Percolation phase transition in weight-dependent random connection models

Joint work with Peter Gracar and Peter Mörters

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There exists a percolation phase transition if there exists a critical edge density $\beta_c \in (0, \infty)$ such that almost surely

- if $\beta < \beta_c \implies \mathscr{G}(\beta)$ does not percolate but
- if $\beta > \beta_c \implies \mathscr{G}(\beta)$ percolates.

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Without loss of generality, we assume $\int_{\mathbb{R}^d} \rho(|x|^d)dx = 1$.
Because then the degree distribution only depends on the kernel g (and β).

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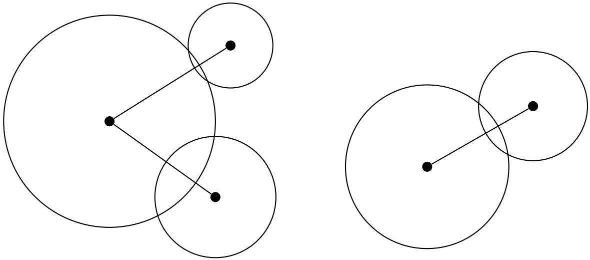
  Leads to heavy-tailed degree distribution with power-law exponent $\tau = 1 + {}^{1}\!/\!{\gamma}$.



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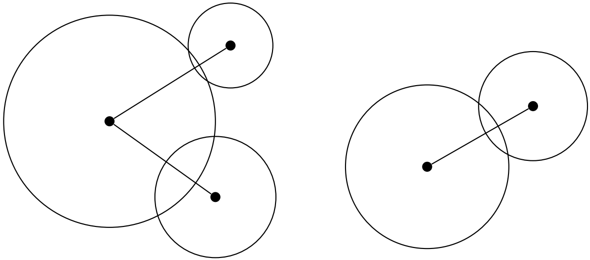

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Theorem (Deijfen et al (2018), Deprez and Wüthrich (2019))

If $\gamma < {}^1/_2$, then $\beta_c > 0$, but if $\gamma > {}^1/_2$, then $\beta_c = 0$.

Further interesting models

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Again, heavy tailed degree-distribution with power-law $\tau = 1 + 1/\gamma$.

Main Result

Theorem (Gracar, L, Mörters (2020))

For the weight-dependent connection model with preferential attachment kernel, g^{pa}, or sum kernel, g^{sum}, or min kernel, g^{min}, and parameters $\delta > 1$ and $\gamma \in (0,1)$ we have

(a) *if $\gamma < \delta/(\delta+1)$, then $\beta_c > 0$.*

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Remark: The theorem also applies for the plain kernel, g^{plain}, i.e. $\gamma = 0$, or profile functions decaying faster then any polynomial for $\delta \to \infty$. Hence, it includes the previous shown results about the classical Boolean Model and the long range percolation model.

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Theorem (Deijfen et al (2018), Deprez and Wüthrich (2019))

For the product kernel, g^{prod} and $\delta > 1$ and $\gamma \in (0,1)$, we have

(c) *If $\gamma \leq 1/2$ or $\tau \geq 3$, then $\beta_c > 0$.*

(d) *If $\gamma > 1/2$ or $\tau < 3$, then $\beta_c = 0$.*

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- $\mathscr{G}_0$ is the empty graph without any vertices or edges.
- Vertices arrive successively after exponential waiting times and are placed uniformly at random on $\mathbb{T}_1^d$.
- Given $\mathscr{G}_{t-}$ a new vertex born at time t and with position x is connected by an edge to each already existing vertex at y and born at time s independently with probability

$$\rho\Big(\frac{t\ \mathrm{d}(x,y)^d}{\beta\,(t/s)^\gamma}\Big).$$

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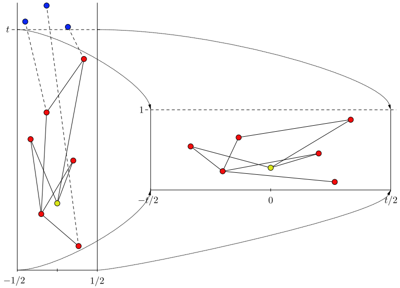
Theorem

The network $(\mathscr{G}_t)_{t\geq 0}$ is robust if $\gamma > \delta/(\delta+1)$ but non-robust if $\gamma < \delta/(\delta+1)$.

