Decay of covariances, uniqueness of ergodic component and scaling limit for a class of $\nabla\phi$ systems with non-convex potential

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Abstract

We consider a gradient interface model on the lattice with interaction potential which is a non-convex perturbation of a convex potential. Using a technique which decouples the neighbouring vertices sites into even and odd vertices, we show for a class of non-convex potentials: the uniqueness of ergodic component for $\nabla\phi$- Gibbs measures, the decay of covariances, the scaling limit and the strict convexity of the surface tension.

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1 Introduction

Phase separation in $\mathbb{R}^{d+1}$ can be described by effective interphase models. In this setting we ignore overhangs and for $x \in \mathbb{Z}^d$, we denote by $\phi(x) \in \mathbb{R}$ the height of the interface above or below the site x. Following a Gibbs formalism, the finite Gibbs distribution ν_Λ^ψ on $\mathbb{R}^{\mathbb{Z}^d}$ of $(\phi(x))_{x\in\mathbb{Z}^d}$

$$\nu_\Lambda^\psi(\mathrm{d}\phi) = \frac{1}{Z_\Lambda^\psi} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} \mathrm{d}\phi_\Lambda, \tag{1}$$

with boundary condition $\phi(x) = \psi(x)$, if $x \in \partial\Lambda$ and normalizing constant Z_Λ^ψ is characterized by the inverse temperature $\beta > 0$ and the Hamiltonian H_Λ^ψ on Λ, which we assume to be of gradient type:

$$H_\Lambda^\psi(\phi) = \sum_{i\in I}\sum_{x\in\Lambda} V(\nabla_i\phi(x)), \tag{2}$$

where

$$I = \{-d, -d+1, \ldots, -1, 1, 2, \ldots, d\}. \tag{3}$$

and where we introduced for each $x \in \mathbb{Z}^d$ and each $i \in I$, the discrete gradient

$$\nabla_i\phi(x) = \phi(x + e_i) - \phi(x),$$

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that is, the interaction depends only on the differences of neighboring heights. We thus have a massless model with a continuous symmetry! Our state space $\mathbb{R}^{\mathbb{Z}^d}$ being unbounded, such models are facing delocalization in lower dimensions $d = 1, 2$, and no infinite Gibbs state exists in these dimensions. Instead of looking at the Gibbs measures of the $(\phi(x))_{x\in\mathbb{Z}^d}$, Funaki and Spohn proposed to consider the distribution of the gradients $(\nabla_i\phi(x))_{i\in I, x\in\mathbb{Z}^d}$ under ν, the so-called gradient Gibbs measure, which in view of the Hamiltonian (2), can also be given in terms of a Dobrushin-Landford-Ruelle description.

Assuming strict convexity of V:

$$0 < C_1 \le V'' \le C_2 < \infty \tag{4}$$

Funaki and Spohn showed in [14], the existence and uniqueness of ergodic gradient Gibbs measures for every tilt $u \in \mathbb{R}^d$, see also Sheffield [21]. Moreover, they also also proved that the corresponding free energy, or surface tension, $\sigma \in C^1(\mathbb{R}^d)$ is strictly convex. Both results are essential for their derivation of the hydrodynamical limit of the Ginzburg Landau model.

In fact under strict the convexity assumption (4) of V, much is known for the gradient field. At large scales it behaves much like the harmonic crystal or gradient free fields which is a gaussian field with quadratic V. In particular Naddaf and Spencer [20] showed that the rescaled gradient field converges weakly as $\epsilon \searrow 0$ to a continuous homogeneous gaussian field, that is

$$S_\epsilon(f) = \epsilon^{d/2} \sum_{x\in\mathbb{Z}^d} \big(\nabla_i\phi(x) - u_i\big) f(\epsilon x) \to N(0, \sigma_u^2(f)) \ \text{ as } \ \epsilon \to 0, \qquad f \in C_0^\infty(\mathbb{R}^d)$$

(see also Giacomin et al. [16] and Biskup and Spohn [4] for similar results). This scaling limit theorem derived at standard scaling $\epsilon^{d/2}$, is far from trivial, since, as shown in Delmotte and Deuschel [9], the gradient field has slowly decaying, non absolutely summable covariances, of the algebraic order

$$|\mathrm{cov}_\nu(\nabla_i\phi(x), \nabla_j\phi(y))| \sim \frac{C}{1 + \|x-y\|^d}. \tag{5}$$

The aim of this paper is to relax the strict convexity assumption (4). Our potential is of the form

$$V(\nabla_i\phi(x)) = V_0(\nabla_i\phi(x)) + g(\nabla_i\phi(x))$$

where $V_0, g \in C^2(\mathbb{R})$ are such that V_0 satisfies (4) and

$$-C_0 \le g_0'' \le 0, \qquad \|g''\|_{L^1(\mathbb{R})} < \infty, \tag{6}$$

with $C_0 > C_2$. Our main result shows that if the inverse temperature β is sufficiently small, that is if:

$$\sqrt{\frac{\beta}{C_1}}\|g''\|_{L^1(\mathbb{R})} \le \frac{C_1}{2C_2\sqrt{d}}, \tag{7}$$

then the results known in the strick convex case hold. In particular we have uniqueness of the ergodic component at any tilt $u \in \mathbb{R}^d$, strict convexity of the surface tension, scaling limit theorem and decay of covariances. As stated above, the hydrodynamical limit for the corresponding Ginzburg-Landau model, should then essentially follow from these results.

Note that uniqueness of the ergodic measures is not true at any β for this type of models: Biskup and Kotecky give an example of non convex V which can be described as the mixture of two gaussians with two different variances, where two ergodic gradient Gibbs measures coexists at $u = 0$ tilt, cf. Biskup and Kotecky [3] (see also Figure 4: Example (a) below). At this particular $\beta = \beta_c$ one expects a non strictly convex free energy Biskup [2] (personal communication). The situation at lower temperature (i.e. large β) is again quite different: using renormalization group techniques, Adams et al. show the strict convexity for small tilt u, cf. [1].

In a previous paper with S. Mueller, cf. [8], we have proved strict convexity of the surface tension for moderate β in a regime similar to (5). The method used in [8], based on two scale decomposition of the free field, gives less sharp estimates for the temperature, however it is more general and could be applied to non bipartite graphs. In this paper we use a different technique, which relies on the bipartite property of our model. We consider the distribution of the even gradient (that is of $\phi(y) - \phi(x)$ where both x, y are even): which is again a gradient field and show that under the condition (5), that the resulting Hamiltonian is strictly convex. The main idea, similar to [8], is that convexity can be gained via integration (see also Bracamp et al. [6] for previous use of the even/odd representation). In fact we show more: the Hamiltonian associated to the even variables admits an random walk representation, cf. Helffer and Sjöstrand [17] or Deuschel [11], which is the key tool in deriving covariance estimates such as (7) and scaling limit theorems. The other ingredient is the fact, that given the even gradients, the conditional law of the odd variables is simply a product law. Of course this is a special feature of our bipartite model, in particular it would be quite challanging to iterate the procedure, a scheme which could possibly lower the temperature towards the transition β_c. Note that iterating the scheme is an interesting open problem.

The rest of the paper is presented as follows: in Section 1 we define the model and recall the definition of gradient Gibbs measures. Section two presents the odd/even characterization of the gradient field, in particular our main result, Theorem 10, shows that the random walk representation holds for the even sites under the condition (5). Section 2 also presents a few examples, in particular we show that our criteria gets very close to the Biskup-Kotecky transition, cf. example 2.3.2. In section 3, we give a proof of the uniqueness of the ergodic component. In view of the product law for conditional distribution of the odd sites given the even gradient, this follows immediately from the uniqueness of the even gradient ergodic measures. Here we adapt the dynamical coupling argument of [14] to our situation. Section 4 deals with the decay of covariances, the proof is based on the random walk representation for the even sites which allows us to use the result of [9]. Section 5 shows the scaling theorem, here again we focus on the even variables and apply the random walk representation idea of [16]. Finally section 6 proves the strict convexity of the surface tension, or free energy, which follows from the convexity of the Hamiltonian for the even gradient. We also show a few useful equalities dealing with the derivative of σ, since they play an important role for the hydrodynamic limits of the Ginzburg Landau model.

2 General Definitions and Notations

2.1 The Hamiltonian

For all $x, y \in \mathbb{Z}^d$, let

$$\|x - y\|_= \sum_{i=1}^{d} |x_i - y_i|, \tag{8}$$

Let

$$I = \{-d, -d+1, \ldots, -1, 1, 2, \ldots, d\}. \tag{9}$$

For all $i \in I$, we define e_i in $\mathbb{Z}^d$ by

$$(e_i)_j = \delta_{ij}, \text{ for all } i = -1, -2, \ldots, -d \text{ and all } j = 1, 2, \ldots, d \tag{10}$$

and

$$e_i = -e_{-i}, \text{ for all } i = -1, -2, \ldots, -d. \tag{11}$$

For each $x \in \mathbb{Z}^d$ and all $i \in I$, we denote

$$\nabla_i \phi(x) = \phi(x + e_i) - \phi(x). \tag{12}$$

Let Λ be a finite set in $\mathbb{Z}^d$ with boundary

$$\partial\Lambda := \{x \notin \Lambda,\ ||x - y|| = 1 \text{ for some } y \in \Lambda\} \tag{13}$$

and with given boundary condition ψ such that $\phi(x) = \psi(x)$ for $x \in \partial\Lambda$. We consider a gradient interface model with Hamiltonian

$$H_\Lambda^\psi(\phi) = \sum_{i \in I} \sum_{\substack{x \in \Lambda \\ x+e_i \in \Lambda \cup \partial\Lambda}} U_i(\nabla_i \phi(x)) = \sum_{i \in I} \sum_{\substack{x \in \Lambda \\ x+e_i \in \Lambda \cup \partial\Lambda}} \left[V_i(\nabla_i \phi(x)) + g_i(\nabla_i \phi(x))\right], \tag{14}$$

where $\phi(x + e_i) = \psi(x + e_i)$, if $x \in \Lambda$ and $x + e_i \in \partial\Lambda$.

For all $i \in I$, $U_i \in C^2(\mathbb{R})$ are functions with quadratic growth at infinity:

$$U_i(\eta) \geq A|\eta|^2 - B, \qquad \eta \in \mathbb{R} \tag{15}$$

for some $A > 0, B \in \mathbb{R}$. We assume that for all $i \in I$, $V_i, g_i \in C^2(\mathbb{R})$ satisfy the following conditions:

$$C_1 \leq V_i'' \leq C_2, \quad \text{where } 0 < C_1 < C_2 \tag{16}$$

and

$$-C_0 \leq g_i'' \leq 0, \quad \text{where } C_0 > C_1. \tag{17}$$

Remark 1 Note that we can extend the results to the case where we have a perturbation with compact support. More precisely, assume that $U_i = Y_i + h_i$, where U_i satisfies (15), $D_1 \leq Y_i'' \leq D_2$ and $-D_0 \leq h_i'' \leq 0$ on $[a, b]$ and $0 < h_i'' < D_3$ on $\mathbb{R} \setminus [a, b]$, with $a, b \in \mathbb{R}$ and $h_i''(a) = h_i''(b) = 0$. Set

$$g_i(s) = h_i(s) 1_{\{s \in [a,b]\}} + \left[h_i(b) + h_i'(b)(s - b)\right] 1_{\{s > b\}} + \left[h_i(a) + h_i'(a)(s - a)\right] 1_{\{s < a\}} \tag{18}$$

and

$$V_i(s) = Y_i(s) + h_i(s) 1_{\{s \notin [a,b]\}} - \left[h_i(b) + h_i'(b)(s - b)\right] 1_{\{s > b\}} - \left[h_i(a) + h_i'(a)(s - a)\right] 1_{\{s < a\}}. \tag{19}$$

Thus, we have $V_i, g_i \in C^2(\mathbb{R})$, with $-D_0 \leq h_i''(s) = g_i''(s) \leq 0$ for $s \in [a, b]$ and $g_i''(s) = 0$ for $s \in \mathbb{R} \setminus [a, b]$ and $D_1 \leq V_i''(s) = Y_i''(s) + h_i''(s) 1_{\{s \notin [a,b]\}} \leq D_2 + D_3$. Note that this procedure can also be extended to the case where h_i'' changes sign more than once.

2.2 ϕ-Gibbs Measures

Let $\beta > 0$. For a finite region $\Lambda \subset \mathbb{Z}^d$, the Gibbs measure for the field of height variables $\phi \in \mathbb{R}^\Lambda$ over Λ is defined by

$$\nu_\Lambda^\psi(\mathrm{d}\phi) = \frac{1}{Z_\Lambda^\psi} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} \mathrm{d}\phi_\Lambda, \tag{20}$$

with boundary condition ψ and

$$Z_\Lambda^\psi = \int_{\mathbb{R}^\Lambda} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} \mathrm{d}\phi_\Lambda, \tag{21}$$

where

$$\mathrm{d}\phi_\Lambda = \prod_{x \in \Lambda} \mathrm{d}\phi(x) \tag{22}$$

is the Lebesgue measure over $\mathbb{R}^\Lambda$. For $A \in \mathbb{Z}^d$, we shall denote by $\mathcal{F}_A$ the σ-field genereated of $\mathbb{R}^{\mathbb{Z}^d}$ generated by $\{\phi(x) : x \in A\}$.

Definition 2 *The probability measure $\nu \in P(\mathbb{R}^{\mathbb{Z}^d})$ is called a Gibbs measure for the ϕ-field (ϕ-Gibbs measure for short), if its conditional probability of $\mathcal{F}_{\Lambda^c}$ satisfies the DLR equation*

$$\nu(\,\cdot\,|\mathcal{F}_{\Lambda^c})(\psi) = \nu_\Lambda^\psi(\cdot), \quad \nu - a.e.\ \psi,$$

for every $\Lambda \subset \mathbb{Z}^d$.

It is known that the ϕ-Gibbs measures exist when the dimension $d \geq 3$, but not for $d = 1, 2$, where the field "delocalizes" as $\Lambda \nearrow \mathbb{Z}^d$, c.f. [13]. An infinite volume limit (thermodynamic limit) for ν_Λ^ψ and $\Lambda \nearrow \mathbb{Z}^d$ exists only when $d \geq 3$.

2.3 $\nabla\phi-$Gibbs Measures

2.3.1 Notation on $\mathbb{Z}^d$

Let

$$\mathcal{B}_{\mathbb{Z}^d} := \Big\{ b = (x_b, y_b) \mid x_b, y_b \in \mathbb{Z}^d, \|x_b - y_b\| = 1, b \text{ directed from } x_b \text{ to } y_b \Big\}, \tag{23}$$

$$\mathcal{B}_{\mathbb{Z}^d}^\Lambda := \mathcal{B}_{\mathbb{Z}^d} \cap (\Lambda \times \Lambda), \quad \bar{\Lambda} := \Lambda \cup \partial\Lambda, \tag{24}$$

$$\partial\mathcal{B}_{\mathbb{Z}^d}^\Lambda := \Big\{ b = (x_b, y_b) \mid x_b \in \mathbb{Z}^d \setminus \Lambda, y_b \in \mathcal{B}_{\mathbb{Z}^d}^\Lambda, \|x_b - y_b\| = 1 \Big\} \tag{25}$$

and

$$\mathcal{B}_{\mathbb{Z}^d}^{\bar{\Lambda}} := \Big\{ b = (x_b, y_b) \in \mathcal{B}_{\mathbb{Z}^d} \mid x_b \in \Lambda \text{ or } y_b \in \Lambda \Big\}. \tag{26}$$

The height variables $\phi = \{\phi(x); x \in \mathbb{Z}^d\}$ on $\mathbb{Z}^d$ automatically determins a field of height differences $\nabla\phi = \{\nabla\phi(b); b \in \mathcal{B}_{\mathbb{Z}^d}\}$. One can therefore consider the distribution μ of $\nabla\psi$-field under the ϕ-Gibbs measure μ. We shall call μ the $\nabla\psi$-Gibbs measure. in fact, it is possible to define the $\nabla\psi$-Gibbs measures directly by means of the DLR equations and, in this sence, $\nabla\psi$-Gibbs measures exist for all dimensions $d \geq 1$.

