Non-triviality of the phase transition in inhomogeneous long-range percolation in dimension one

joint work with Peter Gracar and Christian Mönch

Lukas Lüchtrath, 16/03/2022

Percolation

- Let $(\mathscr{G}(\beta) : \beta > 0)$ be a family of locally finite graphs with vertex set given by a Poisson process (on $\mathbb{R}^d$) where β is an edge-density parameter such that larger β leads to more edges on average

- The connection mechanism depends on the spatial distance of the vertices in a way that short edges are more likely than long edges

- Standard question in percolation theory: Is there a $\beta_c \in (0,\infty)$ such that

    - if $\beta > \beta_c$ then $\mathscr{G}(\beta)$ contains an infinite connected component but

    - if $\beta < \beta_c$ then does not contain an infinite connected component?

- Pete's talk: Is it possible to have $\beta_c = 0$ (when combining long-range edges and heavy-tailed degree distributions)?

Percolation

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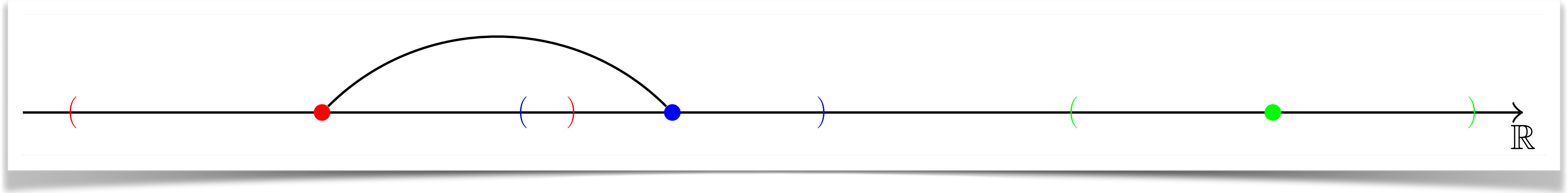
  - if $\beta > \beta_c$ then $\mathcal{G}(\beta)$ contains an infinite connected component but
  - if $\beta < \beta_c$ then does not contain an infinite connected component?
- Pete's talk: Is it possible to have $\beta_c = 0$ (when combining long-range edges and heavy-tailed degree distributions)?
- In this talk we consider $d = 1$ and ask whether there is a supercritical phase or $\beta_c < \infty$.

Percolation

- In $d = 1$ the situation is special because of spatial restrictions
  - Question: Is $\beta_c < \infty$ possible?
- Models without supercritical phase:
  - 1-d-Gilbert's disc model: Connect two vertices whenever their distance is beneath a threshold β
  - Any model with bounded edge length and some independence in the connection mechanism
  - (Poisson) Boolean model: Assign i.i.d. and integrable radii to the vertices and connect two vertices when the associated balls intersect.



Percolation

- In $d = 1$ the situation is special because of spatial restrictions
  - Question: Is $\beta_c < \infty$ possible?
- Long-range percolation model (random connection model):
  - Given the points connect any pair of vertices independently with probability proportional to $(|x - y|/\beta)^{-\delta}$ for $\delta > 1$
  - If $\delta \in (1,2]$ then $\beta_c < \infty$ but
  - If $\delta > 2$ then $\beta_c = \infty$.

Weight-dependent random connection model

- The vertex set is a Poisson point process on $\mathbb{R} \times (0,1)$
- Connect two vertices (x, t) and (y, s) (independently) with probability

