Percolation of the SINR Secrecy Graph (SSG)

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In this talk, we follow the paper [VI14].

1 Definitions and Main Theorem

Definition 1.1. Let the points in $\Phi, \Phi_E \subset \mathbb{R}^2$ be distributed according to independent poisson point processes of intensity λ, λ_E. We call Φ the set of legitimate nodes and Φ_E the set of eavesdropper nodes and define

$$\mathrm{SINR}_{xy} := \frac{l(d_{xy})}{\gamma \sum_{z\in\Phi, z\neq x} l(d_{zy}) + 1}$$

for all $x, y \in \Phi$ and

$$\mathrm{SINR}_{xe} := \frac{l(d_{xe})}{\gamma_E \sum_{z\in\Phi, z\neq x} l(d_{ze}) + 1}$$

for all $x \in \Phi, e \in \Phi_E$, with $d_{xy} := \|x - y\|_2$, the signal attenuation function $l : [0, \infty) \to [0, \infty)$ and the interference suppression parameter $\gamma \in [0, 1]$. Then, the maximum rate of secure communication [Wyn75] between $x, y \in \Phi$ is given by

$$R_{xy}^{\mathrm{SINR}} := 0 \vee \min_{e\in\Phi_E} \log_2 \left(\frac{1 + \mathrm{SINR}_{xy}}{1 + \mathrm{SINR}_{xe}} \right).$$

Definition 1.2. We say that the signal attenuation function $l : [0, \infty) \to [0, \infty)$ fulfills standard assumptions if l is strictly decreasing on its support and $\int_0^\infty x l(x)\,\mathrm{d}x < \infty$. Furthermore, we say that l fulfills the additional decay condition if for all $c > 0$ there is $M > 0$ such that $\forall\, x \geq 0 : \; l(x + M) \leq c l(x)$.

Definition 1.3. For $\theta \geq 0$ we define the SINR secrecy graph $\mathrm{SSG}(\theta) := \{\Phi, \mathcal{E}\}$, where $\mathcal{E} := \{(x, y) : R_{xy}^{\mathrm{SINR}} > \theta\}$. We call $x \in \Phi$ connected to $y \in \Phi$ if $(x, y) \in \mathcal{E}$. If there is a sequence of edges from $x \in \Phi$ to $z \in \Phi$ we speak of a path from x to z and write $x \to z$. The connected component, also called cluster, of $x \in \Phi$ is given by $C_x := \{z \in \Phi : x \to z\}$.

Remark 1.4. In the following, we will only consider $\mathrm{SSG} := \mathrm{SSG}(0)$ with edge set $\mathcal{E} := \{(x, y) : \mathrm{SINR}_{xy} > \mathrm{SINR}_{xe}\ \forall e \in \Phi_E\}$.

Theorem 1.5. *Let P^0 be the palm distribution of Φ and Φ_E with respect to $0 \in \Phi$. Let l be a signal attenuation function fulfilling standard assumptions. For all $\lambda_E \in (0,\infty)$ and $\gamma_E \in [0,1]$,*

1. *there is $\lambda_1 \in (0,\infty), \gamma_1 \in (0,1)$ such that $\forall\, \lambda > \lambda_1, \gamma < \gamma_1 :\ P^0(|C_0| = \infty) > 0$,*
2. *if l satisfies the additional decay condition, there is $\lambda_2 \in (0,\infty)$ such that $\forall\, \lambda < \lambda_2, \gamma \in [0,1] :\ P^0(|C_0| = \infty) = 0$.*

2 Proof of Part 1 of Theorem 1.5

For the proof of the first part, it is sufficient to consider the case of $\gamma_E = 0$.