A sequence of bonds $\mathcal{C} = \{b^{(1)}, b^{(2)}, \dots, b^{(n)}\}$ is called a chain connecting y and x, $x, y \in \mathbb{Z}^d$, if $y_{b_1} = y, x_{b^{(i)}} = y_{b^{(i+1)}}$ for $1 \leq i \leq n-1$ and $x_{b^{(n)}} = x$. The chain is called a closed loop if $x_{b^{(n)}} = y_{b^{(1)}}$. A plaquette is a closed loop $\mathcal{A} = \{b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}\}$ such that $\{x_{b^{(i)}}, i = 1, \dots, 4\}$ consists of 4 different points.

The field $\eta = \{\eta(b)\} \in \mathbb{R}^{\mathcal{B}_{\mathbb{Z}^d}}$ is said to satisfy the plaquette condition if

$$\eta(b) = -\eta(-b) \text{ for all } b \in \mathcal{B}_{\mathbb{Z}^d} \text{ and } \sum_{b \in \mathcal{A}} \eta(b) = 0 \text{ for all plaquettes in } \mathbb{Z}^d, \tag{27}$$

where $-b$ denotes the reversed bond of b. Let χ be the set of all $\eta \in \mathbb{R}^{\mathcal{B}_{\mathbb{Z}^d}}$ which satisfy the plaquette condition and let $L_r^2, r > 0$ be the set of all $\eta \in \mathbb{R}^{\mathcal{B}_{\mathbb{Z}^d}}$ such that

$$|\eta|_r^2 := \sum_{b \in \mathcal{B}_{\mathbb{Z}^d}} |\eta(b)|^2 e^{-2r|x_b|} < \infty. \tag{28}$$

We denote $\chi_r = \chi \cap L_r^2$ equipped with the norm $|\cdot|_r$. For $\phi = (\phi(x))_{x \in \mathbb{Z}^d}$ and $b \in \mathcal{B}_{\mathbb{Z}^d}$, we define the height differences

$$\eta^\phi(b) := \nabla\phi(b) = \phi(y_b) - \phi(x_b). \tag{29}$$

Then $\nabla\phi = \{\nabla\phi(b)\}$ satisfies the plaquette condition. Conversely, the heights $\phi^{\eta,\phi(0)} \in \mathbb{R}^{\mathbb{Z}^d}$ can be constructed from height differences η and the height variable $\phi(0)$ at $x = 0$ as

$$\phi^{\eta,\phi(0)}(x) := \sum_{b\in\mathcal{C}_{0,x}} \eta(b) + \phi(0), \tag{30}$$

where $\mathcal{C}_{0,x}$ is an arbitrary chain connecting 0 and x. Note that $\phi^{\eta,\phi(0)}$ is well-defined if $\eta = \{\eta(b)\} \in \chi$.

2.3.2 Definition of $\nabla\phi$-Gibbs measures

We next define the finite volume $\nabla\phi$-Gibbs measures. For every $\xi \in \chi$ and $\Lambda \subset \mathbb{Z}^d$ the space of all possible configurations of height differences on $\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}$ for given boundary condition ξ is defined as

$$\chi_{\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d},\xi} = \{\eta = (\eta(b))_{b\in\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}}; \eta \vee \xi \in \chi\}, \tag{31}$$

where $\eta \vee \xi \in \chi$ is determined by $(\eta \vee \xi)(b) = \eta(b)$ for $b \in \mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}$ and $= \xi(b)$ for $b \notin \mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}$. The finite volume $\nabla\phi$-Gibbs measure in Λ (or more precisely, in $\mathcal{B}^{\Lambda}_{\mathbb{Z}^d}$) with boundary condition ξ is defined by

$$\mu^{\xi}_{\Lambda}(\mathrm{d}\eta) = \frac{1}{Z_{\lambda.\xi}} \exp\left\{ -\beta \sum_{b\in\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}} \sum_{i\in I} U_i(\eta(b)) \right\} \mathrm{d}\eta_{\Lambda,\xi} \in P(\chi_{\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d},\xi}), \tag{32}$$

where $\mathrm{d}\eta_{\Lambda,\xi}$ denotes a uniform measure on the affine space $\chi_{\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d},\xi}$ and $Z_{\Lambda,\xi}$ is the normalization constant. Let $P(\chi)$ be the set of all probability measures on χ and let $P_2(\chi)$ be those $\mu \in P(\chi)$ satisfying $E^{\mu}[|\eta(b)|^2] < \infty$ for each $b \in \mathcal{B}_{\mathbb{Z}^d}$.

For every $\xi \in \chi$ and $a \in \mathbb{R}$, let $\psi = \phi^{\xi,a}$ be defined by (30) and consider the measure ν^{ψ}_{Λ}. Then μ^{ξ}_{Λ} is the image measure of ν^{ψ}_{Λ} under the map

$$\{\phi(x)\}_{x\in\Lambda} \to \{\eta(b) := \nabla(\phi \vee \psi)(b)\},\ b \in \mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}. \tag{33}$$

Note that the image measure is determined only by ξ and is independent of the choice of a.

Definition 3 *The probability measure $\mu \in P(\chi)$ is called a Gibbs measure for the height differences ($\nabla\phi$-Gibbs measure for short), if it satisfies the DLR equation*

$$\mu(\cdot\,|\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\setminus\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}})(\xi) = \mu^{\xi}_{\Lambda}(\cdot), \quad \mu - a.e.\ \xi,$$

for every $\Lambda \subset \mathbb{Z}^d$, where $\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\setminus\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}}$ stands for the σ-field of χ generated by $\{\eta(b); b \in \mathcal{B}_{\mathbb{Z}^d} \setminus \mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}\}$.

We will define by

$$\begin{aligned}\mathcal{G}(H) := \{\mu \in P_2(\chi) : \mu \text{ shift invariant} \nabla\phi - \text{Gibbs measure such that} \\ \mu^{\xi}_{\Lambda} \text{ has Hamiltonian } H^{\xi}_{\Lambda}\}.\end{aligned} \tag{34}$$

3 Even/Odd Representation

3.1 Notation on the Even Subset of $\mathbb{Z}^d$

As $\mathbb{Z}^d$ is a bipartite graph, we will label the vertices of $\mathbb{Z}^d$ as **even** and **odd** vertices, such that every **even** vertex has only **odd** nearest neighbor vertices and vice-versa. Let

$$\mathcal{E}^d := \left\{ a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d \mid \sum_{i=1}^{d} a_i = 2p, p \in \mathbb{Z} \right\}, \tag{35}$$

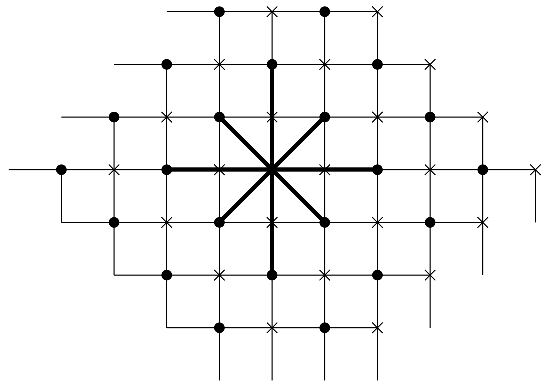

Figure 1: The bonds of 0 in $\mathcal{E}^2$

$$\mathcal{O}^d := \left\{ a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d \mid \sum_{i=1}^{d} a_i = 2p+1, p \in \mathbb{Z} \right\}, \tag{36}$$

$$\mathcal{E}^d_\Lambda := \mathcal{E}^d \cap \Lambda, \quad \mathcal{O}^d_\Lambda := \mathcal{O}^d \cap \Lambda \ \text{ and } \ \mathcal{O}^d_{\bar{\Lambda}} := \mathcal{O}^d \cap \bar{\Lambda}. \tag{37}$$

We will next define the bonds in $\mathcal{E}^d$ in a similar fashion to the definitions for bonds on $\mathbb{Z}^d$. For all $x, y \in \mathbb{Z}^d$, let

$$\mathcal{B}_{\mathcal{E}^d} := \left\{ b = (x_b, y_b) \mid x_b, y_b \in \mathcal{E}^d, \|x_b - y_b\| = 2, b \text{ directed from } x_b \text{ to } y_b \right\}, \tag{38}$$

$$\mathcal{B}^\Lambda_{\mathcal{E}^d} := \mathcal{B}_{\mathcal{E}^d} \cap (\Lambda \times \Lambda), \quad \partial\mathcal{B}^\Lambda_{\mathcal{E}^d} := \left\{ b = (x_b, y_b) \mid x_b \in \mathcal{E}^d \setminus \Lambda, y_b \in \mathcal{E}^d_\Lambda, \|x_b - y_b\| = 2 \right\} \tag{39}$$

and

$$\partial\mathcal{E}^d_\Lambda := \left\{ y \in \mathcal{E}^d_\Lambda \mid y = y_b \text{ for some } b \in \partial\mathcal{B}^\Lambda_{\mathcal{E}^d} \right\}. \tag{40}$$

An **even** plaquette is a closed loop $\mathcal{A}^\mathcal{E} = \{b^{(1)}, b^{(2)}, \ldots, b^{(n)}\}$, where $n \in \{3, 4\}$, such that $\{x_{b^{(i)}}, i = 1, \ldots, n\}$ consists of n different points. The field $\eta = \{\eta(b)\} \in \mathbb{R}^{\mathcal{B}_{\mathcal{E}^d}}$ is said to satisfy the **even** plaquette condition if

$$\eta(b) = -\eta(-b) \text{ for all } b \in \mathcal{B}_{\mathcal{E}^d} \text{ and } \sum_{b \in \mathcal{A}^\mathcal{E}} \eta(b) = 0 \text{ for all } \textbf{even} \text{ plaquettes in } \mathcal{E}^d. \tag{41}$$

Let $\chi^\mathcal{E}$ be the set of all $\eta \in \mathbb{R}^{\mathcal{B}_{\mathcal{E}^d}}$ which satisfy the even plaquette condition. For each $b \in \mathcal{B}_{\mathcal{E}^d}$ we define the even height differences

$$\eta^\mathcal{E}(b) := \nabla^\mathcal{E}\phi(b) = \phi(y_b) - \phi(x_b). \tag{42}$$

The heights $\phi^{\eta^\mathcal{E}, \phi(0)}$ can be constructed from the height differences $\eta^\mathcal{E}$ and the height variable $\phi(0)$ at $a = 0$ as

$$\phi^{\eta^\mathcal{E}, \phi(0)}(a) := \sum_{b \in C^\mathcal{E}_{0,a}} \eta^\mathcal{E}(b) + \phi(0), \tag{43}$$

where $a \in \mathcal{E}^d$ and $C^\mathcal{E}_{0,a}$ is an arbitrary path in $\mathcal{E}^d$ connecting 0 and a. Note that $\phi^{\eta,\phi(0)}(a)$ is well-defined if $\eta^\mathcal{E} = \{\eta^\mathcal{E}(b)\} \in \chi^\mathcal{E}$. We also define $\chi^\mathcal{E}_r$ similarly as we define χ_r. As on $\mathbb{Z}^d$, let $P(\chi^\mathcal{E})$ be the set of all probability measures on $\chi^{\mathcal{E}}$ and let $P_2(\chi^{\mathcal{E}})$ be those $\mu \in P(\chi^{\mathcal{E}})$ satisfying $E^{\mu}[|\eta^{\mathcal{E}}(b)|^2] < \infty$ for each $b \in \mathcal{B}_{\mathcal{E}^d}$. We will define by

$$\begin{aligned} \mathcal{G}^{\mathcal{E}}(H) := \{\mu \in P_2(\chi^{\mathcal{E}}) : \mu \text{ shift invariant} \nabla\phi - \text{Gibbs measure such that} \\ \text{for all } \Lambda \in \mathbb{Z}^d,\ \mu^{\xi}_{\mathcal{E}^d_\Lambda} \text{ has Hamiltonian } H^{\xi}_{\mathcal{E}^d_\Lambda}\}. \end{aligned} \tag{44}$$

3.2 Restriction of a $\nabla\phi$-Gibbs measure to $\mathcal{E}^d$

For simplicity of calculations, we will assume for the remainder of the paper that for all $i \in I$, $U_i = U_{-i}$ and that U_i are symmetric, though this condition is not necessary. Note that throught the paper we will use the notation

$$\nabla_i\phi(x) = \phi(x + e_i) - \phi(x). \tag{45}$$

Let

$$\theta(x) = (\phi(x + e_1), \dots, \phi(x + e_d), \phi(x - e_1), \dots, \phi(x - e_d)). \tag{46}$$

Lemma 4 *Let $\mu \in \mathcal{G}(H)$, where H has the form in (14). Then $\mu|_{\mathcal{E}^d} \in \mathcal{G}^{\mathcal{E}}(H^{(2)})$, where*

$$H^{(2)}_{\mathcal{E}^d_\Lambda}(\phi) := \sum_{x \in \mathcal{O}^d_{\bar{\Lambda}}} F_x(\theta(x)), \tag{47}$$

where

$$F_x(\theta(x)) = -\log \int_{\mathbb{R}} e^{-\beta \sum_{i\in I} U_i(\nabla_i\phi(x))}\, \mathrm{d}\phi(x). \tag{48}$$

Proof. Let

$$\mathcal{F}_{\mathbb{Z}^d} := \sigma\left(\phi(x)), x \in \mathbb{Z}^d\right) \text{ and } \mathcal{F}_{\mathcal{E}^d} := \sigma\left(\phi(x), x \in \mathcal{E}^d\right) \tag{49}$$

and

$$\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}} := \sigma\left(\eta(b), b \in \mathcal{B}_{\mathbb{Z}^d}\right) \text{ and } \mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}} := \sigma\left(\eta^{\mathcal{E}}(b), b \in \mathcal{B}_{\mathcal{E}^d}\right). \tag{50}$$

Since $\mu \in \mathcal{G}(H)$, for all Λ finite sets in $\mathbb{Z}^d$ and for all $A \in \mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}}$ we have

$$\mu(A|\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\backslash\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}})(\xi) \quad = \quad \mu_{\Lambda,\xi}(A) = \frac{1}{Z_{\Lambda,\xi}} \int_A e^{-\beta H^{\xi}_\Lambda(\phi)} \prod_{x\in\Lambda} \mathrm{d}\phi(x) \prod_{y\in\mathbb{Z}^d\backslash\Lambda} \left(\delta(\phi(y) - \xi(y))\right), \tag{51}$$

where $Z_{\Lambda,\xi}$ is the normalizing constant. Let $\Lambda^{\mathcal{E}}$ be a finite set in $\mathcal{E}^d$. We will construct from $\Lambda^{\mathcal{E}}$ a finite set $\Lambda \in \mathbb{Z}^d$ as follows: if $x \in \Lambda^{\mathcal{E}}$, then $x + e_i \in \Lambda$ for all $i \in I$, $\Lambda \cap \mathcal{E}^d = \Lambda^{\mathcal{E}}$ and such that Λ and $\Lambda^{\mathcal{E}}$ have the same boundary conditions, that is only even vertices are boundary points (see Figures 2 and 3). Then, since μ is a $\nabla\phi$-Gibbs measure and since for every $\Lambda \in \mathbb{Z}^d$, $\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}\backslash\mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}} \subset \mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\backslash\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}}$, it follows that for every $A \in \mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}}$

$$\begin{aligned} \mu(A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}\backslash\mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}})(\xi) &= \mathbf{E}_\mu\left(\mathbf{E}_\mu(1_A|\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\backslash\mathcal{B}^{\bar{\Lambda}}_{\mathbb{Z}^d}})|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}\backslash\mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}}\right)(\xi) = \mathbf{E}_\mu\left(\mu_{\Lambda,\xi}(A)\big|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}\backslash\mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}}\right)(\xi) \\ &= \mathbf{E}_\mu\left(\int_A e^{-\beta H_\Lambda(\phi)} \prod_{x\in\Lambda} \mathrm{d}\phi(x) \prod_{y\in\mathbb{Z}^d\backslash\Lambda} \left(\delta(\phi(y) - \xi(y))\right)\big|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}\backslash\mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}}\right)(\xi), \end{aligned} \tag{52}$$

where $\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}\setminus\mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}} = \sigma\left(\eta(b), b \in \mathcal{B}_{\mathcal{E}^d} \setminus \mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}\right)$. In particular, from (52), by integrating out the odd points and due to the boundary conditions on Λ, we have for every $A \in \mathcal{F}_{\mathcal{B}^{\Lambda}_{\mathcal{E}^d}}$

$$\begin{aligned} &\mu(A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}\setminus\mathcal{B}_{\bar{\Lambda}^{\mathcal{E}}}})(\xi) \\ &= \frac{1}{Z_{\Lambda,\xi}} \int_A e^{-\sum_{x\in\mathcal{O}^d_\Lambda} F^{\Lambda}_x(\phi(x+e_1),\ldots,\phi(x-e_d))} \prod_{i\in I} \prod_{\substack{x\in\mathcal{O}^d_\Lambda:\\ x+e_i\in\mathcal{E}^d_\Lambda}} \mathrm{d}\phi(x+e_i) \prod_{y\in\mathcal{E}^d\setminus\Lambda^{\mathcal{E}}} (\delta(\phi(y)-\xi(y)) \end{aligned} \quad (53)$$

Therefore $\mu|_{\mathcal{E}^d}$ satisfies the DLR equations, so the proof is finished. □

Remark 5 Note that for any constant $C_{2d} = (C, C, \ldots, C) \in \mathbb{R}^{2d}$, we have

$$F_x(\theta(x)) = F_x(\theta(x) + C_{2d}). \quad (54)$$

In particular, this means that for any $i \in I$

$$F_x(\theta(x)) = F_x(\phi(x+e_1) - \phi(x+e_i), \ldots, \phi(x-e_d) - \phi(x+e_i)). \quad (55)$$

Therefore we are still dealing with a gradient system, even though this is no longer a two-body gradient system.