$$\rho(\beta^{-1} g(s, t) \, |x - y|)$$

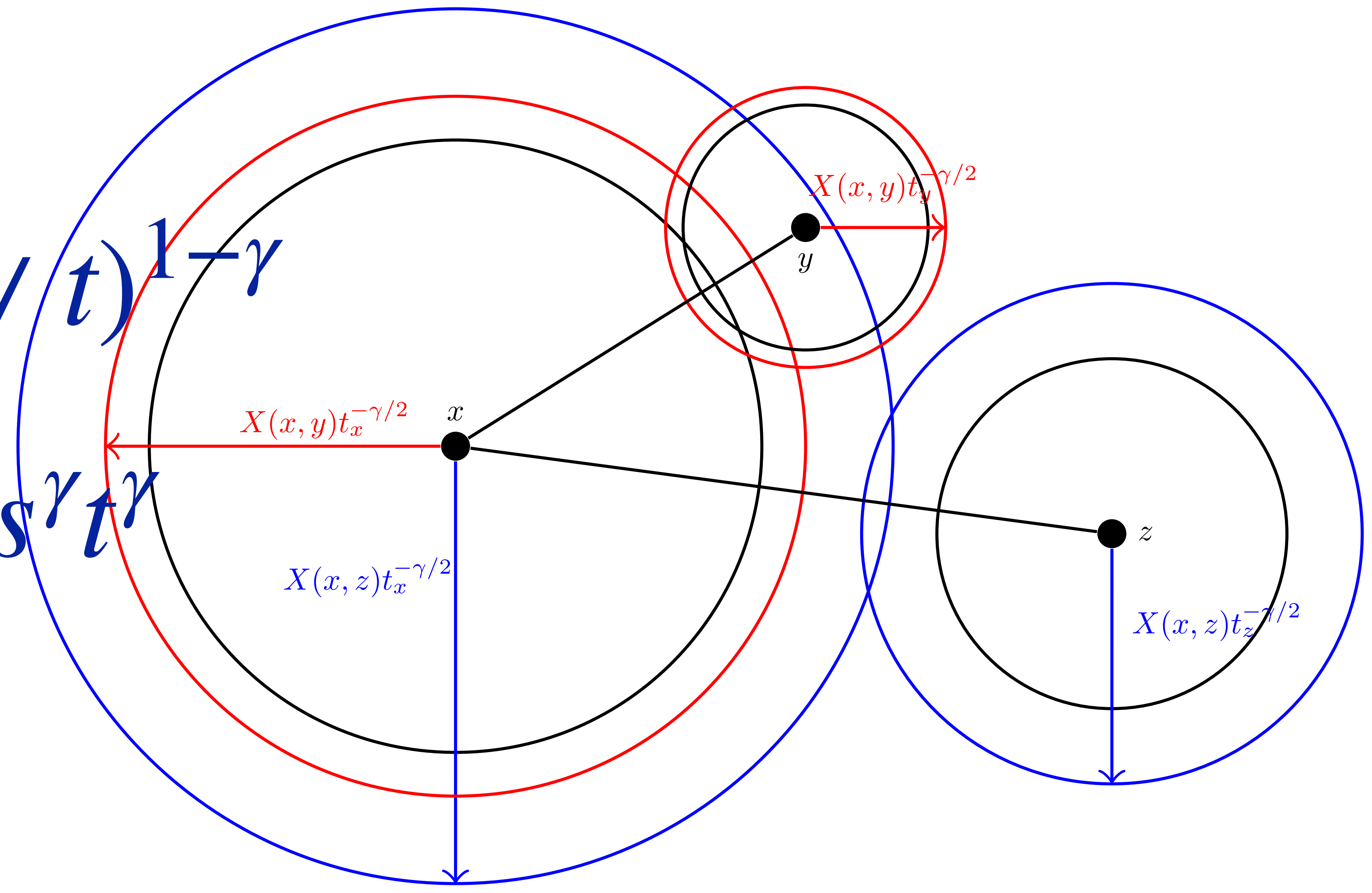
Non-increasing profile function $\rho : \mathbb{R}_+ \to [0,1]$ with $\rho(x) \sim x^{-\delta}$

Non-decreasing, symmetric kernel function $g : (0,1) \times (0,1) \to \mathbb{R}_+$

- Preference is given to short edges and to vertices with small marks.
- Assume $\int \rho(|x|)dx = 1$ since then the degree distribution only depends on the kernel g and β

Interesting kernels and models

- Random connection model: $g^{plain}(s,t) = 1$
- Boolean model: $g^{sum}(s,t) = (s^{-\gamma} + t^{-\gamma})^{-1}$ or $g^{min}(s,t) = (s \wedge t)^{\gamma}$
  - With $\rho(x) = \mathbf{1}_{[0,1/2]}(x)$: Standard Poisson Boolean model
  - With $\rho(x) \sim x^{-\delta}$: Soft Boolean model.
- Age-dependent rcm: $g^{pa}(s,t) = (s \wedge t)^{\gamma}(s \vee t)^{1-\gamma}$
- Scale-free percolation model: $g^{prod}(s,t) = s^{\gamma}t^{\gamma}$