Proof idea



(1) Rescale the graph. The rescaled graph has the same law as $\mathscr{G}^t$ that is constructed on a Poisson process on $\mathbb{T}_t^d \times (0,1)$ and connection probability $\rho\big(\frac{1}{\beta} g^{\mathrm{pa}}(s,t)\mathrm{d}(x,y)^d\big)$.

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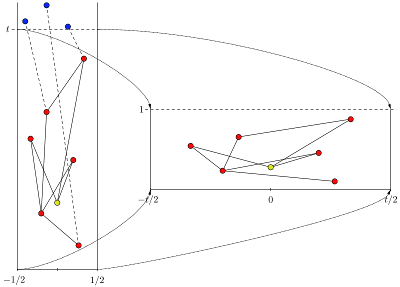

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(3) Use the following weak law of large numbers which is an adoption of Penrose and Yukich (2003)

Theorem (Jacob, Mörters (2015))

Let $(A_t)_{t\geq 0}, A_\infty$ events that depend on a graph and a given root vertex such that

$$\mathbf{1}_{\{(\mathbf{0},\mathscr{G}_0^t)\in A_t\}} \xrightarrow{t\to\infty} \mathbf{1}_{\{(\mathbf{0},\mathscr{G}_0(\beta))\in A_\infty\}} \quad \textit{in probability}$$

where $\mathbf{0}$ is an additional vertex at the origin that is added to $\mathscr{G}^t$ resp. $\mathscr{G}(\beta)$. Then

$$\frac{1}{t}\sum_{\mathbf{x}\in\mathscr{G}_t} \mathbf{1}_{\{(\mathbf{x},\theta_{\mathbf{x}}\mathscr{G}_t)\in A_t\}} \xrightarrow{t\to\infty} \mathbb{P}_0\{(\mathbf{0},\mathscr{G}(\beta))\in A_\infty\} \quad \textit{in probability.}$$

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(4) Show that

$$\frac{\#\text{ vertices in } \mathscr{G}_t \text{ connected to the oldest vertex}}{t} \overset{t\to\infty}{\longrightarrow} \mathbb{P}_0\{\mathbf{0} \leftrightarrow \infty \text{ in } \mathscr{G}(\beta)\}$$

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and

$$\frac{\#\text{ vertices with components } \le k}{t} \xrightarrow{t\to\infty} \mathbb{P}_0\{\text{component of } \mathbf{0} \text{ is of size } \le k\}$$

For $k \to \infty$ the left hand side is the proportion of vertices in finite components and the right hand side equals $1 - \mathbb{P}\{\mathbf{0} \leftrightarrow \infty\}$.

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- **Scale-free percolation**: Deprez and Wüthrich (2018): For $\delta > 2$ and $\gamma < 1/2$ it holds $\beta_c = \infty$.

Our result

Theorem

Let $\delta > 2$, $\gamma \in (0,1)$ and $d = 1$ then

(a) *for the soft Boolean model, g^{sum}, g^{min} and the age-depended random connection model, g^{pa}, it holds*

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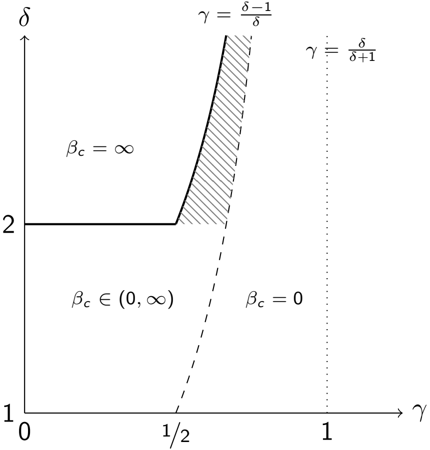
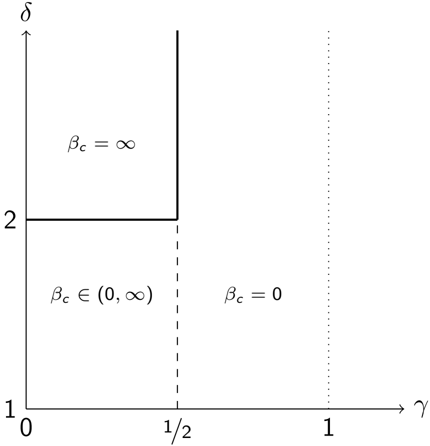
Phase diagram soft Boolean vs scale-free percolation





Thank you for your attention