4. Assume now $ab \notin L^2(0, 1)$, which means $\|\chi_k ab\|_2 \rightarrow \infty$. We define

$$u_k = \frac{1}{\|\chi_k b\|_2} \chi_k b \quad \text{and} \quad w_k = \frac{1}{\|\chi_k ab\|_2} \chi_k ab, \quad \text{giving } \|u_k\|_2 = \|w_k\|_2 = 1.$$

First we have $u_k \rightarrow \frac{1}{\|b\|_2} b$ in $L^2(0, 1)$, and second $w_k \rightarrow 0$ in $L^2(0, 1)$. Indeed,

$$\int_0^1 w_k (\chi_m v) dx = \frac{1}{\|\chi_k ab\|_2} \int_0^1 \chi_m ab v dx \xrightarrow{k \rightarrow \infty} 0 \quad \forall m \in \mathbb{N} \text{ and } v \in L^2(0, 1),$$

since $\chi_k \chi_m = \chi_m$ for $k \geq m$ and $\|\chi_k ab\|_2 \rightarrow \infty$. Because the functions $\chi_m v$ are dense in $L^2(0, 1)$, the proof of $w_k \rightarrow 0$ is complete.

Rearranging the terms in $H(a, b, \varphi_k)$ gives

$$H(a, b, \varphi_k)^2 = \int_0^1 \chi_k a^2 b^2 dx - \frac{\left(\int_0^1 \chi_k ab^2 dx\right)^2}{\int_0^1 \chi_k b^2 dx} = \|\chi_k ab\|_2^2 \left(1 - \left(\int_0^1 u_k w_k dx\right)^2\right).$$

Using $\|\chi_k ab\|_2^2 \rightarrow \infty$ and $\int_0^1 u_k w_k dx \rightarrow 0$ we conclude that $H(a, b, \varphi_k) \rightarrow \infty$, which is the desired lower estimate $\sup_\varphi H(a, b, \varphi) = \infty = M(a, b)$. \square

In the case $d = d_{\text{sq}}$ the situation is again better, as we have λ^ϕ -convexity along our generalized geodesic curves.

PROPOSITION 5.5. *Let $\xi(z) = \sqrt{z}$ and let ϕ and W satisfy (4.2), (3.4), and (4.4). Then, the slope $|\partial\phi|(u)$ as given in Theorem 5.3 is a strong upper gradient and is d_{sq} -lower semicontinuous.*

Proof. We slightly modify the proof of [1, Theorem 2.4.9] to prove that

$$(5.8) \quad |\partial\phi|(v) = \sup_{\substack{w \neq v \\ \phi(w) \leq \phi(v)+1}} \left(\frac{\phi(v) - \phi(w)}{d_{\text{sq}}(v, w)} + \frac{\lambda^\phi}{2} d_{\text{sq}}(v, w) \right)^+,$$

where λ^ϕ is given by (4.6) with $M = \phi(v) + 1$. Once this is established the d_{sq} -lower semicontinuity and strong upper gradient properties can be easily established by following the proof of [1, Corollary 2.4.10].

Let $\phi(w) \leq \phi(v) + 1$ and $v_s := U_{\text{sq}}(s; v, w)$; then (4.6) and a straightforward computation imply

$$\left(\frac{\phi(v) - \phi(v_s)}{d_{\text{sq}}(v, v_s)} \right)^+ \geq \left(\frac{\phi(v) - \phi(w)}{d_{\text{sq}}(v, w)} + \frac{1}{2} \lambda^\phi (1 - s) d_{\text{sq}}(v, w) \right)^+ \frac{s d_{\text{sq}}(v, w)}{d_{\text{sq}}(v, v_s)}.$$

Letting $g(s) = d_{\text{sq}}(v_s, v_0)^2$, by (3.19b) and (3.19c) we have

$$\begin{aligned} g(s) &= g(0) + g'(0)s + \frac{1}{2}g''(0)s^2 + o(s^2) = \frac{1}{2}g''(0)s^2 + o(s^2) \\ &\leq \frac{1}{2} 2 d_{\text{sq}}(v_0, v_1)^2 s^2 + o(s^2) \\ &= (d_{\text{sq}}(v_0, v_1)^2 + o(1))s^2. \end{aligned}$$

Hence,

$$d_{\text{sq}}(v_s, v_0) = \left(\sqrt{d_{\text{sq}}(v_0, v_1)^2 + o(1)} \right) s = d_{\text{sq}}(v_0, v_1)s + o(s).$$

This implies

$$\begin{aligned} |\partial\phi|(v) &\geq \limsup_{s \rightarrow 0} \left(\frac{\phi(v) - \phi(v_s)}{d(v, v_s)} \right)^+ \\ &\geq \left(\frac{\phi(v) - \phi(w)}{d(v, w)} + \frac{1}{2}\lambda^\phi d(v, w) \right)^+. \end{aligned}$$

Taking the supremum over w we obtain a lower bound in (5.8).