Main result

Define the effective decay exponent $\delta_{\text{eff}} := -\lim_{n\to\infty} \frac{\log \int_{1/n}^{1} \int_{1/n}^{1} \rho(g(s,t)n)dsdt}{\log n}$ $(\leq \delta)$

Theorem (Mönch, L, Gracar (2022))

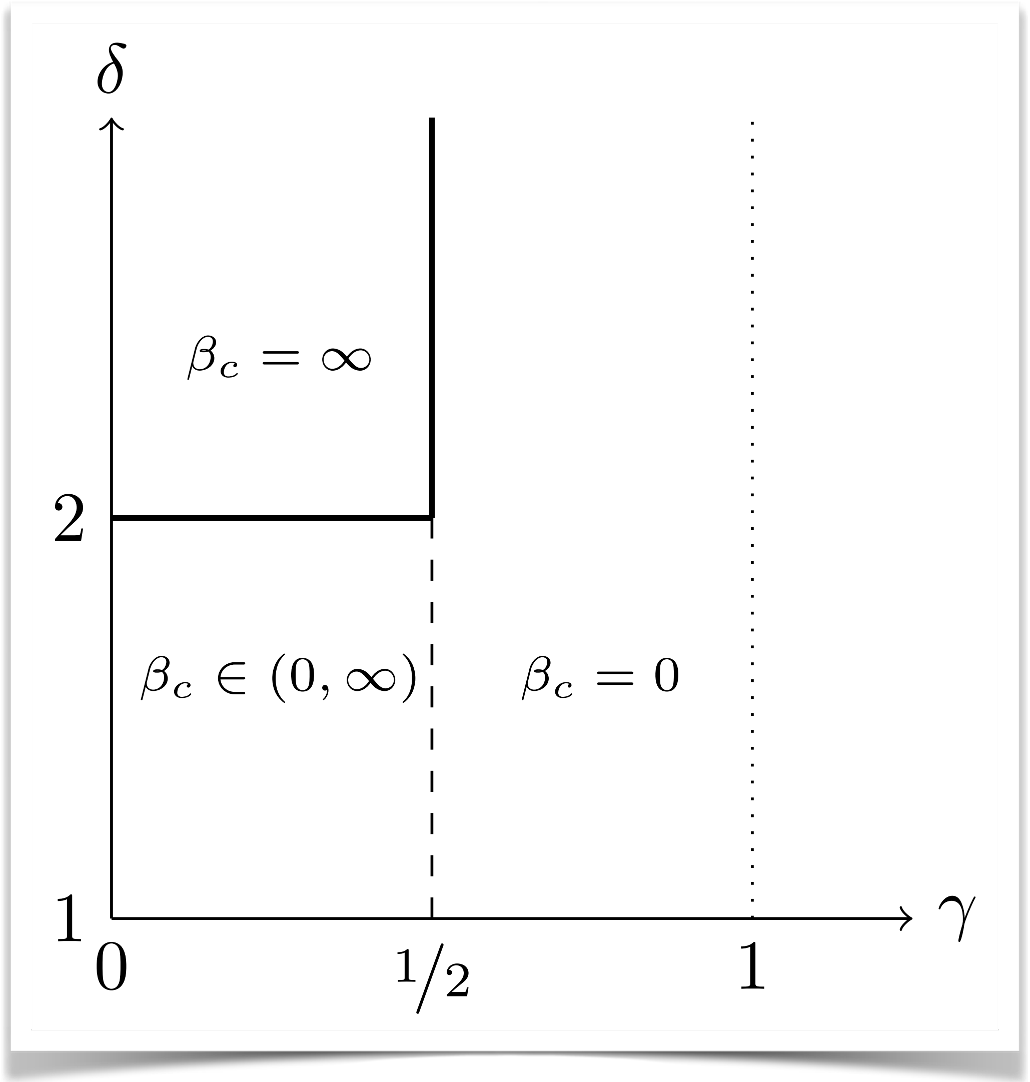
A. If $\delta_{\text{eff}} < 2$, then $\beta_c < \infty$ and
B. if $\delta_{\text{eff}} > 2$, then $\beta_c = \infty$.