Remark 6 Note that the new Hamiltonian $H^{(2)}$ depends on β through the functions F_x.

Lemma 7 *The conditional law of $(\phi(x))_{x\in\mathcal{O}^d}$ given $\mathcal{F}_{\mathcal{E}^d}$ is just the product law with density at $x \in \mathcal{O}^d$*

$$\mu_x(\,\mathrm{d}\phi(x)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}) = \frac{1}{Z(\theta(x))} \exp\left(-\beta \sum_{i\in I} U_i(\phi(x+e_i) - \phi(x))\right) \mathrm{d}\phi(x) \quad (56)$$

with

$$Z(\theta(x)) = \int \exp\left(-\beta \sum_{i\in I} U_i(\phi(x+e_i) - \phi(x))\right) \mathrm{d}\phi(x). \quad (57)$$

PROOF. First note that for $\Lambda = x \in \mathcal{O}^d$, (52) becomes

$$\mu(\,\cdot\,|\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\setminus\mathcal{B}^x_{\mathbb{Z}^d}})(\xi) = \mu^{\xi}_x. \quad (58)$$

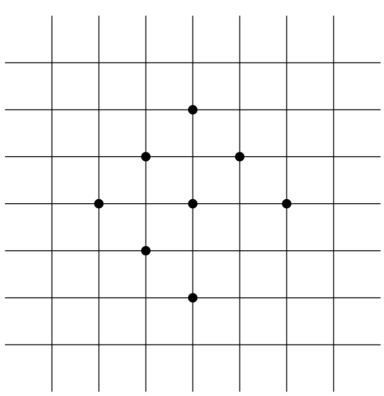

Figure 2: The graph of $\Lambda^{\mathcal{E}}$

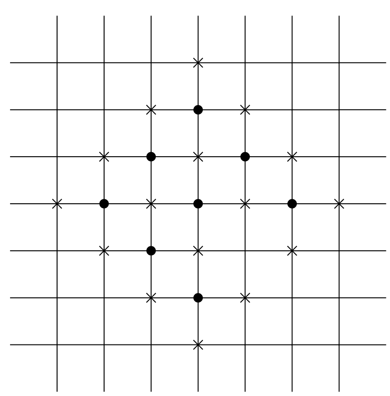

Figure 3: The graph of Λ

Note now that because $\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}} \subset \cap_{x\in\mathcal{O}^d}\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\setminus\mathcal{B}^x_{\mathbb{Z}^d}}$, we can reason as in (52) to get

$$\mu(\,\cdot\,|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})(\xi) = \otimes_{x\in\mathcal{O}^d}\mu(\,\cdot\,|\mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}\setminus\mathcal{B}^x_{\mathbb{Z}^d}})(\xi) = \otimes_{x\in\mathcal{O}^d}\mu^{\xi}_x. \tag{59}$$

The statement of the lemma follows now from (59). □

As an immediate consequence of Lemma 7 and of Remark 5, we have

Corollary 8 *The conditional law of $(\nabla\phi(x))_{x\in\mathcal{O}^d}$ given $\mathcal{F}_{\mathcal{E}^d}$ is just the product law with density at $x\in\mathcal{O}^d$*

$$\begin{aligned}&\tilde{\mu}_x(\,\mathrm{d}\phi(x+e_k) - \,\mathrm{d}\phi(x)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})\\ &\qquad = \frac{1}{\tilde{Z}(\nabla\theta(x))}\exp\left(-\beta\sum_{i\in I}U_i(\phi(x)-(\phi(x+e_k)-\phi(x+e_i)))\right)\mathrm{d}\phi(x),\end{aligned} \tag{60}$$

which depends only on the even gradients $\nabla^{\mathcal{E}}\phi$, where we defined

$$\tilde{Z}(\nabla\theta(x)) := Z(\phi(x+e_1)-\phi(x+e_k),\ldots,\phi(x-e_d)-\phi(x+e_k)), \tag{61}$$

□

3.3 Random Walk Representation

3.3.1 Definition and Theorems

Definition 9 *Let $x\in\mathcal{O}^d$. We say that F_x satisfies* ***the random walk representation****, if there exists $c_1, c_2 > 0$ such that for all $i,j\in I$*

$$D^{i,i}F_x = -\sum_{j:j\neq i}D^{i,j}F_x \text{ and } c_1 \le -D^{i,j}F_x \le c_2 \text{ for } i\neq j, \tag{62}$$

where for $i\in I$, we denoted by

$$D^iF(y_1,\ldots,y_d,y_{-1},\ldots,y_{-d}) := \frac{\partial}{\partial y_i}F(y_1,\ldots y_d,y_{-1},\ldots,y_{-d}). \tag{63}$$

The main result of this section is:

Theorem 10 *(Random Walk Representation) For all $i\in I$, let $U_i\in C^2(\mathbb{R})$ be such that they satisfy (15). We also assume that, for all $i\in I$, $V_i, g_i\in C^2(\mathbb{R})$ satisfy (16) and (17). Then, if*

$$\sqrt{\frac{\beta}{C_1}}\sup_{i\in I}||g''_i||_{L^1(\mathbb{R})} < \frac{C_1}{2C_2\sqrt{d}}, \tag{64}$$

there exists $c_1, c_2 > 0$ such that for all $x\in\mathcal{O}^d_\Lambda$, F_x satisfies the random walk representation.

Lemma 11 *Suppose $x\in\mathcal{O}^d_\Lambda$. Then for all $j\in I$ such that $x+e_j\in\mathcal{E}^d_\Lambda$, we have*

$$D^jF_x(\theta(x)) = -\sum_{i\in I, i\neq j}D^iF_x(\theta(x)), \tag{65}$$

$$D^{j,j}F_x(\theta(x)) = -\sum_{i\in I, i\neq j}D^{i,j}F_x(\theta(x)), \tag{66}$$

and for all $i\in I, i\neq j$ such that $x+e_i\in\mathcal{E}^d_\Lambda$

$$D^{i,j}F_x(\theta(x)) = -\beta^2\mathrm{cov}_{\mu_x}\left(U'_i(\nabla_i\phi(x)), U'_j(\nabla_j\phi(x))\right), \tag{67}$$

where μ_x is given by (56) and $\mathbf{E}_{\mu_x}$ and cov_{μ_x} are the expectation, respectively the covariance, with respect to the measure μ_x.

PROOF. For all $j \in I$, from (55) we have

$$
\begin{aligned}
D^j F_x(\theta(x)) &= \frac{\partial}{\phi(x+e_j)} F_x(\phi(x+e_1)-\phi(x+e_j),\ldots,\phi(x-e_d)-\phi(x+e_j)) \\
&= -\sum_{i\in I, i\neq J} D^i F_x(\phi(x+e_1)-\phi(x+e_j),\ldots,\phi(x-e_d)-\phi(x+e_j)) \qquad (68)
\end{aligned}
$$

and for $i \neq j$

$$
\begin{aligned}
D^i F_x(\theta(x)) &= \frac{\partial}{\phi(x+e_i)} F_x(\phi(x+e_1)-\phi(x+e_j),\ldots,\phi(x-e_d)-\phi(x+e_j)) \\
&= D^i F_x(\phi(x+e_1)-\phi(x+e_j),\ldots,\phi(x-e_d)-\phi(x+e_j)). \qquad (69)
\end{aligned}
$$

It follows now from (68) and (69) that

$$
D^j F_x(\theta(x)) = -\sum_{i\in I, i\neq j} D^i F_x(\theta(x)). \qquad (70)
$$

By simple differentiation in F_x, we have for for all $i, j \in I, i \neq j$

$$
D^{i,j} F_x(\theta(x)) = -\beta^2 \mathrm{cov}_{\mu_x}\left(U_i'(\nabla_i\phi(x)), U_j'(\nabla_j\phi(x))\right). \qquad (71)
$$

(66) now follows immediately from (70) and (71). □

The following lemma is elementary to prove by using Taylor expansion and will be needed for the proof of Theorem 10:

Lemma 12 *(Representation of Covariances)*
Let $k \in \mathbb{N}$. For all functions $F, G \in C^1(\mathbb{R}^k;\mathbb{R})$ and for all measures $\mu \in P(\mathbb{R}^k)$, we have

$$
\begin{aligned}
\mathrm{cov}_\mu(F,G) &= \frac{1}{2}\iint [F(\phi)-F(\psi)]\,[G(\phi)-G(\psi)]\,\mu(\,\mathrm{d}\phi)\mu(\,\mathrm{d}\psi) \\
&= \frac{1}{2}\iint [(\phi-\psi)\cdot DF(\phi,\psi)]\,[(\phi-\psi)\cdot DG(\phi,\psi)]\,\mu(\,\mathrm{d}\phi)\mu(\,\mathrm{d}\psi)
\end{aligned}
\qquad (72)
$$

where we denoted by

$$
DF(\phi,\psi) := \int_0^1 DF\left(\psi + t(\phi-\psi)\right)\,\mathrm{d}t, \quad DG(\phi,\psi) := \int_0^1 DG\left(\psi + s(\phi-\psi)\right)\,\mathrm{d}s \qquad (73)
$$

and by

$$
DF(\phi) := \left(D^1F(\phi),\ldots,D^kF(\phi)\right). \qquad (74)
$$

Proof of Theorem 10 It follows from Lemma 11 that, in order to be able to use the random walk representation, all we need is to estimate the covariances:

$$
\mathrm{cov}_{\mu_x}\left(U_i'(\nabla_i\phi(x)), U_j'(\nabla_j\phi(x))\right).
$$

We have $U_i = V_i + g_i$ and $U_j = V_j + g_j$, where $C_1 \leq V_i'', V_j'' \leq C_2$. Then using Lemma 12 for V_i' and V_j', we see that

$$
\begin{aligned}
0 \leq C_1^2 \mathrm{var}_{\mu_x}(\phi(x)) \leq C_1 \mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))) &\leq \mathrm{cov}_{\mu_x}(V_i'(\nabla_i\phi(x)), V_j'(\nabla_j\phi(x))) \\
&\leq C_2 \mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))). \qquad (75)
\end{aligned}
$$

Since $g_i'', g_j'' < 0$ and since we have

$$\mathrm{cov}_{\mu_x}(g_i'(\nabla_i\phi(x)), g_j'(\nabla_j\phi(x))) = \mathrm{cov}_{\mu_x}(-g_i'(\nabla_i\phi(x)), -g_j'(\nabla_j\phi(x))), \tag{76}$$

we can also estimate $\mathrm{cov}_{\mu_x}(g_i'(\nabla_i\phi(x)), g_j'(\nabla_j\phi(x)))$ using Lemma 12 to get

$$0 \leq \mathrm{cov}_{\mu_x}(g_i'(\nabla_i\phi(x)), g_j'(\nabla_j\phi(x) \leq C_0^2 \mathrm{var}_{\mu_x}(\phi(x)). \tag{77}$$

We now have to find an upper bound for $\mathrm{var}_{\mu_x}(\phi(x))$. From (75), it is sufficient to find an upper bound for $\mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))$. Note now that from (75), we have

$$\mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x) \leq \frac{1}{2dC_1}\mathrm{cov}_{\mu_x}\left(V_j'(\nabla_j\phi(x)), \sum_{i\in I} V_i'(\nabla_i\phi(x))\right). \tag{78}$$

Using integration by parts, we have

$$\begin{aligned}\mathrm{cov}_{\mu_x}\left(V_j'(\nabla_j\phi(x)), \sum_{i\in I} V_i'(\nabla_i\phi(x)\right) &= \frac{1}{\beta}\mathbf{E}_{\mu_x}\left(V_j''(\nabla_j\phi(x))\right) \\ &\quad -\mathrm{cov}_{\mu_x}\left(V_j'(\nabla_j\phi(x)), \sum_{i\in I} g_i'(\nabla_i\phi(x))\right).\end{aligned} \tag{79}$$

It now remains to estimate $\mathrm{cov}_{\mu_x}(V_i'(\nabla_i\phi(x)), g_j'(\nabla_j\phi(x))$. By using Lemma 12, we get that

$$0 \leq -\mathrm{cov}_{\mu_x}(V_j'(\nabla_j\phi(x)), \sum_i g_i'(\nabla_i\phi(x)) < 2dC_0\mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))), \tag{80}$$

which has the wrong sign. But by using the Cauchy-Schwarz inequality and (75), we have

$$\begin{aligned}-\mathrm{cov}_{\mu_x}(V_i'(\nabla_i\phi(x)), g_j'(\nabla_j\phi(x)) &\leq \sqrt{\mathrm{var}_{\mu_x}(V_i'(\nabla_i\phi(x))}\sqrt{\mathrm{var}_{\mu_x}(g_j'(\nabla_j\phi(x))} \\ &\leq \sqrt{C_2\mathrm{cov}_{\mu_x}(\phi(x), V_i'(\nabla_i\phi(x)))}\sqrt{\mathrm{var}_{\mu_x}(g_j'(\nabla_j\phi(x))}.\end{aligned} \tag{81}$$

Then we estimate $\mathrm{var}_{\mu_x}(g_j'(\nabla_j\phi(x))$ by applying Lemma 12 to get

$$\begin{aligned}&\mathrm{var}_{\mu_x}(g_j'(\nabla_j\phi(x)) \\ &= \frac{1}{2}\iint(\phi(x)-\psi(x))^2\left[\int_0^1 g_j''\left(\psi(x)+t(\phi(x)-\psi(x))-\phi(x+e_j)\right)\mathrm{d}t\right]^2\mu_x(\mathrm{d}\phi)\mu_x(\mathrm{d}\psi) \\ &= \frac{1}{2}\iint\left[\int_{\psi(x)-\phi(x+e_j)}^{\phi(x)-\phi(x+e_j)} g_j''(s)\,\mathrm{d}s\right]^2\mu_x(\mathrm{d}\phi)\mu_x(\mathrm{d}\psi) \leq \frac{1}{2}||g_j''||^2_{L^1(\mathbb{R})}.\end{aligned} \tag{82}$$