Conversely, the upper bound is an immediate consequence of the definition of the slope. \square

5.3. Curves of maximal slope. Following [1], let \mathcal{S} be a metric space with distance d and let $\phi : \mathcal{S} \rightarrow \mathbb{R} \cup \{+\infty\}$. An absolutely continuous curve $u : (0, T) \rightarrow \mathcal{S}$ is a curve of maximal slope for the functional ϕ with respect to an upper gradient $g : \mathcal{S} \rightarrow \mathbb{R} \cup \{+\infty\}$ if

$$(5.9) \quad \frac{d}{dt}\phi(u(t)) \leq -\frac{1}{2}|u'(t)|^2 - \frac{1}{2}g^2(u(t))$$

for a.e. $t \in (0, T)$, where $|u'|$ denotes the metric derivative,

$$(5.10) \quad |u'(t)| := \lim_{s \rightarrow t} \frac{d(u(s), u(t))}{|s - t|},$$

which exists for a.e. $t \in (0, T)$.

Next we show that all solutions of the $(\text{EVI})_\lambda$ are in fact curves of maximal slope. This can again be done for general ξ . Note that also for d_ξ different from d_{sq} we may have solutions of $(\text{EVI})_\lambda$, e.g., by assuming that $u'(t, x)$ takes only finitely many values; cf. [29].

THEOREM 5.6. *Assume that $d = d_{\text{sq}}$ and W satisfies (3.4), (3.5), and (4.4). Then the solution u to $(\text{EVI})_\lambda$ is a curve of maximal slope, that is, $\phi \circ u \in AC_{\text{loc}}$, and*

$$(5.11) \quad \frac{d}{dt}\phi(u(t)) \leq -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial\phi|^2(u(t)) \quad \text{for a.e. } t > 0.$$

Proof. From Proposition 5.5 we know that $|\phi|$ is a strong upper gradient and d_{sq} -lower semicontinuous. Since we know from Theorem 5.2 that the unique solution u to $(\text{EVI})_\lambda$ is a minimizing movement, we can apply Theorem 2.3.3 in [1] to obtain that $u \in AC_{\text{loc}}^2$, $\phi \circ u \in AC$, and u is a curve of maximal slope. \square

5.4. Weak solutions of one-dimensional viscoelasticity. Finally in this section we show that curves of maximal slope give rise to weak solutions of the partial differential equation of one-dimensional viscoelasticity.

THEOREM 5.7. *Let $\xi(z) = z^\alpha$ with $\alpha \in [1/2, 1]$ and assume that W satisfies (3.4), (3.5), and (4.4). Also assume that $u(t) \in C^1([0, T]; C^1[0, 1]) \cap AC^2([0, T]; \mathcal{S})$, that $u'(t, x) \geq \delta > 0$ for all $(x, t) \in [0, 1] \times [0, T]$, and that u is a curve of maximal slope for the functional ϕ with respect to its upper gradient $|\partial\phi|$. Then, u solves the partial differential equation*

$$(5.12) \quad (W'(u'(t, x)) + (\xi'(u'(t, x)))^2 \partial_t u'(t, x))' = 0.$$

Proof. We apply the usual “trick” to show the equivalence of curves of maximal slopes and gradient flows on Hilbert spaces. Dropping the dependence of u on x for notational simplicity, we begin by computing a bound on the metric derivative defined by (5.10):

$$(5.13) \quad \begin{aligned} |u'| (t) &= \liminf_{s \rightarrow t} \left(\int_0^1 \frac{(\xi(u'(s)) - \xi(u'(t)))^2}{|s - t|^2} dx \right)^{1/2} \\ &\geq \left(\int_0^1 \liminf_{s \rightarrow t} \left(\frac{(\xi(u'(s)) - \xi(u'(t)))^2}{|s - t|} \right) dx \right)^{1/2} \\ &= \left(\int_0^1 (\partial_t \xi(u'(t)))^2 dx \right)^{1/2} = \|[\xi'(u'(t))]^{-1} \partial_t u(t)\|_2. \end{aligned}$$