Intuition:

- Take two sets of n vertices at distance n. The minimum mark in each set is roughly $1/n$ so the integral is the probability of two randomly picked vertices being connected.
- (Ignoring correlations) the number of edges connecting both sets is binomial with n^2 trials and success probability $n^{-\delta_{eff}}$

Examples

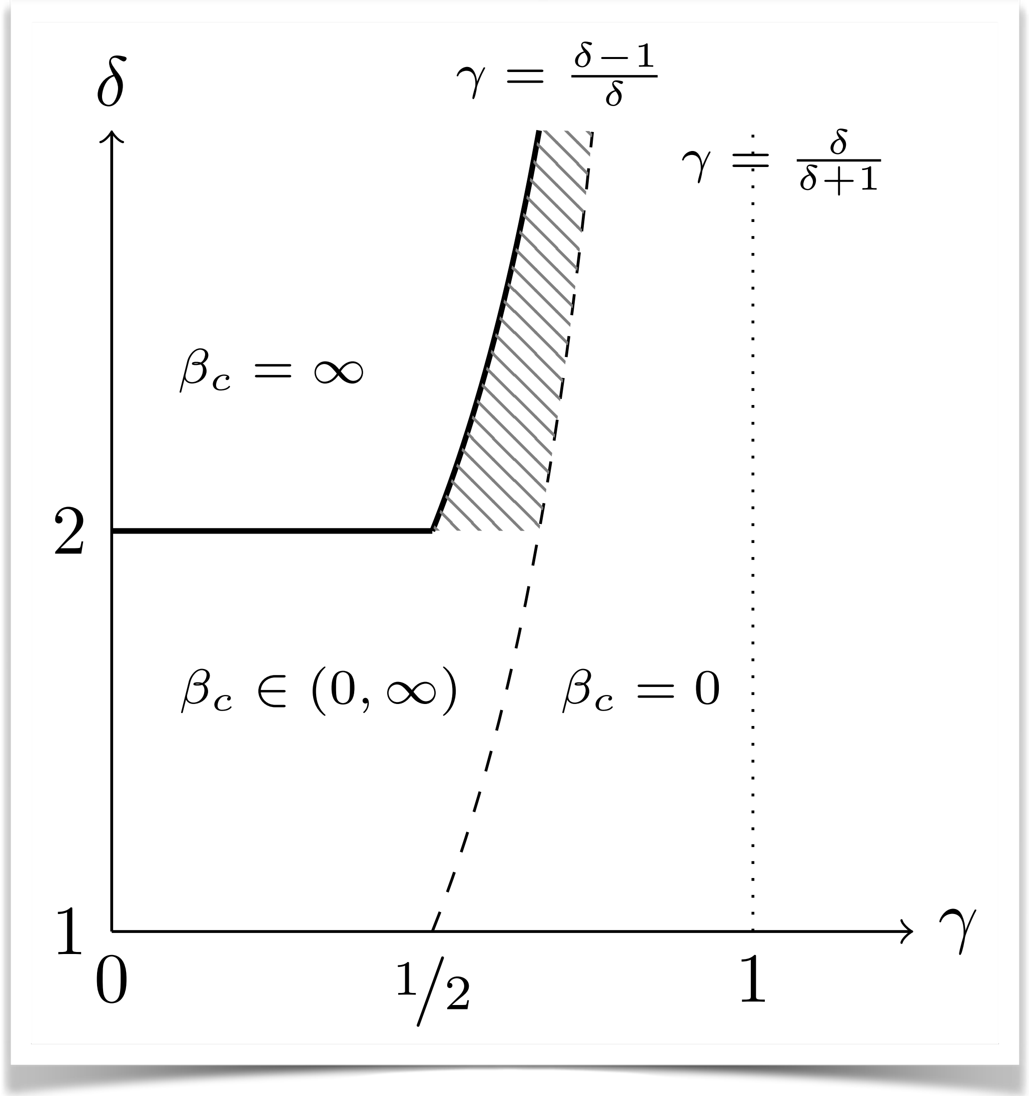
Scale free percolation



Deijfen et. al (2013), Deprez, Wüthrich (2018)