From (78), (79) and (82), we now get the upper bound

$$\mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))) \leq \frac{C_2}{2d\beta C_1} + \frac{\sqrt{C_2}}{C_1\sqrt{2}}\sup_{i\in I}||g_i''||_{L^1(\mathbb{R})}\sqrt{\mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x)))}, \tag{83}$$

from which we get

$$\mathrm{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))) \leq \left(\frac{\sqrt{C_2}}{2\sqrt{2}C_1}\sup_{i\in I}||g_i''||_{L^1(\mathbb{R})} + \frac{1}{2}\sqrt{\frac{C_2}{2C_1^2}\sup_{i\in I}||g_i''||^2_{L^1(\mathbb{R})} + 2\frac{C_2}{d\beta C_1}}\right)^2. \tag{84}$$

Also, by using (75), we get from (84)

$$\text{var}_{\mu_x}(\phi(x)) \leq \frac{1}{C_1}\left(\frac{\sqrt{C_2}}{2\sqrt{2}C_1}\sup_{i\in I}||g_i''||_{L^1(\mathbb{R})} + \frac{1}{2}\sqrt{\frac{C_2}{2C_1^2}\sup_{i\in I}||g_i''||^2_{L^1(\mathbb{R})} + 2\frac{C_2}{d\beta C_1}}\right)^2 := \sigma^2. \tag{85}$$

The upper bound now follows from (75), (77), (84) and (85). To find a lower bound, note now that from (75) we get

$$\text{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))) \geq \frac{1}{2dC_2}\text{cov}_{\mu_x}\left(V_j'(\nabla_j\phi(x)), \sum_{i\in I} V_i'(\nabla_i\phi(x))\right). \tag{86}$$

By using (79) and (80), we get

$$\text{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x))) \geq \frac{C_1}{2dC_2\beta}. \tag{87}$$

From (75), (77) and (82), we get

$$\begin{aligned}&\text{cov}_{\mu_x}\left(U_i'(\nabla_i\phi(x)), U_j'(\nabla_j\phi(x))\right)\\ \geq\;& \sqrt{\text{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x)))}\left[C_1\sqrt{\text{cov}_{\mu_x}(\phi(x), V_j'(\nabla_j\phi(x)))} - \sqrt{C_2/2}\sup_{i\in I}||g_i''||_{L^1(\mathbb{R})}\right].\end{aligned} \tag{88}$$

The lower bound now follows from (87) and (88). □

Remark 13 Note that condition (64) can be replaced by other conditions. For example

(a)

$$(\beta)^{1/4}\sup_{i\in I}||g_i''||_{L^2(\mathbb{R})} < \frac{(C_1)^{3/2}}{(C_2)^{3/4}d^{1/4}}. \tag{89}$$

We obtain this condition by using the Cauchy-Schwartz inequality and (75)

$$\begin{aligned}&\left|\text{cov}_{\mu_x}(V_i'(\nabla_i\phi(x)), g_j'(\nabla_j\phi(x))\right|\\ \leq\;& \sqrt{\text{var}_{\mu_x}(V_i'(\nabla_i\phi(x))}\sqrt{\text{var}_{H_x}(g_j'(\nabla_j\phi(x))}\\ \leq\;& \sqrt{C_2\text{cov}_{\mu_x}(\phi(x), V_i'(\nabla_i\phi(x)))}\sqrt{\text{var}_{\mu_x}(g_j'(\nabla_j\phi(x))}.\end{aligned} \tag{90}$$

Then we estimate $\text{var}_{\mu_x}(g_j'(\nabla_j\phi(x))$ by applying Lemma 12 and Jensen's inequality to get

$$\begin{aligned}&\text{var}_{\mu_x}(g_j'(\nabla_j\phi(x))\\ =\;& \frac{1}{2}\iint(\phi(x)-\psi(x))^2\left[\int_0^1 g_j''\left(\psi(x)+t(\phi(x)-\psi(x))-\phi(x+e_j)\right)\,\mathrm{d}t\right]^2\mu_{\mu_x}(\mathrm{d}\phi)\mu_{\mu_x}(\mathrm{d}\psi)\\ \leq\;& \frac{1}{2}\iint(\phi(x)-\psi(x))^2\int_0^1\left[g_j''\left(\psi(x)+t(\phi(x)-\psi(x))-\phi(x+e_j)\right)\right]^2\mathrm{d}t\;\mu_{\mu_x}(\mathrm{d}\phi)\mu_{\mu_x}(\mathrm{d}\psi)\\ =\;& \frac{1}{2}\iint(\phi(x)-\psi(x))\int_{\psi(x)-\phi(x+e_j)}^{\phi(x)-\phi(x+e_j)}\left[g_j''(s)\right]^2\mathrm{d}s\;\mu_{\mu_x}(\mathrm{d}\phi)\mu_{\mu_x}(\mathrm{d}\psi)\\ \leq\;& \frac{1}{2}||g_j''||^2_{L^2(\mathbb{R})}\iint|\phi(x)-\psi(x)|\,\mu_{\mu_x}(\mathrm{d}\phi)\mu_{\mu_x}(\mathrm{d}\psi)\\ \leq\;& \frac{1}{2}||g_j''||^2_{L^2(\mathbb{R})}\sqrt{\iint(\phi(x)-\psi(x))^2\,\mu_{\mu_x}(\mathrm{d}\phi)\mu_{\mu_x}(\mathrm{d}\psi)}\\ =\;& \frac{1}{\sqrt{2}}||g_j''||^2_{L^2(\mathbb{R})}\sqrt{\text{var}_{H_x}(\phi(x))} \leq ||g_j''||^2_{L^2(\mathbb{R})}\sqrt{\frac{\text{cov}_{\mu_x}(\phi(x)), V_i'(\nabla_i\phi(x))}{2C_1}}.\end{aligned} \tag{91}$$

The rest of the argument to obtain the bound in (64) follows the same steps as in proof of Theorem 10.

(b) Another possible condition is

$$(\beta)^{3/4} \sup_{i\in I} ||g_i'||_{L^2(\mathbb{R})} \le \frac{(C_1)^{3/2}}{(C_2)^{5/4}} \frac{1}{(2d)^{3/4}}, \tag{92}$$

obtained by using the Cauchy-Schwarz inequality, (75) and Lemma 14 below, to get

$$\begin{aligned} &\left|\text{cov}_{\mu_x}(V_i'(\nabla_i\phi(x)), g_j'(\nabla_j\phi(x))\right| \\ \le\; & \sqrt{\text{var}_{\mu_x}(V_i'(\nabla_i\phi(x))}\sqrt{\text{var}_{\mu_x}(g_j'(\nabla_j\phi(x))} \\ \le\; & (C_2)^{3/4}(2d\beta)^{1/4}\sqrt{\text{cov}_{\mu_x}(\phi(x)), V_i'(\nabla_i\phi(x))) \sup_{i\in I}||(g_i')^2||_{L^1(\mathbb{R})}}. \end{aligned} \tag{93}$$

Lemma 14 *If $h \in L^1(\mathbb{R})$, then we have*

$$|\mathbf{E}_{\mu_x}(h)| \le \sqrt{2d\beta C_2}||h||_{L^1(\mathbb{R})}. \tag{94}$$

Proof. Using integration by parts and Cauchy-Schwartz, we have

$$\begin{aligned} |\mathbf{E}_{\mu_x}(h)| &= \left|\mathbf{E}_{\mu_x}\left(\frac{\partial}{\partial y}\left(\int_{-\infty}^{y} h(z)\,\mathrm{d}z\right)\right)\right| = \left|\mathbf{E}_{\mu_x}\left(H_x'(y)\left(\int_{-\infty}^{y} h(z)\,\mathrm{d}z\right)\right)\right| \\ &\le \mathbf{E}_{\mu_x}^{1/2}\left((H_x')^2\right)\mathbf{E}_{\mu_x}^{1/2}\left(\left(\int_{-\infty}^{y} h(z)\,\mathrm{d}z\right)^2\right) = \mathbf{E}_{\mu_x}^{1/2}(H_x'')\mathbf{E}_{\mu_x}^{1/2}\left(\int_{-\infty}^{y} h(z)\,\mathrm{d}z\right)^2 \\ &\le \sqrt{2d\beta C_2}||h||_{L^1(\mathbb{R})}. \end{aligned} \tag{95}$$

Note that we also used property (15) and integration by parts in the above formula. □

Remark 15 As we mentioned before, we can adapt all the reasoning in Section 3.2 and Section 3.3 to the case where U_i do not fullfil the assumption that for all $i \in I$, $U_i = U_{-i}$ and U_i symmetric. In this more general case,

$$F_x(\theta(x)) = -\log \int_{\mathbb{R}} e^{-\beta\sum_{i\in I}[U_i(\nabla\phi(x))+U_{-i}(-\nabla\phi(x))]}\,\mathrm{d}\phi(x). \tag{96}$$

Remark 16 Note that if we consider the case where for all $i \in I$, U_i are strictly convex such that $C_1 \le U_i'' \le C_2$, in view of (75), (87) and (83) applied to the case with $g = 0$

$$\frac{C_1^2}{2d\beta C_2} \le \text{cov}_{\mu_x}\left(U_i'(\nabla_i\phi(x)), U_j'(\nabla_j\phi(x))\right) \le \frac{C_2^2}{2d\beta C_1}. \tag{97}$$

3.3.2 Examples

(a) Let $p \in (0,1)$ and $0 < k_2 < k_1$. Let

$$U(s) = -\log\left(pe^{-k_1\frac{s^2}{2}} + (1-p)e^{-k_2\frac{s^2}{2}}\right).$$

Set $a = \frac{k_1}{k_2}$. Take $p < a^{-1}$ in order that the potential U is non-convex. If

$$0 < (\beta)^{3/4}p(1-p)^{1/4}(a-1)^{1/4} \le \frac{1}{2(2d)^{3/4}(\pi)^{1/4}}, \tag{98}$$

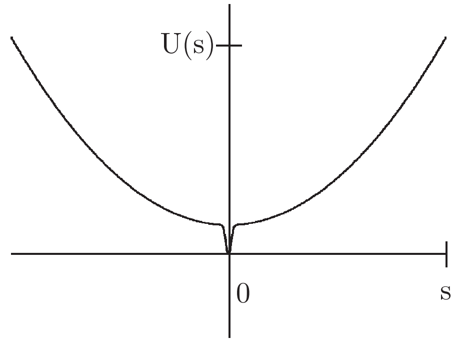

Figure 4: Example (a)

then (92) is satisfied and the RW representation holds. If $\beta = 1$ and $k_1 \gg k_2$, the above condition is equivalent to $p < p_0$, where $p_0 \approx \frac{a^{-1/4}}{2(2d)^{3/4}\pi^{1/4}}$. This is close to, for $d = 2$, the critical point p_c, such that $\frac{p_c}{1-p_c} = a^{-1/4}$, of [3], where uniqueness of ergodic states is violated for this example of potential U.

The computations follow. Take

$$V''(s) = \frac{pk_1 e^{-k_1\frac{s^2}{2}} + (1-p)k_2 e^{-k_2\frac{s^2}{2}}}{pe^{-k_1\frac{s^2}{2}} + (1-p)e^{-k_2\frac{s^2}{2}}}$$

and

$$g''(s) = -\frac{p(1-p)(k_1-k_2)^2 s^2}{p^2 e^{-(k_1-k_2)\frac{s^2}{2}} + 2p(1-p) + (1-p)^2 e^{(k_1-k_2)\frac{s^2}{2}}}.$$

We have

$$k_2 \le V''(s) \le pk_1 + (1-p)k_2 \quad \text{and} \quad -\frac{p(k_1-k_2)}{1-p} \le g''(s) \le 0,$$

where the lower bound inequality for $g''(s)$ follows from the fact that $g''(s)$ attains its minimum for $s \ge \sqrt{\frac{2}{k_1-k_2}}$. Then

$$||g'(s)||_{L^2(\mathbb{R})} \le \frac{2p}{1-p}(k_1-k_2)^{1/4}(\pi)^{1/4} + o\left(\frac{2p}{1-p}(k_1-k_2)^{1/4}(\pi)^{1/4}\right).$$

By using condition (92), the RW representation holds.

(b) $U(s) = s^2 + a - \log(s^2 + a)$, where $0 < a < 1$. Let $0 < \beta < \frac{a}{4d\left(2+\frac{2}{25a}\right)^2}$. Then the RW representation holds.

Then, using the notation from Remark 1, take $Y(s) = s^2$ and $h(s) = -\log(s^2 + a)$. We have $Y''(s) = 2$, so $D_1 = D_2 = 2$; also $h''(s) = 2\frac{s^2-a}{(s^2+a)^2}$, with $-\frac{2}{a} \le h''(s) \le 0$ for $s \in [-\sqrt{a}, \sqrt{a}]$ and $0 < h''(s) \le \frac{2}{25a}$ otherwise. Then $C_0 = \frac{2}{a}$, $C_1 = 2$,$C_2 = 2 + \frac{2}{25a}$ and $||g''(s)||_{L^1(\mathbb{R})} = \frac{2}{\sqrt{a}}$. By using condition (64), the RW representation holds.

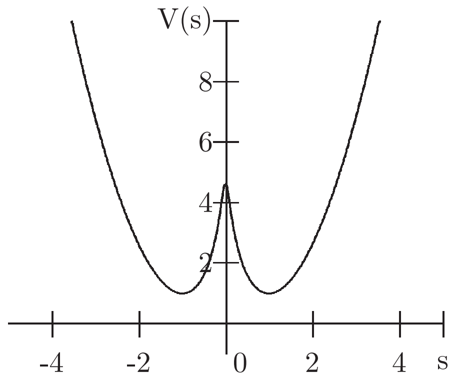

Figure 5: Example (b)

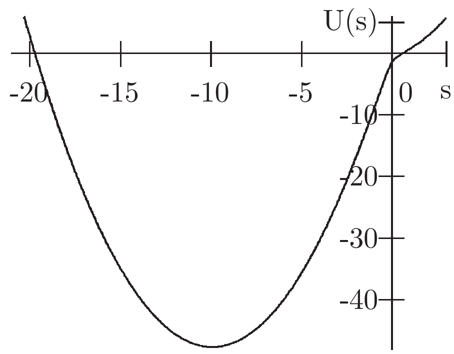

Figure 6: Example (c)

(c) Let $0 < a < 1$ and

$$U(s) = \begin{cases} \frac{s^2}{2} + \log(s+a) & \text{if} \quad s \geq 0 \\ \frac{s^2}{2} + \frac{2s}{a} + \log(a-s) & \text{if} \quad s < 0. \end{cases}$$

Then

$$U'(s) = \begin{cases} s + \frac{1}{s+a} & \text{if} \quad s \geq 0 \\ s + \frac{2}{a} + \frac{1}{s-a} & \text{if} \quad s < 0 \end{cases} \quad \text{and} \quad U''(s) = \begin{cases} 1 - \frac{1}{(s+a)^2} & \text{if} \quad s \geq 0 \\ 1 - \frac{1}{(s-a)^2} & \text{if} \quad s < 0. \end{cases}$$

U is non-convex on the interval $a - 1 < s < 1 - a$ and achieves its unique minimum on $-\infty < s < a - 1$. Let $0 < \sqrt{\beta} < \frac{a}{4\sqrt{d}}$. Then the RW representation holds.

Then take $V(s) = s^2/2$ and $g(s) = \log(s+a)$ if $s \geq 0$ and $g(s) = 2s/a + \log(a-s)$ if $s < 0$. We have $V''(s) = 1$, so $C_1 = C_2 = 1$; also $g''(s) = -1/(s+a)^2$ if $s \geq 0$ and $g''(s) = -1/(s-a)^2$ for $s < 0$; also $||g''(s)||_{L^1(\mathbb{R})} = \frac{2}{a}$. By using condition (64), the RW representation holds.

Note that by using (89), the condition on β becomes $0 < \sqrt{\beta} < \frac{3a^3}{2\sqrt{d}}$ and by using (92), it becomes $0 < \sqrt{\beta} \leq \frac{a^{1/3}}{2^{1/3}\sqrt{2d}}$.