Definition 2.1. Let $\mathbf{S}$ be the square lattice with side $s > 0$ with a vertex at the origin and $\mathbf{S}' := \mathbf{S} + (s/2, s/2)$ be the dual lattice. For an edge $\mathbf{a}$ of $\mathbf{S}$ let $\mathbf{a}'$ be the edge of $\mathbf{S}'$ which crosses $\mathbf{a}$. Choose $\alpha(s) > 0$ such that $l(3s) < \frac{l(\sqrt{5}s)}{1+\alpha(s)}$. For an edge $\mathbf{a}$ of $\mathbf{S}$ let $S_1(\mathbf{a})$ and $S_2(\mathbf{a})$ be its two adjent squares and $Y(\mathbf{a})$ the $7s \times 8s$ rectangle of $\mathbf{S}$ which contains a $3s$ surrounding of $S_1(\mathbf{a}) \cup S_2(\mathbf{a})$.[1]

Definition 2.2. For any edge $\mathbf{a}$ of $\mathbf{S}$ consider indicator variables $A(\mathbf{a}), B(\mathbf{a}), C(\mathbf{a})$ given by

1. $A(\mathbf{a}) = 1$ iff $S_1(\mathbf{a}) \cap \Phi \neq \emptyset$ and $S_2(\mathbf{a}) \cap \Phi \neq \emptyset$,
2. $B(\mathbf{a}) = 1$ iff $Y(\mathbf{a}) \cap \Phi_E = \emptyset$,
3. $C(\mathbf{a}) = 1$ iff for all $x, y \in (S_1(\mathbf{a}) \cup S_2(\mathbf{a})) \cap \Phi$ we have $I_{xy} := \sum_{z\in\Phi, z\neq x} l(d_{zy}) \leq \frac{\alpha(s)}{\gamma}$.

Then $\mathbf{a}$ and $\mathbf{a}'$ are defined to be open edges if $D(\mathbf{a}) := A(\mathbf{a})B(\mathbf{a})C(\mathbf{a}) = 1$ and closed edges otherwise.

Lemma 2.3. *If an edge $\mathbf{a}$ of $\mathbf{S}$ is open, then $(x, y) \in \mathcal{E}$ for all $x, y \in (S_1(\mathbf{a}) \cup S_2(\mathbf{a})) \cap \Phi$.*

Theorem 2.4. [Gri99, page 284][Kes82, page 386] *Any finite open cluster of $\mathbf{S}$ is surrounded by a closed circuit of $\mathbf{S}'$.*

Lemma 2.5. *Let $\{\mathbf{a}_i\}_{1\leq i\leq n}$ be a collection of distinct edges in $\mathbf{S}$. Then,*

1. *$P^0(A(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {p_1}^n$ where $p_1 := \sqrt[7]{1 - (1 - \exp(-\lambda s^2))^2}$,*
2. *$P^0(B(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {p_2}^n$ where $p_2 := \sqrt[449]{1 - \exp(-56 s^2 \lambda_E)}$,*
3. *$P^0(C(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq {p_3}^n$ where $p_3 := \exp\left(\frac{2\lambda}{K}\int_0^\infty x l(x)\,\mathrm{d}x + \frac{l(0)}{K} - \frac{\alpha(s)}{\gamma K}\right)$ and $K > 0$ only depends on l and s,*
4. [DFM+06] *$P^0(D(\mathbf{a}_i) = 0\ \forall\, 1 \leq i \leq n) \leq q^n$ where $q := \sqrt{p_1} + \sqrt[4]{p_2} + \sqrt[4]{p_3}$.*

Lemma 2.6. *For small enough $q > 0$, the probability of having a closed circuit in S' surrounding the origin is lower than 1.*

[1] $S_1(\mathbf{a})$, $S_2(\mathbf{a})$ and $Y(\mathbf{a})$ are defined to be topologically closed.

3 Proof of Part 2 of Theorem 1.5

For the proof of the second part, it is sufficient to consider the case of $\gamma = 0$ and $\gamma_E = 1$.