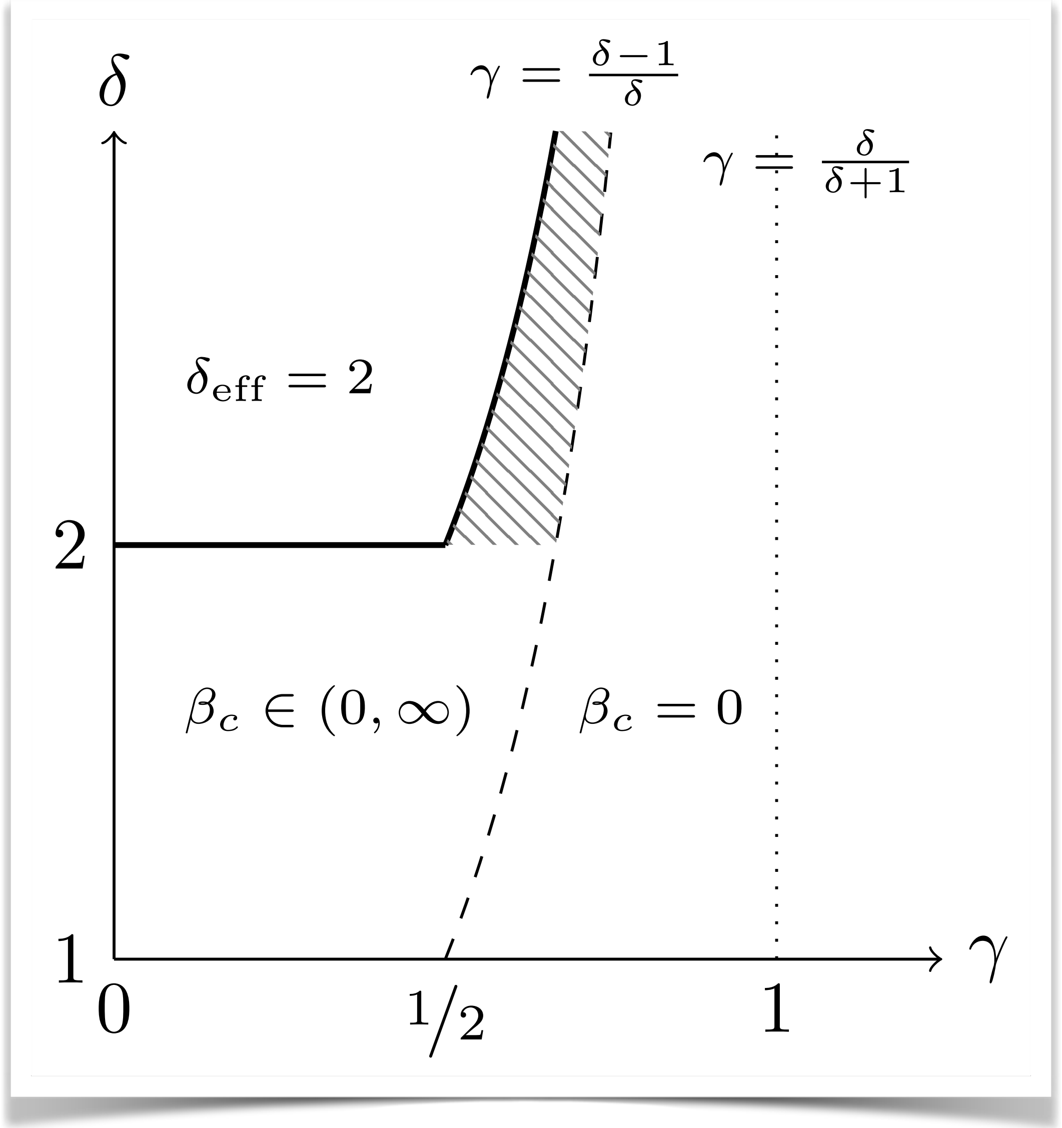
$$\delta_{eff} \begin{cases} > 2, & \gamma < 1/2 \\ = 2, & \gamma = 1/2 \\ < 2, & \gamma > 1/2 \end{cases}$$

Soft Boolean



$$\delta_{eff} \begin{cases} > 2, & \gamma < 1 - 1/\delta \\ = 2, & \gamma = 1 - 1/\delta \\ < 2, & \gamma > 1 - 1/\delta \end{cases}$$