Provided that $W'(u'(t)) \partial_t u'(t) \in L^1(0, 1)$ we have

$$\frac{d}{dt} \phi(u(t)) = \int_0^1 \frac{d}{dt} W(u') dx = \int_0^1 W'(u') \partial_t u'(t) dx.$$

However, we know that

$$\begin{aligned} \int_0^1 W'(u') \partial_t u' dx &= \int_0^1 (W'(u') - C) [\xi'(u'(t))]^{-1} \xi'(u'(t)) \partial_t u'(t) dx \\ &\leq \frac{1}{2} \| (W'(u') - C) [\xi'(u'(t))]^{-1} \|_2^2 + \frac{1}{2} \| \xi'(u'(t)) \partial_t u'(t) \|_2^2 \\ &\leq \frac{1}{2} |\partial\phi|^2(u(t)) + \frac{1}{2} |u'|^2(t). \end{aligned}$$

Since $u' \geq 0$ for a.e. $x \in (0, 1)$, using the Cauchy–Schwarz and Young’s inequalities we can continue by estimating

$$\begin{aligned} \frac{d}{dt} \phi(u(t)) &= \int_0^1 (W'(u') - C) [\xi'(u'(t))]^{-1} \xi'(u'(t)) \partial_t u'(t) dx \\ &\geq - \| (W'(u') - C) [\xi'(u'(t))]^{-1} \|_2 \| \xi'(u'(t)) \partial_t u'(t) \|_2 \\ &\geq - \frac{1}{2} \| (W'(u') - C) [\xi'(u'(t))]^{-1} \|_2^2 - \frac{1}{2} \| \xi'(u'(t)) \partial_t u'(t) \|_2^2. \end{aligned}$$

Applying Theorem 5.3 and (5.13) we obtain

$$\begin{aligned} \frac{d}{dt} \phi(u(t)) &\geq - \frac{1}{2} \| (W'(u') - C) [\xi'(u'(t))]^{-1} \|_2^2 - \frac{1}{2} \| \xi'(u'(t)) \partial_t u'(t) \|_2^2 \\ &\geq - \frac{1}{2} |\partial\phi|^2(u(t)) - \frac{1}{2} |u'|^2(t) \geq \frac{d}{dt} \phi(u(t)), \end{aligned}$$

where in the last inequality we used (5.11). Hence, all inequalities that we employed in this proof are in fact equalities yielding

$$(W'(u') - C) [\xi'(u'(t))]^{-1} = \pm(\xi'(u'(t)) \partial_t u'(t)).$$

Since the energy is decreasing along the trajectory it follows that, in fact,

$$W'(u') - C + (\xi'(u'(t)))^2 \partial_t u'(t) = 0,$$

as required. \square

Appendix A. The one-dimensional case with a Neumann boundary condition. Throughout this paper, we considered gradient flows for deformations with prescribed Dirichlet boundary conditions $u(t, 0) = 0$ and $u(t, 1) = 1$. In this appendix, we briefly summarize the much simpler case of a free boundary, i.e., we keep the boundary condition $u(t, 0) = 0$ and leave $u(t, 1)$ free, giving rise to the natural Neumann boundary condition.

We work with the same setup as in Section 3. For a strictly increasing, continuous function $\xi : [0, \infty) \rightarrow [0, \infty)$ with $\xi(0) = 0$ we consider the metric d_ξ introduced in (3.1) and choose the state space

$$\mathcal{S}_\xi^{\text{free}} := \{u \in W^{1,1}([0, 1]) \mid u(0) = 0, \xi(u') \in L^2(0, 1)\}.$$

As in the Dirichlet case, if ξ satisfies the lower bound

$$(A.1) \quad \xi(0) = 0 \quad \text{and} \quad \exists C > 0 \forall z \geq 0: \xi(z) \geq \frac{1}{C}\sqrt{z} - C,$$

then the metric space $(\mathcal{S}_\xi^{\text{free}}, d_\xi)$ will be complete, but now it is even a geodesic space, i.e., between each two points there exists a geodesic curve. In fact, given $u_0, u_1 \in \mathcal{S}_\xi^{\text{free}}$, we define the connecting curve $u_s = U_\xi^{\text{N}}(s; u_0, u_1)$ via

$$(A.2) \quad u_s(x) = \int_0^x \xi^{-1}((1-s)\xi(u'_0(r)) + s\xi(u'_1(r))) \, dr,$$

and a straightforward calculation yields $d_\xi(u_s, u_t) = |t-s|d_\xi(u_0, u_1)$, i.e., $s \mapsto u_s$ is a constant-speed geodesic; see [1]. The completeness of $(\mathcal{S}_\xi^{\text{free}}, d_\xi)$ as well as geodesic 1-convexity of d_ξ^2 are obtained in the following result.

