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ANALYSIS OF A STREAMLINE DIFFUSION FINITE ELEMENT METHOD FOR THE STOKES AND NAVIER–STOKES EQUATIONS*

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Abstract. For the Stokes equations with convection and the incompressible Navier–Stokes equations, the authors analyze a streamline diffusion finite element method that is capable of balancing both the convection and the pressure, thus allowing the use of arbitrary pairs of velocity-pressure spaces. For the linear problem, the authors obtain for all mesh–Peclet numbers optimal error estimates in natural norms including, in particular, the L^2-norm of the pressure. The same holds for the nonlinear problem, which close to a regular branch of solutions, i.e., the linearized operator, is an isomorphism of the norm of the inverse of which still depends on the Reynolds number. Consequently, the dependence of the error constants on the Reynolds number is not completely resolved in this case.

Key words. streamline diffusion method, a priori error estimates, Stokes equations with convection, Navier–Stokes equations

AMS subject classifications. 65N30, 76D05

1. Introduction.

The streamline diffusion finite element method (SDFEM) for solving problems in computational fluid dynamics is one of the most successfully used methods. Hughes and Brooks [13] first proposed the SDFEM for solving the Navier–Stokes equations but without giving an error analysis. Johnson and Saranen [16] have analyzed this method in the context of the streamfunction-vorticity formulation in the two-dimensional case. They used solenoidal velocity approximations and gave an error analysis for the nonstationary case. Hansbo and Szepessy [12] extended the method by adding a least squares term for the incompressibility condition and gave an analysis for the weak velocity-pressure formulation. For the stationary Navier–Stokes equations, some modifications of the SDFEM have been analyzed independently in [17], [19], and [20]. All the results for the stationary problem, however, are restricted to the uniqueness case, i.e., small Reynolds numbers. An analysis of the general stationary nonlinear problem including the treatment of regular branches of solutions is missing in the literature. The results of [7] and [6] for the nonstationary problem cannot be used for the stationary problem since they heavily rely on an L^2-control of the velocity, which can only be obtained for the nonstationary case thanks to Gronwall's lemma. Moreover they only treat the linearized problem, do not obtain error estimates for the pressure with respect to the L^2-norm, and use regularity assumptions that are not realistic for the stationary Navier–Stokes equations.

The main result of this analysis is that in the velocity-pressure formulation for the stationary case the SDFEM stabilizes two different phenomena of instabilities, namely the instability based on the dominance of convective terms *and* the instability caused by using bad combinations of velocity and pressure approximations, which do not fulfill the Babuška–Brezzi condition. In contrast to scalar convection-diffusion problems, the size of the streamline diffusion is independent of the ratio of meshsize and diffusivity unless equal order interpolation is used. This is due to the additional pressure terms or, equivalently, the incompressibility condition. For the linear problem, we obtain optimal error estimates for the velocity and the pressure with respect to natural norms using arbitrary velocity and pressure approximations.

For the nonlinear problem, we have to assume that we are close to a regular branch of solutions of the continuous problem and that the meshsize is sufficiently small. Moreover, we

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