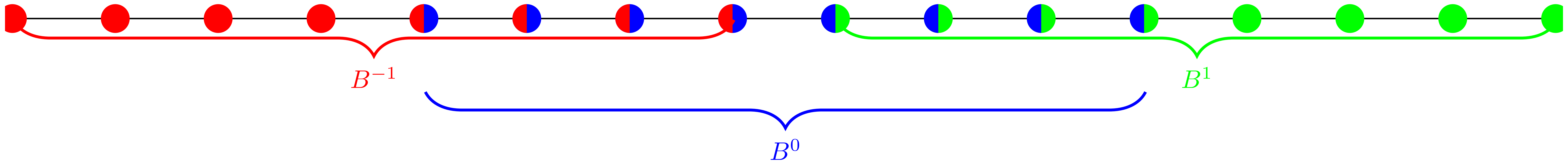
Age-dependent



$$\delta_{eff} \begin{cases} = 2, & \gamma \leq 1 - 1/\delta \\ < 2, & \gamma > 1 - 1/\delta \end{cases}$$

Proof idea

- Generalise a renormalisation scheme of Duminil-Copin, Garban, Tassion (2020).
- Write $\eta_0 = \{\mathbf{X}_i = (X_i, T_i) : i \in \mathbb{Z} : X_0 = 0, X_j < X_\ell \ (j < \ell)\}$
- Define blocks: $B^i_{K_n} := \{\mathbf{X}_{K_n(i-1)}, \ldots, \mathbf{X}_{K_n i}, \ldots, \mathbf{X}_{K_n(i+1)-1}\}$, for the scales $K_n = (n!)^3 K^n$



- Say a block is ϑ-good, if it contains a connected component of density at least ϑ. Define

$$p_\beta(K_n, \vartheta) = \mathbb{P}\{B^0_{K_n} \text{ is } \vartheta\text{-bad}\}$$

Proof idea

Lemma

Assume $\delta_{eff} < 2$. Let $3/4 < \vartheta < 1$. Then $\exists M \in \mathbb{N}$ such that it holds for all $K > M$ and $n \geq 2$

$$p_\beta(K_n, \vartheta - \varepsilon_n) \leq \frac{1}{100} p_\beta(K_{n-1}, \vartheta) + 8\varepsilon_n^{-2} p_\beta(K_{n-1}, \vartheta)^2$$

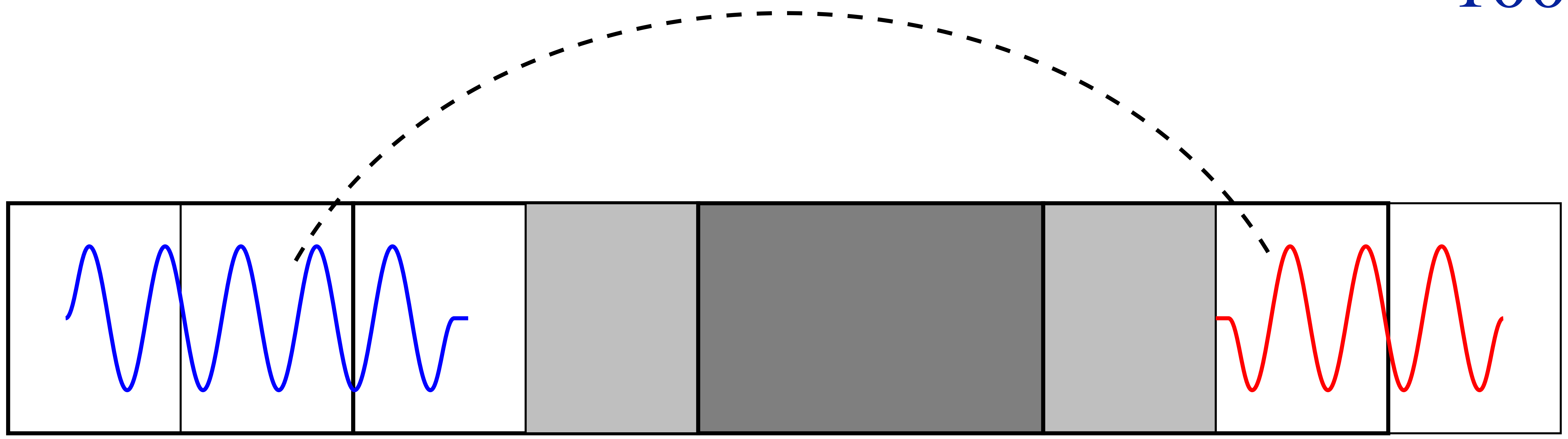
where $\varepsilon_n = 2/(n^3 K)$.