PROPOSITION A.1. *If $\xi : [0, \infty) \rightarrow \mathbb{R}$ is continuous and strictly increasing and satisfies (A.1), then the following statements are true:*

- (i) $(\mathcal{S}_\xi^{\text{free}}, d_\xi)$ is a complete metric space.
- (ii) The mapping $\varphi : \mathcal{S}_\xi^{\text{free}} \rightarrow L^2_{\geq 0}(0, 1) := \{u \in L^2(0, 1) \mid u(x) \geq 0 \text{ a.e.}\}$, $\varphi(u) = \xi \circ u'$, is bijective and metric preserving if $L^2_{\geq 0}(0, 1)$ is equipped with the standard L^2 -norm.
- (iii) Let $\{u_s\}_{0 \leq s \leq 1}$ be the geodesic curve defined in (A.2) and $v \in \mathcal{S}_\xi^{\text{free}}$; then

$$d_\xi(u_s, v)^2 = (1-s)d_\xi(u_0, v)^2 + s d_\xi(u_1, v)^2 - \frac{1}{2}s(1-s)d_\xi(u_0, u_1)^2.$$

Proof. (ii) This statement follows directly from the properties of ξ and the definition of $\mathcal{S}_\xi^{\text{free}}$. Note that (A.1) implies $\xi^{-1}(y) \leq C(1+y^2)$, hence $y \in L^2(0, 1)$ implies $\xi^{-1} \circ y \in L^1(0, 1)$ and the inverse mapping φ^{-1} is well-defined via $u = \varphi^{-1}(y) : x \mapsto \int_0^x \xi^{-1}(y(r)) \, dr$.

(i) The completeness follows from (ii) and the completeness of $(L^2_{\geq 0}(0, 1), \|\cdot\|_{L^2})$.

(iii) This identity is an immediate consequence of the definition of d_ξ and u_s . Of course, the relation also follows from the Hilbert space structure of $(L^2_{\geq 0}(0, 1), \|\cdot\|_{L^2})$ and the metric-preserving mapping φ . \square

Remark A.2. The conditions (A.1) can be generalized considerably by assuming that ξ is a continuous, strictly increasing bijection between two closed intervals I and J in \mathbb{R} , like the function $\xi(z) = \log z$ with $I = (0, \infty)$ and $J = \mathbb{R}$ considered in [29]. Then, the metric space $(\mathcal{S}_\xi^{\text{free}}, d_\xi)$ will no longer be complete. Under suitable coercivity conditions on the energy density W for the energy functional ϕ , it is easy to show that all sublevels $\{u \mid \phi(u) \leq C\}$ are contained in a closed subset \mathcal{A}_C . Because of the energy decay of the time-discrete and time-continuous gradient flows the whole analysis can be done in the complete metric space (\mathcal{A}_C, d_ξ) .

Let $\phi : \mathcal{S}_\xi^{\text{free}} \rightarrow [0, +\infty]$ be of the form (4.2) with W satisfying the coercivity condition (3.3), the lower semicontinuity condition (3.4), and the λ^{W_ξ} -convexity condition (4.4). Then, we immediately see that ϕ is geodesically λ^{W_ξ} convex, i.e., $s \mapsto \phi(U_\xi^N(s; u_0, u_1))$ is λ^{W_ξ} -convex as a mapping from $[0, 1]$ to \mathbb{R} . Moreover, let

$$\Phi(\tau, v; u) := \frac{1}{2\tau} d_\xi^2(u, v) + \phi(u) \quad \text{for } u, v \in \mathcal{S}_\xi^{\text{free}} \text{ and } \tau > 0.$$

Then Proposition A.1(iii) and (iv) immediately imply that

$$\Phi(\tau, v; u_s) \leq (1 - s)\Phi(\tau, v; u_0) + s\Phi(\tau, v; u_1) - \left(\lambda^{W_\xi} + \frac{1}{2\tau}\right) s(1 - s)d_\xi^2(u_0, u_1).$$