4 Uniqueness of ergodic component

In this section, we extend to a class of non-convex potentials, the uniqueness of ergodic component result, proved for strictly convex potentials in [14].

We denote by S the class of all shift invariant $\mu \in P_2(\chi)$ which are stationary and by ext S those $\mu \in S$ which are ergodic with respect to shifts. For each $u \in \mathbb{R}^d$, we denote by $(\text{ext } S)_{\tilde{u}}$ the family of all $\mu \in \text{ext } S$ such that $\mathbf{E}_\mu(\eta(e_\alpha)) = u_\alpha, \alpha = 1, 2 \ldots d$, where we denoted by e_α the bond $(e_\alpha, 0)$. We will prove that

Theorem 17 *Let $U_i = V_i + g_i$, where U_i satisfy (15) and V_i and g_i satisfy (16), (17) and (64). Then for every $u \in \mathbb{R}^d$, there exists at most one ergodic $\mu \in \mathcal{G}(H)$ such that $\mathbf{E}_\mu(\eta_t(e_\alpha)) = u_\alpha, \alpha = 1, 2 \ldots d$.*

The proof will be done in 2 steps: first, we will prove the uniqueness of ergodic component on $\mathcal{E}^d$ and then we will use this result combined with the properties of the $\nabla\phi$-Gibbs measure to extend the result to μ.

4.1 Uniqueness of ergodic component for the even

Let $F \in C^2(\mathbb{R}^{2d}; \mathbb{R})$ be such that for all $(a_1, a_2, \ldots, a_d, a_{-1}a_{-d}) \in \mathbb{R}^d$ and for all $c \in \mathbb{R}$

$$F(a_1, \ldots, a_d, a_{-1}, \ldots a_{-d}) = F(a_1 + c, \ldots, a_d + c, a_{-1} + c, \ldots, a_{-d} + c). \tag{99}$$

Note that from property (99), by the same reasoning as in Lemma 11 we have that for all $j \in I$, (65) and (66) hold. Assume that there exist $c_- > 0$ and $c_+ > 0$ such that for all $(a_1, a_2, \ldots, a_d, a_{-1}a_{-d}) \in \mathbb{R}^d$

$$c_- \leq D^{i,j} F(a_1, a_2, \ldots, a_d, a_{-1}a_{-d}) \leq c_+. \tag{100}$$

Let

$$\mathcal{L} = \left\{ F \in C^2(\mathbb{R}^{2d}; \mathbb{R}) \mid F \text{ satisfies (99) and (100)} \right\}. \tag{101}$$

The proofs in this section follow very closely the arguments from [14]. To make the current paper self-contained, we will sketch proofs for all the theorems in the section. There are three main ingredients necessary in proving uniqueness of ergodic component for a Hamiltonian satisfying (99) and (100). These steps are: the study of the dynamics of the height variables, which dynamics are generated by SDE, a coupling argument and the use of the ergodicity.

4.1.1 Dynamics

Suppose the dynamics of the **even** height variables $\phi_t = \{\phi_t(a)\} \in \mathbb{R}^{\mathcal{E}^d}$ are generated by the SDE

$$\mathrm{d}\phi_t(a) \quad = \quad - \sum_{x \in \mathcal{O}^d_{\bar{\Lambda}}, |x-a|=1} \frac{\partial}{\partial \phi(a)} F_x(\phi_t(x+e_1), \ldots, \phi_t(x+e_{-d}))\,\mathrm{d}t + \sqrt{2} dW_t(a), \tag{102}$$

where for all $x \in \mathcal{O}^d_{\bar{\Lambda}}$, $F_x \in \mathcal{L}$ and $\{W_t(a), a \in \mathcal{E}^d\}$ is a family of independent Brownian motions. Note that in (102), for each $x \in \mathcal{O}^d_{\bar{\Lambda}}$ such that $|x - a| = 1$, there exists $i \in I$ such that $a = x + e_i$.

The dynamics for the **even** height differences $\eta^{\mathcal{E}} = \{\eta^{\mathcal{E}}_t(b)\} \in \mathcal{B}_{\mathcal{E}^d}$ are determined by the SDE

$$\begin{aligned} \mathrm{d}\eta^{\mathcal{E}}(b) \quad &= \quad \mathrm{d}\phi_t(x_b) - \mathrm{d}\phi_t(y_b) = - \sum_{x \in \mathcal{O}^d_{\bar{\Lambda}}, |x-x_b|=1} \frac{\partial}{\partial \phi(x_b)} F_x(\phi_t(x+e_1), \ldots, \phi_t(x+e_{-d}))\,\mathrm{d}t \\ &+ \sum_{x \in \mathcal{O}^d_{\bar{\Lambda}}, |x-y_b|=1} \frac{\partial}{\partial \phi(y_b)} F_x(\phi_t(x+e_1), \ldots, \phi_t(x+e_{-d}))\,\mathrm{d}t \\ &+ \sqrt{2} d[W_t(x_b) - W_t(y_b)], \end{aligned} \tag{103}$$

where $b = (x_b, y_b) \in \mathcal{B}^{\Lambda}_{\mathcal{E}^d}$.

Lemma 18 *(a) The solution of (103) satisfies* $\eta^{\mathcal{E}}_t \in \chi^{\mathcal{E}}$ *for all* $t > 0$.

(b) *If* ϕ_t *is a solution of (102), then* $\eta^{\mathcal{E}}_t := \nabla^{\mathcal{E}}\phi_t$ *is a solution of (103).*

(c) *If* $\eta^{\mathcal{E}}_t$ *is a solution of (103) and we define* $\phi_t(0)$ *through (102) for* $x = 0$ *and* $\nabla^{\mathcal{E}}\phi_t(b) = \eta^{\mathcal{E}}_t(b)$, *with* $\phi_0(0) \in \mathbb{R}$, *then* $\phi_t := \phi^{\eta^{\mathcal{E}}_t, \phi_t(0)}$ *is a solution of (102).*

(d) *For each* $\eta^{\mathcal{E}} \in \chi^{\mathcal{E}}_r, r > 0$ *the SDE (103) has a unique* $\chi^{\mathcal{E}}_r$*-valued continuous solution* $\eta^{\mathcal{E}}_t$ *starting at* $\eta^{\mathcal{E}}_0 = \eta^{\mathcal{E}}$.

Proof. The proofs for (a), (b) and (c) are immediate, so we will concentrate on the proof for (d). For every $\theta_t(x)$ and $\bar{\theta}_t(x)$, by expanding $D^j F_x(\theta_t(x))$ in Taylor series around $\bar{\theta}_t(x)$ to get

$$D^j F_x(\theta_t(x)) - D^j F_x(\bar{\theta}_t(x)) = \sum_{k\in I} \tilde{\phi}_t(x+e_k) \int_0^1 D^{j,k} F_x\left(\bar{\theta}_t(x) + y\left(\theta_t(x) - \bar{\theta}_t(x)\right)\right) \mathrm{d}y. \tag{104}$$

By using now the fact that $F_x \in \mathcal{F}$, we obtain global Lipschitz continuity in $\chi^{\mathcal{E}}_r$ of the drift term of the SDE in (103), from which existence and uniqueness of the solution in (103) follows. □

First, we will prove

Lemma 19 *Let* ϕ_t *and* $\bar{\phi}_t$ *be two solutions for (102) and set* $\tilde{\phi}_t(a) := \phi_t(a) - \bar{\phi}_t(a)$, *where* $a \in \mathcal{E}^d$. *Then for every* $\Lambda \in \mathbb{Z}^d$, *we have*

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{a\in\mathcal{E}^d_\Lambda}(\tilde{\phi}_t(a))^2 \;=\; & -2\sum_{x\in\mathcal{O}^d_\Lambda}\;\sum_{\substack{\{j\in I|\\ x+e_j\in\mathcal{E}^d_\Lambda\}}}\Big[D^j F_x(\phi_t(x+e_1),\ldots,\phi_t(x+e_{-d})) \\ & -D^j F_x(\bar{\phi}_t(x+e_1),\ldots,\bar{\phi}_t(x+e_{-d}))\Big]\tilde{\phi}_t(x+e_j),\end{aligned} \tag{105}$$

where

$$\frac{\partial}{\partial t}\sum_{a\in\mathcal{E}^d_\Lambda}(\tilde{\phi}_t(a))^2 \le -c_- \sum_{b\in\mathcal{B}^\Lambda_{\mathcal{E}^d}}\left[\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right]^2 + 4c_+\sum_{b\in\partial\mathcal{B}^\Lambda_{\mathcal{E}^d}}|\phi_t(y_b)|\left|\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right|. \tag{106}$$

Proof. From (102), we have

$$\begin{aligned}\frac{\partial}{\partial t}(\tilde{\phi}_t(a))^2 \;=\; & -2\sum_{x\in\mathcal{O}^d_\Lambda,|x-a|=1}\Big[\frac{\partial}{\partial\phi(a)}F_x(\phi_t(x+e_1),\ldots,\phi_t(x+e_{-d})) \\ & -\frac{\partial}{\partial\phi(a)}F_x(\bar{\phi}_t(x+e_1),\ldots,\bar{\phi}_t(x+e_{-d}))\Big]\tilde{\phi}_t(a).\end{aligned} \tag{107}$$

By summing now over all $a \in \mathcal{E}^d_\Lambda$ in (107), we get (105). For simplicity of notation, we will denote as before by $\theta_t(x) := (\phi_t(x+e_1),\ldots,\phi_t(x+e_{-d}))$ and by $\bar{\theta}_t(x) := \left(\bar{\phi}_t(x+e_1),\ldots,\bar{\phi}_t(x+e_{-d})\right)$. To find an upper bound for S_1, we expand now $D^j F_x(\theta_t(x))$ in Taylor series around $\bar{\theta}_t(x)$ to get

$$D^j F_x(\theta_t(x)) - D^j F_x(\bar{\theta}_t(x)) = \sum_{k\in I} \tilde{\phi}_t(x+e_k) \int_0^1 D^{j,k} F_x\left(\bar{\theta}_t(x) + y\left(\theta_t(x) - \bar{\theta}_t(x)\right)\right) \mathrm{d}y. \tag{108}$$

By using now (100), (108) and the equality $D^{k,j}F_x = D^{j,k}F_x$ in (105), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\sum_{a\in\mathcal{E}^d_\Lambda}(\tilde{\phi}_t(a))^2 \\
= &-2\sum_{x\in\mathcal{O}^d_\Lambda}\sum_{\substack{\{j\in I,\\ x+e_j\in\mathcal{E}^d_\Lambda\}}}\sum_{k\in I}\tilde{\phi}_t(x+e_k)\tilde{\phi}_t(x+e_j)\int_0^1 D^{j,k}F_x\left(\bar{\phi}_t(x)+y\left(\phi_t(x)-\bar{\phi}_t(x)\right)\right)\,\mathrm{d}y \\
&= 2\sum_{x\in\mathcal{O}^d_\Lambda}\sum_{\substack{\{j\in I,\\ x+e_j\in\mathcal{E}^d_\Lambda\}}}\sum_{k\in I,k\neq j}\left[\tilde{\phi}_t^{\,2}(x+e_j)-\tilde{\phi}_t(x+e_k)\tilde{\phi}_t(x+e_j)\right] \\
&\qquad\qquad\int_0^1 D^{j,k}F_x\left(y\phi_t(x)+(1-y)\bar{\phi}_t(x)\right)\,\mathrm{d}y \\
&= \sum_{x\in\mathcal{O}^d_\Lambda}\sum_{\substack{\{j,k\in I,j\neq k|\\ x+e_j,x+e_k\in\mathcal{E}^d_\Lambda\}}}\left[\tilde{\phi}_t(x+e_j)-\tilde{\phi}_t(x+e_k)\right]^2\int_0^1 D^{j,k}F_x\left(y\phi_t(x)+(1-y)\bar{\phi}_t(x)\right)\,\mathrm{d}y \\
&+2\sum_{x\in\mathcal{O}^d_\Lambda}\sum_{\substack{\{j\in I,\\ x+e_j\in\mathcal{E}^d_\Lambda\}}}\sum_{\substack{\{k\in I|\\ x+e_j\in\partial\mathcal{E}^d_\Lambda\}}}\left[\tilde{\phi}_t^{\,2}(x+e_j)-\tilde{\phi}_t(x+e_k)\tilde{\phi}_t(x+e_j)\right] \\
&\qquad\qquad\int_0^1 D^{j,k}F_x\left(y\phi_t(x)+(1-y)\bar{\phi}_t(x)\right)\,\mathrm{d}y \\
&\leq -c_-\sum_{b\in\mathcal{B}^\Lambda_{\mathcal{E}^d}}\left[\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right]^2+4c_+\sum_{b\in\partial\mathcal{B}^\Lambda_{\mathcal{E}^d}}|\phi_t(y_b)|\left|\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right|.
\end{aligned}
\tag{109}
$$

□

4.1.2 Coupling Argument

We will call

$$\mathcal{N}=\{f^0_{ij}\mid f^0_{ij}=e_i+e_j,\ \text{where } i,j\in I,j\neq -i,i\leq j\} \tag{110}$$

the set of neighbours of 0 in $\mathcal{E}^d$. Let

$$\mathcal{N}_+=\{f^0_{ij}\mid f^0_{ij}=e_i+e_j,\ \text{where } i,j\in I,j\neq -i,i\leq j,j\geq 1\} \tag{111}$$

Let us define now a generator set in $\mathcal{E}^d$:

$$e^{\mathcal{E}}_1=e_1+e_2,\ldots,e^{\mathcal{E}}_{d-1}=e_{d-1}+e_d,e^{\mathcal{E}}_d=e_d-e_1\ \text{ if }\ d\text{ even} \tag{112}$$

and

$$e^{\mathcal{E}}_1=e_1+e_2,\ldots,e^{\mathcal{E}}_{d-1}=e_{d-1}+e_d,e^{\mathcal{E}}_d=e_d+e_1\ \text{ if }\ d\text{ odd.} \tag{113}$$

For each $u\in\mathbb{R}^d$, let $\tilde{u}_\alpha,\alpha=1,2,\ldots,d$ be defined as follows:

$$\tilde{u}_\alpha=u_\alpha+u_{\alpha+1},\ \alpha=1,2,\ldots,d-1\text{ and }\tilde{u}_d=u_d-u_1\text{ if }d\text{ even} \tag{114}$$

and

$$\tilde{u}_\alpha=u_\alpha+u_{\alpha+1},\ \alpha=1,2,\ldots,d-1\text{ and }\tilde{u}_d=u_d+u_1\text{ if }d\text{ odd.} \tag{115}$$

For $x \in \mathcal{B}_{\mathcal{E}^d}$, we define the **even** shift operators $\sigma_x^{\mathcal{E}} : \mathbb{R}^{\mathcal{E}^d} \to \mathbb{R}^{\mathcal{E}^d}$, for **even** heights by $\sigma_x^{\mathcal{E}}(y) = \phi(y-x)$ for $y \in \mathcal{E}^d$ and $\phi \in \mathbb{R}^{\mathcal{E}^d}$ and for **even** height differences by $(\sigma_x^{\mathcal{E}}\eta)(b) = \eta(b-x)$, for $b \in \mathcal{B}_{\mathcal{E}^d}$ and $\eta \in \chi^{\mathcal{E}}$. We denote by $S^{\mathcal{E}}$ the class of all shift invariant (with respect to the **even** shifts) $\mu \in P_2(\chi^{\mathcal{E}})$ which are stationary for the SDE (102) and by ext $S^{\mathcal{E}}$ those $\mu^{\mathcal{E}} \in S^{\mathcal{E}}$ which are ergodic with respect to the **even** shifts. For each $u \in \mathbb{R}^d$, we denote by $\left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{u}}$ the family of all $\mu^{\mathcal{E}} \in \text{ext } S^{\mathcal{E}}$ such that $\mathbf{E}_{\mu^{\mathcal{E}}}\left(\eta_t^{\mathcal{E}}(e_\alpha^{\mathcal{E}})\right) = \tilde{u}_\alpha, \alpha = 1, 2 \ldots d$. We will prove that μ is unique.