Proof idea

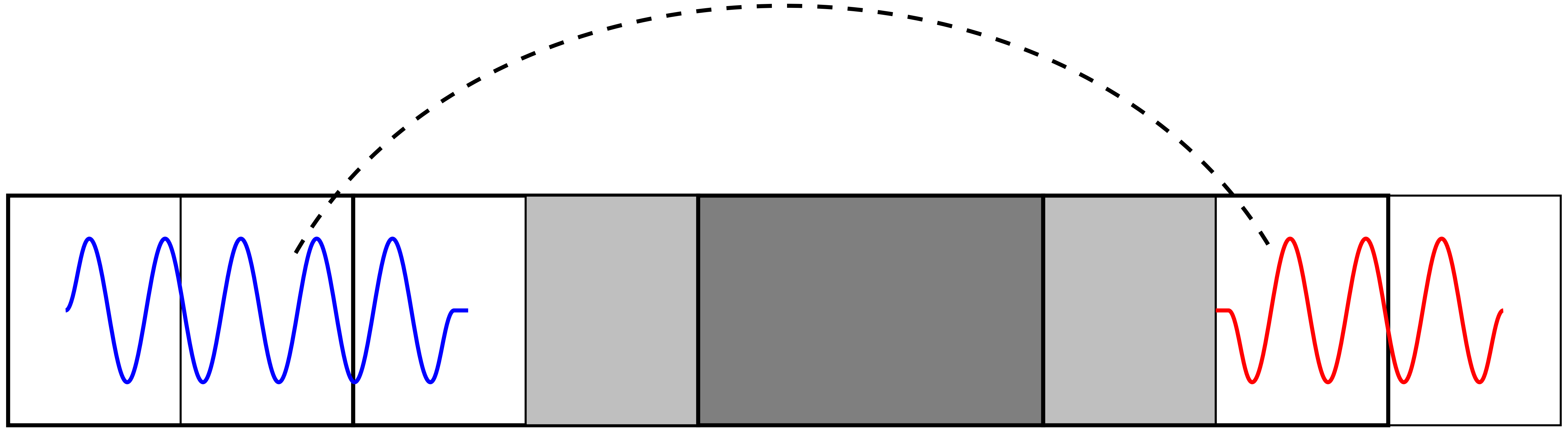
$$p_\beta(K_n, \vartheta - \varepsilon_n) \le \frac{1}{100} p_\beta(K_{n-1}, \vartheta) + 8\varepsilon_n^{-2} p_\beta(K_{n-1}, \vartheta)^2$$

- Block $B^0_{K_n}$ is formed by $4/\varepsilon_n$ of the $B_{K_{n-1}}$ blocks. If all are ϑ-good, then $B^0_{K_n}$ is good as well
- Either, there are at least two disjoint bad blocks (second summand) or
- One bad block and all blocks disjoint from it are good. Then we need to show

$$\sum_i \mathbb{P}(B^0_{K_n} \text{ is } (\vartheta - \varepsilon_n)\text{-bad} \mid B_{K^i_{n-1}} \text{ is } \vartheta\text{-bad, all disjoint are good}) \le \frac{1}{100}.$$



Proof idea



$\mathbb{P}(\mathscr{L} \nsim \mathscr{R} \mid B_{K^i_{n-1}}$ is ϑ-bad, all disjoint are good)

$$\leq \exp\Big(-\text{const } K_{n-1}^2 \int_{K_{n-1}^{\mu-1}}^1 \int_{K_{n-1}^{\mu-1}}^1 \rho(\beta^{-1} g(s,t) K_n) ds dt\Big)$$

$$\leq \exp\big(-K_{n-1}^{2-\delta_{eff}-\epsilon}\big).$$

Proof idea

1. $p_\beta(K_n, \vartheta - \varepsilon_n) \leq \dfrac{1}{100} p_\beta(K_{n-1}, \vartheta) + 8\varepsilon_n^{-2} p_\beta(K_{n-1}, \vartheta)^2$ for large enough K

2. $p_\beta(K_1, \vartheta) \leq \exp(-\text{const } K)$ for large enough $\beta \gg K$.

$\blacksquare$

- For the non-existence of a supercritical phase we use a similar approach to generalize to count the number of edges crossing the origin.

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