Thus, we can now directly apply the results of [1, Section 4] to obtain existence and uniqueness of solutions of the evolutionary variational inequality (compare with the beginning of Section 5)

$$(A.3) \quad \frac{1}{2} \frac{d^+}{dt} d_\xi^2(u(t), v) + \frac{\lambda^{W_\xi}}{2} d_\xi^2(u(t), v) \leq \phi(v) - \phi(u(t))$$

with $u : [0, \infty) \rightarrow \mathcal{S}_\xi^{\text{free}}$ absolutely continuous and $u(0) = u_0 \in \mathcal{S}_\xi^{\text{free}}$. In particular, we obtain Lipschitz continuity of the semiflow, i.e., for any two solutions u_1 and u_2 and $0 \leq s < t$ we have

$$d_\xi(u_1(t), u_2(t)) \leq e^{\lambda^{W_\xi}(t-s)} d_\xi(u_1(s), u_2(s)).$$

We emphasize that for these results it is sufficient to assume that the constitutive functions W and ξ are merely continuous.

If W and ξ are both differentiable, all results of Sections 5.2, 5.3, and 5.4 are readily extended to the present setting (indeed some arguments can be considerably simplified). The slope of ϕ is again a strong upper gradient and is now given by

$$|\partial\phi|(u) = \begin{cases} \|W'(u')/\xi'(u')\|_2 & \text{for } W'(u')/\xi'(u') \in L^2(0, 1), \\ \infty & \text{otherwise.} \end{cases}$$

(The constant C_u in Theorem 5.3 is removed due to the absence of the mean-zero condition $\int_0^1 u' - v' dx = 0$ for all $u, v \in \mathcal{S}_\xi^{\text{free}}$.) The solution to the variational inequality (A.3) is again a curve of maximal slope (cf. Theorem 5.6) and under suitable regularity assumptions solves the boundary value problem (5.12).

In particular, the solutions $u : [0, \infty) \rightarrow \mathcal{S}_\xi^{\text{free}}$ can be understood as weak solutions of the partial differential equation

$$(A.4) \quad \begin{aligned} 0 &= (W'(u'(t, x)) + (\xi'(u'(t, x)))^2 \partial_t u'(t, x))', \\ u(t, 0) &= 0, \quad (W'(u'(t, x)) + (\xi'(u'(t, x)))^2 \partial_t u'(t, x))|_{x=1} = 0. \end{aligned}$$

In fact, the transformation $y = \varphi(u) : x \mapsto \xi(u'(x))$ maps the metric gradient system $(\mathcal{S}_\xi^{\text{free}}, \phi, d_\xi)$ bijectively into the gradient system $(L_{\geq 0}^2(0, 1), \psi, \|\cdot\|_{L^2})$, where the transformed energy ψ is given by $\psi(y) = \int_0^1 W_\xi(y(x)) dx$. Hence, the classical L^2 gradient flow gives the partial differential equation

$$\partial_t y(t, x) = W'_\xi(y(t, x)),$$

which is in fact an ordinary differential equation for each x . Clearly the latter equation transforms into (A.4) using the transformation φ^{-1} .