Next, for clarity purposes, we will sketch the coupling argument used in [14] to prove uniqueness of μ. Suppose that there exist $\mu^{\mathcal{E}} \in \left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{u}}$ and $\bar{\mu}^{\mathcal{E}} \in \left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{v}}$ for $u, v \in \mathbb{R}^d$. Let us construct two independent-$\chi_r^{\mathcal{E}}$ valued random variables $\eta^{\mathcal{E}} = \{\eta^{\mathcal{E}}(b)\}$ and $\bar{\eta}^{\mathcal{E}} = \{\bar{\eta}^{\mathcal{E}}(b)\}$ on a common probability space (Ω, F, P) in such a manner that $\eta^{\mathcal{E}}$ and $\bar{\eta}^{\mathcal{E}}$ are distributed by $\mu^{\mathcal{E}}$ and $\bar{\mu}^{\mathcal{E}}$ respectively. We define $\phi_0 = \phi^{\eta,0}$ and $\bar{\phi}_0 = \phi^{\bar{\eta},0}$. Let ϕ_t and $\bar{\phi}_t$ be two solutions of the SDE (102) with common Brownian motions having initial data ϕ_0 and $\bar{\phi}_0$. Since $\mu^{\mathcal{E}}, \bar{\mu}^{\mathcal{E}} \in S^{\mathcal{E}}$, we conclude that $\eta_t^{\mathcal{E}} = \left(\eta^{\mathcal{E}}\right)^{\phi_t}$ and $\bar{\eta}^{\mathcal{E}} = \left(\eta^{\mathcal{E}}\right)^{\bar{\phi}_t}$ are distributed by $\mu^{\mathcal{E}}$ and $\bar{\mu}^{\mathcal{E}}$ respectively, for all $t \geq 0$. Let $\tilde{u}$ and $\tilde{v}$ be such that $\tilde{u}_\alpha = \mathbf{E}_{\mu^{\mathcal{E}}}\left(\eta_t^{\mathcal{E}}(e_\alpha^{\mathcal{E}})\right)$ and $\tilde{v}_\alpha = \mathbf{E}_{\bar{\mu}^{\mathcal{E}}}\left(\bar{\eta}_t^{\mathcal{E}}(e_\alpha^{\mathcal{E}})\right)$. We claim that:

Lemma 20 *There exists a constant $C > 0$ independent of $\tilde{u}, \tilde{v} \in \mathbb{R}^d$ such that*

$$\overline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T \sum_{\alpha=1}^d \mathbf{E}^P \left[\left(\eta_t^{\mathcal{E}}(e_\alpha^{\mathcal{E}}) - \bar{\eta}_t^{\mathcal{E}}(e_\alpha^{\mathcal{E}})\right)^2\right] \mathrm{d}t \leq C|\tilde{u} - \tilde{v}|^2. \tag{116}$$

Proof. **Step 1.** For simplicity of notation, we will label for this proof the d^2 elements of $\mathcal{N}_+$ as $f_1^0, f_2^0, \ldots, f_{d^2}^0$. By applying Lemma 19 to the differences $\{\tilde{\phi}_t(x) := \phi_t(x) - \bar{\phi}_t(x)\}$, where $x \in \mathcal{E}^d$, one obtains just as in [14] for every $T > 0$, $\Lambda = \Lambda_N$, where $N \in \mathbb{N}$

$$\int_0^T g(t)\,\mathrm{d}t \leq \frac{2d^2}{c_-|\mathcal{B}_{\mathcal{E}^d}^\Lambda|}\mathbf{E}^P\left[\sum_{x\in\mathcal{E}_\Lambda^d}(\tilde{\phi}_0(x))^2\right] + \frac{(2c'_+c_0)^2d^2}{(c_-N)^2}\int_0^T \sup_{y\in\partial\mathcal{E}_\Lambda^d}\|\tilde{\phi}_t(y)\|^2|_{L^2(P)}\,\mathrm{d}t, \tag{117}$$

where $c'_+ = 4c_+ + 2C$,

$$g(t) = \sum_{\alpha=1}^{d^2}\mathbf{E}^P\left[(\nabla\tilde{\phi}_t(f_\alpha))^2\right] \text{ and } c_0 := \sup_{\{N\geq 1\}}\left\{N\frac{|\partial\mathcal{B}_{\mathcal{E}^d}^\Lambda|}{|\mathcal{B}_{\mathcal{E}^d}^\Lambda|}\right\} < \infty. \tag{118}$$

Step 2. Next we derive, just as in [14], the following bound on the boundary term: For each $\epsilon > 0$ there exists an $l_0 \in \mathbb{N}$ such that

$$\sup_{y\in\partial\mathcal{E}_\Lambda^d}\|\tilde{\phi}_t(y)\|_{L^2(P)}^2 \leq C_1\left(\epsilon^2N^2 + N^2|\tilde{u}-\tilde{v}|^2 + N^{-2}t\int_0^t g(s)\,\mathrm{d}s\right) \tag{119}$$

for every $t > 0$ and $l \geq l_0$, where $C_1 > 0$ is a constant independent of ϵ, l, and t.

To this end, because $\mathcal{E}^d$ is a subalgebra, we can use the mean ergodic theorem for cocycles (see for example [5], [19] or [18]) and apply it to $\mu^{\mathcal{E}} \in (\text{ext } \mathcal{S}^{\mathcal{E}})_u$ to obtain

$$\lim_{\substack{\|x\|\to\infty,\\ x\in\mathcal{E}^d}} \frac{1}{\|x\|}\|\phi^{\nu,0}(x) - x\cdot\tilde{u}\|_{L^2(\mu^{\mathcal{E}})} = 0 \tag{120}$$

In order to apply the proof from Step 2 in [14], we will need the result proved below in (123). By using (105) and the reasoning used to derive (70), we have

$$\begin{aligned}\frac{\partial}{\partial t}\left\{\sum_{a\in\Lambda'}\tilde{\phi}_t(a)\right\} &= -2\sum_{x\in\Lambda'\cap\mathcal{O}^d}\sum_{\substack{\{j\in I|\\ x+e_j\in\mathcal{E}_\Lambda^d\}}}\left[D^jF_x\left(\theta_t(x)\right) - D^jF_x\left(\bar{\theta}_t(x)\right)\right]\\ &= -2\sum_{\substack{\{x\in\Lambda'\cap\mathcal{O}^d, j\in I|x+e_j\in\mathcal{E}_\Lambda^d\\ \exists i\in I, i\neq j \text{ such that } x+e_i\notin\Lambda'\}}}\left[D^jF_x\left(\theta_t(x)\right) - D^jF_x\left(\bar{\theta}_t(x)\right)\right],\end{aligned} \tag{121}$$

where $\Lambda' = \Lambda_{[N/2]} \cap \mathcal{E}^d_\Lambda$, $\theta_t(x+e_i) = (\phi_t(x+e_i), \ldots, \phi_t(x+e_i)) \in \mathbb{R}^d$ and $\bar{\theta}_t(x+e_i) = (\bar{\phi}_t(x+e_i), \ldots, \bar{\phi}_t(x+e_i)$. By using Taylor's expansion and (100) in (121), we get

$$|D^j F_x(\theta_t(x)) - D^j F_x(\bar{\theta}_t(x))| \le c_+ \sum_{k\in I, k\neq j} \left|\tilde{\phi}(x+e_k) - \tilde{\phi}(x+e_j)\right|. \tag{122}$$

Then

$$\sum_{\substack{\{x\in\Lambda'\cap\mathcal{O}^d, j\in I | x+e_j\in\mathcal{E}^d_\Lambda \\ \exists i\in I, i\neq j \text{ such that } x+e_i\notin\Lambda'\}}} ||D^j F_x(\theta_t(x) - \theta_t(x+e_i)) - D^j F_x(\bar{\theta}_t(x) - \bar{\theta}_t(x+e_i))||_{L^2(P)}$$

$$\le \frac{c_+|\partial_B\Lambda'|}{d^2} \sum_{\alpha=1}^{d^2} ||\nabla\tilde{\phi}_t(f^0_\alpha)||_{L^2(P)}, \tag{123}$$

where we define

$$\partial_B\Lambda' := \{b = (x_b, y_b) \mid x_b \in \mathcal{E}^d \setminus \Lambda, y_b \in \mathcal{E}^d$$
$$\|x_b - y_b\| = 2, \exists z \in \mathcal{O}^d_\Lambda \text{ with } \|x_b - z\| = \|y_b - z\| = 1\}. \tag{124}$$

With these estimates and knowing that $\frac{|\partial_B\Lambda'|}{|\Lambda'|} \le \frac{C}{N}$, the proof from Step 2 in [14] follows now immediately.

Step 3 The proof for Step 3 follows the same way as in [14]. The desired estimate follows now from the fact that

$$\int_0^T \sum_{\alpha=1}^d \mathbf{E}^P\left[\left(\eta^{\mathcal{E}}_t(e^{\mathcal{E}}_\alpha) - \bar{\eta}^{\mathcal{E}}_t(e^{\mathcal{E}}_\alpha)\right)^2\right] \mathrm{d}t \le \int_0^T g(t)\,\mathrm{d}t. \tag{125}$$

□

Theorem 21 *For every $u \in \mathbb{R}^d$, there exists at most one $\mu^{\mathcal{E}} \in \left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{u}}$.*

Proof. By using Lemma 20, the proof follows the same arguments as in [14], so it will be omitted. □

4.2 Proof of Theorem 17

Proof.

Note first that any $\mu \in \mathcal{G}(H)$ is reversible under the dynamics η_t defined by the (103). In particular, $\mathcal{G} \subset \mathcal{S}$.

Suppose now that there exist $\mu, \bar{\mu} \in \mathcal{G}(H)$ ergodic such that $\mathbf{E}_\mu(\eta_t(e_\alpha)) = u_\alpha, \alpha = 1, 2 \ldots d$ for $u \in \mathbb{R}^d$. Note first that $\mathbf{E}_\mu\left(\eta^{\mathcal{E}}_t(e^{\mathcal{E}}_\alpha)\right) = \mathbf{E}_{\bar{\mu}}\left(\eta^{\mathcal{E}}_t(e^{\mathcal{E}}_\alpha)\right) = \tilde{u}_\alpha, \alpha = 1, 2 \ldots d$. Hence, from Lemma 4, we get that $\mu|_{\mathcal{E}^d}, \bar{\mu}|_{\mathcal{E}^d} \in \mathcal{G}^{\mathcal{E}}(H^{(2)})$ such that for all finite $\Lambda \in \mathbb{Z}^d$, we have $H^{(2)}_{\mathcal{E}^d_\Lambda} \in \mathcal{L}$.

Since for all $\eta^{\mathcal{E}} \in \chi^{\mathcal{E}}$, with $\eta^{\mathcal{E}}(b) = \phi(y_b) - \phi(x_b)$, $b \in \mathcal{B}_{\mathcal{E}^d}$, we can write $\eta^{\mathcal{E}}(b) = \eta(b_1) + \eta(b_2)$, $b_1, b_2 \in \mathcal{B}_{\mathbb{Z}^d}$, ergodicity and invariance under the even shifts for $\mu|_{\mathcal{E}^d}, \bar{\mu}|_{\mathcal{E}^d}$ follow immediately from the similar properties for $\mu, \bar{\mu}$. We also have reversibility for the even (see for example [15]).

Therefore $\mu|_{\mathcal{E}^d}, \bar{\mu}|_{\mathcal{E}^d} \in \left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{u}}$, so we can apply Theorem 21 to get $\mu|_{\mathcal{E}^d} = \bar{\mu}|_{\mathcal{E}^d}$. Then for any $A \in \mathcal{F}_{\mathcal{B}_{\mathbb{Z}^d}}$, we have $\mathbf{E}_\mu(1_A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}), \mathbf{E}_{\bar{\mu}}(1_A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}) \in \mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}$. From Lemma 7, we have

$$\mathbf{E}_\mu(1_A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}) = \mathbf{E}_{\bar{\mu}}(1_A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}) \tag{126}$$

and thus

$$\begin{aligned}\mu(A) &= \mathbf{E}_\mu(1_A) = \mathbf{E}_\mu(\mathbf{E}_\mu(1_A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})) = \mathbf{E}_{\bar\mu}(\mathbf{E}_\mu(1_A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})) \\ &= \mathbf{E}_{\bar\mu}(\mathbf{E}_{\bar\mu}(1_A|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})) = \mathbf{E}_{\bar\mu}(A) = \bar\mu(A). \end{aligned} \tag{127}$$

□

Remark 22 Note that as an immediate consequence of the results in Sections 3.2, 3.3 and of Remarks 15 and 16, we can also adapt the reasoning above to get the uniqueness of the ergodic component in the case where, for all $i \in I$, U_i are strictly convex and non-symmetric.

5 Covariance

We will extend in this section the covariance estimates of [9] to the class of non-convex potentials $U_i = V_i + g_i$ which satisfy (15) such V_i and g_i satisfy (16), (17) and (64).

Let $\mu_u \in P_2(\chi)$ be the unique shift invariant measure, ergodic with respect to shifts such that $\mathbf{E}_{\mu_u}(\eta(e_\alpha)) = u_\alpha, \alpha = 1, 2 \dots d$.

Let

$$\mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d} := \left\{ b = (x_b, y_b), b \in \mathcal{B}_{\mathbb{Z}^d} \text{ such that } x_b \in \mathcal{O}^d \right\}. \tag{128}$$

Let $F \in C^1_b(\mathbb{R}^{\mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}})$. We will denote for every $i \in I$ and for every $b = (x, x+e_i) \in \mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}$ by

$$\partial_b F = \partial_{(x,x+e_i)} F = \frac{\partial}{\partial \nabla_{e_i}\phi(x)} F(\nabla\phi) \text{ and } ||\partial_{x,x+e_i} F||_\infty = \sup_{\nabla\phi} |\partial_{x,x+e_i} F(\nabla\phi)|. \tag{129}$$

Theorem 23 *Let $F, G \in C^1_b(\mathbb{R}^{\mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}})$. Then there exists $C > 0$ such that*

$$|\mathrm{cov}_{\mu_u}(F(\nabla\phi), G(\nabla\phi))| \le C \sum_{b,b' \in \mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}} \frac{||\partial_b F||_\infty ||\partial_{b'} G||_\infty}{1 + ||b_1 - b'_1||^d}. \tag{130}$$

Proof. We have

$$\begin{aligned}\mathrm{cov}_{\mu_u}(F(\nabla\phi), G(\nabla\phi)) &= \mathbf{E}_{\mu_u}\left[\mathrm{cov}_{\mu_u}(F(\nabla\phi), G(\nabla\phi)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})\right] \\ &+ \mathrm{cov}_{\mu_u}\left(\mathbf{E}_{\mu_u}[F(\nabla\phi)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}], \mathbf{E}_{\mu_u}[G(\nabla\phi)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}]\right), \end{aligned} \tag{131}$$

where

$$\mathbf{E}_{\mu_u}(F(\nabla\phi)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}) = \int F(\nabla\phi) \prod_{y \in \mathcal{O}^d} (\tilde{\mu_u})_y(\phi(y+e_k) - \phi(y)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})\, \mathrm{d}\phi(y), \tag{132}$$

since the conditional measure is a product measure; a similar formula holds for G. Note that under $\mu(\,\cdot\,|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})$, the gradients $(\nabla\phi_i(x), x \in \mathcal{O}^d, i \in I)$ are independent. Thus

$$\begin{aligned}\left|\mathrm{cov}_{\mu_u}(F(\nabla\phi), G(\nabla\phi)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})\right| &\le \sum_{b \in \mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}} ||\partial_b F||_\infty ||\partial_b G||_\infty \mathrm{var}_{\,\mu_u}(\nabla\phi(b)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}) \\ &\le \sigma^2 \sum_{b \in \mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}} ||\partial_b F||_\infty ||\partial_b G||_\infty, \end{aligned} \tag{133}$$

where for the first inequality we used a result in [10] and for the last inequality we used (85). Next, in view of [9]

$$\left|\mathrm{cov}_{\mu_u}(\hat{F},\hat{G})\right| \leq c \sum_{b,b'\in\mathcal{B}_{\mathcal{E}^d}} \frac{||\partial_b \hat{F}||_\infty ||\partial_{b'}\hat{G}||_\infty}{1+\|b_1-b_1'\|^d}, \tag{134}$$

where

$$\hat{F} = \mathbf{E}_{\mu_u}[F(\nabla\phi)|\tilde{\mathcal{F}}_{\mathcal{B}_{\mathcal{E}^d}}] \text{ and } \hat{G} = \mathbf{E}_{\mu_u}[G(\nabla\phi)|\tilde{\mathcal{F}}_{\mathcal{B}_{\mathcal{E}^d}}]. \tag{135}$$

