Problem sheet 1

Problem 1.1 (WLLN for Bernoulli)**.** Let $X_{i,j}^{(n)}$ be independent Bernoulli random variables with expectation $p_{i,j}^{(n)}$, such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} p_{i,j}^{(n)} = c.$$

Show that, in probability,

$$\frac{1}{n} \sum_{i,j=1}^{n} X_{i,j}^{(n)} \longrightarrow c.$$

Problem 1.2 (Coupling)**.**

(a) Let X and Y be two real random variables with distribution function F_X and F_Y, such that $F_X(x) \leq F_Y(x)$ for all $x \in \mathbb{R}$. Show that there exists a probability space, on which X and Y can jointly be defined such that $X \geq Y$ almost surely. We call this *coupling*.

(b) Let $(\mathcal{G}_n)$ and $(\overline{\mathcal{G}}_n)$ be two sequences of inhomogeneous random graphs with kernels κ_n and $\overline{\kappa}_n$, respectively. Assume that $\kappa_n(x_i, x_j) \leq \overline{\kappa}_n(x_i, x_j)$ for all $n \in \mathbb{N}$ and $x_i, x_j \in \mathscr{S}$. Show that both graph sequences can be coupled in a way that $\mathcal{E}_n \subset \overline{\mathcal{E}}_n$, i.e., each edge appearing in $\mathcal{G}_n$ also appears in $\overline{\mathcal{G}}_n$.

Problem 1.3. Let Λ be a positive random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let X be a random variable that is, conditioned on Λ, Poisson distributed with mean Λ. Put differently, X is mixed-Poisson distributed with mixing distribution $\mathbb{P} \circ \Lambda^{-1}$

(a) Assume that Λ has a density f satisfying

$$cx^{-\tau} \leq f(x) \leq Cx^{-\tau}, \qquad \text{for all } x \geq A,$$

where $\tau > 2$ and $0 < c < C < \infty$ and $A > 0$ is some bound. Show that there exist constants $0 < c' < C' < \infty$ such that

$$c' x^{-\tau} \leq \mathbb{P}(X = x) \leq C' x^{-\tau}.$$

<u>Hint:</u> The Γ-function is defined as $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \mathrm{d}x$ and it has the following useful property $\frac{\Gamma(k-\tau)}{\Gamma(k)} \sim k^{-\tau}$, as $k \to \infty$.

(b) Assume that there are $\lambda_1, \dots, \lambda_m > 0$ such that $\mathbb{P}(\Lambda \in \{\lambda_1, \dots, \lambda_m\}) = 1$. Show that X is light-tailed.

Problem 1.4. Prove or falsify the following statements:

(a) $\log^p n = o(n^{1/p})$, as $n \to \infty$, for any $p > 1$.

(b) $3^n = O(2^n)$, as $n \to \infty$.

(c) $\sum_{i=0}^{n} i! \asymp n!$, as $n \to \infty$.

(d) Let $X^{(n)}$ be a binomial random variable with n trials and success probability $p^{(n)} = 9/n^2$, then $\mathbb{E}X^{(n)} = O(1/n)$.

(e) $\sin(1/x) \sim 1/x$, as $x \to \infty$.