Definition 3.1. For initially arbitrary $m > 0$ and $c > 0$ fix $M(m, c) > 9m$ such that $l\big(d + \frac{1}{9}M(m, c)\big) \leq \frac{l(d)}{1+c}$ for all $d \geq M(m, c)$. Let $\mathbf{M}$ be the square lattice with side $M(m, c)$ with a vertex at the origin and $\mathbf{M}'$ be the dual lattice. For an edge $\mathbf{a}$ of $\mathbf{M}$ let $S_1(\mathbf{a})$ and $S_2(\mathbf{a})$ be its two adjent squares and $T_i(\mathbf{a})$ be the square with side m with the same center as $S_i(\mathbf{a})$.[2]

Definition 3.2. For any edge $\mathbf{a}$ of $\mathbf{M}$ consider indicator variables $\tilde{A}(\mathbf{a}), \tilde{B}(\mathbf{a}), \tilde{C}(\mathbf{a})$ given by

1. $\tilde{A}(\mathbf{a}) = 1$ iff $T_1(\mathbf{a}) \cap \Phi_E \neq \emptyset$ and $T_2(\mathbf{a}) \cap \Phi_E \neq \emptyset$,
2. $\tilde{B}(\mathbf{a}) = 1$ iff $(S_1(\mathbf{a}) \cup S_2(\mathbf{a})) \cap \Phi = \emptyset$,
3. $\tilde{C}(\mathbf{a}) = 1$ iff for all $e \in (T_1(\mathbf{a}) \cup T_2(\mathbf{a})) \cap \Phi_E$ we have $I_e := \sum_{z \in \Phi} l(d_{ze}) \leq c$.

Then $\mathbf{a}$ and $\mathbf{a}'$ are defined to be open edges iff $\tilde{D}(\mathbf{a}) := \tilde{A}(\mathbf{a})\tilde{B}(\mathbf{a})\tilde{C}(\mathbf{a}) = 1$.

Lemma 3.3. *Edges of* SSG *cannot cross open edges of* $\mathbf{M}$.

Lemma 3.4. *Let* $\{\mathbf{a}_i\}_{1 \leq i \leq n}$ *be a collection of distinct edges in* $\mathbf{M}$ *which do not contain the origin. Then,*

1. $P^0(\tilde{A}(\mathbf{a}_i) = 0\ \forall 1 \leq i \leq n) \leq {r_1}^n$ *where* $r_1 := \sqrt[7]{1 - (1 - \exp(-\lambda_E m^2))^2}$,
2. $P^0(\tilde{B}(\mathbf{a}_i) = 0\ \forall 1 \leq i \leq n) \leq {r_2}^n$ *where* $r_2 := \sqrt[7]{1 - \exp(-2\lambda M^2)}$,
3. $P^0(\tilde{C}(\mathbf{a}_i) = 0\ \forall 1 \leq i \leq n) \leq {r_3}^n$ *where* $r_3 := \exp\left(\frac{4\lambda\pi}{K}\int_0^\infty x l(x)\,\mathrm{d}x + \frac{l(0)}{K} - \frac{c}{K}\right)$ *and* $K > 0$ *only depends on* l *and* M,
4. $P^0(\tilde{D}(\mathbf{a}_i) = 0\ \forall 1 \leq i \leq n) \leq {r_s}^n$ *where* $r_s := \sqrt{r_1} + \sqrt[4]{r_2} + \sqrt[4]{r_3}$.

Lemma 3.5. *For small enough* $q > 0$*, the probability of having an open circuit in* $\mathbf{M}$ *surrounding the origin is equal to* 1.

[2] $S_1(\mathbf{a})$, $S_2(\mathbf{a})$, $T_1(\mathbf{a})$ and $T_2(\mathbf{a})$ are defined to be topologically closed.

References

[DFM+06] Olivier Dousse, Massimo Franceschetti, Nicolas Macris, Ronald Mester, and Patrick Thiran. Percolation in the Signal to Interference Ratio Graph. *J. Appl. Probab.*, 43:552–562, 2006.

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