We need to estimate now $\partial_b \hat{F}$ and $\partial_b \hat{G}$. Suppose that $b=(x+e_k, x+e_j)$ for some $x\in\mathcal{O}^d$ and $j,k\in I$. But

$$\begin{aligned}
\partial_b \hat{F} &= \partial_b \mathbf{E}_{\mu_u}[F(\nabla\phi)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}] = \partial_b\left[\int F(\nabla\phi)\prod_{y\in\mathcal{O}^d}(\tilde{\mu_u})_y(\phi(y+e_k)-\phi(y)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}})\,\mathrm{d}\phi(y)\right] \\
&= \partial_b\left[\int F(\nabla\phi)\prod_{y\in\mathcal{O}^d}\frac{1}{\tilde{Z}(\nabla\theta(y))}\exp\left(-\sum_{i\in I}U_i(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))\right)\mathrm{d}\phi(y)\right] \\
&= \int \partial_b F(\nabla\phi)\prod_{y\in\mathcal{O}^d}\frac{1}{\tilde{Z}(\nabla\theta(y))}\exp\left(-\sum_{i\in I}U_i(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))\right)\mathrm{d}\phi(y) \\
&\quad -\mathrm{cov}_{\mu_u}\left(F(\nabla\phi),\partial_b\left(\sum_{y\in\mathcal{O}^d}\sum_{i\in I}U_i(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))\right)\Bigg|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}\right).
\end{aligned} \tag{136}$$

To estimate the above equation, we will use now the following simple change of variables

$$\phi(x+e_k)+\phi(x+e_j) = a \text{ and } \phi(x+e_k)-\phi(x+e_j)=b, \tag{137}$$

therefore $\phi(x+e_k)=\frac{a+b}{2}$ and $\phi(x+e_j)=\frac{a-b}{2}$. Then

$$\partial_b F(\nabla\phi) = \sum_{s\in\{k,l\}}\sum_{\{z\in\mathcal{E}^d:||z-(x+e_s)||=2\}}\frac{\partial F(\nabla\phi)}{\partial^{(z,x+s)}\phi}\frac{\partial(\phi(z)-\phi(x+e_s))}{\partial b}, \tag{138}$$

where we denoted by $\frac{\partial F(\nabla\phi)}{\partial^{(z,x+e_s)}}$ the partial derivative $D^l F$ such that l is the indice which gives the position in F of $\phi(z)-\phi(x+e_s)$. Since a similar formula holds for the term differentiated in the covariance, from (136) we obtain

$$\begin{aligned}
|\partial_b\hat{F}| \leq\; & \frac{1}{2}\sum_{s\in\{k,l\}}\sum_{\{z\in\mathcal{E}^d:||z-(x+e_s)||=2\}}\left|\frac{\partial F(\nabla\phi)}{\partial^{(z,x+e_s)}\phi}\right| \\
&+\left|\mathrm{cov}_{\mu_u}\left(F(\nabla\phi),\sum_{i\in I}\sum_{\substack{\{y\in\mathcal{O}^d:\\ y+e_i\in b \text{ or } y+e_k\in b\}}}U_i'(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))\Big|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}\right)\right|
\end{aligned} \tag{139}$$

By using (133) applied to $F(\nabla\phi)$ and to $\sum_{i\in I}\sum_{\substack{\{y\in\mathcal{O}^d:\\ y+e_i\in b \text{ or } y+e_k\in b\}}}U_i'(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))$, by using the fact that $|U_i''|\leq C_0+C_2$ and a similar formula to to the one used to derive (138), we get

$$\left|\mathrm{cov}_{\mu_u}\left(F(\nabla\phi),\sum_{i\in I}\sum_{\substack{\{y\in\mathcal{O}^d:\\ y+e_i\in b \ or \ y+e_k\in b\}}} U_i'(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))\Big|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}\right)\right|$$
$$\leq 8d^2(C_0+C_2)\sum_{\substack{b\in\mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}\\ y+e_k\in b \ or \ y+e_i\in b}}||\partial_b F||_\infty \mathrm{var}_{\mu_u}(\nabla\phi(b)|\mathcal{F}_{\mathcal{B}_{\mathcal{E}^d}}) \leq 8d^2\sigma^2(C_0+C_2)\sum_{\substack{b\in\mathcal{B}^{\mathcal{O}}_{\mathbb{Z}^d}\\ y+e_k\in b \ or \ y+e_i\in b}}||\partial_b F||_\infty. \tag{140}$$

We also used in (140) the fact that $\partial_b\left(\sum_{i\in I}\sum_{\substack{\{y\in\mathcal{O}^d:\\ y+e_i\in b \ or \ y+e_k\in b\}}} U_i'(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))\right)=0$ if neither $y+e_k$, nor $y+e_i$ are vertices of the bond b.

Note now that

$$\delta_x(\tilde{F})\leq\sum_{i\in I}||\partial_{(x,x+e_i)}F||_\infty \text{ and } \delta_x(\tilde{G})\leq\sum_{i\in I}||\partial_{(x,x+e_i)}G||_\infty, \text{ with } \delta_x(\tilde{F})=\sup_\phi\left|\frac{\partial}{\partial\phi(x)}\tilde{F}(\phi)\right|. \tag{141}$$

The statement of the theorem follows now from (139), (140), (141), (133) and (134). □

6 Scaling Limit

We will extend next the scaling limit results results from [16] to a class of non-convex potentials.

Theorem 24 *Let $u\in\mathbb{R}^d$. Let $\mu_u\in\mathcal{G}(H)$ ergodic with*

$$\mathbf{E}_{\mu_u}[\eta(e_i)]=u_i$$

Assume $U_i=V_i+g_i$, where U_i satisfy (15) and V_i and g_i satisfy (16), (17) and (64). Set

$$S_\epsilon(f)=\epsilon^{d/2}\sum_{x\in\mathbb{Z}^d}f(x\epsilon)(\phi(x+e_i)-\phi(x)-u_i), \tag{142}$$

where $f\in C_0^\infty(\mathbb{R}^d)$. Then

$$S_\epsilon(f)\to N(0,\sigma_u^2(f)) \quad as \quad \epsilon\to 0, \tag{143}$$

where $\sigma_u^2(f)>0$.

Proof. Let us write E_d for the even sites and O_d for the odd sites of $\mathbb{Z}^d$. Then

$$\begin{aligned} S_\epsilon(f) &= \epsilon^{d/2}\sum_{x\in\mathbb{Z}^d}f(x\epsilon)(\phi(x+e_i)-\phi(x)-u_i)=\epsilon^{d/2}\sum_{x\in\mathcal{E}^d}f(x\epsilon)(\phi(x+2e_i)-\phi(x)-2u_i)\\ &\quad -\epsilon^{d/2}\sum_{x\in\mathcal{E}^d}f(x\epsilon)(\phi(x+2e_i)-\phi(x+e_i)-u_i)+\epsilon^{d/2}\sum_{x\in\mathcal{O}^d}f(x\epsilon)(\phi(x+e_i)-\phi(x)-u_i)\\ &= \epsilon^{d/2}\sum_{x\in\mathcal{E}^d}f(x\epsilon)(\phi(x+2e_i)-\phi(x)-2u_i)\\ &\quad +\epsilon^{d/2}\sum_{x\in\mathcal{E}^d}\Big(f((x+e_i)\epsilon)-f(x\epsilon)\Big)(\phi(x+2e_i)-\phi(x+e_i)-u_i)=S^e_\epsilon(f)+R_\epsilon(f). \end{aligned} \tag{144}$$

Now we can show CLT for $S^e_\epsilon(f)$ since the summation is concentrated on the even sites; the proof uses the same arguments as in [16] and is based on the RW representation. Also, since by Theorem 23

$$\left|\mathrm{cov}_{\mu_u}(\nabla_i\phi(x),\nabla_j\phi(y))\right| \le \frac{C}{(\|x-y\|+1)^d}, \tag{145}$$

we have

$$\begin{aligned}\mathrm{var}_{\mu_u}(R_\epsilon(f)) &\le \epsilon^d \sum_{x,y\in\mathcal{E}^d} |\nabla_i f(x\epsilon)||\nabla_i f(y\epsilon)||\mathrm{cov}_{\mu_u}(\phi(x+e_i)-\phi(x),\phi(y+e_i)-\phi(y))| \\ &\le \epsilon^d \sum_{x,y\in\mathcal{E}^d} |\nabla_i f(x\epsilon)||\nabla_i f(y\epsilon)|\frac{C}{(\|x-y\|+1)^d}, \end{aligned}\tag{146}$$

where $\nabla_i f(x\epsilon) = f((x+e_i)\epsilon) - f(x\epsilon)$. Expanding $f((x+e_i)\epsilon)$ in Taylor expansion around $x\epsilon$, we have

$$\nabla_i f(x\epsilon) = D^i f(a)\epsilon, \tag{147}$$

for some $a\in\mathbb{R}^d$. As $f\in C_0^\infty(\mathbb{R}^d)$, there exist $M,N>0$ such that for all $x\in\mathbb{R}^d$ with $|\epsilon x|\le N$ we have $f(\epsilon x)\le M$, $|D^i f(\epsilon x)|\le M$ and both functions equal to 0 for $|\epsilon x|>N$. Therefore

$$\mathrm{var}_{\mu_u}(R_\epsilon(f)) \le \epsilon^{d+2}M^2C \sum_{\substack{x,y\in\mathcal{E}^d,\\ |\epsilon x|\le N, |\epsilon x|\le N}} \frac{1}{(\|x-y\|+1)^d}. \tag{148}$$

It now remains to evaluate the sum $\sum_{\substack{x,y\in\mathcal{E}^d,\\ |\epsilon x|\le N,|\epsilon x|\le N}} \frac{1}{(|x-y|+1)^d}$. But

$$\begin{aligned}\sum_{\substack{x,y\in\mathcal{E}^d,\\ |\epsilon x|\le N,|\epsilon x|\le N}} \frac{1}{(\|x-y\|+1)^d} &\le \sum_{\substack{y\in\mathcal{E}^d,\\ |\epsilon y|\le N}} \int_{-\frac{N+1}{\epsilon}}^{\frac{N+1}{\epsilon}} \cdots \int_{-\frac{N+1}{\epsilon}}^{\frac{N+1}{\epsilon}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_d}{\left(\sum_{i=1}^d |x_i-y_i|+1\right)^d} \\ &\le \epsilon^2 C(d,N)\log(1+2dN/\epsilon) \le 2dNC(d,N)\epsilon, \end{aligned}\tag{149}$$

where $C(d,N)$ is a positive constant depending on d and N. It follows that $R_\epsilon(f)\to 0$ as $\epsilon\to 0$ in probability. □

7 Surface tension

7.1 Surface tension on the torus

We will extend here to the family of non-convex potentials satisfying (15), (16), (17) and (64), the surface tension strict convexity result from [14] and [12]. Additionally, in Theorem 28 we prove a series of surface tension equalities, which are important for the derivation of the hydrodynamic limit.

To study the convexity properties of the surface tension (as a function of the tilt u) for non-convex gradient models on a lattice, we will work on the torus, instead of the finite box on the lattice $\mathbb{Z}^d$. Thus, let $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d = \mathbb{Z}^d \bmod (N)$ be the lattice torus in $\mathbb{Z}^d$ and let $u\in\mathbb{R}^d$. Then, we define the surface tension on the torus $\mathbb{T}_N^d$ as

$$\sigma^\beta_{\mathbb{T}_N}(u) = -\frac{1}{|\mathbb{T}_N^d|^d}\log\frac{Z^\beta_{\mathbb{T}_N}(u)}{Z^\beta_{\mathbb{T}_N}(0)}, \tag{150}$$

where

$$Z^{\beta}_{\mathbb{T}_N}(u) = \int_{\mathbb{R}^{\mathbb{T}^d_N}} \exp(-\beta H_{\mathbb{T}_N}(\phi + \langle \cdot, u\rangle)) \prod_{x\in\mathbb{T}^d_N\setminus\{0\}} \mathrm{d}\phi(x) \tag{151}$$

and where $H_{\mathbb{T}_N}$ is the Hamiltonian on the torus $\mathbb{T}^d_N$ given by

$$H_{\mathbb{T}_N}(\phi) = \sum_{i\in I} \sum_{\substack{x\in\mathbb{T}^d_N \\ x+e_i\in\mathbb{T}^d_N}} U_i(\nabla_i\phi(x)) = \sum_{i\in I} \sum_{\substack{x\in\mathbb{T}^d_N \\ x+e_i\in\mathbb{T}^d_N}} \left[V_i(\nabla_i\phi(x)) + g_i(\nabla_i\phi(x))\right]. \tag{152}$$

Note that we define $u_{-i} = -u_i$ for $i = 1, 2, \ldots, d$. Just as before, let us label the vertices of the torus as odd and even; let the set of odd vertices be $\mathbb{O}^d_N$ and the set of even vertices be $\mathbb{E}^d_N$. Then we can of course first integrate all the odd coordinate first and then:

$$\begin{aligned} Z^{\beta}_{\mathbb{T}_N}(u) &= \int_{\mathbb{R}^{\mathcal{E}^d_N}} \left(\int_{\mathbb{R}^{\mathbb{O}^d_N}} \exp(-\beta H_{\mathbb{T}_N}(\phi + \langle \cdot, u\rangle) \prod_{x\in\mathbb{O}^d_N} \mathrm{d}\phi(x) \right) \prod_{x\in\mathbb{E}^d_N\setminus\{0\}} \mathrm{d}\phi(x) \\ &= \int_{\mathbb{R}^{\mathbb{E}^d_N}} \exp(-\beta H_{\mathbb{E}^d_N}(\phi, u)) \prod_{x\in\mathbb{E}^d_N\setminus\{0\}} d\phi(x), \end{aligned} \tag{153}$$

where $H_{\mathbb{E}_N}(\phi, u)$ is the induced Hamiltonian on the even. It is easy to see that

$$H_{\mathbb{E}_N}(\phi, u) = H_{\mathbb{E}_N}(\phi + \langle \cdot, u\rangle, 0). \tag{154}$$

Then, defining the **even** surface tension on $\mathbb{E}^d_N$ as

$$\sigma^{\beta}_{\mathbb{E}_N}(u) = -\frac{1}{|\mathbb{E}^d_N|} \log \frac{Z^{\beta}_{\mathbb{E}_N}(u)}{Z^{\beta}_{\mathbb{E}_N}(0)}, \tag{155}$$

where

$$Z^{\beta}_{\mathbb{E}_N}(u) = \int_{\mathbb{R}^{\mathbb{E}^d_N}} \exp(-\beta H_{\mathbb{E}_N}(\phi + \langle \cdot, u\rangle, 0)) \prod_{x\in\mathbb{E}^d_N\setminus\{0\}} \mathrm{d}\phi(x), \tag{156}$$

we obtain the following result by integrating out the odds

Lemma 25

$$\sigma^{\beta}_{\mathbb{E}_N}(u) = \frac{1}{2}\sigma^{\beta}_{\mathbb{T}_N}(u). \tag{157}$$

We will next prove strict convexity for the **even** surface tension, uniformly in N.