REFERENCES

- [1] L. AMBROSIO, N. GIGLI, AND G. SAVARÉ, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures Math. ETH Zürich, Birkhäuser, Basel, 2005.
- [2] S. S. ANTMAN, *Physically unacceptable viscous stresses*, Z. Angew. Math. Phys., 49 (1998), pp. 980–988.
- [3] S. S. ANTMAN, *Nonlinear Problems of Elasticity*, Appl. Math. Sci. 107, Springer-Verlag, New York, 2004.
- [4] J. M. BALL, *Constitutive inequalities and existence theorems in nonlinear elastostatics*, in Nonlinear Analysis and Mechanics: Heriot-Watt Symposium (Edinburgh, 1976), vol. I, Res. Notes Math. 17, Pitman, London, 1977, pp. 187–241.
- [5] J. M. BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal., 63 (1977), pp. 337–403.
- [6] J. M. BALL, *Minimizers and the Euler-Lagrange equations*, in Trends and Applications of Pure Mathematics to Mechanics (Palaiseau, 1983), Lecture Notes in Phys. 195, Springer, Berlin, 1984, pp. 1–4.
- [7] J. M. BALL, *Some open problems in elasticity*, in Geometry, Mechanics, and Dynamics, P. Newton, P. Holmes, and A. Weinstein, eds., Springer, New York, 2002, pp. 3–59.
- [8] J. M. BALL AND Y. ŞENGÜL, *Quasistatic nonlinear viscoelasticity and gradient flows*, submitted.
- [9] C. M. DAFERMOS, *The mixed initial-boundary value problem for the equations of nonlinear one-dimensional viscoelasticity*, J. Differential Equations, 6 (1969), pp. 71–86.
- [10] S. DANERI AND G. SAVARÉ, *Eulerian calculus for the displacement convexity in the Wasserstein distance*, SIAM J. Math. Anal., 40 (2008), pp. 1104–1122.
- [11] S. DANERI AND G. SAVARÉ, *Lecture Notes on Gradient Flows and Optimal Transport*, arXiv:1009.3737v1, 2010.
- [12] S. DEMOULINI, *Weak solutions for a class of nonlinear systems of viscoelasticity*, Arch. Ration. Mech. Anal., 155 (2000), pp. 299–334.
- [13] G. FRANCFORT AND A. MIELKE, *Existence results for a class of rate-independent material models with nonconvex elastic energies*, J. Reine Angew. Math., 595 (2006), pp. 55–91.
- [14] G. FRIESECKE AND G. DOLZMANN, *Implicit time discretization and global existence for a quasi-linear evolution equation with nonconvex energy*, SIAM J. Math. Anal., 28 (1997), pp. 363–380.
- [15] G. FRIESECKE, R. D. JAMES, AND S. MÜLLER, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*, Comm. Pure Appl. Math., 55 (2002), pp. 1461–1506.
- [16] M. GIAQUINTA, G. MODICA, AND J. SOUČEK, *Cartesian Currents in the Calculus of Variations: Variational Integrals*, Springer-Verlag, Berlin, 1998.
- [17] J. M. GREENBERG, R. C. MACCAMY, AND V. J. MIZEL, *On the existence, uniqueness, and stability of solutions of the equation $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$* , J. Math. Mech., 17 (1968), pp. 707–728.
- [18] K. KUTTLER AND D. HICKS, *Initial-boundary value problems for the equation $u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f$* , Quart. Appl. Math., 66 (1988), pp. 393–407.

- [19] A. MAINIK AND A. MIELKE, *Global existence for rate-independent gradient plasticity at finite strain*, J. Nonlinear Sci., 19 (2009), pp. 221–248.
- [20] A. MIELKE, *Necessary and sufficient conditions for polyconvexity of isotropic functions*, J. Convex Anal., 12 (2005), pp. 291–314.
- [21] A. MIELKE, *Formulation of thermoelastic dissipative material behavior using GENERIC*, Contin. Mech. Thermodyn., 23 (2011), pp. 233–256.
- [22] A. NOVICK-COHEN AND R. L. PEGO, *Stable patterns in a viscous diffusion equation*, Trans. Amer. Math. Soc., 324 (1991), pp. 331–351.
- [23] H. C. ÖTTINGER, *Beyond Equilibrium Thermodynamics*, John Wiley, New York, 2005.
- [24] R. L. PEGO, *Phase transitions in one-dimensional nonlinear viscoelasticity: Admissibility and stability*, Arch. Ration. Mech. Anal., 97 (1987), pp. 353–394.
- [25] M. POTIER-FERRY, *The linearization principle for stability in quasilinear parabolic equations I*, Arch. Ration. Mech. Anal., 77 (1981), pp. 301–320.
- [26] M. POTIER-FERRY, *On the mathematical foundations of elastic stability theory I*, Arch. Ration. Mech. Anal., 78 (1982), pp. 55–72.
- [27] P. RYBKA, *Dynamical modelling of phase transitions by means of viscoelasticity in many dimensions*, Proc. Roy. Soc. Edinburgh Sect. A, 121 (1992), pp. 101–138.
- [28] P. RYBKA, *A priori estimates for the gradient of the solution to the system of viscoelasticity in several dimensions*, Topol. Methods Nonlinear Anal., 3 (1994), pp. 235–256.
- [29] Y. ŞENGÜL, *Well-Posedness of Dynamics of Microstructure in Solids*, Ph.D. thesis, University of Oxford, Oxford, UK, 2010.
- [30] M. ŠILHAVÝ, *Semiconvexity of invariant functions of rectangular matrices*, Calc. Var. Partial Differential Equations, 17 (2003), pp. 75–84.
- [31] B. TVEDT, *Quasilinear equations for viscoelasticity of strain-rate type*, Arch. Ration. Mech. Anal., 189 (2008), pp. 237–281.
- [32] S. J. WATSON, *Unique global solvability for initial-boundary problems in one-dimensional nonlinear thermoviscoelasticity*, Arch. Ration. Mech. Anal., 153 (2000), pp. 1–37.