Lemma 26 *Suppose that $V_i, g_i \in C^2(\mathbb{R})$ such that they satisfy (16), (17) and (64). Then, for all $N = 2k$, we have*

$$D^2\sigma^{\beta}_{\mathbb{E}_N}(u) \geq C|\mathbb{E}^d_N| Id, \ \forall\ u \in \mathbb{R}^d. \tag{158}$$

That is, the ***even*** *surface tension is uniformly convex, uniformly in N.*

Proof. First note that if $N = 2k$, we can write $H_{\mathbb{E}_N}(\phi, u)$ as

$$H_{\mathbb{E}_N}(\phi, u) = \sum_{x\in\mathbb{O}^d_N} F_x(\theta, u), \tag{159}$$

where for all $x \in \mathbb{O}^d_N$

$$F_x(\theta(x), u) = -\log \int_{\mathbb{R}} e^{-\beta\sum_{i\in I} U_i(\nabla_i\phi(x)+u_i)}\, \mathrm{d}\phi(x) \tag{160}$$

and where, just as in (46)

$$\theta(x) = (\phi(x+e_1), \dots, \phi(x-e_d)). \tag{161}$$

Note that for all $i \in I$, we have $u_{-i} = -u_i$. Then

$$\sigma^{\beta}_{\mathbb{E}_N}(u) = -\frac{1}{|\mathbb{E}^d_N|} \log \frac{\int_{\mathbb{E}^d_N} e^{-\sum_{x\in\mathbb{O}^d_N} F_x(\theta(x),u)} \prod_{i\in I}\prod_{x+e_i\in\mathbb{E}^d_N} \mathrm{d}\phi(x+e_i)}{\int_{\mathcal{E}^d_N} e^{-\sum_{x\in\mathbb{O}^d_N} F_x(\theta(x),0)} \prod_{i\in I}\prod_{x+e_i\in\mathbb{E}^d_N} \mathrm{d}\phi(x+e_i)}. \tag{162}$$

As the denominator of $\sigma_{\mathbb{E}_N}(u)$ doesnt depend on u, it is enough to focuss on the term

$$R_{\mathbb{E}_N}(u) := \log \int_{\mathbb{E}^d_N} e^{-\sum_{x\in\mathbb{O}^d_N} F_x(\theta(x),u)} \prod_{i\in I}\prod_{x+e_i\in\mathbb{E}^d_N} \mathrm{d}\phi(x+e_i). \tag{163}$$

Note now that by Theorem 10, we have that for each $x \in \mathcal{O}^d_\Lambda$, F_x is convex, that is

$$\left(D^2F_x(\theta)(\bar\theta)\right)(\bar\theta) \geq c_1 \sum_{\substack{i,j\in I,\\ i\neq j}} \left|\bar\theta(x+e_i) - \bar\theta(x+e_j)\right|^2. \tag{164}$$

Because by Theorem 10 the F_x fullfil the random walk representation, we can apply to R_N Lemma 3.2 in [8], (164) and the fact that for all $i \in I$, we have $u_{-i} = -u_i$, to get the statement of the lemma. □

We consider the finite volume Gibbs measures $\tilde\mu_{N,u} \in P(\chi_{\mathbb{T}^d_N})$ with periodic boundary conditions which, for each $u \in \mathbb{R}^d$, are defined by

$$\tilde\mu_{N,u}(\,\mathrm{d}\tilde\eta) = Z^{-1}_{N,u} \exp\left[-\frac{1}{2}\sum_{b\in\mathcal{B}_{\mathbb{Z}^d}{}^N} V(\tilde\eta(b)+u_b)\right] \mathrm{d}\tilde\eta_N \in P(\chi_{\mathcal{B}_{\mathbb{Z}^d}{}^N}). \tag{165}$$

Here $\mathrm{d}\tilde\eta_N$ is the uniform measure on the affine space $\chi_{\mathbb{T}^d_N}$ and $Z_{N,u}$ is the normalizing constant. The law of $\{\eta(b) := \tilde\eta(b) + u_b\}$ under $\tilde\mu_{N,u}$ is denoted by $\mu_{N,u}$.

Lemma 27 *$\mu_{N,u}$ converges weakly to $\mu_u \in \operatorname{ext}\mathcal{G}$.*

Proof. Tightness of the family $\{\mu_{N,u}\}_N$ is known for non-convex potentials with quadratic growth at ∞ (see Remark 4.4 page 152 in [15]). Therefore a limiting measure exists by taking $N \to \infty$ along a suitable subsequence. From now on and using Theorem 17, the proof follows the same reasoning as the proof of Theorem 3.2 in [14]. In particular, because of the uniqueness of ergodic gradient Gibbs measures for each u, $\mu_{N,u}$ converges weakly to μ_u. □

Let

$$\nabla\sigma_{\mathbb{T}^d_N} = \left(D^1\sigma_{\mathbb{T}^d_N}, \dots, D^d\sigma_{\mathbb{T}^d_N}\right), \quad \text{where} \quad D^i\sigma_{\mathbb{T}^d_N} = \frac{\partial\sigma_{\mathbb{T}^d_N}}{\partial u_i}, i = 1, \dots d. \tag{166}$$

Theorem 28 *Suppose that $V_i, g_i \in C^2(\mathbb{R})$ such that they satisfy (16), (17) and (64) and such that for all $i \in I$, U_i are symmetric. Then we have*

(a)

$$\lim_{N\to\infty} \sigma^{\beta}_{\mathbb{T}_N}(u) = \sigma_T(u), \quad \sigma_T \in C^1(\mathbb{R}^d); \tag{167}$$

(b) σ_T *is strictly convex as a function of* u*;*

(c) $\mathbf{E}_{\mu_u}[\eta(b)] = u_b$*;*

(d) $\mathbf{E}_{\mu_u}[U_i'(\eta(e_i)] = D^i\sigma_T(u)$*, for all* $i = 1, \dots d$*;*

(e) $\mathbf{E}_{\mu_u}[\sum_{i=1}^d \eta(e_i)U_i'(\eta(e_i)] = u \cdot \nabla\sigma_T(u) + 1$*, for all* $i = 1, \dots d$*;*

(f) $|\nabla\sigma(u) - \nabla\sigma(v)| \le C|u - v|$ *for some* $C > 0$.

PROOF.

(a) Noting from Lemma 27 that $\tilde{\mu}_{N,u}$ converges weakly to $\tilde{\mu}_u$ as $N \to \infty$ and using the tightness of the family $\{\mu_{N,u}\}_N$, the proof now follows the same steps as the proof of Theorem 3.4.(0) in [14].

(b) Since by (a)

$$\lim_{N\to\infty} \sigma^{\beta}_{\mathbb{T}_N}(u) = \sigma_T(u), \tag{168}$$

every subsequence of $\sigma^{\beta}_{\mathbb{T}_N}(u)$ will converge to $\sigma_T(u)$, in particular for $N = 2k$. The statement of the theorem follows immediately by using now Lemma 25 and Lemma 26 applied to the subsequence $\left(\sigma^{\beta}_{\mathbb{T}_N}(u)\right)_N$, with $N = 2k$.

(c) , (d) and (e) follow just as in [14], so their proofs will be omitted.

(f) Let $N = 2k$. Define

$$\mu^{\mathbb{E}}_{N,u}(\,\mathrm{d}\phi^{\mathbb{E}}) = \frac{1}{Z^{\mathbb{E}}_{N,u}} \exp\left[-\sum_{x\in\mathbb{O}_N} F_x(\theta(x), u)\right] \mathrm{d}\phi^{\mathbb{E}_N}, \tag{169}$$

where $\mathrm{d}\phi^{\mathbb{E}_n} = \prod_{x\in\mathbb{E}^d_N\setminus\{0\}} \phi(x)$, $Z^{\mathbb{E}}_{N,u}$ is the normalizing constant and F_x and $\theta(x)$ are defined as in (160) and (46), respectively. Due to the fact that the random walk representation holds on the set of the evens and to Theorem 21, one can show as in [14] that for $N = 2k$, $\mu^{\mathbb{E}}_{N,u}$ converges weakly to $\mu^{\mathbb{E}}_u \in \left(\text{ext } S^{\mathbb{E}}\right)_{\tilde{u}}$, where the same notations as in the uniqueness of ergodic component section apply. Note now from (160) that

$$\mathbf{E}_{\mu_{N,u}}\left[U_i'(\nabla_i\phi(x)\right] = \mathbf{E}_{\mu^{\mathbb{E}}_{N,u}}\left[D^iF_x(\theta(x), u)\right], \tag{170}$$

where $x \in \mathbb{O}^d_N$. Using now (d), the weak convergence of $\mu_{N,u}$ to μ_u and the weak convergence of $\mu^{\mathbb{E}}_{N,u}$ to $\mu^{\mathbb{E}}_u$, we get

$$\mathbf{E}_{\mu^{\mathbb{E}}_u}\left[D^iF_x(\theta(x), u)\right] = D^i\sigma_T(u). \tag{171}$$

Uisng the random walk representation and Taylor expansion, we have

$$\left|D^iF_x(\theta(x)) - D^iF_x(\bar{\theta}(x))\right| \le c_+ \sum_{k\in I} \left|\phi(x + e_k) - \bar{\phi}(x + e_k)\right|. \tag{172}$$

The bound in (f) is now a simple consequence of (171), (172) and Lemma 20.

□

7.2 Equality between the surface tension on the torus and the surface tension on the box

Let $\Lambda_N = [-N,N]^d \cap \mathbb{Z}^d$ be a cube of side length $2N+1$ in $\mathbb{Z}^d$ with boundary $\partial\Lambda_N = \{x = (x_1,\dots,x_n) || x_\alpha = N$ for at least one $\alpha, \alpha = 1,\dots d\}$. We enforce a tilt $u = (u_1,\dots u_d) \in \mathbb{R}^d$ by setting $\phi(x) = x \cdot u$ for $x \in \partial\Lambda_N$. Then, we define the surface tension on Λ_N as

$$\sigma^\beta_{\Lambda_N}(u) = -\frac{1}{|\Lambda_N|^d}\log\frac{Z^\beta_{\Lambda_N}(u)}{Z^\beta_{\Lambda_N}(0)}, \quad \text{where} \tag{173}$$

$$Z^\beta_{\Lambda_N}(u) = \int_{\mathbb{R}^{\Lambda^d_N}} \exp(-\beta H_{\Lambda_N}(\phi)) \prod_{x\in\Lambda_N} \mathrm{d}\phi(x) \prod_{y\in\partial\Lambda_N} \delta(\phi(y) - u\cdot y), \tag{174}$$

and where H_{Λ_N} is the Hamiltonian given by (14).

Noting that $C_1 - C_0 \le U_i'' \le C_2$, the proof for the existence of the surface tension on the box follows the same steps as in [14]. We thus have

Theorem 29

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \sigma_\Lambda(u) = \sigma_B(u) \tag{175}$$

exists and is independent of the chosen sequence Λ.

To prove the equality between the surface tension on the torus and the surface tension on the box, we follow the same steps as Lemma II.2 in [14]. Note that we will also use the same reasoning as from our covariance section in order to estimate the variance needed for the proof.

Lemma 30 *Let U_i, $i \in I$, be symmetric potentials. Then we have*

$$\sigma_B(u) = \sigma_T(u), \tag{176}$$

where $\sigma_T(u)$ is the one from Theorem 28.

8 Appendix

8.1 Counter-Example to Random Walk Representation

We will next outline a method to check if the RW representation holds for large β; we will also provide a counter-example when the RW doesnt hold for very large β.
Let $\mu(\mathrm{d}x) = \frac{1}{\beta}\exp(-\beta H(s))\,\mathrm{d}s$. We assume that there exists a unique minimum $\bar{s}$ such that

$$H(\bar{s}) = \min_s H(s), \quad \text{where} \quad H''(\bar{s}) > 0. \tag{177}$$

Let $U_1, U_2 \in C^2(\mathbb{R})$, with $U_i'(\bar{s}) \neq 0$, for $i = 1,2$. Note that

$$\mathrm{cov}_{\mu_\beta}(U_1,U_2) = \frac{1}{2}\iint F(s_1,s_2)\mu_\beta(\mathrm{d}s_1)\mu_\beta(\mathrm{d}s_2), \tag{178}$$

with $F(s_1,s_2) = [U_1(s_1) - U_1(s_2)]\,[U_2(s_1) - U_2(s_2)]$. Note now that

$$\begin{aligned} F(s_1,s_2) \;&=\; F(\bar{s},\bar{s}) + \partial_{s_1}F(\bar{s},\bar{s})(s_1-\bar{s}) + \partial_{s_2}F(\bar{s},\bar{s})(s_2-\bar{s}) + \frac{1}{2}\big(\partial^2_{s_1}F(\bar{s},\bar{s})(s_1-\bar{s})^2 \\ &\quad +2\partial_{s_1}\partial_{s_2}F(\bar{s},\bar{s})(s_1-\bar{s})(s_2-\bar{s}) + \partial^2_{s_2}F(\bar{s},\bar{s})(s_2-\bar{s})^2\big) + o(1), \end{aligned} \tag{179}$$

where

$$F(\bar{s},\bar{s})=0,\ \partial^2_{s_1}F(\bar{s},\bar{s})=\partial^2_{s_2}F(\bar{s},\bar{s})=2U_1'(\bar{s})U_2'(\bar{s})\text{ and }\partial_{s_1}\partial_{s_2}F(\bar{s},\bar{s})=-2U_1'(\bar{s})U_2'(\bar{s}).\tag{180}$$

Also $H(s)=H(\bar{s})+\frac{1}{2}H''(\bar{s})(s-\bar{s})^2+o(1)$. For large β, we can ignore the rest $o(1)$ and using $\int_{\mathbb{R}}(s-\bar{s})\mu_\beta(\,\mathrm{d}s)=o(1)$, we get

$$\iint(s_1-\bar{s})(s_2-\bar{s})\mu_\beta(\,\mathrm{d}s_1)\mu_\beta(\,\mathrm{d}s_2)=o(1).\tag{181}$$

Then

$$\beta\int(s_1-\bar{s})^2\mu_\beta(\,\mathrm{d}s_1)=\frac{\sqrt{2\pi}}{H''(\bar{s})}+o(1)\tag{182}$$

and we get for large β

$$\beta\mathrm{cov}_{\mu_\beta}(U_1,U_2)=\frac{\sqrt{2\pi}}{H''(\bar{s})}2U_1'(\bar{s})U_2'(\bar{s})+o(1).\tag{183}$$

Take $U(s)$ as in Example (c) from the Random Walk Representation section, $H(s)=2U(s-1)+2U(s+3)$ and $a=0.72$. Take $d=2$, β large and the vector of the even vertices $\theta=(1,1,-3,-3)$. We will show using (183) that for large β

$$\beta\mathrm{cov}(U'(s-1),U'(s+3))\le C\frac{\sqrt{2\pi}U''(\bar{s}-1)U''(\bar{s}+3)}{H''(\bar{s})}<0,\tag{184}$$

where $\bar{s}$ is the minimum of $H(s)$. For this, it is enough to show that the minimum will fall in the region where $U''(s-1)$ is strictly convex and where $U''(s+3)$ is non-convex. Note first that for $a=0.8$, we have

$$H''(s)=2-\frac{1}{a^2}-\frac{1}{(a+4)^2}>0.\tag{185}$$

We have 3 cases to consider

(1) $s\ge 1$. Then $H'(s)>0$, so the minimum can not be found on this interval.

(2) $-3\le s\le 1$. Then $H''(s)$ is increasing on $[-3,-1]$ and decreasing on $[-1,1]$. In view of (185), $H''(s)>0$ on $[-3,1]$.

(3) $s<-3$. Then $H''(s)$ decreasing on $(-\infty,-3)$ and $H''(s)>0$ because of (185).

It follows now from all the 3 cases that $H''(s)>0$ on $\mathbb{R}$. Therefore H has a unique minimum $\bar{s}$. As $H'(-3)<0$ and $H'(-2.68)>0$, we have that $\bar{s}\in(-3,-2.68)$, on which interval $H''(s+3)<0$ and $H''(s-1)>0$.

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